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Inventiones mathematicae

Multipliers for the convolution algebra of left and right K-finite compactly supported smooth functions on a semi-simple Lie group

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0. Introduction

Let G be a real semi-simple Lie group, connected, with finite center, and K a maximal compact subgroup of G. In this paper, we study multipliers of the convolution algebra $\mathscr{D}(G)_{(K)}$ of smooth, compactly supported functions on G, which are left and right K-finite. By a multiplier we mean a linear endomorphism commuting with the left and right actions of the algebra. Essentially we construct a subalgebra of the algebra of multipliers of $\mathscr{D}(G)_{(K)}$ (Th. 3). This result was originally proved by Arthur (cf. [1], Theorem III.4.2), but his proof rests on a Paley-Wiener theorem for real semi-simple Lie groups, the proof of which is very difficult (cf. [1], Theorem III.4.1). Our construction of multipliers for $\mathscr{D}(G)_{(K)}$ is simple and elementary. Let us explain our argument in more detail.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ, $\mathfrak{g}_{\mathbb{C}}$ the complexified Lie algebra of $\mathfrak{g}$. We set $\mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{p}$, $\mathfrak{q}=i\mathfrak{u}$. Then $\mathfrak{g}_{\mathbb{C}}=\mathfrak{u}\oplus\mathfrak{q}$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ (viewed as a real Lie algebra).

Let $\mathfrak{h}_\phi$ be the Lie algebra of a maximally split θ-stable Cartan subgroup of G. Then $\mathfrak{h}_\phi=\mathfrak{t}_\phi\oplus\mathfrak{a}_\phi$, where $\mathfrak{t}_\phi=\mathfrak{h}_\phi\cap\mathfrak{k}$, $\mathfrak{a}_\phi=\mathfrak{h}_\phi\cap\mathfrak{p}$. Moreover $\mathfrak{a}=i\mathfrak{t}_\phi\oplus\mathfrak{a}_\phi$ is a Cartan subspace of $\mathfrak{q}$, and $(\mathfrak{h}_\phi)_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We denote by $W_{\mathbb{C}}$ the Weyl group of the pair $(\mathfrak{g}_{\mathbb{C}}, (\mathfrak{h}_\phi)_{\mathbb{C}})$ which acts on $\mathfrak{a}$.

Now we denote by $G_{\mathbb{C}}$ the connected, simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ and by U the analytic subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{u}$.

Let $\mathscr{E}(G_{\mathbb{C}}/U)$ (resp. $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$) be the space of smooth functions on $G_{\mathbb{C}}/U$ (resp. the space of compactly supported distributions on $G_{\mathbb{C}}$, biinvariant under U).

From the spherical Paley-Wiener theorem (cf. [4]), for each τ in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ (compactly supported, $W_{\mathbb{C}}$-invariant distribution on $\mathfrak{a}$) there exists a unique $\tilde{\tau}$ in $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$ the spherical Fourier transform of which is equal to the usual Fourier transform of τ, $\hat{\tau}$. The right convolution by $\tilde{\tau}$ determines a continuous endomorphism T_τ of $\mathscr{E}(G_{\mathbb{C}}/U)$ which commutes with the left translations by elements of $G_{\mathbb{C}}$. We show in Theorem 1 that every such map is a right convolution by an element of $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$, i.e. is one of the T_τ. Now, from the Flensted-Jensen correspondence between certain functions on dual symmetric spaces (cf. [2]), we know that there exists an injection η of $\mathscr{D}(G)_{(K)}$ in $\mathscr{E}(G_{\mathbb{C}}/U)$, with remarkable properties. It is easy to show that each T_τ leaves stable the image of η, hence $T_\tau^\eta=\eta^{-1}\circ T_\tau\circ\eta$ is a well defined endomorphism of $\mathscr{D}(G)_{(K)}$. From the properties of η, it is easy to see that T_τ^η commutes with the left and right actions of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ on $\mathscr{D}(G)_{(K)}$.

We show in Theorem 2 that this suffices to ensure that T_τ^η is a multiplier for $\mathscr{D}(G)_{(K)}$. Finally, we have defined a map $(\tau\to T_\tau^\eta)$ from $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ into the algebra of multipliers of the algebra $\mathscr{D}(G)_{(K)}$. Now we identify $Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$, with $S(\mathfrak{a})^{W_{\mathbb{C}}}$. Then we show in Theorem 3 that, for any element φ of $\mathscr{D}(G)_{(K)}$ and any principal series representation (π, H_π) of G with infinitesimal character $\chi_\nu(\nu\in\mathfrak{a}_{\mathbb{C}}^*)$, $\pi(T_\tau^\eta\varphi)=\hat{\tau}(\nu)\pi(\varphi)$. This concludes the comparison with the multipliers constructed by Arthur in [1], Th. III.4.2. Notice that this theorem is an analogue of a Bernstein's result for p-adic groups.

In paragraph 1, we introduce the general conventions.

In paragraph 2, we introduce the Flensted-Jensen correspondence and establish some of its properties needed in the sequel.

In paragraph 3, we study the $G_{\mathbb{C}}$-endomorphisms of $\mathscr{E}(G_{\mathbb{C}}/U)$ (Th. 1).

In paragraph 4, we construct certain multipliers for the convolution algebra $\mathscr{D}(G)_{(K)}$ and establish some of their properties.

1. Preliminaries and notations

1.1. If E is a vector space over $\mathbb{R}$ or $\mathbb{C}$, we denote by E^* its algebraic dual. If E is a real vector space we denote by $E_{\mathbb{C}}$ its complexification and by $S(E)$ the symmetric algebra of $E_{\mathbb{C}}$ which will be identified with the algebra of polynomial functions on $E_{\mathbb{C}}^*$. Sometimes, in this paper, we will complexify vector spaces which are already defined over $\mathbb{C}$, viewing them as real vector spaces. In particular, if E is a real vector space with an automorphism σ, denoting by $\bar{X}$ the conjugate of X in $E_{\mathbb{C}}$ with respect to the real form E of $E_{\mathbb{C}}$ and also by σ the complexification of σ, the complexification of the $\mathbb{R}$-linear map from $E_{\mathbb{C}}$ into $E_{\mathbb{C}}\times E_{\mathbb{C}}$ defined by $X\to(X,\overline{\sigma(X)})$ extends to an isomorphism of complex vector spaces from $(E_{\mathbb{C}})_{\mathbb{C}}$ into $E_{\mathbb{C}}\times E_{\mathbb{C}}$ denoted by $\tilde{\sigma}$.

1.2. If $\mathfrak{l}$ is a real Lie algebra, we denote by $U(\mathfrak{l})$ the enveloping algebra of the complex Lie algebra $\mathfrak{l}_{\mathbb{C}}$, and by $Z(\mathfrak{l})$ the center of $U(\mathfrak{l})$. If $\mathfrak{l}$ is already a complex Lie algebra, we regard it as a real one and use the same notation. In particular, if $\mathfrak{l}$ is a real Lie algebra, with an automorphism σ, the $\mathbb{C}$-linear isomorphism $\tilde{\sigma}$ from $\mathfrak{l}_{\mathbb{C}}$ into $\mathfrak{l}_{\mathbb{C}}\times\mathfrak{l}_{\mathbb{C}}$ is an isomorphism of Lie algebras which gives rise to an isomorphism of algebras denoted also by $\tilde{\sigma}$ from $U(\mathfrak{l}_{\mathbb{C}})$ onto $U(\mathfrak{l})\otimes U(\mathfrak{l})$.

1.3. If L is a group, we denote by $Z(L)$ its center. If $\mathscr{F}$ is an L-module, we will denote by $\mathscr{F}_{(L)}$ the space of L-finite vectors in L. If $\mathscr{M}$ is an $L\times L$-module, by abuse of notations we will often denote by $\mathscr{M}_{(L)}$ the space of $L\times L$-finite vectors in $\mathscr{M}$, instead of $\mathscr{M}_{(L\times L)}$.

Now suppose that $\mathscr{F}$ is an L-module and δ a finite dimensional simple L-module, then we denote by $\mathscr{F}^\delta$ the subspace of elements in $\mathscr{F}_{(L)}$ which generate an L-module isomorphic to a multiple of δ. $\mathscr{F}^\delta$ is the isotypic component of $\mathscr{F}$ of type δ. If $\mathscr{M}$ is an $L\times L$-module and δ, γ are finite dimensional simple representations of L, we denote by $\mathscr{M}^{\delta\gamma}$ the isotypic component of $\mathscr{M}$ of type $\delta\otimes\gamma$.

1.4. We will say that a linear map from a topological vector space into another is a topological embedding if and only if it is injective, has a closed image and is bicontinuous on its image. From the closed graph theorem for Fréchet spaces, an injective continuous linear map between Fréchet spaces is a topological embedding iff it has a closed image.

1.5. If X is a differentiable manifold, we denote by $\mathscr{D}(X)$ (resp. $\mathscr{E}(X)$, resp. $\mathscr{E}'(X)$) the space of compactly supported smooth functions (resp. smooth functions, resp. compactly supported distributions) on X endowed with its usual (strong) topology.

2. On the Flensted-Jensen correspondence between functions on dual semi-simple symmetric spaces

2.1. Let G be a real semi-simple Lie group, connected, with finite center, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ. Let $G_{\mathbb{C}}$ be the simply connected, connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let K be the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Let K_0, $K_{\mathbb{C}}$, U be the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{k}$, $\mathfrak{k}_{\mathbb{C}}$, $\mathfrak{u}=\mathfrak{k}\oplus i\mathfrak{p}$. Notice that U is a maximal compact subgroup of $G_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}=\mathfrak{u}\oplus\mathfrak{q}$ is a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$ (where $\mathfrak{q}=i\mathfrak{u}$). Let $\mathfrak{a}_\phi$ be a Cartan subspace in $\mathfrak{p}$, M_ϕ the centralizer of $\mathfrak{a}_\phi$ in K and $\mathfrak{t}_\phi$ a Cartan subalgebra of the Lie algebra $\mathfrak{m}_\phi$ of M_ϕ. Then $\mathfrak{a}=i\mathfrak{t}_\phi\oplus\mathfrak{a}_\phi$ is a Cartan subspace in $\mathfrak{q}$. Let $\mathfrak{t}=i\mathfrak{a}$, $\mathfrak{h}=\mathfrak{t}\oplus\mathfrak{a}$, $\mathfrak{h}_\phi=\mathfrak{t}_\phi\oplus\mathfrak{a}_\phi$. Then $\mathfrak{h}$ (resp. $\mathfrak{h}_\phi$) is a Cartan subalgebra of the complex (resp. real) Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{g}$). Let $W_{\mathbb{C}}$ be the Weyl group of the pair $(\mathfrak{g}_{\mathbb{C}},\mathfrak{h})$. It is also the Weyl group of the pair $(\mathfrak{g},\mathfrak{h}_\phi)$ ("complex" Weyl group of $\mathfrak{g}$). It acts on $\mathfrak{h}$, $\mathfrak{a}$. Notice that $W_{\mathbb{C}}$ is the ("small") Weyl group of $G_{\mathbb{C}}$. Let A_ϕ (resp. A, H, T) be the analytic subgroup of G (resp. $G_{\mathbb{C}}$) with Lie algebra $\mathfrak{a}_\phi$ (resp. $\mathfrak{a}$, $\mathfrak{h}$, $\mathfrak{t}$) and $P_\phi=M_\phi A_\phi N_\phi$ (resp. $B=HN$) a minimal parabolic subgroup of G with nilradical N_ϕ (resp. a Borel subgroup of $G_{\mathbb{C}}$ with nilradical N) such that the Lie algebra $\mathfrak{n}_\phi$ of N_ϕ is contained in the Lie algebra $\mathfrak{n}$ of N. We will denote by $\|\ \|$ the norm on $\mathfrak{q}$ derived from the restriction of the Killing form of $\mathfrak{g}_{\mathbb{C}}$ (regarded as a real Lie algebra) to $\mathfrak{q}$. This induces a norm on $\mathfrak{p}$ which is the norm derived from the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{p}$ (up to a multiplicative constant).

For $r\geqq 0$, we set:

$$B_r=\{X\in\mathfrak{a}\mid \|X\|\leqq r\},$$

$$C_r=\{X\in\mathfrak{q}\mid \|X\|\leqq r\},$$

$$D_r=\{X\in\mathfrak{p}\mid \|X\|\leqq r\}.$$

2.2. From 1.2, we have a canonical isomorphism $\tilde{\theta}$ from $U(\mathfrak{g}_{\mathbb{C}})$ onto $U(\mathfrak{g})\otimes U(\mathfrak{g})$ derived from the Cartan involution θ of $\mathfrak{g}$. The inverse map will be denoted by η in the sequel. Denote by $\mathbf{D}(G_{\mathbb{C}}/U)$ the algebra of $G_{\mathbb{C}}$-invariant differential operators on $G_{\mathbb{C}}/U$. We have an isomorphism, denoted also by η, from $Z(\mathfrak{g})$ onto $\mathbf{D}(G_{\mathbb{C}}/U)$. Using the well known Harish-Chandra homomorphism, we identify $\mathbf{D}(G_{\mathbb{C}}/U)$ with $S(\mathfrak{a})^{W_{\mathbb{C}}}$. For all of this, see [2], §§ 2, 7.

For $\nu\in\mathfrak{a}_{\mathbb{C}}^*$, we will denote by χ_ν the corresponding character of $S(\mathfrak{a})^{W_{\mathbb{C}}}$, $\mathbf{D}(G_{\mathbb{C}}/U)$ and $Z(\mathfrak{g})$.

2.3. Let $\tilde{G}$ be the universal covering group of G, and $\tilde{K}$ the analytic subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{k}$. Then $\tilde{K}$ is the iniversal covering group of K and K_0. We denote by π_0 the canonical projection of $\tilde{K}$ on K_0 and by Z_0 the kernel of π_0. Then Z_0 is central in $\tilde{G}$. We set

$$Z_1=\{(z,z)\mid z\in Z_0\} \quad \text{and} \quad K_1^{\sim}=\tilde{K}\times\tilde{K}/Z_1.$$

From 1.2, we have a canonical isomorphism between $(\mathfrak{k}\times\mathfrak{k})_{\mathbb{C}}$ and $(\mathfrak{k}_{\mathbb{C}})_{\mathbb{C}}$ (associated to the identity automorphism of $\mathfrak{k}$). So we have a natural one-one correspondence between finite dimensional representations of $\mathfrak{k}\times\mathfrak{k}$ and $\mathfrak{k}_{\mathbb{C}}$. On the group level, this gives a natural one-one correspondence between finite dimensional representations of $K_1^{\sim}$ and $K_{\mathbb{C}}$ (cf. [2], proof of Theorem 2.3).

On the level of functions, this gives a canonical linear bijection η_0 between the space $\mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}$ of left $K_1^{\sim}$-finite smooth functions on $K_1^{\sim}$ and the space $\mathscr{E}(K_{\mathbb{C}})_{(K_{\mathbb{C}})}$ of left $K_{\mathbb{C}}$-finite smooth functions on $K_{\mathbb{C}}$ (cf. [2], proof of Th. 2.3 and Th. 7.1).

One can define η_0 in the following way: Let φ be in $\mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}$ and V be the finite dimensional $K_1^{\sim}$-submodule of $\mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}$ generated by φ. Let δ_e be the element of the dual V^* of V defined by $\langle\delta_e,\psi\rangle=\psi(e)$ for all ψ in V. By what has been said previously, we have also a canonical action of $K_{\mathbb{C}}$ on V. Then we define $(\eta_0(\varphi))(k)=\langle\delta_e,k^{-1}\varphi\rangle$ for all k in $K_{\mathbb{C}}$. Clearly $\eta_0(\varphi)$ is in $\mathscr{E}(K_{\mathbb{C}})_{(K_{\mathbb{C}})}$. It is easy to deduce from this definition that, if V_1 is a finite dimensional $K_1^{\sim}$-submodule of $\mathscr{E}(K_1^{\sim})$ and δ_e is again the restriction to V_1 of the Dirac measure at the origin, for all φ in V_1 and k in $K_{\mathbb{C}}$ we have $(\eta_0(\varphi))(k)=\langle\delta_e,k^{-1}\varphi\rangle$ where V_1 has been endowed with its natural structure of $K_{\mathbb{C}}$-module.

2.4. Let us define:

$$i\colon \mathscr{E}(\tilde{G})\to\mathscr{E}(\tilde{K}\times\tilde{K})\hat{\otimes}\mathscr{E}(\mathfrak{p}) \quad (=\mathscr{E}(\tilde{K}\times\tilde{K}\times\mathfrak{p})),$$

by:

$$\forall\varphi\in\mathscr{E}(\tilde{G}),\ \forall k_1,k_2\in\tilde{K},\ \forall X\in\mathfrak{p},\quad (i\varphi)(k_1,k_2,X)=\varphi(k_1\cdot\exp X\cdot k_2^{-1}).$$

Then i is obviously continuous. From the Cartan decomposition, $\tilde{G}=\tilde{K}\exp\mathfrak{p}$, it follows that i is injective. It is an easy consequence of [2] (§ 7 and Lemma 2.1) that i has a closed image. Therefore, from 1.4, i is a topological embedding.

Moreover, Z_0 being central in $\tilde{G}$, for each function φ on $\tilde{G}$, we have $f(zgz^{-1})=f(g)$ for all g in G and z in Z_0. This implies that the image of i is contained in $\mathscr{E}(K_1^{\sim})\hat{\otimes}\mathscr{E}(\mathfrak{p})$, when we regard $\mathscr{E}(K_1^{\sim})$ as a (closed) subspace of $\mathscr{E}(\tilde{K}\times\tilde{K})$. Therefore i is in fact a topological embedding:

$$i\colon\ \mathscr{E}(\tilde{G})\to\mathscr{E}(K_1^{\sim})\hat{\otimes}\mathscr{E}(\mathfrak{p}).$$

Similarly, let us define:

$$j\colon\ \mathscr{E}(G_{\mathbb{C}}/U)\to\mathscr{E}(K_{\mathbb{C}})\hat{\otimes}\mathscr{E}(\mathfrak{p})\qquad(=\mathscr{E}(K_{\mathbb{C}}\times\mathfrak{p}))$$

by:

$$\forall\psi\in\mathscr{E}(G_{\mathbb{C}}/U),\ \forall k\in K_{\mathbb{C}},\ \forall X\in\mathfrak{p},\qquad (j(\psi))(k,X)=\psi(k(\exp X)U).$$

The map J is obviously continuous. It follows from [2], Lemma 2.1, that j has a closed image. As $G_{\mathbb{C}}=K_{\mathbb{C}}(\exp\mathfrak{p})U$, J is injective, hence it is a topological embedding (cf. 1.4).

From the definitions we deduce easily:

$$i(\mathscr{E}(\tilde{G})_{(\tilde{K})})\subset\mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}\otimes\mathscr{E}(\mathfrak{p}),$$

$$j(\mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})})\subset\mathscr{E}(K_{\mathbb{C}})_{(K_{\mathbb{C}})}\otimes\mathscr{E}(\mathfrak{p}).$$

Here $\mathscr{E}(\tilde{G})_{(\tilde{K})}$ is the subspace of left and right $\tilde{K}$-finite elements in $\mathscr{E}(\tilde{G})$.

2.5. It follows from [2], Theorem 2.3, Theorem 7.1, that there exists a unique linear map $\tilde{\eta}$ from $\mathscr{E}(\tilde{G})_{(\tilde{K})}$ into $\mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$ such that the following diagram is commutative

$$\begin{array}{ccc}\mathscr{E}(\tilde{G})_{(\tilde{K})} & \xrightarrow{\ i\ } & \mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}\otimes\mathscr{E}(\mathfrak{p})\\ \Big\downarrow{\scriptstyle\tilde{\eta}} & & \Big\downarrow{\scriptstyle\eta_0\otimes\mathrm{Id}}\\ \mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})} & \xrightarrow{\ j\ } & \mathscr{E}(K_{\mathbb{C}})_{(K_{\mathbb{C}})}\otimes\mathscr{E}(\mathfrak{p}).\end{array}$$

Moreover, $\tilde{\eta}$ is a linear isomorphism, and if we endow $\mathscr{E}(\tilde{G})_{(\tilde{K})}$ (resp. $\mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$) with its natural structure of $U(\mathfrak{g})\otimes U(\mathfrak{g})$ (resp. $U(\mathfrak{g}_{\mathbb{C}})$)-module derived from the left and right regular action of $\tilde{G}$ (resp. left regular action of $G_{\mathbb{C}}$) we have:

$$\forall D\in Z(\mathfrak{g})\cup U(\mathfrak{g})\otimes U(\mathfrak{g}),\ \forall\varphi\in\mathscr{E}(\tilde{G})_{(\tilde{K})},\qquad (D\varphi)^{\tilde{\eta}}=D^{\eta}\varphi^{\tilde{\eta}}.$$

2.6. Now we embed $\mathscr{E}(G)_{(K)}$ in $\mathscr{E}(\tilde{G})_{(\tilde{K})}$. This subspace of $\mathscr{E}(\tilde{G})_{(\tilde{K})}$ is the space of smooth functions on $\tilde{G}$, which generate a finite dimensional $\tilde{K}\times\tilde{K}$-submodule of $\mathscr{E}(\tilde{G})$ which factors through the quotient to $K\times K$. Now let δ, γ be in $\hat{K}$ (the set of equivalence classes of finite dimensional irreducible representations of K) such that $\mathscr{E}(G)^{\delta\gamma}$ is non zero. Then the lift of $\delta\otimes\gamma$ to $\tilde{K}\times\tilde{K}$ factors through the quotient by Z_1 in a representation of $K_1^{\sim}$ (see 2.4). Let us denote by (δ,γ) the corresponding representation of $K_{\mathbb{C}}$ (see 2.3), which is simple, as is $\delta\otimes\gamma$. We denote by $\breve{K}_{\mathbb{C}}$ the set of classes of equivalence of irreducible representations of $K_{\mathbb{C}}$ obtained in this way: a generic element in $\breve{K}_{\mathbb{C}}$ will be denoted by (δ,γ) (with δ, γ in $\hat{K}$).

Proposition 1. *Denote by η the restriction of $\tilde{\eta}$ to $\mathscr{E}(G)_{(K)}$ embedded in $\mathscr{E}(\tilde{G})_{(\tilde{K})}$ and $\mathscr{E}(G)^{\eta}_{(K)}$ the image of η in $\mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$. Then:*

(i) *The linear map η is a bijection between $\mathscr{E}(G)_{(K)}$ and $\mathscr{E}(G)^{\eta}_{(K)}$, and is the unique linear map making commutative the following diagram:*

$$\text{(D)}\qquad \begin{array}{ccc} \mathscr{E}(G)_{(K)} & \xrightarrow{\;i\;} & \mathscr{E}(K_1^{\sim})_{(K_1^{\sim})}\otimes\mathscr{E}(\mathfrak{p}) \\ \big\downarrow{\scriptstyle\tilde{\eta}} & & \big\downarrow{\scriptstyle\eta_0\otimes \mathrm{Id}} \\ \mathscr{E}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})} & \xrightarrow{\;j\;} & \mathscr{E}(K_{\mathbb{C}})_{(K_{\mathbb{C}})}\otimes\mathscr{E}(\mathfrak{p}) \end{array}$$

(*see* 2.3 *for the definition of* η_0, *and* 2.4 *for the definitions of i and J*).

(ii) $\forall D\in(U(\mathfrak{g})\otimes U(\mathfrak{g}))\cup Z(\mathfrak{g})$, $\forall\varphi\in\mathscr{E}(G)_{(K)}$, $(D\varphi)^{\eta}=D^{\eta}\varphi^{\eta}$.

(iii) *For each δ, γ in $\hat{K}$ such that $\mathscr{E}(G)^{\delta\gamma}$ is non zero, $\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$ is closed in $\mathscr{E}(G_{\mathbb{C}}/U)$ and η is a topological isomorphism between $\mathscr{E}(G)^{\delta\gamma}$ and $\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$.*

(iv) $\mathscr{E}(G)^{\eta}_{(K)}=\bigoplus_{(\delta,\gamma)\in\widehat{K_{\mathbb{C}}}}\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$.

Proof. (i) and (ii) follow from the properties of $\tilde{\eta}$ quoted in 2.5. Now, retain the notations of (iii). We have seen that $\delta\otimes\gamma$, when lifted to $\tilde{K}\times\tilde{K}$ factors through the quotient by Z_1 in a representation of $K_1^{\sim}$, also denoted by $\delta\otimes\gamma$.

It is clear that $\mathscr{E}(K_1^{\sim})^{\delta\otimes\gamma}$ is isomorphic to $\mathscr{E}(K)^{\delta}\otimes\mathscr{E}(K)^{\gamma}$, the spaces $\mathscr{E}(K)^{\delta}$ and $\mathscr{E}(K)^{\gamma}$ being finite dimensional by the Peter-Weyl theorem for compact groups. On the other hand, we deduce from the definition of η_0 (cf. 2.3):

$$\mathscr{E}(K_{\mathbb{C}})^{(\delta,\gamma)}=\eta_0(\mathscr{E}(K_1^{\sim})^{\delta\otimes\gamma}).$$

Hence $\mathscr{E}(K_{\mathbb{C}})^{(\delta,\gamma)}$ is finite dimensional and this implies that $\mathscr{E}(K_{\mathbb{C}})^{(\delta,\gamma)}\otimes\mathscr{E}(\mathfrak{p})$ is closed in $\mathscr{E}(K_{\mathbb{C}}\times\mathfrak{p})$. As j is a topological embedding of $\mathscr{E}(G_{\mathbb{C}}/U)$ in $\mathscr{E}(K_{\mathbb{C}}\times\mathfrak{p})$ (cf. 2.4), we deduce, from the obvious equality:

$$j(\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)})=j(\mathscr{E}(G_{\mathbb{C}}/U))\cap(\mathscr{E}(K_{\mathbb{C}})^{(\delta,\gamma)}\otimes\mathscr{E}(\mathfrak{p})),$$

that $\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$ is closed in $\mathscr{E}(G_{\mathbb{C}}/U)$. Then (iii) and (iv) follow easily from the commutativity of (D) and from the properties of i and j.

2.7. Denote by $\mathscr{D}(G)$ (resp. $\mathscr{D}_{K_{\mathbb{C}}}(G_{\mathbb{C}}/U)$ the space of compactly supported (resp. compactly supported modulo $K_{\mathbb{C}}$) smooth functions on G (resp. $G_{\mathbb{C}}/U$) endowed with its natural inductive topology. As every compact subset of G (resp. compact subset modulo $K_{\mathbb{C}}$ of $G_{\mathbb{C}}/U$) is contained in $K\exp D_r$ (resp. $K_{\mathbb{C}}(\exp D_r)U$) for some $r\geqq 0$, setting

$$\mathscr{D}_r(G)=\{\varphi\mid\varphi\in\mathscr{D}(G),\operatorname{Supp}\varphi\subset K\exp D_r\}$$

and

$$\mathscr{D}_{r,K_{\mathbb{C}}}(G_{\mathbb{C}}/U)=\{\psi\mid\psi\in\mathscr{D}_{K_{\mathbb{C}}}(G_{\mathbb{C}}/U),\operatorname{Supp}\psi\subset K_{\mathbb{C}}(\exp D_r)U\},$$

we have:

$$\mathscr{D}(G)=\bigcup_{r\geqq 0}\mathscr{D}_r(G),\qquad \mathscr{D}_{K_{\mathbb{C}}}(G_{\mathbb{C}}/U)=\bigcup_{r\geqq 0}\mathscr{D}_{r,K_{\mathbb{C}}}(G_{\mathbb{C}}/U).$$

Moreover $\mathscr{D}(G)$ (resp. $\mathscr{D}_{K_{\mathbb{C}}}(G_{\mathbb{C}}/U)$) is the inductive limit of the $\mathscr{D}_r(G)$ (resp. $\mathscr{D}_{r,K_{\mathbb{C}}}(G_{\mathbb{C}}/U)$) endowed with the topology induced from $\mathscr{E}(G)$ (resp. $\mathscr{E}(G_{\mathbb{C}}/U)$).

We will denote by $\mathscr{D}_r(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$ (resp. $\mathscr{D}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$) the space of left $K_{\mathbb{C}}$-finite elements in $\mathscr{D}_{r,K_{\mathbb{C}}}(G_{\mathbb{C}}/U)$ (resp. $\mathscr{D}_{K_{\mathbb{C}}}(G_{\mathbb{C}}/U)$).

Attention. Notice that $\mathscr{D}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}$ is not the space of left $K_{\mathbb{C}}$-finite elements in $\mathscr{D}(G_{\mathbb{C}}/U)$ which is reduced to zero.

Similarly, for $(\delta,\gamma)\in\widehat{K_{\mathbb{C}}}$, we set

$$\mathscr{D}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}=\mathscr{D}(G_{\mathbb{C}}/U)_{(K_{\mathbb{C}})}\cap\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}.$$

Proposition 2. (i) *Let φ be in $\mathscr{D}(G)_{(K)}$. Then* $\operatorname{Supp}\varphi$ *is included in $K\exp D_r$ if and only if* $\operatorname{Supp}\varphi^{\eta}$ *is included in $K_{\mathbb{C}}(\exp D_r)U$.*

(ii) *η is a topological isomorphism between $\mathscr{D}(G)^{\delta\gamma}$ and $\mathscr{D}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$, for all δ, γ in $\hat{K}$ such that $\mathscr{D}(G)^{\delta\gamma}$ is non zero.*

(iii) $\mathscr{D}(G)^{\eta}_{(K)}=\bigoplus_{(\delta,\gamma)\in\widehat{K_{\mathbb{C}}}}\mathscr{D}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$.

Proof. We have diffeomorphisms: $K\times\mathfrak{p}\to G$ (resp. $i\mathfrak{k}\times\mathfrak{p}\times U\to G_{\mathbb{C}}$) defined by:

$$(k;X)\to k\exp X\ (\text{resp.}\ (X,Y,u)\to(\exp X)(\exp Y)u)$$

(see e.g. [2], 2.1). Then (i) is an easy consequence of Proposition 1(i). (ii) follows from (i) and Proposition 1(iii). We deduce (iii) from (i) and Proposition 1(iv).

3. Commuting algebra of the left regular action of $G_{\mathbb{C}}$ on $\mathscr{D}(G_{\mathbb{C}}/U)$

3.1. In this part, we will study the continuous linear endomorphisms of $\mathscr{D}(G_{\mathbb{C}}/U)$ which commute with the action of $G_{\mathbb{C}}$. We will denote the algebra of such maps by $\mathscr{X}(G_{\mathbb{C}}/U)$. We will show (Theorem 1) that $\mathscr{X}(G_{\mathbb{C}}/U)$ is canonically isomorphic to the convolution algebra $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ of $W_{\mathbb{C}}$-invariant, compactly supported distributions on $\mathfrak{a}$.

3.2. Let $\nu\in\mathfrak{a}^*_{\mathbb{C}}$. Consider the one dimensional representation of $B=TAN$ which is trivial on N and T, whose differential restricted to $\mathfrak{a}$ is equal to ν. Denote by π_ν the representation of $G_{\mathbb{C}}$ smoothly induced from this representation of B (the so-called spherical principal series with parameter ν). We will use the compact realization of π_ν, namely the space of π_ν will be $\mathscr{E}(U/T)$, denoted by $\mathscr{H}$ in the sequel. When $\nu\in i\mathfrak{a}^*$, π_ν extends to a unitary representation of $G_{\mathbb{C}}$ in $L^2(U/T,d\dot{u})$, where $d\dot{u}$ is the U-invariant measure on U/T with total mass one.

3.3. Let $\mathbb{1}$ denote the function in $\mathscr{H}$ which is identically equal to one on U/T. The spherical Fourier transform $\hat{\varphi}$ of an element of $\mathscr{E}'(G_{\mathbb{C}}/U)$ is the map:

$$\hat{\varphi}\colon\ \mathfrak{a}^*_{\mathbb{C}}\to\mathscr{H},\qquad \text{defined by } \hat{\varphi}(\nu)=\pi_\nu(\varphi)\mathbb{1}.$$

3.4. Let $\mathscr{E}(U\backslash G_{\mathbb{C}}/U)$ (resp. $\mathscr{D}(U\backslash G_{\mathbb{C}}/U)$, resp. $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$) be the space of smooth functions (resp. compactly supported smooth functions, resp. compactly supported distributions) on $G_{\mathbb{C}}$ which are biinvariant under U with its usual topology. $\mathscr{D}(U\backslash G_{\mathbb{C}}/U)$ and $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$ are convolution subalgebras of $\mathscr{E}'(G_{\mathbb{C}})$.

If φ is in $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$ and ν in $\mathfrak{a}_{\mathbb{C}}^*$, $\hat{\varphi}(\nu)$ is just a constant function on U/T and $\hat{\varphi}(\nu)$ is identified with this constant. The spherical Paley-Wiener theorem (cf. [4]) asserts that the space of spherical Fourier transforms of elements in $\mathscr{D}(U\backslash G_{\mathbb{C}}/U)$ is exactly the space of usual Fourier transforms of elements of $\mathscr{D}(\mathfrak{a})^{W_{\mathbb{C}}}$. This gives rise to a topological isomorphism of algebras between $\mathscr{D}(U\backslash G_{\mathbb{C}}/U)$ and the convolution algebra $\mathscr{D}(\mathfrak{a})^{W_{\mathbb{C}}}$. It is obvious that this isomorphism extends to an isomorphism between $\mathscr{E}'(\mathfrak{a})$ and $\mathscr{E}'(U\backslash G_{\mathbb{C}}/U)$, denoted by $\tau\to\tilde{\tau}$, such that the usual Fourier transform of τ is equal to he spherical Fourier transform of $\tilde{\tau}$. Moreover, τ in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ has support in B_r, if and only if $\tilde{\tau}$ has support in $(\exp C_r)U(=U\exp C_r=U(\exp B_r)U)$.

3.5. **Theorem 1.** (i) *Let τ be in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ and let T_τ be the continuous linear endomorphism of $\mathscr{E}(G_{\mathbb{C}}/U)$ defined by: $\forall\varphi\in\mathscr{E}(G_{\mathbb{C}}/U)$, $T_\tau\varphi=\varphi*\tilde{\tau}$. Then T_τ commutes with the left action of $G_{\mathbb{C}}$ and leaves stable $\mathscr{D}(G_{\mathbb{C}}/U)$. Its restriction to $\mathscr{D}(G_{\mathbb{C}}/U)$, denoted also by T_τ, is a continuous endomorphism of $\mathscr{D}(G_{\mathbb{C}}/U)$ commuting with the left $G_{\mathbb{C}}$-action. In other words T_τ is in $\mathscr{L}(G_{\mathbb{C}}/U)$.*

(ii) *The map $\tau\to T_\tau$ from the convolution algebra $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ into the algebra (under composition of endomorphisms) $\mathscr{L}(G_{\mathbb{C}}/U)$ is an isomorphism of algebras.*

Proof. (i) is clear and the only assertion which is not obvious in (ii) is to show that the map $\tau\to T_\tau$ is surjective. Let us prove this. Let T be an element of $\mathscr{L}(G_{\mathbb{C}}/U)$. Proceeding as in [6] Chap. VI, Theorem X and the remark following the proof of this theorem, we get immediately that T is a right convolution by a compactly supported distribution on $G_{\mathbb{C}}$ which is invariant under U. From the spherical Paley-Wiener theorem, this distribution is of the form $\tilde{\tau}$ for some τ in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$. Hence T is of the form T_τ and the theorem is proved.[1]

3.6. *Remark.* A similar result holds for any Riemannian symmetric space of non compact type. The spherical Paley-Wiener theorem for these spaces is needed. Then the proof is exactly the same. We have not written down the proof in the general case, first by economy of notations and references, secondly because it can be easily deduced from our Theorem 3.

3.7. With the notations of 2.2, let ν be in $\mathfrak{a}_{\mathbb{C}}^*$ and denote by $\mathscr{E}_\nu(G_{\mathbb{C}}/U)$ the space of joint eigenfunctions under $\mathbf{D}(G_{\mathbb{C}}/U)$ on $G_{\mathbb{C}}/U$, with joint eigenvalue χ_ν, which is a $G_{\mathbb{C}}$-module for the left regular action of $G_{\mathbb{C}}$.

Proposition 3. *Let τ be in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ and ν in $\mathfrak{a}_{\mathbb{C}}^*$. If $\mathscr{E}_\nu(G_{\mathbb{C}}/U)$ is irreducible under $G_{\mathbb{C}}$, then for all φ in $\mathscr{E}_\nu(G_{\mathbb{C}}/U)$, $T_\tau\varphi$ is equal to $\hat{\tau}(\nu)\varphi$.*

Proof. In $\mathscr{E}_\nu(G_{\mathbb{C}}/U)$ there is a unique biinvariant function under U with value one at the origin, the so-called zonal spherical function with parameter ν, denoted by φ_ν in the sequel. From the properties of φ_ν and of the spherical Fourier transform, we see that: $\varphi_\nu*\tilde{\tau}=\tilde{\tau}*\varphi_\nu=\hat{\tau}(\nu)\varphi_\nu$.

In other words:

$$T_\tau\varphi_\nu=\hat{\tau}(\nu)\varphi_\nu.$$

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[1] I am grateful to M. Duflo and F. Rouviere for having suggested this proof of (ii) which is simpler than my original one.

As T_τ commutes with the action of $G_{\mathbb{C}}$ and is continuous on $\mathscr{E}(G_{\mathbb{C}}/U)$, we deduce from this that $T_\tau \varphi$ is equal to $\hat{\tau}(v)\varphi$ for all φ in the closed subspace of $\mathscr{E}_v(G_{\mathbb{C}}/U)$ generated by the orbit of φ_v under $G_{\mathbb{C}}$. If $\mathscr{E}_v(G_{\mathbb{C}}/U)$ is irreducible, this subspace is the whole space and this concludes the proof of the proposition.

3.8. **Lemma 1.** (i) *Let* X, Y *be in* $\mathfrak{q}$ (resp. $\mathfrak{p}$), *then there exists a unique* Z *in* $\mathfrak{q}$ (resp. $\mathfrak{p}$) *such that:*

$$(\exp X)(\exp Y) \in (\exp Z)\, U \qquad (\text{resp.} (\exp Z)\, K).$$

Moreover:

$$\|Z\| \leqq \|X\| + \|Y\|.$$

(ii) *Let* Z' *be in* $\mathfrak{q}$. *There exists a unique* (X', Y') *in* $i\mathfrak{k} \times \mathfrak{p}$ *such that:*

$$\exp Z' \in \exp X' \exp Y'\, U.$$

Moreover:

$$\|Z'\|^2 \leqq \|X'\|^2 + \|Y'\|^2.$$

Proof. (i) We introduce the geodesic distance d on the Riemannian symmetric space $G_{\mathbb{C}}/U$ (resp. G/K). It is well known that d is invariant under $G_{\mathbb{C}}$ (resp. G).

Moreover: $\forall X \in \mathfrak{q}$ (resp. $\mathfrak{p}$) $d(\dot{e}, \widehat{\exp X}^{\boldsymbol{\cdot}}) = \|X\|$. Here, for g in $G_{\mathbb{C}}$ (resp. G) we denote by $\dot{g}$ the class of g modulo U (resp. K). The existence of Z in (i) follows from the Cartan decomposition of $G_{\mathbb{C}}$ (resp. G) with respect to U (resp. K). Then we have:

$$\|Z\| = d(\dot{e}, \widehat{\exp X \exp Y}^{\boldsymbol{\cdot}}) = d(\widehat{\exp - X}^{\boldsymbol{\cdot}}, \widehat{\exp Y}^{\boldsymbol{\cdot}}),$$

hence, by the triangular inequality:

$$\|Z\| \leqq d(\widehat{\exp - X}^{\boldsymbol{\cdot}}, \dot{e}) + d(\dot{e}, \widehat{\exp Y}^{\boldsymbol{\cdot}}) = \|X\| + \|Y\|.$$

So (i) is proved.

(ii) As the map $i\mathfrak{k} \times \mathfrak{p} \times U \to G_{\mathbb{C}}$ defined by $(X, Y, u) \to \exp X \exp Yu$ is a diffeomorphism (cf. e.g. [2], 2.1), the existence of (X', Y') in (ii) is clear. Now $G_{\mathbb{C}}/U$ is a Riemannian space with negative curvature and the exponential map at any point in this Riemannian space is a diffeomorphism. Then it follows from [3], Corollary 13.2, that for any geodesic triangle in $G_{\mathbb{C}}/U$, ABC, with AB perpendicular to AC we have:

$$d(A, B)^2 + d(A, C)^2 \leqq d(B, C)^2 \qquad (\text{Pythagorian theorem in } G_{\mathbb{C}}/U).$$

As $\mathfrak{p}$ is orthogonal in $i\mathfrak{k}$ in $\mathfrak{q}$ we can apply this to the triangle

$$(\dot{e}, \widehat{\exp - X'}^{\boldsymbol{\cdot}}, \widehat{\exp Y'}^{\boldsymbol{\cdot}}) \qquad \text{in } G_{\mathbb{C}}/U$$

and we get:

$$\|X'\|^2 + \|Y'\|^2 \geqq (d(\widehat{\exp(-X')}^{\boldsymbol{\cdot}}, \widehat{\exp Y'}^{\boldsymbol{\cdot}}))^2.$$

From the $G_{\mathbb{C}}$ invariance of d we also have:

$$d(\widehat{\exp - \dot{X}'}, \widehat{\exp \dot{Y}'}) = d(\dot{e}, \widehat{(\exp X')(\exp \dot{Y}')}) = \|Z'\|$$

and (ii) follows.

3.9. **Proposition 4.** *Let τ be in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ with support in B_{r_0}. Then:*

(i) *If φ in $\mathscr{E}(G_{\mathbb{C}}/U)$ is supported by $K_{\mathbb{C}}(\exp D_r)U$ for some r, $T_\tau\varphi$ is supported by $K_{\mathbb{C}}(\exp D_{r+r_0})U$.*

(ii) *For each (δ, γ) in $\widehat{K_{\mathbb{C}}}$, T_τ leaves stable $\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$ and $\mathscr{D}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$. Moreover T_τ induces a continuous endomorphism of these spaces, when they are endowed with their natural topologies.*

Proof. (i) As T_τ is a right convolution by the distribution $\tilde{\tau}$, which has its support contained in $\exp C_{r_0}U$ (cf. 3.4), it follows that $T_\tau\varphi$ has its support contained in:

$$F = K_{\mathbb{C}}(\exp D_r)U \exp C_{r_0} U.$$

But, as $K_{\mathbb{C}} = (\exp i\mathfrak{k})K$, we have:

$$K_{\mathbb{C}}(\exp D_r)U = (\exp i\mathfrak{k})(\exp D_r)U.$$

We have also:

$$U \exp C_{r_0} = (\exp C_{r_0})U.$$

Hence: $F = \exp i\mathfrak{k} \exp D_r \exp C_{r_0} U$, and if x is in F, we have:

$$\exists (X, Y, Z, u) \in (i\mathfrak{k}) \times D_r \times C_{r_0} \times U, \qquad x = (\exp X)(\exp Y)(\exp Z)u.$$

From Lemma 1(i), it follows that:

$$\exists Z_1 \in \mathfrak{q}, \quad \|Z_1\| \leqq \|Y\| + \|Z\|, \quad (\exp Z_1)U = (\exp X)(\exp Y)u$$

and from Lemma 1(ii) we get:

$$\exists X_1 \in i\mathfrak{k}, \quad \exists Y_1 \in \mathfrak{p}, \quad \|X_1\|^2 + \|Y_1\|^2 \leqq \|Z_1\|^2, \quad (\exp Z_1)U = (\exp X_1)(\exp Y_1)U.$$

But $\|Y\| \leqq r$, $\|Z\| \leqq r_0$ implies $\|Z_1\| \leqq r + r_0$, thus $\|Y_1\| \leqq r + r_0$, and x is in $K_{\mathbb{C}} \exp D_{r+r_0} U$ and (i) is proved.

(ii) As T_τ commutes with the left translations by $K_{\mathbb{C}}$, and is continuous on $\mathscr{E}(G_{\mathbb{C}}/U)$, it is clear that T_τ is a continuous endomorphism of $\mathscr{E}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$. From this and (i), we deduce immediately that T_τ is a continuous endomorphism of $\mathscr{D}(G_{\mathbb{C}}/U)^{(\delta,\gamma)}$.

4. Multipliers for the algebra $\mathscr{D}(G)_{(K)}$

4.1. The convolution algebra $\mathscr{D}(G)_{(K)}$ has a structure of left and right $\mathscr{D}(G)_{(K)}$ (resp. $U(\mathfrak{g})$-module). Let us denote by $\mathscr{Z}(G, K)$ (resp. $Z(G, K)$) the algebra of endomorphisms of $\mathscr{D}(G)_{(K)}$ which are commuting with these actions of $\mathscr{D}(G)_{(K)}$

(resp. $U(\mathfrak{g})$) and which are continuous on $\mathscr{D}(G)^{\delta\gamma}$ for all δ, γ in $\hat{K}$. In [1], it is stated that $\mathscr{Z}(G, K)$ is equal to $Z(G, K)$. We will deduce this equality from the abstract Plancherel theorem.

4.2. **Lemma 2.** (i) *Let $\mathscr{M}$ be a complete, locally convex space, with a smooth G-action such that $Z(\mathfrak{g})$ acts on $\mathscr{M}$ by a character. Let $\mathscr{N}$ be a K-stable subspace of $\mathscr{M}_{(K)}$ such that $\mathscr{N}^{\delta}$ is closed in $\mathscr{M}$, for all δ in $\hat{K}$. Let $\bar{\mathscr{N}}$ be the closure of $\mathscr{N}$ in $\mathscr{M}$. Then the following properties are equivalent:*

(a) *$\bar{\mathscr{N}}$ is G-invariant.*

(b) *$\mathscr{N}$ is $\mathscr{D}(G)_{(K)}$-invariant.*

(c) *$\mathscr{N}$ is $U(\mathfrak{g})$-invariant.*

If one of these properties is true, $\mathscr{N} = \bar{\mathscr{N}} \cap \mathscr{M}_{(K)}$.

(ii) *Let $\mathscr{M}_1, \mathscr{M}_2$ be complete, locally convex spaces with smooth G-actions, and let T be a linear map from $(\mathscr{M}_1)_{(K)}$ into $(\mathscr{M}_2)_{(K)}$, continuous on $\mathscr{M}_1^{\delta}$ for all δ in $\hat{K}$. If T satisfies one of the following properties:*

(a′) *T is a $\mathscr{D}(G)_{(K)}$-morphism,*

(b′) *T is a $U(\mathfrak{g})$-morphism,*

then T is a K-map.

Moreover, if $Z(\mathfrak{g})$ acts by a character on both $\mathscr{M}_1$ and $\mathscr{M}_2$, the properties (a′) *and* (b′) *are equivalent.*

Proof. (i) First assume (b). Then $\bar{\mathscr{N}}$ is obviously invariant under $\mathscr{D}(G)$, hence G-invariant and (b) implies (a).

Now, assume (c). An easy adaptation of [7], Theorem 3.23 to the case of complete, locally convex spaces implies that $\bar{\mathscr{N}}$ is stable under G which shows that (c) implies (a).

Finally, assume (a). As $\mathscr{N}$ is K-stable, it is obvious that $(\bar{\mathscr{N}})^{\delta}$ is equal to the closure of $\mathscr{N}^{\delta}$.

From the hypothesis on $\mathscr{N}$ we deduce that $(\bar{\mathscr{N}})_{(K)} = \mathscr{N}$ and this implies that (b) and (c) are satisfied. This finishes the proof of (i).

(ii) Let us show that, if one of the properties (a′) or (b′) is satisfied, T is a K-map.

First assume (a′). If δ is in $\hat{K}$ we denote by χ_δ the normalized character of δ and view it as a distribution on G, supported by K. We can approximate, in $\mathscr{E}'(G)$, the Dirac measure at any point k of K by a sequence (φ_n^k) in $\mathscr{D}(G)$. Denote by π_i $(i=1,2)$ the representation of G in $\mathscr{M}_i$. Then:

$$\forall \delta \in \hat{K}, \quad \forall v \in \mathscr{M}_i, \quad \pi_i(\chi_\delta) v = v,$$

and:

$$\forall \delta \in \hat{K}, \quad \forall v \in \mathscr{M}_i^{\delta}, \quad \forall k \in K, \quad \pi_i(k) v = \lim_{n \to \infty} \pi_i(\chi_\delta * \varphi_n^k * \chi_\delta) v.$$

Now, for all n in $\mathbb{N}$, $\chi_\delta * \varphi_n^k * \chi_\delta$ is in $\mathscr{D}(G)^{\delta\delta}$. From this, the assumption (a′) and the continuity of T on $\mathscr{M}_1^{\delta}$, we get:

$$\forall \delta \in \hat{K}, \quad \forall v \in \mathscr{M}_1^{\delta}, \quad \forall k \in K, \quad T(\pi_1(k) v) = \lim_{n \to \infty} \pi_2(\chi_\delta * \varphi_n^k * \chi_\delta) T v.$$

Hence:

$$\forall \delta \in \hat{K}, \quad \forall v \in \mathcal{M}_1^\delta, \quad \forall k \in K, \quad T(\pi_1(k)v) = (\pi_2(k))(Tv)$$

and T is a K-map if (a′) is true.

Now assume (b′). Then, as K is connected and T is a $\mathfrak{k}$-map continuous on $\mathcal{M}_1^\delta$ for all δ in $\hat{K}$, it is clear that T is a K-map. Then, if $Z(\mathfrak{g})$ acts by the same character on both $\mathcal{M}_1$ and $\mathcal{M}_2$, the equivalence of (a′) and (b′) is easily deduced from the equivalence of (b) and (c) in (i) applied to the graph of T.

4.3. Let χ be a character of $Z(\mathfrak{g})$ and I_χ its kernel. Let $\bar{J}_\chi$ be the closure of $I_\chi \mathcal{D}(G) = \mathcal{D}(G) I_\chi$ in $\mathcal{D}(G)$ and $J_\chi = \bar{J}_\chi \cap \mathcal{D}(G)_{(K)}$. Then obviously J_χ (resp. $\bar{J}_\chi$) is a two-sided ideal in $\mathcal{D}(G)_{(K)}$ (resp. $\mathcal{D}(G)$), stable by left and right translations by elements of K. Moreover $\bar{J}_\chi$ is stable by left and right translations by elements of G (as $I_\chi \mathcal{D}(G) = \mathcal{D}(G) I_\chi$ is) and, for all δ, γ in $\hat{K}$, $J_\chi^{\delta\gamma} (= \bar{J}_\chi \cap \mathcal{D}(G)^{\delta\gamma})$ is the closure in $\mathcal{D}^{\delta\gamma}(G)$ of $I_\chi \mathcal{D}(G)^{\delta\gamma}$. We also have:

$$J_\chi = \bigoplus_{\delta, \gamma \in \hat{K}} J_\chi^{\delta\gamma}.$$

Now let us consider the $G \times G$-module $\mathcal{D}(G)/\bar{J}_\chi$. It is clear that $\mathcal{D}(G)_{(K)}/J_\chi$ embeds in it and is its subspace of $K \times K$ finite vectors and we have a topological isomorphism between $(\mathcal{D}(G)/\bar{J}_\chi)^{\delta\gamma}$ and $\mathcal{D}(G)^{\delta\gamma}/J_\chi^{\delta\gamma}$. Moreover $Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$ acts by $\chi \otimes \chi$ on $\mathcal{D}(G)/\bar{J}_\chi$.

4.4. **Lemma 3.** *Let T be in $Z(G, K)$. Then, for any character χ of $Z(\mathfrak{g})$:*

(i) *T leaves stable J_χ.*

(ii) *Denoting by T_χ the quotient map T_χ: $\mathcal{D}(G)_{(K)}/J_\chi \to \mathcal{D}(G)_{(K)}/J_\chi$, T_χ commutes with the right and left actions of $\mathcal{D}(G)_{(K)}$.*

Proof. As $T(I_\chi \mathcal{D}(G)_{(K)}) = I_\chi(T\mathcal{D}(G)_{(K)})$, (i) follows from the properties of J_χ and the continuity of T on each $\mathcal{D}(G)^{\delta\gamma}$.

On the other hand, (ii) follows from Lemma 2(ii) (applied to $G \times G$ and T_χ which is a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ map) and from the properties of $\mathcal{D}(G)/\bar{J}_\chi$ quoted above.

4.5. **Lemma 4.** *The intersection of the $\bar{J}_\chi$ over the characters χ of $Z(\mathfrak{g})$ is reduced to zero.*

Proof. Every element φ in this intersection is in the annihilator of any G-module with an infinitesimal character. As the smooth G-module of smooth vectors of any irreducible unitary representation of G has an infinitesimal character, φ annihilates any irreducible unitary representation of G. From Plancherel's abstract formula for G, this implies that φ is the zero function and the lemma is proved.

4.6. **Theorem 2.** *The algebra $Z(G, K)$ of $U(\mathfrak{g})$-endomorphisms of $\mathcal{D}(G)_{(K)}$ is equal to the algebra $\mathcal{Z}(G, K)$ of multipliers of $\mathcal{D}(G)_{(K)}$.*

Proof. Let T be in $\mathcal{Z}(G, K)$. From Lemma 2(i), T is a $K \times K$-map. One sees easily that T admits a closure $\bar{T}$ whose graph is $G \times G$-invariant. From which it follows that T is in $Z(G, K)$. Thus we have proved that $\mathcal{Z}(G, K)$ is contained in $Z(G, K)$.

Let us prove the reversed inclusion. Let T' be in $Z(G, K)$. Then, from Lemma 3(ii), we know that:

$$\forall \chi \in Z(\mathfrak{g})\hat{}, \quad \forall \varphi, \psi \in \mathscr{D}(G)_{(K)}, \quad \varphi * (T'\psi) - T'(\varphi * \psi) \in J_\chi$$

and

$$\forall \chi \in Z(\mathfrak{g})\hat{}, \quad \forall \varphi, \psi \in \mathscr{D}(G)_{(K)}, \quad (T'\varphi) * \psi - T'(\varphi * \psi) \in J_\chi.$$

From Lemma 4 one deduces that it implies:

$$\forall \varphi, \psi \in \mathscr{D}(G)_{(K)}, \quad \varphi * (T'\psi) = T'(\varphi * \psi) = (T'\varphi) * \psi.$$

Hence T is in $\mathscr{Z}(G, K)$ and this finishes the proof of the theorem.

4.7. We now turn to the main result of this paper (which we recall has been proved first by J. Arthur (cf. [1], Theorem 4.1)).

Let τ be in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$. It follows from Propositions 1, 2, 4 that $T\tau$ leaves stable $\eta(\mathscr{E}(G)_{(K)})$ and $\eta(\mathscr{D}(G))_K$, hence $T_\tau^\eta = \eta^{-1} \circ T_\tau \circ \eta$ is a well defined endomorphism of $\mathscr{E}(G)_{(K)}$ and $\mathscr{D}(G)_{(K)}$.

Theorem 3. (i) *For each τ in $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$, the endomorphism T_τ^η of $\mathscr{D}(G)_{(K)}$ (resp. $\mathscr{E}(G)_{(K)}$) commutes with the left and right actions of $U(\mathfrak{g})$ and $\mathscr{D}(G)_{(K)}$.*

(ii) *If τ has support in B_{r_0}, T_τ^η sends $\mathscr{D}_r(G)_{(K)}$ into $\mathscr{D}_{r+r_0}(G)_{(K)}$.*

(iii) *Identifying $Z(\mathfrak{g})$ with $S(\mathfrak{a})^{W_{\mathbb{C}}}$ as in 2.2, for all $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ and any principal series representation (π, H_π) of G with infinitesimal character χ_ν we have:*

$$\forall \varphi \in \mathscr{D}(G)_{(K)} \quad \pi(T_\tau^\eta \varphi) = \hat{\tau}(\nu)\, \pi(\varphi).$$

(iv) *The mapping $\tau \to T_\tau^\eta$ is an algebra homomorphism from $\mathscr{E}'(\mathfrak{a})^{W_{\mathbb{C}}}$ into $\mathscr{Z}(G, K)$.*

Proof. (i) From the properties of T_τ and η it follows easily that T_τ^η, as an endomorphism of $\mathscr{D}(G)_{(K)}$ is in $Z(G, K)$; hence, from Theorem 2, T_τ^η is in $\mathscr{Z}(G, K)$. By a continuity argument, this implies that T_τ^η, as an endomorphism of $\mathscr{E}(G)_{(K)}$, commutes with the left and right actions of $U(\mathfrak{g})$ and $\mathscr{D}(G)_{(K)}$ and (i) is proved.

(ii) is a consequence of the Propositions 2 and 4.

(iii) Let σ be a (finite dimensional) irreducible unitary representation of M_ϕ, λ an element of $(\mathfrak{a}_\phi)_{\mathbb{C}}^*$ and $(\pi_{\sigma,\lambda}, H_{\sigma,\lambda})$ the corresponding principal series obtained by inducing from $M_\phi A_\phi N_\phi$ to G the representation $\sigma \otimes e^\lambda \otimes 1_{N_\phi}$.

The space $\mathscr{E}_{\sigma,\lambda}(G)$ of coefficients of the K-finite vectors of $H_{\sigma,\lambda}$ is a $\mathscr{D}(G)_{(K)}$ invariant subspace of $\mathscr{E}(G)_{(K)}$. If e (resp. f) is a K-finite vector in $H_{\sigma,\lambda}$ (resp. in the topological dual of $H_{\sigma,\lambda}$), denote by $c_{e,f}$ the corresponding coefficient. Then we have:

$$\forall \varphi \in \mathscr{D}(G)_{(K)}, \quad \varphi * c_{e,f} = c_{\pi_{\sigma,\lambda}(\varphi)e, f}.$$

On the other hand, following [2], Remark 7.2, $\eta(\mathscr{E}_{\sigma,\lambda})$ is contained in $\mathscr{E}_{\nu_{\sigma,\lambda}}(G_{\mathbb{C}}/U)$, where $\nu_{\sigma,\lambda}$ is in $\mathfrak{a}_{\mathbb{C}}^*$ and $\chi_{\nu_{\sigma,\lambda}}$ is the infinitesimal character of $\pi_{\sigma,\lambda}$

(with the identification of $Z(\mathfrak{g})$ with $S(\mathfrak{a})^{W_{\mathbb{C}}}$, see 2.2). From the irreducibility criteria of the $G_{\mathbb{C}}$-module $\mathscr{E}_{\nu_{\sigma,\lambda}}(G_{\mathbb{C}}/U)$ (cf. [5]), it is easy to find a real affine subspace E of $(\mathfrak{a}_\phi)^*_{\mathbb{C}}$, with the same dimension as $\mathfrak{a}_\phi$, such that for all λ in E, $\mathscr{E}_{\nu_{\sigma,\lambda}}$ is irreducible.

Then from Proposition 3 we get:

$$\forall \lambda \in E, \quad \forall \psi \in \mathscr{E}_{\nu_{\sigma,\lambda}}(G), \quad T^\eta_\tau \psi = \hat{\tau}(\nu_{\sigma,\lambda})\psi.$$

Hence, for all K-finite vectors e (resp. f) in $H_{\sigma,\lambda}$ (resp. $H^*_{\sigma,\lambda}$), with λ in E, we have:

$$\forall \varphi \in \mathscr{D}(G)_{(K)}, \quad \varphi * (T^\eta_\tau c_{e,f}) = \hat{\tau}(\nu_{\sigma,\lambda})\, \varphi * c_{e,f}.$$

But: $\varphi * c_{e,f} = c_{\pi_{\sigma,\lambda}(\varphi)e,f}$.

Moreover, as T^η_τ is in $\mathscr{Z}(G,K)$, we have also:

$$\varphi * (T^\eta_\tau c_{e,f}) = T^\eta_\tau(\varphi * c_{e,f}).$$

From the properties of T^η_τ proved in (i) and its continuity, we deduce:

$$T^\eta_\tau(\varphi * c_{e,f}) = (T^\eta_\tau \varphi) * c_{e,f} = c_{\pi_{\sigma,\lambda}(T^\eta_\tau \varphi)e,f}.$$

Finally we get:

$$\forall \varphi \in \mathscr{D}(G)_{(K)}, \quad \forall \lambda \in E, \quad \forall e \in (H_{\sigma,\lambda})_{(K)}, \quad \forall f \in (H^*_{\sigma,\lambda})_{(K)},$$
$$c_{\pi_{\sigma,\lambda}(T^\eta_\tau \varphi)e,f} = \hat{\tau}(\nu_{\sigma,\lambda})\, c_{\pi_{\sigma,\lambda}(\varphi)e,f}.$$

$(H_{\sigma,\lambda})_{(K)}$ (resp. $H^*_{\sigma,\lambda}$) being dense in $H_{\sigma,\lambda}$ (resp. $H^*_{\sigma,\lambda}$), it follows that:

$$\forall \varphi \in \mathscr{D}(G)_{(K)}, \quad \forall \lambda \in E, \quad \pi_{\sigma,\lambda}(T^\eta_\tau \varphi) = \hat{\tau}(\nu_{\sigma,\lambda})\, \pi_{\sigma,\lambda}(\varphi).$$

On the other hand $\hat{\tau}$ is analytic, and $\pi_{\sigma,\lambda}(T^\eta_\tau \varphi)$, $\pi_{\sigma,\lambda}(\varphi)$ have a well known property of analyticity in λ.

Then, by analytic continuation of the equality above, we get (iii).

(iv) is clear from (i) and Theorem 1. This concludes the proof of the theorem.

Remark. Notice that in general the multipliers of $\mathscr{D}(G)_{(K)}$ do not extend in a continuous multiplier of $\mathscr{D}(G)$. In fact, proceeding as in [6], VI. Theorem X, and the remark following this theorem, it is easy to show that a multiplier of $\mathscr{D}(G)$ is a convolution by a compactly supported distribution, invariant by conjugacy. The only compact conjugacy classes of G being the conjugacy classes of the elements of the center of G (if G has no compact factors), it follows that a multiplier of $\mathscr{D}(G)$ is the composition of a translation by an element of the center of G and of a convolution with an element of $Z(\mathfrak{g})$.

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