A TWISTED INVARIANT PALEY-WIENER THEOREM FOR REAL REDUCTIVE GROUPS

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Abstract

Let G^+ be the group of real points of a possibly disconnected linear reductive algebraic group defined over $\mathbb{R}$ which is generated by the real points of a connected component G'. Let K be a maximal compact subgroup of the group of real points of the identity component of this algebraic group. We characterize the space of maps $\pi \mapsto \mathrm{tr}(\pi(f))$, where π is an irreducible tempered representation of G^+ and f varies over the space of smooth, compactly supported functions on G' which are left and right K-finite. This work is motivated by applications to the twisted Arthur-Selberg trace formula.

Contents

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1. Introduction

Let G^+ be the group of real points of a possibly disconnected linear reductive algebraic group $\underline{G}^+$ defined over $\mathbb{R}$, and let G be the group of real points of the identity component of $\underline{G}^+$. We assume that $\underline{G}^+$ is generated by one of its connected components, whose set of real points G' is assumed to be nonempty. Then, G' generates G^+. Let K^+ be a maximal compact subgroup of G^+, and let K be the maximal compact subgroup $K^+ \cap G$ of G. The intersection $G' \cap K^+$ is nonempty. We may therefore fix $\sigma \in G' \cap K^+$ so that $G' = G\sigma$. In fact, we may choose σ so that it fixes a minimal parabolic subgroup $P_m = M_m A_m N_m$.

The present article has four principal goals.

(A) It seeks to classify the irreducible tempered representations π of G which are σ-stable (i.e., equivalent to their σ-conjugates π^σ). This is equivalent to classifying the irreducible representations of G^+ whose restrictions to G are irreducible and tempered. The set of equivalence classes of irreducible tempered σ-stable representations are denoted by $\hat{G}^\sigma_{\text{temp}}$

(B) It seeks to classify (using goal (A)) the irreducible admissible representations of G which are σ-stable.

(C) It seeks to construct a canonical operator S_π that intertwines π^σ with π for every irreducible tempered σ-stable representation π. We denote by Θ^σ_π the distribution on G defined by

$$\Theta^\sigma_\pi(f) = \text{tr}\big(\pi(f)S_\pi\big), \quad f \in C_c^\infty(G).$$

(D) It seeks to characterize the functions $F : \hat{G}^\sigma_{\text{temp}} \to \mathbb{C}$ which are σ-twisted invariant Fourier transforms of some left and right K-finite elements $f \in C_c^\infty(G)$, that is, functions of the form

$$\pi \mapsto \Theta^\sigma_\pi(f), \quad \pi \in \hat{G}^\sigma_{\text{temp}}.$$

This characterization is required for the general proof of the invariant trace formula as given in [A2, page 505].

Let us briefly recall previous works in the case where $\underline{G}^+$ is connected (i.e., when $G = G^+$). Goal (A) was achieved by Knapp and Zuckerman [KZ] using limits of discrete series (see also [S] for the case when G is disconnected as a Lie group) and by Vogan [V] using minimal K-types. Goal (B) is known as the Langlands classification (see [L]). When $G = G^+$, goal (C) is trivial. Goal (D) is the work of Clozel and Delorme [CD1], [CD2]. These two articles used the work of Knapp and Zuckerman and the work of Vogan. The result of Delorme in [Dl] on Harish-Chandra homomorphisms and transitional spaces is also a crucial ingredient, as is Arthur's Paley-Wiener theorem, [A1, Theorems III.4.1, III.4.2].

Turning to the case of $G \neq G^+$, goals (A) – (D) were solved in [D2] for complex groups G and σ equal to complex conjugation relative to a quasi-split inner form. The solution relied mainly on a very nice classification for goal (A) and a restriction theorem from [CD2, proposition A.1], which is generalized in the present work.

Later, Mezo [M1] treated the case of $G = \mathrm{GL}(n, \mathbb{R})$ and some automorphisms in the case of $G = \mathrm{SL}(n, \mathbb{R})$, exhibiting some new phenomena. More recently, Mezo [M2] treated the case of general G w hen σ is an involution. The proof assumed that all R-groups are trivial but included an approach to the general case. This work is the starting point of the present article.

We point out that Bouaziz [B, Section 7] has proved an invariant Paley-Wiener theorem and that Renard [R, Section 17] has proved a twisted invariant Paley-Wiener theorem for functions that are not K-finite; neither of the results may be deduced from the other. To apply the Paley-Wiener theorem to the Arthur-Selberg trace formula (see [A2], [A3]), K-finite functions are needed. This is our main motivation in studying K-finite functions.

Our solutions to goals (A) – (D) unfold as follows. We obtain the classification of goal (A) (see Theorem 1) by combining the classifications of Vogan [V] and Knapp and Zuckerman [KZ] with the characterization of σ-stable representations given in [M2, Section 7]. The latter characterization uses automorphisms of G which are attached to generalized principal series representations. The solution of goal (B) follows from goal (A) and the Langlands classification (see [M2, Proposition 3.1]).

Any generalized principal series representation of goal (A) depends on a continuous parameter. The canonical intertwining operators of goal (C) (see Lemma 6) are conjugates of operators that are independent of these continuous parameters (see Lemma 5). The operators of conjugation are normalized intertwining operators as in [KS, Part 1]. This independence from the continuous parameters plays a key role in the growth estimates of the twisted invariant Fourier transforms of compactly supported functions on G (see Lemma 8).

A slight change in perspective carries us from twisted invariant Fourier transforms to twisted characters (see (5.2)). The relations between the twisted characters arising from goals (A) and (C) are studied systematically in Proposition 2. These relations are sufficient for us to formulate our twisted invariant Paley-Wiener theorem (see Theorem 3). The proof of our theorem generalizes [CD2, démonstration du théorème 1] to the twisted context. This generalization presents several obstacles. One must correlate the properties of the R-groups with those of the automorphisms attached to the generalized principal series of goal (A). One must also generalize a restriction theorem for polynomials invariant under an automorphism of a Dynkin diagram (see Theorem 5). The heart of Theorem 3 is Proposition 4, where the relations of Proposition 2 again play an important part.

Theorem 3 has a corollary (Theorem 4) motivated by the twisted Arthur-Selberg trace formula (see [A2, page 505], [A3, Section 11]). We do not need to list the relations of Proposition 2 to state it, but we do need to introduce some notation. Let $P = MAN$ be a parabolic subgroup of G, and let P' be the intersection of its normalizer P^+ in G^+ with G'. We assume in the following that P' is nonempty. Similarly, let L^+ be the intersection of the normalizer of $L = MA$ in G^+ with P^+. Let M^+ be the subgroup of L^+ generated by τ and M. Then, $L^+ = M^+A$, $P^+ = L^+N$, and N and A are normal subgroups of P^+ and L^+, respectively. Note that A is not necessarily in the center of L^+. Let $\mathfrak{a}$ be the Lie algebra of A.

We repeatedly use the convention that if a group J acts on a vector space E, and $X \subset J$, then E^X denotes the space of elements of E fixed by all of the elements of X. Now, given a tempered unitary representation ε^+ of M^+ whose restriction to M is irreducible, and given $\lambda \in i\mathfrak{a}^{*L^+}$, then $\varepsilon^+ \otimes e^\lambda \otimes 1_N$ is a unitary representation of P^+. We denote by $\pi^{P^+}_{\varepsilon^+,\lambda}$ the corresponding unitarily induced representation from P^+ to G^+.

THEOREM

Let ϕ be a complex-valued function defined on the tempered dual $\hat{G}^+_{\mathrm{temp}}$ of G^+, which is nonzero only on the subset $\hat{G}'_{\mathrm{temp}}$ of (equivalence classes) of representations of G^+ whose restrictions to G are irreducible and tempered. We identify ϕ with its $\mathbb{Z}$-linear extension to the set of tempered representations of G^+ of finite length. Then, there exists a left and right K-finite $f \in C^\infty_c(G)$ such that

$$\operatorname{tr}\bigl(\pi^+(f)\pi^+(\sigma)\bigr) = \phi(\pi^+), \quad \pi^+ \in \hat{G}^+_{\mathrm{temp}}$$

if and only if ϕ satisfies the following conditions.

(i) *There exists a finite subset Γ of the unitary dual $\hat{K}$ such that $\phi(\pi^+) = 0$ if the restriction of π^+ to G does not contain any K-type in Γ.*

(ii) *If π^+ and $\pi'^+ \in \hat{G}'_{\mathrm{temp}}$ have the same restriction to G, then $\pi^+(\sigma) = c\,\pi'^+(\sigma)$, where c is a root of unity. In this case, $\phi(\pi^+) = c\,\phi(\pi'^+)$.*

(iii) *Let $Q = M_Q A_Q N_Q$ be a parabolic subgroup of G with Q' nonempty. Assume that ε^+ is a representation of M^+_Q whose restriction to M_Q is* *tempered and irreducible, and assume that* $\lambda \in \mathfrak{a}^{*M_Q^+}$*. Then,* $\lambda \mapsto \phi(\pi^{Q^+}_{\varepsilon^+,\lambda})$ *is the Fourier transform of a function in* $C_c^\infty(\mathfrak{a}_Q^{M_Q^+})$*.*

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Theorem 4 is a more precise version of this theorem, taking into account the size of the support of the function f.

2. Preliminaries

2.1. Generalized principal series

Given a group J, elements $g, x \in J$, and $X \subset J$, we set $g \cdot x := gxg^{-1}$ and $g \cdot X := gXg^{-1}$. If π is a map defined on a subgroup H of J, and if H is normalized by some $x \in G$, then π^x denotes the map defined on H by $\pi^x(h) = \pi(x^{-1} \cdot h)$ for $h \in H$. Note that π may be a representation of H. Recall from the introduction that E^J is the subspace of J-fixed elements in a vector space E. If J is a Lie group, then $\mathfrak{j}$ denotes its Lie algebra and Ad denotes the adjoint representation of J on $\mathfrak{j}$.

Let us reconsider the objects defined from G^+ as in the introduction. The set G' generates G^+; indeed, for any $\sigma \in G'$, one has $G^+ = \bigcup_{i=1}^n \sigma^i G$, where n is the least positive integer such that $\sigma^n \in G$. We choose a maximal compact subgroup K^+ of G^+, which is the fixed-point group of some Cartan involution θ, in the sense of [BH, Proposition 1.10]. Set $K = K^+ \cap G$. According to [BH, Proposition 1.10], K is a maximal compact subgroup of G, and $K' = K^+ \cap G'$ is nonempty. Clearly, $\sigma \cdot K = K$ for any $\sigma \in K'$. We fix such a σ for the remainder of this article.

Let ${}^0G$ be the intersection of the kernels of the continuous characters of G with values in $\mathbb{R}^{*+}$. Let A_G be the analytic subgroup of G whose Lie algebra is the subspace of the anti-invariant elements under θ in the center of the Lie algebra $\mathfrak{g}$. Actually, with our definition of a Cartan involution, A_G is just the identity component of the group of real points of a maximally split torus in the center of G. One calls A_G the *split component of the center of* G.

We fix a symmetric bilinear form B on $\mathfrak{g}$ which is invariant under the adjoint group Ad G as well as under θ. In addition, we may assume that the quadratic form $\|X\|^2 = -B(X, \theta X)$ is positive definite.

Suppose that P is a parabolic subgroup of G. Then, $L = P \cap \theta(P)$ is its θ-stable Levi subgroup. The decomposition $P = MAN$, where $M = {}^0L$, $A = A_L$, and N is the unipotent radical of P, is called the *Langlands decomposition of* P. We fix a minimal parabolic subgroup $P_m = M_m A_m N_m$, and we recall that a parabolic subgroup is said to be standard if it contains P_m.

Let $W(A)$ be the quotient of the normalizer of A in K by its centralizer. The group $W(A)$ acts naturally on A and on the (equivalence classes of) representations of M.

If δ is a unitary representation of M and if $\lambda \in i\mathfrak{a}^*$, then we denote by W_δ (resp., W_λ) the stabilizer in $W(A)$ of δ (resp., λ). We also define $W_{\delta,\lambda} := W(A)_\delta \cap W(A)_\lambda$.

Let ρ_P be the half-sum of the roots of $\mathfrak{a}^*$ determined by the root spaces in $\mathfrak{n}$. We denote by $I^P_{\delta,\lambda}$ the space of measurable functions φ from G to the space of δ, such that

$$\varphi(gman) = a^{-\lambda-\rho_P}\delta(m^{-1})\varphi(g), \quad g \in G,\ m \in M,\ a \in A,\ n \in N,$$

and the integral $\|\varphi\|^2 := \int_K |\varphi(k)|^2\, dk$ is finite. This space is endowed with the scalar product defined from $\|\cdot\|$. The group G acts unitarily on this space by left translations, and the corresponding representation is denoted by $\tilde{\pi}^P_{\delta,\lambda}$.

Let $I(\delta)$ be the space formed by the restriction of the elements of $I^P_{\delta,\lambda}$ to K. Observe that $I(\delta)$ is independent of λ. The restriction is bijective, and the representation obtained from $\tilde{\pi}^P_{\delta,\lambda}$ by "transport de structure" is denoted by $\pi^P_{\delta,\lambda}$. This version of $\tilde{\pi}^P_{\delta,\lambda}$ is called the *compact realization of* $\tilde{\pi}^P_{\delta,\lambda}$. For any $\varphi \in I(\delta)$, we take φ_λ to be the unique element of $I^P_{\delta,\lambda}$ whose restriction to K is φ. The equivalence class of $\pi^P_{\delta,\lambda}$, $\lambda \in i\mathfrak{a}^*$, does not depend on P with Levi subgroup MA (see [KS, Section 8]). As a result, we sometimes write $\pi^{MA}_{\delta,\lambda}$ instead of $\pi^P_{\delta,\lambda}$.

2.2. *A review of the classification of irreducible tempered representations: A dictionary between two points of view*

Let $P = MAN$ be a parabolic subgroup of G, let δ be a discrete series representation of M, and let $\lambda \in i\mathfrak{a}^*$. Define $A(\delta)$ to be the set of minimal K-types of $I(\delta)$ (see [V], [D3, references following (1.8)] for G not connected as a Lie group). For $\mu \in A(\delta)$, let $I^\mu(\delta)$ be the corresponding isotypic component of μ in $I(\delta)$. It is irreducible as a representation of K. We fix an element μ_0 of $A(\delta)$.

We make extensive use of the intertwining operators of Knapp and Stein [KS, Part 1] and of the particular normalization introduced by Delorme [Dl, Section 1], [D3, Section 1] for G not connected as a Lie group. For a parabolic subgroup $Q = MAV$ with Levi subgroup MA, $A(Q, P, \delta, \lambda)$ is an analytic family of unitary operators in $\lambda \in i\mathfrak{a}^*$, intertwining $\pi^P_{\delta,\lambda}$ with $\pi^Q_{\delta,\lambda}$. In addition, for any $\mu \in A(\delta)$, there exists $c^\mu_\delta(Q, P) \in \mathbb{C}$ with $c^{\mu_0}_\delta(Q, P) = 1$, such that

$$\text{for all } \lambda \in i\mathfrak{a}^*,\ A(Q, P, \delta, \lambda)_{|I^\mu(\delta)} \text{ reduces to multiplication by } c^\mu_\delta(Q, P). \tag{2.1}$$

If R is another parabolic subgroup with Levi subgroup MA, then

$$A(\mathrm{R}, Q, \delta, \lambda)A(Q, P, \delta, \lambda) = A(\mathrm{R}, P, \delta, \lambda), \quad \lambda \in i\mathfrak{a}^*. \tag{2.2}$$

Note that we are using the letter A for the normalized intertwining operators instead of $\mathscr{A}$ as in [KS, Part 1] or [Dl, Section 1].

Let $M'A'$ be a Levi subgroup of G with $MA \subset M'A'$. We assume that $\theta(V) \cap N \subset M'$. Let $\mathfrak{a}'^{\perp}$ be the orthogonal complement of $\mathfrak{a}'$ in $\mathfrak{a}$. We may decompose $\lambda \in i\mathfrak{a}^*$ as

$$\lambda = \lambda' + \lambda'', \quad \lambda' \in \mathfrak{a}'^*, \ \lambda'' \in (\mathfrak{a}'^{\perp})^*.$$

We require the fact that

$$A(Q, P, \delta, \lambda) \text{ depends only on } \lambda''. \tag{2.3}$$

Indeed, by [KS, Theorems 7.6, 8.4], it suffices to prove (2.3) in the case where P and Q are adjacent. In that case (see [KS, Proposition 7.5, Section 8]), the unnormalized operators, as well as the normalizing factors, depend only on λ'', so (2.3) follows.

For $w \in W_\delta$, $A(P, w, \delta, \lambda)$ is an analytic family of unitary operators in $\lambda \in i\mathfrak{a}^*$, intertwining $\pi^P_{\delta,\lambda}$ with $\pi^P_{\delta,w\lambda}$. In addition, for any $\mu \in A(\delta)$, there exists a character χ_μ of W_δ such that

for all $\lambda \in i\mathfrak{a}^*$ and $w \in W_\delta$, we have the fact that $A(P, w, \delta, \lambda)_{|I^\mu(\delta)}$ is multiplication by $\chi_\mu(w)$ and χ_{μ_0} is trivial. (2.4)

$$A(P, w, \delta, w'\lambda)A(P, w', \delta, \lambda) = A(P, ww', \delta, \lambda), \quad \lambda \in i\mathfrak{a}^*, \ w, w' \in W_\delta. \tag{2.5}$$

Let $\tilde{w}$ be a representative in K of $w \in W_\delta$ and let $u_{\tilde{w}}$ be a unitary operator intertwining $\delta^{\tilde{w}}$ with δ. Then there exists $c \in \mathbb{C}$ of modulus one such that

$$A(P, w, \delta, \lambda) = cu_{\tilde{w}}\mathrm{R}_{\tilde{w}}A(w^{-1}Pw, P, \delta, \lambda), \quad \lambda \in i\mathfrak{a}^*. \tag{2.6}$$

Indeed, both sides of this equation are unitary self-intertwining operators of $\pi^P_{\delta,\lambda}$, and they are analytic in λ. When $\pi^P_{\delta,\lambda}$ is irreducible, these operators have to be proportional. This is true for λ in an open and dense set of $i\mathfrak{a}^*$. Furthermore, the two families are constant on minimal K-types. Hence, the proportionality factor is constant on this set, and the assertion follows by analytic continuation.

Recall (see [CD2, section 2]) that W^0_δ is the normal subgroup of W_δ consisting of those elements w for which $A(P, w, \delta, 0)$ is trivial. Because of this, each character χ_μ as mentioned above is a character of W_δ which is trivial on W^0_δ.

Recall also that W^0_δ is the Weyl group of a root system Δ_δ in $\mathfrak{a}$, and recall that P determines a set of positive roots Δ^+_δ of Δ_δ. The positive roots Δ^+_δ determine a unique chamber C_δ in $\mathfrak{a}$ (i.e., the set of elements in $\mathfrak{a}$ on which the elements of Δ^+_δ are greater than zero). If Δ^+_δ is empty, it is equal to $\mathfrak{a}$. We denote its closure by $\bar{C}_\delta$.

Let R_δ^c be the subgroup of W_δ leaving Δ_δ^+ invariant. The group W_δ is the semidirect product of R_δ^c and W_δ^0. Consequently, every character $\chi_\mu \in \hat{W}_\delta$ described above may be identified with a character of R_δ^c. Moreover, the map $\mu \mapsto \chi_\mu$ is a bijection between $A(\delta)$ and $\hat{R}_\delta^c$.

For $\lambda \in i\mathfrak{a}^*$, let $W_{\delta,\lambda}^0$ be the subgroup of elements w of $W_{\delta,\lambda}$ for which $A(P, w, \delta, \lambda)$ is trivial. (It is already trivial on the K-type μ_0.) It is the Weyl group of the root system $\Delta_{\delta,\lambda} = \{\alpha \in \Delta_\delta \mid (\alpha, \lambda) = 0\}$. We let $\Delta_{\delta,\lambda}^+$ be the subset of positive roots determined by P. Let $R_{\delta,\lambda}^c$ be the subgroup of $W_{\delta,\lambda}$ preserving $\Delta_{\delta,\lambda}^+$.

We now introduce representations attached to the objects that we have just defined (see [V], [Dl, Section 1] for G connected as a Lie group; see [D3, (1.6)–(1.8), (1.15), (1.16)] for G not connected as a Lie group).

Let $\lambda \in i\overline{C}_\delta$, and let $H = R_{\delta,\lambda}^c$. Let $\chi \in \hat{H}$, and let $\pi_{\delta,H,\chi,\lambda}^P$ be the subrepresentation of $\pi_{\delta,\lambda}^P$ generated by the set of minimal K-types $\mu \in A(\delta)$ so that $\chi_{\mu|H} = \chi$. Then, $\pi_{\delta,H,\chi,\lambda}^P$ is irreducible and $\pi_{\delta,H,\chi,\lambda}^P$ contains the minimal K-type μ if and only if $\chi_{\mu|H} = \chi$. Moreover, (2.7)

$$\pi_{\delta,\lambda}^P = \bigoplus_{\chi \in \hat{H}} \pi_{\delta,H,\chi,\lambda}^P.$$

Every irreducible tempered representation arises as such a $\pi_{\delta,H,\chi,\lambda}^P$ for a standard parabolic subgroup P. The data (M, δ, λ) are determined modulo conjugacy by K. Once δ is given, H and χ are unique, and λ is unique up to the action of R_δ^c.

The notions of (2.7) actually hold in a more general setting:

every subrepresentation of $\pi_{\delta,\lambda}^P$, $\lambda \in i\mathfrak{a}^*$ is characterized by the minimal K-types that it contains. (2.8)

Let $\mathrm{Diag}(\delta)$ be the set of subgroups $\{R_{\delta,\lambda}^c \mid \lambda \in i\overline{C}_\delta\}$ of R_δ^c. For $H \in \mathrm{Diag}(\delta)$, $\chi \in \hat{H}$, and $\lambda \in i\mathfrak{a}^{*H}$, let $\pi_{\delta,H,\chi,\lambda}^P$ be the subrepresentation of $\pi_{\delta,\lambda}^P$ generated by the minimal K-types μ such that $\chi_{\mu|H} = \chi$. The subrepresentation $\pi_{\delta,H,\chi,\lambda}^P$ contains the minimal K-type μ if and only if $\chi_{\mu|H} = \chi$: (2.9)

$$\pi_{\delta,\lambda}^P = \bigoplus_{\chi \in \hat{H}} \pi_{\delta,H,\chi,\lambda}^P, \quad \lambda \in i\mathfrak{a}^{*H}. \tag{2.10}$$

For $H, H' \in \mathrm{Diag}(\delta)$ with $H \subset H'$, one has

$$\pi^P_{\delta,H,\chi,\lambda} = \bigoplus_{\chi' \in \hat{H}', \chi'_{|H} = \chi} \pi^P_{H',\chi',\lambda}, \quad \lambda \in i\mathfrak{a}^{*H'}. \tag{2.11}$$

Let us justify (2.9)–(2.11). From (2.7), one sees that (2.9) and (2.10) are true for $\lambda \in i\overline{C}_\delta$ and $H = \mathrm{R}^c_{\delta,\lambda}$. After decomposing $\pi^P_{\delta,H,\chi,\lambda}$ into irreducible representations and then using (2.7), it is apparent that (2.9) and (2.10) hold also for $\lambda \in i\overline{C}_\delta$ and $H \subset \mathrm{R}^c_{\delta,\lambda}$. Decomposing both sides of (2.11) in a similar manner reveals that (2.11) holds for $\lambda \in i\overline{C}_\delta$. This establishes (2.9)–(2.11) when $\lambda \in i\overline{C}_\delta$. Now, suppose that $\lambda \in i\mathfrak{a}^{*H}$ for some $H \in \mathrm{Diag}(\delta)$. As W^0_δ is the Weyl group of Δ_δ, there exists $w \in W^0_\delta$ such that $\nu := w\lambda \in i\overline{C}_\delta$. Let us first show that $\mathrm{R}^c_{\delta,\lambda} \subset \mathrm{R}^c_{\delta,\nu}$. If $r \in \mathrm{R}^c_{\delta,\lambda}$, then

$$r\nu = rw\lambda = rwr^{-1}\lambda = rwr^{-1}w^{-1}\nu.$$

As W^0_δ is normal in W_δ, the element rwr^{-1} lies in W^0_δ. Therefore, $r\nu$ and ν are conjugate by an element of W^0_δ, and both belong to $i\overline{C}_\delta$. This implies that $r\nu = \nu$, and so the inclusion $\mathrm{R}^c_{\delta,\lambda} \subset \mathrm{R}^c_{\delta,\nu}$ is proved. Now, taking into account that $\pi^P_{\delta,\lambda}$ is equivalent to $\pi^P_{\delta,\nu}$, one sees that $\pi^P_{\delta,H,\chi,\lambda}$ is equivalent to $\pi^P_{\delta,H,\chi,\nu}$. Assertions (2.9)–(2.11) for $\pi^P_{\delta,H,\chi,\lambda}$ are therefore consequences of the parallel statements for $\pi^P_{\delta,H,\chi,\nu}$, which we proved for $\nu \in i\overline{C}_\delta$.

We carry on by listing some facts relating discrete series representations to nondegenerate limits of discrete series (see [CD2, section 2] for references; essentially, see [KZ, Theorem 14.2] for G connected as a Lie group and [S, Section 4.3] for G not connected as a Lie group). Let $\lambda \in i\overline{C}_\delta$, and let A^λ be the fixed-point set in A of $\mathrm{R}^c_{\delta,\lambda}$. The centralizer of A^λ in G admits A^λ as split component and is written as $M^\lambda A^\lambda$. It is the Levi subgroup of a parabolic subgroup of G which does not necessarily contain P or P_m. The element λ may be regarded as an element of $i(\mathfrak{a}^\lambda)^*$, and one has the finite decomposition

$$\pi^P_{\delta,\lambda} \cong \bigoplus_j \pi^{M^\lambda A^\lambda}_{\delta^\lambda_j,\lambda}, \tag{2.12}$$

where each induced representation on the right-hand side is irreducible and

$$\pi^{P\cap M^\lambda}_{\delta,0} = \bigoplus_j \delta^\lambda_j \tag{2.13}$$

is a decomposition into nondegenerate limits of discrete series δ^λ_j of M^λ. The set $\{\delta^\lambda_j \mid \lambda \in i\overline{C}_\delta\}$ is the set of nondegenerate limits of discrete series called *strongly affiliated* to δ. The set of Levi subgroups $M^\lambda A^\lambda$, as λ varies over $i\overline{C}_\delta$, is the set of Levi subgroups called *strongly affiliated* to δ.

Every irreducible tempered representation π of G occurs in a decomposition as above (i.e., each is of the form $\pi_{\delta^\lambda,\lambda}^{M^\lambda A^\lambda}$, where $\lambda \in i\overline{C}_\delta$ and δ^λ is a nondegenerate limit of discrete series of M^λ strongly affiliated to δ). The data $(M^\lambda A^\lambda, \delta^\lambda, \lambda)$ are determined up to conjugacy under K.

Let $M_1 A_1$ be a Levi subgroup strongly affiliated to δ, and let δ_1 be a nondegenerate limit of discrete series of M_1 strongly affiliated to δ. By definition, we may choose an element $\nu \in i\overline{C}_\delta \cap i\mathfrak{a}^*$ so that $A^\nu = A_1$ and $\pi_{\delta_1,\nu}^{M_1 A_1}$ is irreducible. To these data, one may associate a subgroup H of R_δ^c, a character of H, and $\chi \in \hat{H}$ in the following manner. Recall that the dual group $\hat{\mathrm{R}}_\delta^c$ acts simply transitively on $A(\delta)$ and that by fixing μ_0, we may identify $A(\delta)$ with $\hat{\mathrm{R}}_\delta^c$. The set $A(\delta_1)$ of minimal K-types $\pi_{\delta_1,\lambda}^{M_1 A_1}$, $\lambda \in i\mathfrak{a}_1^*$, corresponds to an orbit in $A(\delta)$ of the orthogonal complement $H^\perp$ of H in $\hat{\mathrm{R}}_\delta^c$, where $H = \mathrm{R}_{\delta,\nu}^c$. The elements of this orbit are characterized by their restriction χ to H. This may be seen by applying the results of (2.7) to the irreducible subrepresentation $\pi_{\delta_1,\nu}^{M_1 A_1}$ of $\pi_{\delta,\nu}^P$. For all λ in $i\mathfrak{a}^{*H} = i(\mathfrak{a}^\nu)^* = i\mathfrak{a}_1^*$, one has

$$\pi_{\delta_1,\lambda}^{M_1 A_1} \cong \pi_{\delta,H,\chi,\lambda}^P. \tag{2.14}$$

In fact, both sides are equivalent to subrepresentations of $\pi_{\delta,\lambda}^P$ containing the same minimal K-types, and so the assertion follows from (2.8).

We henceforth fix a set of representatives of conjugacy classes under K of pairs (M, δ) (often abbreviated simply as δ), where MA is the Levi subgroup of a standard parabolic subgroup of G, and δ is a discrete series representation of M. This set is called the *set of discrete data* and denoted by DD. For $(M, \delta) \in$ DD, there exists by definition a unique standard parabolic subgroup with Levi subgroup MA. It is called the *standard parabolic subgroup* of (M, δ). This fixes a choice of Δ_δ^+ and $\overline{C}_\delta$.

3. The classification of irreducible tempered σ-stable representations

3.1. A choice of σ

LEMMA 1

There exists σ in $K' = K^+ \cap G'$ such that

(i) *conjugation by σ commutes with θ and* $\mathrm{Ad}\,\sigma$ *commutes with the differential of θ;*

(ii) *σ normalizes M_m, A_m and P_m;*

(iii) $\mathrm{Ad}\,\sigma$ *is of finite order on $\mathfrak{a}_m$; and*

(iv) *we may choose our bilinear form B to be* $\mathrm{Ad}\,\sigma$*-invariant.*

Proof

The set $K^+ \cap G'$ is nonempty (see [BH, Proposition 1.10]). Let σ' be any element of $K^+ \cap G'$. Then, $\sigma' \cdot P_m$ is a minimal parabolic subgroup of G. Hence, it is of the form

$k \cdot P_m$ for some $k \in K$. Set $\sigma = k^{-1}\sigma'$. Clearly, σ is fixed by θ. This implies assertion (i). By definition, $\sigma \cdot P_m = P_m$, and by (i) it is clear that $M_m A_m = P_m \cap \theta(P_m)$ is preserved by σ. Thus σ preserves M_m and A_m in view of their definitions. This proves (ii). As σ preserves the roots of $\mathfrak{a}_m$, we need only prove that σ has finite order on $\mathfrak{a}_G$ to obtain (iii). As G^+ has finitely many connected components, the element σ^n lies in G for some minimal positive integer n. Hence, $\operatorname{Ad}\sigma^{\mathrm{n}}$ is trivial on the center of G, and (iii) is proved. Finally, recall that B restricted to $[\mathfrak{g}, \mathfrak{g}]$ is a multiple of the Killing form. Hence, it is invariant under $\operatorname{Ad}\sigma$. Also, the center $\mathfrak{z}$ of $\mathfrak{g}$ is orthogonal to $[\mathfrak{g}, \mathfrak{g}]$ with respect to B. After possibly averaging B over the finite group of automorphisms generated by $\operatorname{Ad}\sigma$ restricted to $\mathfrak{z}$, one may assume that B is $\operatorname{Ad}\sigma$-invariant. □

3.2. The definition of DDT

The next lemma is a slight improvement of [M2, Proposition 4.1].

LEMMA 2

Let $(M, \delta) \in$ DD, and let $P = MAN$ be the standard parabolic subgroup of G with Levi subgroup MA. If there exists a σ-stable irreducible subrepresentation of $\pi^P_{\delta,\lambda}$ for some $\lambda \in i\overline{C}_\delta$, then there exists k_δ in K such that

(i) *conjugation by $\tau_\delta := k_\delta^{-1}\sigma$ leaves A_m, A, M, Δ_δ^+, and the equivalence class of δ invariant; and*

(ii) *the automorphism $\operatorname{Ad}\tau_\delta$ is of finite order on $\mathfrak{a}$, commutes with θ, and preserves B.*

Proof

(i) For $\lambda \in i\overline{C}_\delta$, the representation $(\pi^P_{\delta,\lambda})^\sigma$ is equivalent to $\pi^{\sigma\cdot P}_{\delta^\sigma,\sigma\lambda}$ (see [M2, proof of Proposition 3.1], the intertwining map being given by $\varphi \mapsto \varphi^\sigma$ with the notation of Section 2.1 in the compact realizations). The Langlands disjointness theorem [L, pages 149, 150] provides $k \in K$ with $k\sigma \cdot M = M$, $k\sigma \cdot A = A$, and δ^σ equivalent to δ^k. After possibly multiplying by a representative in K of a suitable element of W_δ^0 and then by an element of $K \cap M$, one finds the desired k_δ.

(ii) The automorphism $\operatorname{Ad}\tau_\delta$ preserves B, since the actions of $\operatorname{Ad}k_\delta$ and $\operatorname{Ad}\sigma$ preserve B. The action of $\operatorname{Ad}\tau_\delta$ on the center is equal to that of σ, as $\operatorname{Ad}k_\delta$ acts trivially. Moreover, $\operatorname{Ad}\tau_\delta$ preserves the Killing form. Hence, it commutes with θ and preserves B (see Lemma 1(iv)). The assertion on the finite order of $\operatorname{Ad}\tau_\delta$ on $\mathfrak{a}$ follows from the fact that it permutes the roots of $\mathfrak{a}$ and agrees with $\operatorname{Ad}\sigma$ on the center of $\mathfrak{g}$ (see Lemma 1(iii)). □

We denote by DDT the set of elements $(M, \delta) \in$ DD satisfying the hypotheses of Lemma 2. The Langlands disjointness theorem tells us that an element $(M, \delta) \in$ DD is in DDT if and only if $(\pi^P_{\delta,0})^\sigma$ is equivalent to $\pi^P_{\delta,0}$. For each $(M,\delta) \in$ DDT, we fix a k_δ and τ_δ as in Lemma 2.

Remark 1
There are several choices for DD and DDT. In the case of base change (see [D2, Sections 3 – 5]), the set DDT was chosen so that k_δ was always trivial, and hence, τ_δ was equal to σ. Such a choice may not be possible in general.

3.3. The action of τ_δ on R^c_δ and $A(\delta)$
Fix $\delta \in$ DDT, and let $P = MAN$ be its standard parabolic subgroup. We choose a unitary operator U_δ on the space of δ so that

$$\delta^{\tau_\delta}(m) = U_\delta^{-1}\delta(m)U_\delta, \quad m \in M. \tag{3.1}$$

Recall that we have fixed $\mu_0 \in A(\delta)$ in Section 2.2. For the remainder of this article, we denote the differential of $\mathrm{Ad}\,\tau_\delta$ by τ_δ whenever the action on $\mathfrak{a}$ is required. With the same abuse of notation, we denote $\mathrm{Ad}\,k_\delta$ by k_δ and $\mathrm{Ad}\,\sigma$ by σ.

LEMMA 3
Suppose that $r \in \mathrm{R}^c_\delta$. Then, the element $\tau_\delta(r) := \tau_\delta\, r\, \tau_{\delta|\mathfrak{a}}^{-1}$ belongs to R^c_δ as a group of automorphisms of $\mathfrak{a}$.

Proof
As conjugation by τ_δ preserves K and A (see Lemma 2), it is clear that $\tau_\delta(r)$ is in $W(A)$. Moreover, as τ_δ preserves δ and Δ^+_δ (see Lemma 2), it follows that $\tau_\delta(r)$ preserves them as well. □

LEMMA 4
For every $\mu \in A(\delta)$, one has $\mu^{\tau_\delta} \in A(\delta)$.

Proof
Suppose that $\mathfrak{k}_\mathbb{C}$ is the complexification of the Lie algebra of K, and suppose that $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{k}_\mathbb{C}$ containing the Lie algebra $\mathfrak{t}$ of a Cartan subgroup of K. Let γ be a highest weight of μ with respect to $\mathfrak{b}$ (i.e., a highest weight of some irreducible constituent of μ restricted to the identity component of K). Let ρ_c be the half-sum of the roots of $\mathfrak{t}$ in $\mathfrak{b}$. As Cartan subgroups of K and Borel subalgebras of $\mathfrak{k}_\mathbb{C}$ are conjugate by elements of K, there exists k in K such that $\mathrm{Ad}\,(k\tau_\delta)\mathfrak{b} = \mathfrak{b}$ and $\mathrm{Ad}(k\tau_\delta)\mathfrak{t} = \mathfrak{t}$. It is clear that $k\tau_\delta(\gamma)$ is a highest weight of μ^{τ_δ}. As k and τ_δ preserve B, it follows that

$$\|\gamma + 2\rho_c\| = \|k\tau_\delta\gamma + 2k\tau_\delta\rho_c\| = \|k\tau_\delta\gamma + 2\rho_c\|.$$

From the definition of minimal K-types (see [V, Definition 5.1] for G connected as a Lie group, and see [CD1, page 433] in general), this implies that μ^{τ_δ} is an element of $A(\delta)$. □

Using Lemma 4 and the simply transitive action of $\hat{R}_\delta^c$ on $A(\delta)$, we define $\chi_0 \in \hat{R}_\delta^c$ so that

$$\mu_0^{\tau_\delta} = \chi_0 \cdot \mu_0$$

or, equivalently,

$$\chi_{\mu_0{}^{\tau_\delta}} = \chi_0. \tag{3.2}$$

3.4. τ_δ-stable and σ-stable representations

We are now prepared to determine exactly which of the representations defined in (2.7) are τ_δ-stable or σ-stable.

LEMMA 5

(i) *The following defines a unitary operator T_δ on $I(\delta)$ (see (3.1)):*

$$\big(T_\delta(\varphi)\big)(k) := U_\delta \varphi^{\tau_\delta}(k) = U_\delta \varphi(\tau_\delta^{-1} \cdot k), \quad \varphi \in I(\delta),\ k \in K.$$

(ii) *For all $\lambda \in i\mathfrak{a}^*$, T_δ intertwines $(\pi_{\delta,\lambda}^P)^{\tau_\delta}$ with $\pi_{\delta,\tau_\delta\lambda}^{\tau_\delta \cdot P}$.*

(iii) *For all $\mu \in A(\delta)$, T_δ sends $I^\mu(\delta)$ to $I^{\mu^{\tau_\delta}}(\delta)$.*

(iv) *For $w \in W_\delta$, one has*

$$T_\delta A(P, w, \delta, \lambda) T_\delta^{-1} = \chi_0(w) A\big(\tau_\delta \cdot P, \tau_\delta(w), \delta, \tau_\delta\lambda\big), \quad \lambda \in i\mathfrak{a}^*.$$

(v) *For $\mu \in A(\delta)$, $\chi_{\mu^{\tau_\delta}} = \chi_0 \chi_\mu^{\tau_\delta}$.*

(vi) *If $H \in \mathrm{Diag}(\delta)$ and $\lambda \in i\mathfrak{a}^{*H}$, then the operator T_δ intertwines $(\pi_{\delta,H,\chi,\lambda}^P)^{\tau_\delta}$ with $\pi_{\delta,H',\chi',\tau_\delta\lambda}^{\tau_\delta \cdot P}$, where $H' = \tau_\delta(H)$ and $\chi' = \chi_{0|\tau_\delta(H)} \chi^{\tau_\delta}$.*

Proof

(i) If $\psi = T_\delta(\varphi)$, $k \in K$, and $m \in K \cap M$, then, using (3.1), one has

$$\psi(km) = U_\delta \delta(\tau_\delta^{-1} \cdot m^{-1})\big(\varphi(\tau_\delta^{-1} \cdot k)\big) = \delta(m^{-1}) U_\delta \varphi(\tau_\delta^{-1} \cdot k) = \delta(m^{-1})\psi(k).$$

This proves that T_δ is an operator on $I(\delta)$. It is evident that it is unitary.

(ii) The operator T_λ corresponding to T_δ in the noncompact realization is given by

$$\big(T_\lambda(\varphi)\big)(g) := U_\delta\big(\varphi(\tau_\delta^{-1} \cdot g)\big), \quad \varphi \in I_{\delta,\lambda}^P,\ g \in G.$$

Its image lies in $I^{\tau_\delta \cdot P}_{\delta,\tau_\delta\lambda}$ by computations similar to those in (i), and it intertwines $(\tilde{\pi}^P_{\delta,\lambda})^{\tau_\delta}$ with $\tilde{\pi}^{\tau_\delta \cdot P}_{\delta,\tau_\delta\lambda}$. Assertion (ii) now follows easily.

(iii) Let π_δ be the representation of K on $I(\delta)$. Let (μ, V_μ) be a model for $\mu \in \hat{K}$, and let $T_\mu : V_\mu \to I^\mu(\delta)$ be a unitary intertwining operator. Then, $T_\delta \circ T_\mu$ intertwines $(\mu^{\tau_\delta}, V_\mu)$ with the image of $I^\mu(\delta)$ under T_δ, as may be seen from

$$\pi_\delta(k) T_\delta T_\mu v = T_\delta \pi_\delta\big(\tau_\delta^{-1}(k)\big) T_\mu v = T_\delta T_\mu \mu^{\tau_\delta}(k) v, \quad k \in K,\ v \in V_\mu.$$

(iv) Both sides of the equation to be proved intertwine $\pi^{\tau_\delta \cdot P}_{\delta,\tau_\delta\lambda}$ with $\pi^{\tau_\delta \cdot P}_{\delta,\tau_\delta w\lambda}$. To see this, one may use the intertwining properties of the various operators and the fact that if an operator intertwines π and π', then it also intertwines π^{τ_δ} and π'^{τ_δ}. On the open dense set of $i\mathfrak{a}^*$, where these representations are irreducible, both operators are proportional. To see that they are equal, it is therefore enough to verify that they are equal on the isotypic component of ${\mu_0}^{\tau_\delta}$. This is obvious from (iii), (2.4), and (3.2).

(v) The desired equation follows by restricting the equation of (iv) to the isotypic component of $\mu^{\tau_\delta} \in A(\delta)$, recalling the definitions of χ_μ and $\chi_{\mu^{\tau_\delta}}$, and then applying (iii).

(vi) From (iii), the minimal K-types of $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ are of the form μ^{τ_δ}, where μ runs through the minimal K-types of $\pi^P_{\delta,H,\chi,\lambda}$. Using (v), they are seen to be the elements $\mu' \in A(\delta)$ such that $\chi_{\mu'|\tau_\delta(H)} = \chi_{0|\tau_\delta(H)}\chi^{\tau_\delta}$. The claim follows from (2.7). □

The following theorem is a classification of the σ-stable irreducible tempered representations.

THEOREM 1

(i) *Every irreducible tempered σ-stable representation of G is equivalent to $\pi^P_{\delta,H,\chi,\lambda}$ for some $\delta \in$ DDT, $\lambda \in i\overline{C}_\delta$, $H = \mathrm{R}^c_{\delta,\lambda}$, $\chi \in \hat{H}$ such that*
   (1) *H is τ_δ-stable;*
   (2) *the character χ satisfies $\chi^{\tau_\delta} = \chi_{0|H}\chi$ (the set of such characters of H is denoted by $\hat{H}(\tau_\delta)$); and*
   (3) *there exists $r \in \mathrm{R}^c_\delta$ such that $r\tau_\delta\lambda = \lambda$.*

The set of such subgroups of R^c_δ, corresponding to r, is denoted $\mathrm{Diag}(\tau_\delta, r)$.

(ii) *Two representations $\pi^P_{\delta,H,\chi,\lambda}$ and $\pi^{P'}_{\delta',H',\chi',\lambda'}$ as in (i) are equivalent if and only if $\delta = \delta'$, $H = H'$, $\chi = \chi'$, and there exists $r \in \mathrm{R}^c_\delta$ such that $\lambda' = r\lambda$.*

Proof

(i) By Lemma 2 and Section 2.2, every irreducible tempered σ-stable representation of G is equivalent to $\pi^P_{\delta,H,\chi,\lambda}$ for some $\delta \in$ DDT, $\lambda \in i\overline{C}_\delta$. Since $\tau_\delta = k_\delta\sigma$, the representation $(\pi^P_{\delta,H,\chi,\lambda})^\sigma$ is equivalent to $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$. By Lemma 5(vi), we have the fact that $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ is equivalent to $\pi^{\tau_\delta\cdot P}_{\delta,H',\chi',\tau_\delta\lambda}$, where $H' = \tau_\delta(H)$ and $\chi' = \chi_{0|\tau_\delta(H)}\chi^{\tau_\delta}$. As A is τ_δ-stable (see Lemma 2), the parabolic subgroup $\tau_\delta \cdot P$ has MA as a Levi subgroup, and $\pi^{\tau_\delta\cdot P}_{\delta,H',\chi',\tau_\delta\lambda}$ is equivalent to $\pi^P_{\delta,H',\chi',\tau_\delta\lambda}$ (see [KS, Theorem 8.4]). Thus $\pi^P_{\delta,H,\chi,\lambda}$ is τ_δ-stable if and only if $\pi^P_{\delta,H,\chi,\lambda}$ and $\pi^P_{\delta,H',\chi',\tau_\delta\lambda}$ are equivalent. Using the third sentence following (2.7) and the fact that τ_δ preserves $\overline{C}_\delta$ (see Lemma 2), this is true if and only if $H = H'$, $\chi = \chi'$, and $\tau_\delta\lambda = r\lambda$ for some $r \in \mathrm{R}^c_\delta$.

(ii) The assertion follows from the classification of tempered irreducible representations in Section 2.2. □

The σ-stable representations of Theorem 1 are induced from parabolic subgroups that are not necessarily stable under the actions of any σ, τ_δ, or $r\tau_\delta$. We close this section by relating our σ-stable representations to some representations that are induced from parabolic subgroups that are stable under the action of $r\tau_\delta$.

PROPOSITION 1

Let $(M, \delta) \in \mathrm{DDT}$, let $\lambda \in i\overline{C}_\delta$, and let π be an irreducible subrepresentation of $\pi^P_{\delta,\lambda}$. Suppose that π is equivalent to $\pi^{M_1A_1}_{\delta_1,\lambda}$, where M_1A_1 is equal to $M^\lambda A^\lambda$ and (M_1, δ_1) is a nondegenerate limit of discrete series strongly affiliated to δ, as follows from Section 2.2. Recall that $\mathfrak{a}_1^*$ may be viewed as a subspace of $\mathfrak{a}^*$, and recall that $\lambda \in i\mathfrak{a}_1^*$ (see (2.12)). Let Q be the parabolic subgroup of G whose Lie algebra is the sum of the $\mathfrak{a}$ root spaces for roots α with $\mathrm{Im}(\alpha, \lambda) \geq 0$. Its Levi subgroup M_QA_Q is the centralizer in G of λ and contains M_1A_1. Then, the representation π is σ-stable if and only if there exists $r \in \mathrm{R}^c_\delta$ such that $r\tau_\delta\lambda = \lambda$, $\tilde{r}\tau_\delta$ normalizes M_QA_Q and Q, and $(\pi^{P\cap M_Q}_{\delta_1,0})^{\tilde{r}\tau_\delta}$ is equivalent to $\varepsilon := \pi^{P\cap M_Q}_{\delta_1,0}$, where $\tilde{r}$ is a representative in K of r. Moreover, π is equivalent to $\pi^Q_{\varepsilon,\lambda}$ and $\tilde{r}\tau_\delta \cdot Q = Q$.

Proof

We must determine when π is τ_δ-stable. First, Q contains a parabolic subgroup P_1 with Levi subgroup M_1A_1. Indeed, let $P_1' = M_1A_1N_1'$ be a parabolic subgroup of G with Levi subgroup M_1A_1. Then, $P_1 = (P_1' \cap M_Q)N_Q$ has the required property. Thus $\pi^{M_1A_1}_{\delta_1,\lambda}$ is equivalent to $\pi^{P_1}_{\delta_1,\lambda}$. Applying induction in stages, π is equivalent to $\pi^Q_{\pi^{P_1\cap M_Q}_{\delta_1,O},\lambda}$. Let us show that the conditions of the theorem are sufficient. Suppose that they are satisfied. As $r\tau_\delta\lambda = \lambda$, the parabolic subgroup Q satisfies $\tilde{r}\tau_\delta \cdot Q = Q$. The resulting representation is induced from an $\tilde{r}\tau_\delta$-stable parabolic subgroup and an $\tilde{r}\tau_\delta$-stable representation. This implies that the induced representation is $\tilde{r}\tau_\delta$-stable. It is then also τ_δ- and σ-stable, as $\tilde{r}, k_\delta \in G$. This proves that the conditions are sufficient.

Let us prove that they are necessary. According to Theorem 1(i), there exists $r \in \mathrm{R}^c_\delta$ such that $r\tau_\delta\lambda = \lambda$. It follows from Lemma 2 that τ_δ normalizes $\mathrm{R}^c_{\delta,\lambda}$. Hence, $\tilde{r}\tau_\delta$ normalizes $A_1 = A^\lambda$ (see Section 2.2) and M_1. Therefore, it suffices to determine when $\pi^{M_1A_1}_{\delta_1,\lambda}$ is $\tilde{r}\tau_\delta$-stable. The representation $(\pi^{M_1A_1}_{\delta_1,\lambda})^{\tilde{r}\tau_\delta}$ is easily seen to be equivalent to $\pi^{M_1A_1}_{\delta'_1,\lambda}$, where δ'_1 is equal to $\delta_1^{\tilde{r}\tau_\delta}$. Thus π is τ_δ-stable if and only if $\pi^{M_1A_1}_{\delta'_1,\lambda}$ and $\pi^{M_1A_1}_{\delta_1,\lambda}$ are equivalent. By Section 2.2, this is true if and only if there exists a $w \in N_K(A_1)$ fixing λ and conjugating δ'_1 with δ_1. Finally, if w fixes λ, it is in M_Q by definition, which implies that $(\pi^{P\cap M_Q}_{\delta_1,0})^{\tilde{r}\tau_\delta}$ is equivalent to $\pi^{P\cap M_Q}_{\delta_1,0}$. □

4. The classification of irreducible admissible σ-stable representations

We now turn to the resolution of goal (B), our second principal goal. The results of this section essentially appear in [M2, Section 3]. We include a brief review for the sake of completeness and convenience.

Suppose that $P = MAN$ is a parabolic subgroup, and suppose that ρ is an irreducible tempered representation of M. Suppose further that λ lies in the complexification of $\mathfrak{a}^*$, and suppose that its real part lies in the positive chamber of $\mathfrak{a}^*$ determined by P. Langlands [L, Section 3] has shown that the induced representation $\pi^P_{\rho,\lambda}$ has a unique irreducible quotient $J^P_{\rho,\lambda}$ and that every irreducible admissible representation of G is (infinitesimally) equivalent to some such Langlands quotient. He also proved that two Langlands quotients, $J^P_{\rho,\lambda}$ and $J^{P'}_{\rho',\lambda'}$, are equivalent if and only if there exists $h \in G$ such that $h \cdot P = P'$, ρ^h is equivalent to ρ', and $h\lambda = \lambda'$ (i.e., Langlands quotients are unique up to conjugation). This allows us to restrict our classification to standard parabolic subgroups P, as all parabolic subgroups are conjugate to a (unique) standard parabolic subgroup.

THEOREM 2
Suppose that P is a standard parabolic subgroup of G, and suppose that $J^P_{\rho,\lambda}$ is a Langlands quotient as defined in the preceding paragraph. Then, $J^P_{\rho,\lambda}$ is σ-stable if and only if $\sigma \cdot P = P$, ρ^σ is equivalent to ρ and $\sigma\lambda = \lambda$.

Proof
We provide a sketch, leaving the details to [M2, proof of Proposition 3.1]. Composition by σ provides an intertwining operator between $(\pi^P_{\rho,\lambda})^\sigma$ and $\pi^{\sigma\cdot P}_{\rho^\sigma,\sigma\lambda}$. This equivalence induces an equivalence between the Langlands quotients $(J^P_{\rho,\lambda})^\sigma$ and $J^{\sigma\cdot P}_{\rho^\sigma,\sigma\lambda}$. The "if" part of the theorem is now immediate. For the converse, we see that $J^P_{\rho,\lambda}$ is equivalent to $J^{\sigma\cdot P}_{\rho^\sigma,\sigma\lambda}$, and since $\sigma.P_m = P_m$, the parabolic subgroup $\sigma \cdot P$ remains standard. The theorem then follows from the uniqueness of Langlands quotients up to conjugation. □

Taken together, both Theorems 1 and 2 constitute a classification of irreducible admissible σ-stable representations of G.

5. Intertwining operators and twisted characters

5.1. Representations of G^+ and twisted characters

The group G^+/G is cyclic and generated by the coset of any element $\tau \in G'$. Let us look at an irreducible unitary representation π^+ of G^+ whose restriction to G is an irreducible representation π of G. Let $\tau \in G'$, and let n be the order of the image of τ in G^+/G so that $\tau^n = g \in G$. Then, $T = \pi^+(\tau)$ is an intertwining operator between π^τ and π such that $T^n = \pi^+(\tau^n) = \pi(g)$. Conversely, if T is such an intertwining operator, then it defines a representation of G^+.

If T' is another operator that intertwines π^τ with π, then T'^n is proportional to $\pi(g)$ by Schur's lemma. Thus there are exactly n distinct choices for a constant $c \in \mathbb{C}$ such that $T = cT'$ and $T^n = \pi(g)$.

The τ-twisted character Θ_π^τ of π^+ is the distribution on G defined by

$$\Theta_\pi^\tau(f) = \operatorname{tr}\bigl(\pi(f)\pi^+(\tau)\bigr), \quad f \in C_c^\infty(G).$$

Now, suppose that τ, σ belong to G', and suppose that $\sigma = k\tau$ for $k \in K$. Then, one has

$$\Theta_\pi^\sigma = \mathrm{R}_k \Theta_\pi^\tau, \tag{5.1}$$

where R denotes the right-regular representation of G. We speak of a τ-twisted character of π more generally by replacing $\pi^+(\tau)$ above with any intertwining operator between π^τ and π.

5.2. The definition of the operators $T(\delta, r, H, \chi, \lambda)$

LEMMA 6

In the notation of Theorem 1 and Lemma 5, one has the following.

(i) *The operator $T(\delta, r, \lambda)$, defined by*

$$T(\delta, r, \lambda) := A(P, r, \delta, \tau_\delta\lambda)A(P, \tau_\delta \cdot P, \delta, \tau_\delta\lambda)T_\delta, \quad r \in \mathrm{R}_\delta^c,\ \lambda \in i\mathfrak{a}^{*r\tau_\delta},$$

intertwines $(\pi_{\delta,\lambda}^P)^{\tau_\delta}$ with $\pi_{\delta,r\tau_\delta\lambda}^P$.

(ii) *$T(\delta, r, \lambda)$ sends the isotypic component $I^\mu(\delta)$ to $I^{\mu^{\tau_\delta}}(\delta)$ for every $\mu \in A(\delta)$. It is independent of λ, and it is equal to a scalar multiple of T_δ on each $I^\mu(\delta)$.*

(iii) *Let $\tilde{r}$ be a representative in K of $r \in \mathrm{R}_\delta^c$, and let $u_{\tilde{r}}$ be a unitary intertwining operator between $\tilde{r}\delta$ and δ. There then exists $c \in \mathbb{C}$ such that*

$$T(\delta, r, \lambda) = cu_{\tilde{r}}\mathrm{R}_{\tilde{r}}A(r \cdot P, \tau_\delta \cdot P, \delta, \tau_\delta\lambda)T_\delta, \quad \lambda \in i\mathfrak{a}^{*r\tau_\delta}.$$

(iv) *Let* $T(\delta, r, H, \chi, \lambda)$ *be the restriction of* $T(\delta, r, \lambda)$ *to the space of* $\pi^P_{\delta,H,\chi,\lambda}$ *for each* $H \in \mathrm{Diag}(\tau_\delta, r)$, $\chi \in \hat{H}(\tau_\delta)$, *and* $\lambda \in i\mathfrak{a}^{*r\tau_\delta}$. *Then the operator* $T(\delta, r, H, \chi, \lambda)$ *intertwines* $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ *with* $\pi^P_{\delta,H,\chi,\lambda}$.

Proof
As T_δ intertwines $(\pi^P_{\delta,\lambda})^{\tau_\delta}$ with $\pi^{\tau_\delta \cdot P}_{\delta,\tau_\delta\lambda}$ (see Lemma 5(i), (ii)), assertion (i) follows from the properties of intertwining operators of [KS, Section 8]. The first part of assertion (ii) follows from Lemma 5(iii). The second part follows from the intertwining properties of our normalized intertwining operators (see (2.1), (2.4)). Using equation (2.6) and taking into account the fact that $r = r^{-1}$, one has

$$T(\delta, r, \lambda) = c\, u_{\tilde{r}}\, \mathrm{R}_{\tilde{r}}\, A(r \cdot P, P, \delta, \tau_\delta\lambda)\, A(P, \tau_\delta \cdot P, \delta, \tau_\delta\lambda)\, T_\delta.$$

Assertion (iii) therefore results from the properties of our normalized intertwining operators (see (2.2)). As $\pi^P_{\delta,H,\chi,\lambda}$ is τ_δ-stable, its set of minimal K-types is also τ_δ-stable (see Lemma 4). Assertion (iv) therefore follows from (i), (ii), and the definitions of $\pi^P_{\delta,H,\chi,\lambda}$ and $\hat{H}(\tau_\delta)$ (see Theorem 1(i)). □

Remark 2
In Lemma 6, one may replace P by any other parabolic subgroup with the same Levi subgroup. This is also the case for Lemma 5.

Lemma 6 allows us to define the twisted character

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}(f) = \mathrm{tr}\big(\pi^P_{\delta,H,\chi,\lambda}(f)\, T(\delta, r, H, \chi, \lambda)\big), \quad f \in C^\infty_c(G). \tag{5.2}$$

5.3. Some auxiliary operators
This section is inspired by [M2, Section 5]. We begin with a simple remark:

> let $L \subset L'$ be two Levi subgroups of G which contain A_m. Let $P' = L'N'$ be a parabolic subgroup of G with Levi subgroup L'. Then, there exists a parabolic subgroup P of G with Levi subgroup L such that $P \subset P'$. (5.3)

Indeed, if $P_{L'}$ is any parabolic subgroup of L' with Levi subgroup L, then $P = P_{L'}N'$ has the required property.

Let $(M, \delta) \in \mathrm{DDT}$, let $P = MAN$ be a standard parabolic subgroup of G, let $r \in \mathrm{R}^c_\delta$, and let $H \in \mathrm{Diag}(\tau_\delta, r)$. By the definition of $\mathrm{Diag}(\tau_\delta, r)$, there exists $\lambda \in i\mathfrak{a}^*$ such that $\mathrm{R}^c_{\delta,\lambda} = H$, $r\tau_\delta\lambda = \lambda$, and $\lambda \in i\bar{C}_\delta$. Furthermore, the centralizer of $\mathfrak{a}^H$ is a Levi subgroup $L_1 = M_1A_1$ with $\mathfrak{a}_1 = \mathfrak{a}^H$ (see Section 2.2). Thus, the set O of $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$ with $\mathrm{R}^c_{\delta,\lambda} = H$ is nonempty. Hence, the fixed-point set of $r' \in \mathrm{R}^c_\delta \setminus H$ in $i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$ is a proper subspace. The complement of O in $i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$ is equal to the possibly trivial finite union of these proper subspaces. According to Baire's theorem, the set O is open and dense in $i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$.

The finite group $\langle r\tau_\delta, H\rangle$ preserves C_δ. Therefore, we may average an element of iC_δ under the action of $\langle r\tau_\delta, H\rangle$ to conclude that $iC_\delta \cap i\mathfrak{a}^{\langle r\tau_\delta, H\rangle} \neq \emptyset$. The density of O now implies that $iC_\delta \cap O \neq \emptyset$.

Let us show that there exists $\lambda_0 \in iC_\delta \cap O$ such that for any $\mathfrak{a}$-weight α, α vanishes on $i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$ if and only if it vanishes on λ_0. Suppose that α is a weight of $\mathfrak{a}$ in $\mathfrak{g}$ which does not vanish on $\mathfrak{a}^{\langle r\tau_\delta, H\rangle}$, and let O_α be the complement of the kernel of α in $\mathfrak{a}^{\langle r\tau_\delta, H\rangle}$; it is the complement of a hyperplane and, as such, is open and dense. The intersection of all the sets O_α is dense by Baire's theorem, and it therefore intersects the nonempty open set $iC_\delta \cap O$. Therefore, we may take λ_0 in this intersection. The element λ_0 satisfies the desired property.

Let $Q = M_Q A_Q N_Q$ be the parabolic subgroup of G whose Lie algebra is the sum of the weight spaces of the $\mathfrak{a}$-weights α so that $\mathrm{Im}(\alpha, \lambda_0) \geq 0$. This implies that the Levi subgroup $M_Q A_Q$ is the centralizer in G of λ_0. Since λ_0 is $r\tau_\delta$-invariant, one has

$$\tilde{r}\tau_\delta \cdot Q = Q, \tag{5.4}$$

where $\tilde{r}$ is a representative in K of r. Notice that the set of weights of $\mathfrak{a}$ whose weight spaces lie in $\mathfrak{n}_Q$ contains Δ_δ^+, as $\lambda_0 \in C_\delta$. Notice also that $\mathfrak{a}_Q$ is the intersection of the kernels of those weights that vanish on λ_0. Recalling the main property of λ_0 above, this implies that $\mathfrak{a}_Q$ contains $\mathfrak{a}^{\langle r\tau_\delta, H\rangle}$. Let $M_1 A_1 = M^{\lambda_0} A^{\lambda_0}$ be a strongly affiliated Levi subgroup as defined in Section 2.2. The group $M_1 A_1$ centralizes $\mathfrak{a}^{\langle r\tau_\delta, H\rangle}$, which contains λ_0. As $M_Q A_Q$ is the centralizer of λ_0, one has

$$MA \subset M_1 A_1 \subset M_Q A_Q.$$

By (5.3), there exist parabolic subgroups P' and P_1 of G with Levi subgroups MA and $M_1 A_1$, respectively, such that

$$P' \subset P_1 \subset Q.$$

As $P' \subset Q$ and Q is $\tilde{r}\tau_\delta$-stable, we have

$$\theta(P') \cap \tilde{r}\tau_\delta \cdot P' \subset \theta(Q) \cap \tilde{r}\tau_\delta \cdot Q = \theta(Q) \cap Q = M_Q A_Q. \tag{5.5}$$

LEMMA 7

(i) *The operator*

$$T' := A(P', \delta, r, \tau_\delta\lambda) A(P', \tau_\delta \cdot P', \delta, \lambda) T_\delta, \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle},$$

is independent of $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$ *and intertwines* $(\pi_{\delta,\lambda}^{P'})^{\tau_\delta}$ *with* $\pi_{\delta,\lambda}^{P'}$.

(ii) *The space of* $\pi^{P'}_{\delta,H,\chi,\lambda}$ *is independent of* $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, *and*

$$\pi^{P'}_{\delta,H,\chi,\lambda} = \pi^{Q}_{\varepsilon,\lambda}, \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle},$$

where Q *is* $\tilde{r}\tau_\delta$*-stable and*

$$\varepsilon = \pi^{P'\cap(M_Q A_Q)}_{\delta_1,0}$$

is an $\tilde{r}\tau_\delta$*-stable representation of* M_Q.

(iii) *Let* $T'(\delta, r, H, \chi)$ *be the restriction of* T' *to the space of* $\pi^{P'}_{\delta,H,\chi,\lambda}$. *It intertwines* $(\pi^{P'}_{\delta,H,\chi,\lambda})^{\tau_\delta}$ *with* $\pi^{P'}_{\delta,H,\chi,\lambda}$.

(iv) *There exists* $c \in \mathbb{C}$ *such that for all* $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, *the operator* $T(\delta, r, H, \chi, \lambda)$ *of Lemma 6 is equal to the restriction of the operator* $cA(P', P, \delta, \lambda)^{-1}T'A(P', P, \delta, \lambda)$ *to the space of* $\pi^{P}_{\delta,H,\chi,\lambda}$.

(v) *For* $H \in \mathrm{Diag}(\tau_\delta, r)$, $\chi \in \hat{H}(\tau_\delta)$, *and* $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, *we have*

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}(f) = c\,\mathrm{tr}\big(\pi^{P'}_{\delta,\lambda}(f)T'(\delta, r, H, \chi)\big), \quad f \in C^\infty_c(G).$$

Proof

(i) By Remark 2 and Lemma 6(iii) applied to P', one sees that T' has the desired intertwining property and that

$$A(P', \delta, r, \tau_\delta\lambda)A(P', \tau_\delta\cdot P', \delta, \lambda) = c\,u_{\tilde{r}}\,\mathrm{R}_{\tilde{r}}\,A(r\cdot P', \tau_\delta\cdot P', \delta, \tau_\delta\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}.$$

As $r(\mathfrak{a}^{\langle r\tau_\delta,H\rangle}) \subset r\mathfrak{a}_Q$, the element $\tau_\delta\lambda = r\lambda$ lies in $i(r\mathfrak{a}_Q)^*$ for any $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$. It follows from (2.3) and (5.5) that $A(r\cdot P', \tau_\delta\cdot P', \delta, \tau_\delta\lambda)$ is an intertwining operator that does not depend on $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$. This shows that T' is independent of $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$.

(ii) For $\lambda \in O\cap iC_\delta$, one has $M^\lambda A^\lambda = M_1A_1$ (see Section 2.2). As $P' \subset P_1 \subset Q$, we may apply induction in stages, as in equation (2.14). In the context at hand, this equation takes the form

$$\pi^{P'}_{\delta,H,\chi,\lambda} = \pi^{Q}_{\varepsilon,\lambda}, \quad \lambda \in (O \cap iC_\delta) \subset i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle},$$

where

$$\varepsilon = \pi^{P'\cap(M_Q A_Q)}_{\delta_1,0}.$$

When $\lambda \in iC_\delta \cap O$, the first assertion of (ii) follows from an analysis of the induction in stages in the compact realization. It follows for all $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$ by analytic continuation. The $\tilde{r}\tau_\delta$-stability of Q has been shown in (5.4). For $\lambda = \lambda_0$, finally, the parabolic subgroup Q and the representation ε are as in Proposition 1, and so the $\tilde{r}\tau_\delta$-stability of ε follows.

(iii) This assertion follows from Remark 2 and Lemma 5 applied to P'.

(iv) We make use of the following fact. Let π, π' be two unitary representations of G of finite length, and let A (resp., S) be an invertible intertwining operator between π and π' (resp., π^σ and π). Then, $S' = ASA^{-1}$ is an intertwining operator between π'^σ and π', and

$$\operatorname{tr}\bigl(\pi'(f)S'\bigr) = \operatorname{tr}\bigl(\pi(f)S\bigr), \quad f \in C_c^\infty(G). \tag{5.6}$$

The operators $T(\delta, r, H, \chi, \lambda)$ and $A(P, P', \delta, \lambda)^{-1}T'A(P, P', \delta, \lambda)$ are intertwining operators between $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ and $\pi^P_{\delta,H,\chi,\lambda}$ for $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$. For λ in the nonempty open set $iC_\delta \cap O$ (see (ii) above) of $i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, the representation $\pi^P_{\delta,H,\chi,\lambda}$ is irreducible (see Section 2.2). These two operators are proportional by Schur's lemma. The two operators are independent of λ on each minimal K-type and are proportional to T_δ (see equation (2.1) and Lemma 6(ii)). Therefore, the proportionality factor is independent of λ in this open set. The assertion follows by analytic continuation.

(v) This assertion follows from (iii), (iv), and equation (5.6). □

6. Some properties of twisted characters

The properties that we prove here reappear in Theorem 3 as those properties that characterize functions derived from twisted characters. The first property that we prove harkens back to the classical Paley-Wiener theorem. Let $C_c^\infty(G, K)_t$ be the space of smooth functions on G, which are left and right K-finite and whose support is contained in $K\exp(B_t)K$, where B_t is the closed ball of radius $t > 0$ about the origin in $\mathfrak{a}_m$. Define $C_c^\infty(\mathfrak{a}^{\langle r\tau_\delta,H\rangle})_t$ to be the space of smooth functions on $\mathfrak{a}^{\langle r\tau_\delta,H\rangle}$ with support in the closed ball of radius t, and define $\mathscr{PW}(\mathfrak{a}^{\langle r\tau_\delta,H\rangle})_t$ to be the image of $C_c^\infty(\mathfrak{a}^{\langle r\tau_\delta,H\rangle})_t$ under the Fourier transform.

LEMMA 8

For each $f \in C_c^\infty(G, K)_t$, the function on $i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, defined by

$$\lambda \mapsto F_{\delta,r,H,\chi}(\lambda) := \Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}(f),$$

is an element of the space $\mathscr{PW}(\mathfrak{a}^{\langle r\tau_\delta,H\rangle})_t$.

Proof

Let $f \in C_c^\infty(G, K)_t$. The K-finiteness of f implies that there exists an orthogonal projection p onto a finite sum of isotypic components of K in $I(\delta)$ such that $\pi^{P'}_{\delta,H,\chi,\lambda}(f) = p\,\pi^{P'}_{\delta,H,\chi,\lambda}(f)\,p$. Substituting the right-hand side into the equality of Lemma 7(ii), $F_{\delta,r,H,\chi}$ is seen to be the restriction to $i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$ of a finite sum of coefficients of the matrix Fourier transform of f. The lemma follows from the properties of the Fourier transform of elements of $C_c^\infty(G, K)_t$ as given in [A1, Lemma 3.1] (see also [CD1, section 2.1]). □

PROPOSITION 2

(i) *Let $r \in \mathrm{R}_\delta^c$, and let $H \in \mathrm{Diag}(\tau_\delta, r)$. If $w \in W_\delta^0$ satisfies $w(i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}) = i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$, then*

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,w\lambda} = \Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}, \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

(ii) *Let $r \in \mathrm{R}_\delta^c$. If H, H' are subgroups in $\mathrm{Diag}(\tau_\delta, r)$ with $H \subset H'$, then*

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda} = \sum_{\chi' \in \hat{H}'(\tau_\delta),\, \chi'_{|H} = \chi} \Theta^{\tau_\delta}_{\delta,r,H',\chi',\lambda}, \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in \mathfrak{a}^{\langle r\tau_\delta, H'\rangle}.$$

(iii) *If $r, s \in \mathrm{R}_\delta^c$ and $H \in \mathrm{Diag}(\tau_\delta, s)$ with $rs \in H$, then $i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle} = i\mathfrak{a}^{*\langle s\tau_\delta, H\rangle}$ and*

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda} = \chi(rs)\Theta^{\tau_\delta}_{\delta,s,H,\chi,\lambda}, \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

(iv) *Suppose that $r, s \in \mathrm{R}_\delta^c$, and let $H \in \mathrm{Diag}(\tau_\delta, r)$. Recall that $\tau_\delta(s)$ is the action of τ_δ on s. Then, H is an element of $\mathrm{Diag}(\tau_\delta, rs\tau_\delta(s))$, and*

$$\Theta^{\tau_\delta}_{\delta,rs\tau_\delta(s),H,\chi,s\lambda} = \chi_0\big(\tau_\delta(s)\big)\, \Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}, \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

(The left-hand side is defined, as our hypothesis on λ implies that $s\lambda \in i\mathfrak{a}^{*\langle rs\tau_\delta(s), H\rangle}$.)

Proof

(i) Let $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$. The restriction $A(\lambda)$ of $A(P, w, \delta, \lambda)$ to the space of $\pi_{\delta,H,\chi,\lambda}$ intertwines $\pi^P_{\delta,H,\chi,\lambda}$ with $\pi^P_{\delta,H,\chi,w\lambda}$. By Lemma 6(i), the operator $T(\delta, r, H, \chi, \lambda)$ intertwines $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ with $\pi^P_{\delta,H,\chi,\lambda}$. Taking into account equation (5.6), it suffices to prove that

$$T'(\lambda) := A(\lambda)^{-1} T(\delta, r, H, \chi, w\lambda) A(\lambda)$$

is equal to $T(\lambda) := T(\delta, r, H, \chi, \lambda)$. Both operators intertwine $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ with $\pi^P_{\delta,H,\chi,\lambda}$. As the representation $\pi^P_{\delta,H,\chi,\lambda}$ and its equivalent representation $(\pi^P_{\delta,H,\chi,\lambda})^{\tau_\delta}$ are generated by their minimal K-types, it is enough to prove this equality on the minimal K-types of $\pi^P_{\delta,H,\chi,\lambda}$. On these minimal K-types, $T(\delta, H, \chi, \lambda)$ does not depend on λ (see Lemma 6(ii)). On the other hand, $A(\lambda)$ is trivial on all of the minimal K-types as $w \in W_\delta^0$ (see (2.4)). The equality $T'(\lambda) = T(\lambda)$ follows.

(ii) From decomposition (2.11), one has

$$\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}(f) = \sum_{\chi' \in \hat{H}',\, \chi'_{|H} = \chi} \mathrm{tr}\big(\pi^P_{\delta,H',\chi',\lambda}(f) T(\delta, r, H, \chi, \lambda)_{|I_{\delta,H',\chi',\lambda}}\big),$$

where $I_{\delta,H',\chi',\lambda}$ denotes the space of $\pi^P_{\delta,H',\chi',\lambda}$. For λ in the nonempty open subset $\{\nu \in i\mathfrak{a}^{*\langle r\tau_\delta,H'\rangle} \cap i\overline{C}_\delta \mid \mathrm{R}^c_{\delta,\nu} = H'\}$ of $i\mathfrak{a}^{*\langle r\tau_\delta,H'\rangle}$, the representation $\pi^P_{\delta,H',\chi',\lambda}$ is irreducible. If $\chi' \notin \hat{H}'(\tau_\delta)$, this representation is neither σ-stable (see Theorem 1(i)) nor τ_δ-stable. In this case, the operator $T(\delta, r, \lambda)$ of Lemma 6 sends the space of this irreducible subrepresentation of $\pi^P_{\delta,\lambda}$ to an orthogonal space. Indeed, this irreducible subrepresentation occurs with multiplicity 1 in the decomposition $\pi^P_{\delta,\lambda}$. Hence, the contribution of this χ' to the sum above is zero. As the operators $T(\delta, r, H, \chi, \lambda)$ and $T(\delta, r, H', \chi', \lambda)$, $\chi' \in \hat{H}'(\tau_\delta)$, are restrictions of $T(\delta, r, \lambda)$, assertion (ii) follows.

(iii) Since $rs \in H$, one has $\mathfrak{a}^{*\langle r\tau_\delta,H\rangle} = \mathfrak{a}^{*\langle s\tau_\delta,H\rangle}$. As in (i), it suffices to prove that

$$T(\delta, r, H, \chi, \lambda) = \chi(rs)T(\delta, s, H, \chi, \lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle} = i\mathfrak{a}^{*\langle s\tau_\delta,H\rangle}. \tag{6.7}$$

Since both sides are intertwining operators between $(\pi^P_{\delta,H,\chi,\lambda})^\tau_\delta$ and $\pi^P_{\delta,H,\chi,\lambda}$ and since the sum of the isotypic components of the minimal K-types of $\pi^P_{\delta,H,\chi,\lambda}$ generates this representation of G, one need only check the equality of these two operators on each such component $I^\mu(\delta)$. These $\mu \in A(\delta)$ satisfy $\chi_{\mu|H} = \chi$. Using equations (2.1), (2.4), and Lemma 6(i), (ii), one sees that the restriction of $T(\delta, r, H, \chi, \lambda)$ to such an $I^\mu(\delta)$ is equal to

$$\chi_{\mu^{\tau_\delta}}(r)\, c^{\mu^{\tau_\delta}}(P, \tau_\delta \cdot P)\, T_{\delta|I^\mu(\delta)}.$$

Similarly, the restriction of $T(\delta, s, H, \chi, \lambda)$ to such an $I^\mu(\delta)$ is equal to

$$\chi_{\mu^{\tau_\delta}}(s)\, c^{\mu^{\tau_\delta}}(P, \tau_\delta \cdot P)\, T_{\delta|I^\mu(\delta)}.$$

As $\pi^P_{\delta,H,\chi,\lambda}$ is τ_δ-stable, μ^{τ_δ} is a minimal K-type of $\pi^P_{\delta,H,\chi,\lambda}$ (see Lemma 4). Hence, $\chi_{\mu^{\tau_\delta}\,|H} = \chi$ and $\chi_{\mu^{\tau_\delta}}(r) = \chi(rs)\chi_{\mu^{\tau_\delta}}(s)$. Identity (6.7) follows.

(iv) First, if $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, one has $s\lambda \in i\mathfrak{a}^{*\langle rs\tau_\delta(s),H\rangle}$ by

$$rs\tau_\delta(s)\tau_\delta s\lambda = rs\tau_\delta s\tau_\delta^{-1}\tau_\delta s\lambda = rs\tau_\delta\lambda = s(r\tau_\delta\lambda) = s\lambda.$$

As $H \in \mathrm{Diag}(\tau_\delta, r)$, there exists $\lambda \in i\overline{C}_\delta \cap i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$ with $\mathrm{R}^c_{\delta,\lambda} = H$. Then, $s\lambda$ satisfies $s\lambda \in i\overline{C}_\delta \cap i\mathfrak{a}^{*\langle rs\tau_\delta(s),H\rangle}$ and $\mathrm{R}^c_{\delta,s\lambda} = H$. Hence, $H \in \mathrm{Diag}(\tau_\delta, rs\tau_\delta(s))$.

The restriction $A(\lambda)$ of $A(P, \delta, s, \lambda)$ to the space of $\pi^P_{\delta,H,\chi,\lambda}$ intertwines $\pi^P_{\delta,H,\chi,\lambda}$ with $\pi^P_{\delta,H,\chi,s\lambda}$. Its inverse is the restriction of $A(P, \delta, s, s\lambda)$ to the space of $\pi^P_{\delta,H,\chi,s\lambda}$ (see (2.5)). Using (5.6), assertion (iv) follows once we prove that

$$A(\lambda)^{-1}\, T\big(\delta, rs\tau_\delta(s), H, \chi, \lambda\big)\, A(\lambda) = \chi_0\big(\tau_\delta(s)\big)\, T(\delta, r, H, \chi, \lambda). \tag{6.8}$$

It is enough to prove this equation on the isotypic component of each minimal K-type μ of $\pi^P_{\delta,H,\chi,\lambda}$. Using (2.4), (2.1), and Lemma 6(i), (ii), one sees that the restriction to

$I^\mu(\delta)$ of the operator on the left-hand side of (6.8) is equal to

$$\chi_{\mu^{\tau_\delta}}(s)\,\chi_{\mu^{\tau_\delta}}\big(rs\tau_\delta(s)\big)\,c^{\mu^{\tau_\delta}}\big(P,\tau_\delta(P)\big)\,T_{\delta|I^\mu(\delta)}\,\chi_\mu(s).$$

This operator is equal to

$$\chi_{\mu^{\tau_\delta}}(r)\,\chi_{\mu^{\tau_\delta}}\big(\tau_\delta(s)\big)\,\chi_\mu(s)\,c^{\mu^{\tau_\delta}}(P,\tau_\delta\cdot P)\,T_{\delta|I^\mu(\delta)}. \tag{6.9}$$

On the other hand, by Lemma 5(v), one has, in turn, that

$$\chi_{\mu^{\tau_\delta}}=\chi_0\chi_\mu^{\tau_\delta}$$

and

$$\chi_{\mu^{\tau_\delta}}\big(\tau_\delta(s)\big)=\chi_0\big(\tau_\delta(s)\big)\chi_\mu(s).$$

Therefore, (6.9) is equal to

$$\chi_{\mu^{\tau_\delta}}(r)\,\chi_0\big(\tau_\delta(s)\big)\,c^{\mu^{\tau_\delta}}(P,\tau_\delta\cdot P)\,T_{\delta|I^\mu(\delta)}.$$

The restriction to $I^\mu(\delta)$ of the operator on the right-hand side of (6.8) is seen, by parallel computations, to be equal to the previous expression. □

6.1. σ-twisted characters

PROPOSITION 3

Define $S(\delta,r,H,\chi,\lambda)=\pi^P_{\delta,H,\chi,\lambda}(k_\delta)T(\delta,r,H,\chi,\lambda)$, where k_δ is the element of Lemma 2. For $H\in\mathrm{Diag}(\tau_\delta,r)$, $\chi\in\hat{H}(\tau_\delta)$, and $\lambda\in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}$, the following hold:

(i) *the operator $S(\delta,r,H,\chi,\lambda)$ intertwines $(\pi^P_{\delta,H,\chi,\lambda})^\sigma$ with $\pi^P_{\delta,H,\chi,\lambda}$; and*

(ii) *the corresponding twisted character*

$$\Theta^\sigma_{\delta,r,H,\chi,\lambda}(f)=\mathrm{tr}\big(\pi^P_{\delta,H,\chi,\lambda}(f)\,S(\delta,r,H,\chi,\lambda)\big),\qquad f\in C_c^\infty(G),$$

satisfies both the same relations as $\Theta^{\tau_\delta}_{\delta,r,H,\chi,\lambda}$ does in Proposition 2, and the analogue of Lemma 8.

Proof

(i) This assertion is clear, as $\sigma=k_\delta\tau_\delta$.

(ii) This assertion follows easily from (5.1). □

7. The main theorem

THEOREM 3

Suppose that we are given functions $F_{\delta,r,H,\chi}: i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}\to\mathbb{C}$ for every $\delta\in\mathrm{DDT}$,

$r \in \mathrm{R}_\delta^c$, $H \in \mathrm{Diag}(\tau_\delta, r)$, $\chi \in \hat{H}$, *and let* $t > 0$. *Then, there exists* $f \in C_c^\infty(G, K)_t$ *such that*

$$\Theta^\sigma_{\delta,r,H,\chi,\lambda}(f) = F_{\delta,r,H,\chi}(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle},$$

for all of the above data if and only if the following conditions hold.

(i) *The functions are identically zero, except for a finite number of* $\delta \in \mathrm{DDT}$.

(ii) *Each function* $F_{\delta,r,H,\chi}$ *belongs to* $\mathscr{PW}(\mathfrak{a}^{\langle r\tau_\delta, H\rangle})_t$.

(iii) *Let* $r \in \mathrm{R}_\delta^c$, *and let* $H \in \mathrm{Diag}(\tau_\delta, r)$. *If* $w \in W_\delta^0$ *and* $r \in \mathrm{R}_\delta^c$ *satisfy* $w(i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}) = i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$, *then*

$$F_{\delta,r,H,\chi}(w\lambda) = F_{\delta,r,H,\chi}(\lambda), \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

(iv) *Let* $r \in \mathrm{R}_\delta^c$. *If* H, H' *are elements of* $\mathrm{Diag}(\tau_\delta, r)$ *for* $r \in \mathrm{R}_\delta^c$ *with* $H \subset H'$, *then*

$$F_{\delta,r,H,\chi}(\lambda) = \sum_{\chi' \in \hat{H}'(\tau_\delta),\, \chi'_{|H} = \chi} F_{\delta,r,H',\chi'}(\lambda), \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in \mathfrak{a}^{\langle r\tau_\delta, H'\rangle}.$$

(v) *If* $r, s \in \mathrm{R}_\delta^c$, $H \in \mathrm{Diag}(\tau_\delta, s)$ *with* $rs \in H$, *one has*

$$F_{\delta,r,H,\chi}(\lambda) = \chi(rs) F_{\delta,s,H,\chi}(\lambda), \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle} = i\mathfrak{a}^{*\langle s\tau_\delta, H\rangle}.$$

(vi) *Let* $r, s \in \mathrm{R}_\delta^c$, *and let* $H \in \mathrm{Diag}(\tau_\delta, r)$. *Then,* H *is an element of* $\mathrm{Diag}(\tau_\delta, rs\tau_\delta(s))$ *and*

$$F_{\delta,rs\tau_\delta(s),H,\chi}(s\lambda) = \chi_0\big(\tau_\delta(s)\big) F_{\delta,r,H,\chi}(\lambda), \quad \chi \in \hat{H}(\tau_\delta),\ \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

Proof

The "only if" part of the theorem follows from Propositions 2 and 3, together with Lemma 8.

Therefore, we turn to the proof of the "if" part. In so doing, we follow the inductive reasoning of [CD1, section 2.3] and the following analogue of [CD1, proposition 1]. Let us introduce the transitional spaces $C_c^\infty(G)^{\mu\mu'}$, where μ, μ' are equivalence classes of irreducible unitary representations of K. Let $\theta_\mu, \theta_{\mu'}$ be the complex conjugates of the normalized characters of μ and μ'. We define

$$C_c^\infty(G)^{\mu\mu'} = \theta_\mu * C_c^\infty(G) * \theta_{\mu'}. \tag{7.10}$$

An element f of this space is said to be of type (μ, μ'). For each representation π of G, the operator $\pi(f)$ sends the isotypic component of type μ' to the isotypic component of type μ. It annihilates the other isotypic components.

PROPOSITION 4

Let $t > 0$. *Suppose that* $(F_{\delta,r,H,\chi})$ *is a family of functions satisfying Theorem 3(i)–(vi)* *for a fixed* $\delta \in \mathrm{DDT}$. *Then, there exist functions* $h_\mu \in C_c^\infty(G, K)_t$, $\mu \in A(\delta)$, *of type* (μ, μ^{τ_δ}) *such that*

$$F_{\delta,r,H,\chi}(\nu) = \mathrm{tr}\Big(\pi^P_{H,\chi,\nu}\Big(\sum_{\mu \in A(\delta)} h_\mu\Big) T(r, H, \chi, \nu)\Big), \quad r \in \mathrm{R}^c_\delta,\ \nu \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle},\ \chi \in \hat{H}(\tau_\delta).$$

We postpone the proof of Proposition 4 until Section 8 and continue our proof of Theorem 3 by induction. We perform this proof by induction using a partial ordering on DDT. Define $\delta < \delta'$ to mean that $\|\mu\| < \|\mu'\|$ for all $\mu \in A(\delta)$ and $\mu' \in A(\delta')$ (see [V, Definition 5.1] and [CD1, page 433] for the definition of $\|\mu\|$). Define $\delta \le \delta'$ to mean that either $\delta < \delta'$ or $\delta = \delta'$. Now, suppose that we are given functions satisfying Theorem 3(i)–(vi). Define the support Γ_F of these functions to be the collection of $\delta \in \mathrm{DDT}$ so that $F_{\delta,r,H,\chi}$ does not vanish for some r, H, and χ. Condition (ii) implies that Γ_F is a finite collection of representations. Let $\bar{\Gamma}_F$ be the collection of representations in DDT which are less than or equal to some representation in Γ_F. We prove Theorem 3 by induction on $|\bar{\Gamma}_F|$. If $\bar{\Gamma}_F$ is empty, then $f = 0$ solves the problem. Now, suppose that $\bar{\Gamma}_F$ is not empty, and suppose that $\delta' \in \mathrm{DDT}$ is a maximal element of Γ_F. Clearly, δ' is also a maximal element of $\bar{\Gamma}_F$. Proposition 4 tells us that there exists $h \in C_c^\infty(G, K)_t$, which is a sum of functions of type (μ', μ'^{δ}), $\mu' \in A(\delta')$, such that

$$F_{\delta',r',H',\chi'}(\lambda) = \mathrm{tr}\big(\pi^{P'}_{H',\chi',\lambda}(h)\, T(r', H', \chi')\big), \quad \lambda \in i(\mathfrak{a}')^{*\langle r'\tau_{\delta'}, H'\rangle},$$

for any $r' \in \mathrm{R}^c_{\delta'}$, $H' \in \mathrm{Diag}(\tau_{\delta'}, r')$, and $\chi' \in \hat{H}'(\tau_{\delta'})$. Define a new family function by

$$F'_{\delta,r,H,\chi}(\lambda) = F_{\delta,r,H,\chi}(\lambda) - \mathrm{tr}\big(\pi^P_{H,\chi,\lambda}(h)\, T(r, H, \chi)\big), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}, \tag{7.11}$$

for $\delta \in \mathrm{DDT}$, $r \in \mathrm{R}^c_\delta$, $H \in \mathrm{Diag}(\tau_\delta, r)$, and $\chi \in \hat{H}(\tau_\delta)$.

This new family apparently also satisfies Theorem 3(i)–(vi). In addition, $F'_{\delta',r',H',\chi'}$ vanishes by construction. We wish to show that $\bar{\Gamma}_{F'} \subsetneq \bar{\Gamma}_F$. Suppose that $\delta \in \Gamma_{F'}$. Then, there exist r, H, and χ such that $F'_{\delta,r,H,\chi} \neq 0$. From (7.11), we evidently have that $F_{\delta,r,H,\chi} \neq 0$ or $\mathrm{tr}(\pi^P_{H,\chi,\lambda}(h)\, T(r, H, \chi)) \neq 0$. The former inequality implies that $\delta \in \Gamma_F - \{\delta'\}$. The latter inequality implies that some $\mu' \in A(\delta')$ is a K-type of δ. By definition, any $\mu \in A(\delta)$ satisfies $\|\mu\| \le \|\mu'\|$. If $\|\mu\| = \|\mu'\|$, then $\mu' \in A(\delta)$ and [CD2, proposition D.1] implies that δ is equal to δ', which contradicts $F'_{\delta,r,H,\chi} \neq 0$. In consequence, $\|\mu\| < \|\mu'\|$ for all $\mu \in A(\delta)$, that is, $\delta < \delta'$. This proves, in turn, that $\delta \in \bar{\Gamma}_F - \{\delta'\}$, $\Gamma_{F'} \subsetneq \bar{\Gamma}_F$, and $\bar{\Gamma}_{F'} \subsetneq \bar{\Gamma}_F$. We may now appeal to the induction hypothesis to obtain a function $f' \in C_c^\infty(G, K)_t$ such that

$$F'_{\delta,r,H,\chi}(\lambda) = \operatorname{tr}\big(\pi^P_{H,\chi,\lambda}(f')\,T(r,H,\chi)\big), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

Substituting this equation into (7.11), it is clear that $f = f' + h$ satisfies the desired properties. □

8. Proof of Proposition 4

In this section, we fix $\delta \in$ DDT, and for simplicity, we often drop the lower index δ from much of the previous notation. For example, we write Δ instead of Δ_δ and τ for τ_δ. Moreover, R^c_δ is denoted by R.

8.1. An extension result

LEMMA 9

Suppose that $F_{\delta,r,H,\chi}$ satisfies Theorem 3(ii)–(vi) for every $r \in \mathrm{R}$, $H \in \mathrm{Diag}(\tau_\delta, r)$, and $\chi \in \hat{H}(\tau)$. Then, $F_{\delta,r,H,\chi}$ extends to a W^0-invariant function $\tilde{F}_{\delta,r,H,\chi} \in \mathscr{PW}(\mathfrak{a})_t$.

Proof

Suppose that $\{r_1, \ldots, r_m\}$ is a minimal set of generators for $H \subset \mathrm{R}$. Let E_1 be the subspace of fixed points under r_1, let $\Delta_1 = \{\alpha_{|E_1} \mid \alpha \in \Delta, \alpha_{|E_1} \neq 0\}$, and let Δ_1^+ be the set of elements of Δ_1 which are restrictions of elements of Δ^+ to E_1^*. By [CD2, proposition A.2], Δ_1 is a root system and the Weyl group of Δ_1 is

$$W_1 = \left\{ w_{|E_1} \,\middle|\, w \in W^0,\ w(E_1) = E_1 \right\}.$$

By [CD1, proposition A.1, lemme C.1], the restriction map

$$\mathscr{PW}(\mathfrak{a})_t^{W^0} \to \mathscr{PW}(E_1)_t^{W_1}$$

is surjective. Arguing inductively, we obtain surjections

$$\mathscr{PW}(\mathfrak{a})_t^{W^0_\delta} \to \mathscr{PW}(E_1)_t^{W_1} \to \cdots \to \mathscr{PW}(E_m)_t^{W_m} = \mathscr{PW}(\mathfrak{a}^H)_t^{W_m}, \tag{8.1}$$

where W_m is the Weyl group of the root system of $\Delta_m = \{\alpha_{|E^H} \mid \alpha \in \Delta_\delta, \alpha_{|E^H} \neq 0\}$. Thus $\Delta_m^+ = \{\alpha_{|E^H} \mid \alpha \in \Delta_\delta^+, \alpha_{|E^H} \neq 0\}$ is a set of positive roots of Δ_m. Now, consider the automorphism $r\tau$ of $\mathfrak{a}$. For any $s \in H$, $\tau(s) = \tau^{-1}s\tau$ lies in H, as H is τ-stable, and

$$sr\tau\lambda = r\tau(\tau^{-1}s\tau)\lambda = r\tau(\lambda), \quad \lambda \in \mathfrak{a}^{*H}.$$

This shows that $\mathfrak{a}^{*H}$ is $r\tau$-stable. Since Δ^+ is also τ- and r-stable, Δ_m^+ is $r\tau$-stable. We may apply Corollary 1 of the appendix to conclude that the restriction map

$$\mathscr{PW}(\mathfrak{a}^H)_t^{W_m} \to \mathscr{PW}(\mathfrak{a}^{\langle r\tau,H\rangle})_t^{W'} \tag{8.2}$$

is surjective, where W' is a certain subgroup of

$$W'' := \left\{ w_{|\mathfrak{a}^{*\langle r\tau,H\rangle}} \,\middle|\, w \in W_\delta^0,\ w(\mathfrak{a}^{*\langle r\tau,H\rangle}) = \mathfrak{a}^{*\langle r\tau,H\rangle} \right\}.$$

By Theorem 3(iii), the function $F_{\delta,r,H,\chi}$ is invariant under W''.

As a result, the function $F_{\delta,r,H,\chi}$ belongs to $\mathscr{PW}(\mathfrak{a}^{\langle r\tau,H\rangle})_t^{W'}$. Combining the surjections of (8.1) and (8.2), we obtain a function $\tilde{F}_{\delta,rH,\chi}$, as desired. □

8.2. Some τ-stable subgroups of R

Let R′ be the subgroup of the automorphism group of $\mathfrak{a}^*$ generated by τ and R. It is a finite group, as τ is of finite order on $\mathfrak{a}$ and normalizes R. Given $\lambda \in \mathfrak{a}^*$, define $\mathrm{R}'_\lambda = \{r \in \mathrm{R}' : r\lambda = \lambda\}$.

LEMMA 10

Suppose that $s \in \mathrm{R}$. Then, there exists a unique subgroup $\mathrm{R}'[s\tau]$ of R' which satisfies the following.

(i) *We have $\mathfrak{a}^{*\mathrm{R}'[s\tau]} = \mathfrak{a}^{*s\tau}$.*

(ii) *There exists $\lambda \in \bar{C} \cap \mathfrak{a}^{*s\tau}$ such that $\mathrm{R}'_\lambda = \mathrm{R}'[s\tau]$. (Here $C = C_\delta$.)*

(iii) *The element $s\tau$ belongs to $\mathrm{R}'[s\tau]$.*

(iv) *Suppose that H' is a subgroup of R' such that $s\tau \in H'$ and $\mathrm{R}'_\lambda = H'$ for some $\lambda \in \mathfrak{a}^*$. Then, $\mathrm{R}'[s\tau] \subset H'$.*

Proof

The proof is essentially that of [CD2, lemme C.2] with R replaced by R′. We include it for the sake of completeness. Recall that C is a Weyl chamber of $\Delta = \Delta_\delta$. For each subgroup H of R′, set

$$A_H = \{\lambda \in \bar{C} \cap \mathfrak{a}^{*s\tau} \mid \mathrm{R}'_\lambda = H\}.$$

It is immediate that

$$A_H \subset \mathfrak{a}^{*H} \subset \mathfrak{a}^{*s\tau}$$

and that $\bar{C} \cap \mathfrak{a}^{*s\tau} = \bigcup_{H \subset \mathrm{R}'} A_H$. One may average an element of C over the finite group generated by $s\tau$ to obtain an element in $C \cap \mathfrak{a}^{*s\tau}$ (see Section 5.3). As a result, the set $\bar{C} \cap \mathfrak{a}^{*s\tau}$ has nonempty interior as a subset of $\mathfrak{a}^{*s\tau}$. According to Baire's theorem, one of the subgroups in the above union has an invariant subspace that is open in, and therefore equal to, $\mathfrak{a}^{*s\tau}$. Denote such a subgroup by $\mathrm{R}'[s\tau]$. It is apparent from its definition that $s\tau \in \mathrm{R}'[s\tau]$. This proves assertions (i)–(iii). Now, suppose that $\lambda \in \mathfrak{a}^*$ is as in the hypothesis of (iv). Then, we have $\lambda \in \mathfrak{a}^{*s\tau} = \mathfrak{a}^{*\mathrm{R}'[s\tau]}$, and so $\mathrm{R}'[s\tau] \subset \mathrm{R}'_\lambda = H'$. This proves (iv), from which the uniqueness assertion also follows. □

LEMMA 11
Suppose that $s \in \mathrm{R}$, *and set* $\mathrm{R}[s\tau] = \mathrm{R}'[s\tau] \cap \mathrm{R}$. *Then* $\mathrm{R}[s\tau]$ *is stable under conjugation by* τ.

Proof
Suppose that $r \in \mathrm{R}[s\tau]$. By Lemma 3, $\tau(r) = \tau r \tau^{-1}$ belongs to R. Since R is abelian, the element $\tau r \tau^{-1}$ is equal to $(s\tau)r(s\tau)^{-1}$. The latter element belongs to $\mathrm{R}'[s\tau]$ (see Lemma 10(iii)). □

LEMMA 12
Suppose that $s \in \mathrm{R}$. *Then* $\mathrm{R}[s\tau]$ *belongs to* $\mathrm{Diag}(\tau, s)$.

Proof
In view of Lemma 11, it remains only to show that $\mathrm{R}[s\tau] = \mathrm{R}_\lambda$ for some $\lambda \in \mathfrak{a}^{*s\tau} \cap \bar{C}$. According to Lemma 10, there exists $\lambda \in \mathfrak{a}^{*s\tau} \cap \bar{C}$ such that $\mathrm{R}'_\lambda = \mathrm{R}'[s\tau]$. It is easily verified that

$$\mathrm{R}_\lambda = \mathrm{R}'_\lambda \cap \mathrm{R} = \mathrm{R}'[s\tau] \cap \mathrm{R} = \mathrm{R}[s\tau]. \qquad \square$$

LEMMA 13
Suppose that $s \in \mathrm{R}$, *let* $H \in \mathrm{Diag}(\tau, s)$, *and let* $h \in H$. *Then,* H *contains* $\mathrm{R}[sh\tau]$.

Proof
By the definition of $\mathrm{Diag}(\tau_\delta, s)$, there exists $\lambda \in \mathfrak{a}^{*s\tau} \cap \bar{C}$ such that $\mathrm{R}_\lambda = H$. Since λ is fixed by $sh\tau = hs\tau$, the element $sh\tau$ belongs to R'_λ. According to Lemma 10(iv), the group $\mathrm{R}'[sh\tau]$ is contained in R'_λ. Finally,

$$H = \mathrm{R}_\lambda = \mathrm{R}'_\lambda \cap \mathrm{R} \supset \mathrm{R}'[sh\tau] \cap \mathrm{R} = \mathrm{R}[sh\tau]. \qquad \square$$

8.3. The statement and proof of a key lemma

LEMMA 14
Suppose that $F_{\delta,r,H,\chi}$, $r \in \mathrm{R}$, $H \in \mathrm{Diag}(\tau, r)$, $\chi \in \hat{H}(\tau)$, *is a family of functions satisfying Theorem 3(ii)–(vi). Then, for each* $\alpha \in \hat{\mathrm{R}}$, *there exists a function* $\Phi_\alpha \in$

$\mathscr{PW}(\mathfrak{a})_t^{W^0}$ such that

$$F_{\delta,r,H,\alpha}(\lambda) = \sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} \chi_0(r)\, \alpha^\tau(r)\, \Phi_\alpha(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau, H\rangle},$$

for any $r \in \mathrm{R}$, $H \in \mathrm{Diag}(\tau, r)$, $\chi \in \hat{H}(\tau)$.

Proof

In what follows, we generalize the proof of the similar nontwisted statement [CD1, proposition C.1]. Let $r \in \mathrm{R}$. The subgroup $\mathrm{R}[r\tau]$, defined in Lemma 10, belongs to $\mathrm{Diag}(\delta, r)$ (see Lemma 12). Consequently, the function $F_{\delta,r,\mathrm{R}[r\tau],\eta} \in \mathscr{PW}(\mathfrak{a}^{\langle r\tau, \mathrm{R}[r\tau]\rangle})_t$ is defined for each character $\eta \in \widehat{\mathrm{R}[r\tau]}(\tau)$. By Lemma 9, each of these functions extends to a function $\tilde{F}_{\delta,r,\mathrm{R}[r\tau],\eta} \in \mathscr{PW}(\mathfrak{a})_t^{W^0}$. Define $\varphi_r : i\mathfrak{a}^* \to \mathbb{C}$ by

$$\varphi_r = \sum_{\eta \in \widehat{\mathrm{R}[r\tau]}(\tau)} \tilde{F}_{\delta,r,\mathrm{R}[r\tau],\eta}.$$

For each $\alpha \in \hat{\mathrm{R}}$, define

$$\Phi_\alpha = \frac{1}{|\mathrm{R}|} \sum_{s \in \mathrm{R}} \chi_0(s)\, \alpha^\tau(s)\, \varphi_s.$$

Clearly, the function Φ_α lies in $\mathscr{PW}(\mathfrak{a})_t^{W^0}$. In order to prove that these objects satisfy the lemma, we choose $r \in \mathrm{R}$, $H \in \mathrm{Diag}(\tau, r)$, and $\chi \in \hat{H}(\tau)$. We proceed by rearranging the right-hand side of the identity in the lemma as

$$\begin{aligned}
&\sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} \chi_0(r)\, \alpha^\tau(r)\, \Phi_\alpha(\lambda) \\
&= \sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} \chi_0(r)\alpha^\tau(r) \frac{1}{|\mathrm{R}|} \sum_{s \in \mathrm{R}} \chi_0(s)\alpha^\tau(s)\varphi_s(\lambda) \\
&= \frac{1}{|\mathrm{R}|} \sum_{s \in \mathrm{R}} \varphi_s(\lambda) \sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} (\chi_0 \alpha^\tau)(rs), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.
\end{aligned}$$

Let us consider the inner sum in more detail. Since R is a product of copies of $\mathbb{Z}/2\mathbb{Z}$, we may regard it as a vector space over $\mathbb{Z}/2\mathbb{Z}$. In this view, H is a vector subspace, and we may fix a complementary subspace $\{1, r_1, \ldots, r_\ell\}$. This complementary subspace forms a subgroup of R which is isomorphic to R/H. Then, the map

$$(r_i H, h) \mapsto r_i h, \quad h \in H,$$

is a group isomorphism from $\mathrm{R}/H \times H$ to R. This induces the dual isomorphism $\hat{\mathrm{R}} \cong \widehat{\mathrm{R}/H} \times \hat{H}$, as all of the groups are abelian. Suppose that $rs = r_i h$ for some $h \in H$. Then, the summand $(\chi_0\alpha^\tau)(rs)$ may be decomposed according to the dual isomorphism as $\alpha'(r_i H)(\chi_0\alpha^\tau)(h)$ for some $\alpha' \in \widehat{\mathrm{R}/H}$. Furthermore, since $H \in \mathrm{Diag}(\tau, r)$ and $\chi \in \hat{H}(\tau)$ (see Theorem (i.2)), one has

$$(\chi_0\alpha^\tau)(h) = \chi_0(h)\alpha\big(\tau^{-1}(h)\big) = \chi_0(h)\chi\left(\tau^{-1}(h)\right) = \chi_0(h)\chi^\tau(h)$$
$$= \chi_0(h)\chi_0(h)\chi(h) = \chi(h),$$

so that

$$(\chi_0\alpha^\tau)(r_i h) = \alpha'(r_i H)\chi(h).$$

If $r_i \neq 1$, then the inner sum reduces to

$$\chi(h) \sum_{\alpha' \in \widehat{R/H}} \alpha'(r_i H) = \chi(h) \times 0 = 0,$$

thanks to the orthogonality relations of characters. On the other hand, if $rs = h$, then the sum is equal to $|\widehat{R/H}|\chi(h)$. Taking these identities into account, we continue our earlier computation by writing

$$\begin{aligned}
&\sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} \chi_0(r)\, \alpha^\tau(r)\, \Phi_\alpha(\lambda) \\
&= \frac{1}{|\mathrm{R}|} \sum_{h \in H} |\widehat{\mathrm{R}/H}| \chi(h) \varphi_{rh}(\lambda) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi(h) \varphi_{rh}(\lambda) \qquad (8.3) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi(h) \sum_{\eta \in \widehat{\mathrm{R}[rh\tau]}(\tau)} \tilde{F}_{\delta, rh, \mathrm{R}[rh\tau], \eta}(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.
\end{aligned}$$

Lemma 13 tells us that $H \supset \mathrm{R}[rh\tau]$ for every $h \in H$. As $F_{\delta, rh, \mathrm{R}[rh\tau], \eta}$ satisfies Theorem 3(iv), we have

$$\tilde{F}_{\delta, rh, \mathrm{R}[rh\tau], \eta}(\lambda) = F_{\delta, rh, \mathrm{R}[rh\tau], \eta}(\lambda) = \sum_{\chi' \in \hat{H}(\tau),\, \chi'_{|\mathrm{R}[rh\tau]} = \eta} F_{\delta, rh, H, \chi'}(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

Substituting the expression on the right-hand side into (8.3) and summing over the characters η, we obtain

$$\sum_{\alpha \in \hat{\mathrm{R}}, \alpha_{|H} = \chi} (\chi_0\alpha^\tau)(r)\Phi_\alpha(\lambda) = \frac{1}{|H|} \sum_{h \in H} \chi(h) \sum_{\chi' \in \hat{H}(\tau)} F_{\delta, rh, H, \chi'}(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

Now, Theorem 3(v) tells us that

$$F_{\delta,rh,H,\chi'}(\lambda) = \chi'(h) F_{\delta,r,H,\chi'}(\lambda).$$

Combining this with the orthogonality relations, we conclude that

$$\begin{aligned}\sum_{\alpha\in\hat{\mathrm{R}},\alpha_{|H}=\chi} (\chi_0\alpha^\tau)(r)\Phi_\alpha(\lambda) &= \frac{1}{|H|}\sum_{h\in H}\chi(h)\sum_{\chi'\in\hat{H}(\tau)}\chi'(h)F_{\delta,r,H,\chi'}(\lambda)\\ &= \sum_{\chi'\in\hat{H}(\tau)} F_{\delta,r,H,\chi'}(\lambda)\Big(\frac{1}{|H|}\sum_{h\in H}\chi(h)\chi'(h)\Big)\\ &= F_{\delta,r,H,\chi}(\lambda),\end{aligned}$$

as desired. □

8.4. The behavior of Φ_α *under* R

For the remainder of this article, we use the bijection $\mu\mapsto\chi_\mu$ between $A(\delta)$ and $\hat{\mathrm{R}}$ to define $\Phi_\mu = \Phi_{\chi_\mu}$ for any function Φ_{χ_μ} as defined in Lemma 14. By Lemma 5(v), the equation of Lemma 14 may be rewritten as

$$F_{\delta,r,H,\alpha}(\lambda) = \sum_{\mu\in A(\delta),\chi_\mu|H=\chi} \chi_0(r)\,\chi_\mu^\tau(r)\,\Phi_\mu(\lambda) = \sum_{\mu\in A(\delta),\chi_\mu|H=\chi} \chi_{\mu^\tau}(r)\,\Phi_\mu(\lambda) \quad (8.4)$$

for any $\lambda\in i\mathfrak{a}^{*\langle r\tau,H\rangle}$.

Now, suppose that $s\in\mathrm{R}$, let $t>0$, and suppose that Φ is *any* collection $(\Phi_\mu)_{\mu\in A(\delta)}$ of functions in $\mathcal{PW}(\mathfrak{a})_t^{W^0}$. Then,

$$(s\cdot\Phi)_\mu(\lambda) = \chi_\mu(s)\chi_{\mu^\tau}(s)\,\Phi_\mu(s\lambda), \quad \lambda\in i\mathfrak{a}^*,$$

defines an action on the set of such collections, thanks to the commutativity of R.

LEMMA 15

Suppose that a collection $\Phi = (\Phi_\mu)_{\mu\in A(\delta)}$ *of functions in* $\mathcal{PW}(\mathfrak{a})_t^{W^0}$ *satisfies equation (8.4) of Lemma 14, and let* $s\in\mathrm{R}_\delta^c$*. Then,* $s\cdot\Phi$ *is also a collection of functions in* $\mathcal{PW}(\mathfrak{a})_t^{W^0}$ *which satisfies equation (8.4).*

Proof

By Lemma 5(v), we have

$$\chi_{\mu^\tau}\big(\tau(s)\big) = \chi_0\big(\tau(s)\big)\,\chi_\mu\big(\tau^{-1}(\tau(s))\big) = \chi_0\big(\tau(s)\big)\,\chi_\mu(s).$$

Consequently,

$$\begin{aligned}\chi_{\mu^\tau}(r)\,\chi_\mu(s)\,\chi_{\mu^\tau}(s) &= \chi_\mu(s)\,\chi_{\mu^\tau}(rs)\\ &= \chi_\mu(s)\,\chi_{\mu^\tau}\big(\tau(s)\big)\,\chi_{\mu^\tau}\big(\tau(s)\big)\,\chi_{\mu^\tau}(rs)\\ &= \chi_0\big(\tau(s)\big)\,\chi_{\mu^\tau}\big(rs\tau(s)\big).\end{aligned}$$

Therefore, for $\lambda \in i\mathfrak{a}^{*\langle r\tau,H\rangle}$, we have

$$\sum_{\mu\in A(\delta),\,\chi_{\mu|H}=\chi} \chi_{\mu^\tau}(r)(s\cdot\Phi)_\mu(\lambda) = \chi_0\big(\tau(s)\big) \sum_{\mu\in A(\delta),\,\chi_{\mu|H}=\chi} \chi_{\mu^\tau}\big(rs\tau(s)\big)\Phi_\mu(s\lambda).$$

By hypothesis, the right-hand side is equal to $\chi_0(\tau(s))F_{\delta,rs\tau(s),H,\chi}(s\lambda)$. By Theorem 3(vi), this expression is equal to $F_{\delta,r,H,\chi}(\lambda)$. □

Lemma 15 further shows that if $\Phi = (\Phi_\mu)_{\mu\in A(\delta)}$ satisfies the conclusion of Lemma 14, then this is also the case for $|\mathrm{R}|^{-1}\sum_{s\in\mathrm{R}} s\cdot\Phi$. This being the case, we assume without loss of generality that our Φ satisfies

$$\Phi_\mu(s\lambda) = \chi_\mu(s)\chi_{\mu^\tau}(s)\,\Phi_\mu(\lambda), \quad \lambda \in i\mathfrak{a}^*, \tag{8.5}$$

for any $s \in \mathrm{R}^c_\delta$.

8.5. The conclusion of the proof of Proposition 4

Suppose that the hypotheses of Proposition 4 are satisfied. In Lemma 14, we have shown the existence of functions $\Phi_\mu \in \mathscr{PW}(\mathfrak{a})^{W^0}_t$ satisfying (8.5) for every minimal K-type $\mu \in A(\delta)$ and shown that

$$F_{\delta,r,H,\chi}(\lambda) = \sum_{\mu\in A(\delta),\,\chi_{\mu|H}=\chi} \chi_{\mu^\tau}(r)\,\Phi_\mu(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau,H\rangle}.$$

We wish to express the right-hand side as

$$\mathrm{tr}\bigg(\pi^P_{\delta,H,\chi,\lambda}\Big(\sum_{\mu\in A(\delta)} h_\mu\Big)\,T(\delta,r,H,\chi,\lambda)\bigg)$$

for some functions $h_\mu \in C_c^\infty(G,K)^{\mu\mu^\tau}$. To obtain this expression, it is sufficient to have

$$\mathrm{tr}\big(\pi^P_{\delta,H,\chi,\lambda}(h_\mu)\,T_{r,\mu}\big) = \chi_{\mu^\tau}(r)\Phi_\mu(\lambda), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta,H\rangle}, \tag{8.6}$$

where $T_{r,\mu}$ is the restriction of $T(\delta,r,H,\chi,\lambda)$ to $I^\mu(\delta)$. In fact, it is sufficient for this equation to hold for $r=1$ and for all $\lambda \in i\mathfrak{a}^{*H}$. Indeed, by Lemma 6(i), (ii) and (2.4), the operator $T_{r,\mu}$ is bijective from $I^{\mu}(\delta)$ to $I^{\mu^{\tau_\delta}}(\delta)$, independent of $\lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}$, and

$$T_{r,\mu} = A(P, \delta, r, 0)\, T_{1,\mu} = \chi_{\mu^\tau}(r)\, T_{1,\mu}.$$

Having reduced the proof of Proposition 4 to finding these specific functions h_μ satisfying (8.6), we may take advantage of the action of R on Φ_μ, as in (8.5). For any $t > 0$ and $\eta \in \hat{\mathrm{R}}$, define $(\mathscr{PW}(\mathfrak{a})_t^{W^0})^\eta$ to be the subspace of functions Φ in $\mathscr{PW}(\mathfrak{a})_t^{W^0}$ which satisfy

$$\Phi(s\lambda) = \eta(s)\, \Phi(\lambda), \quad \lambda \in i\mathfrak{a}^*,\ s \in \mathrm{R}_\delta^c.$$

It is obvious from (8.5) that Φ_μ belongs to $(\mathscr{PW}(\mathfrak{a})^{W^0})^{\eta_\mu}$, where $\eta_\mu = \chi_\mu \chi_{\mu^\tau}$. The existence of the h_μ as stated in Proposition 4 is a consequence of the next result with $\mu' = \mu^\tau$. This result appears in [D3, (1.38)] in the required generality. The particular case of $\mu = \mu'$ was proved for G connected as a Lie group in [CD1, proposition 1]. Note that the proof given in [D3, (1.38)] uses [DF, Theorem 2] instead of [A1, Theorem III.4.1]:

> suppose that $t > 0$, let $\mu, \mu' \in A(\delta)$, and let $\eta = \chi_{\mu'}\chi_\mu \in \hat{\mathrm{R}}$; suppose further that $\mathscr{PW}(\mathfrak{a})_t^{\mu\mu'}$ is the space of functions from $i\mathfrak{a}^*$ to $\mathrm{Hom}(I^{\mu'}(\delta), I^{\mu}(\delta))$ defined by
>
> $$\lambda \mapsto \pi^P_{\delta,\lambda}(h)_{|I^{\mu'}(\delta)}, \quad \lambda \in i\mathfrak{a}^*, \tag{8.7}$$
>
> for $h \in C_c^\infty(G, K)_t$ of type (μ, μ'); then,
>
> $$\mathscr{PW}(\mathfrak{a})_t^{\mu\mu'} = \big(\mathscr{PW}(\mathfrak{a})_t^{W^0}\big)^\eta \otimes \mathrm{Hom}\big(I^{\mu'}(\delta), I^{\mu}(\delta)\big).$$

9. A corollary of the main theorem

Let $P = MAN$ be a parabolic subgroup of G, and let P' be the intersection of G' with its normalizer P^+ in G^+. We assume in the following that it is nonempty and contains τ. Thus $\tau \cdot P = P$, which implies that $\sigma \cdot P$ is conjugate under G to P. This implies that there exists $k \in K$ with $k\sigma \in P'$ such that $P' \cap K^+$ is nonempty. We may choose $\tau \in P' \cap K^+$. Hence, the map $P^+/P \to G^+/G$ is surjective. It is bijective, as the normalizer of P in G is P. The normalizer P^+ is generated by P'.

Similarly, let L^+ be the intersection of the normalizer of $L = MA$ in G^+ with P^+. It is an algebraic group and has nonempty intersection L' with G', which generates L^+. In fact, $\tau \in K^+ \cap P'$ is in L^+, as τ normalizes P^+ and $\theta(P^+)$. Let M^+ be the subgroup of L^+ generated by τ and M. Then, $L^+ = M^+A$, and A is a normal subgroup of L^+, but it is not necessarily in the center of L^+.

If B is equal to either K, P, L, or M, then $B' := B \cap G'$ is equal to $B\tau$, B' generates B^+, and the canonical map $B^+/B \to G^+/G$ is surjective. Moreover, $P^+ = L^+N$, and A and N are normal subgroups of L^+ and P^+, respectively. The fixed-point spaces $\mathfrak{a}^{L^+}$ and $\mathfrak{a}^\tau$ are equal.

If ε^+ is a tempered unitary representation of M^+ whose restriction ε to M is irreducible and if $\lambda \in i\mathfrak{a}^{*L^+} = i\mathfrak{a}^{*\tau}$, then $\varepsilon^+ \otimes e^\lambda \otimes 1_N$ is a unitary representation of

P^+. We denote by $\pi^{P^+}_{\varepsilon^+,\lambda}$ the corresponding unitarily induced representation from P^+ to G^+:

the unitarily induced representation $\pi^{P^+}_{\varepsilon^+,\lambda}$ from P^+ to G^+ restricts to G as a representation canonically equivalent to $\pi^{P}_{\varepsilon,\lambda}$. (9.8)

LEMMA 16

Let (Θ_λ) and (Θ'_λ) be two families of σ-twisted characters (see Section 5.1) of representations π_λ and π'_λ of G^+, respectively, for λ in an open connected subset Ω of a finite-dimensional subspace of $i\mathfrak{a}^*$ so that

(i) *the restrictions of π_λ and π'_λ to G are equivalent;*

(ii) *the families of twisted characters are analytic; and*

(iii) *$\Theta_{\lambda_0} = \Theta'_{\lambda_0} \neq 0$ for some $\lambda_0 \in \Omega$ such that $\pi^P_{\delta,\lambda}$ is irreducible in a neighborhood of λ_0.*

Then, the two families of twisted characters are identical.

Proof

Recall from Section 5.1 that n is the least positive integer such that $\sigma^n \in K$. It is also the order of the coset of σ in G^+/G or K^+/K. As mentioned in Section 5.1, there exist n equivalence classes of representations of G^+ with a given irreducible restriction to G. They differ by an nth root of unity on $\sigma \in G'$. Thus there exist nth roots of unity $c(\lambda)$ such that in a connected neighborhood of λ_0,

$$\Theta_\lambda = c(\lambda)\Theta'_\lambda.$$

Since Θ and Θ' are analytic, $c(\lambda)$ is constant. The constant $c(\lambda)$ equals 1, as $c(\lambda_0) = 1$. The lemma follows by analytic continuation. □

THEOREM 4

Let ϕ be a complex-valued function defined on the tempered dual $\hat{G}^+_{\text{temp}}$ of G^+ which is nonzero only on the subset $\hat{G}'_{\text{temp}}$ of equivalence classes of representations of G^+ whose restrictions to G are irreducible and tempered. We also denote by ϕ the $\mathbb{Z}$-linear extension of ϕ to the set of tempered representations of G^+ of finite length. Then, there exists $f \in C^\infty_c(G, K)_t$ with

$$\operatorname{tr}\big(\pi^+(f)\pi^+(\sigma)\big) = \phi(\pi^+), \quad \pi^+ \in \hat{G}^+_{\text{temp}},$$

if and only if ϕ satisfies the following conditions.

(i) *There exists a finite subset Γ of the unitary dual $\hat{K}$ such that $\phi(\pi^+) = 0$ if the restriction π of π^+ to G does not contain any K-type in Γ.*

(ii) *If π^+ and $\pi'^+ \in \hat{G}'_{\text{temp}}$ have the same restriction π to G and $\pi^+(\sigma) = c\pi'^+(\sigma)$ for some root of unity c, then $\phi(\pi^+) = c\phi(\pi'^+)$.*

(iii) *Let $Q = M_Q A_Q N_Q$ be a parabolic subgroup of G with Q' nonempty. Assume that ε^+ is a tempered representation of M_Q^+, and let $\lambda \in \mathfrak{a}^{*M_Q^+}$. Then, $\lambda \mapsto \phi(\pi^{Q^+}_{\varepsilon^+,\lambda})$ is the Fourier transform of a function on $\mathfrak{a}_Q^{M_Q^+}$ of support contained in the closed ball of radius t. In such a case, we say that ϕ is the twisted invariant Fourier transform of f.*

Proof

First, let us show that the conditions are necessary. If ϕ is the twisted invariant Fourier transform of $f \in C_c^\infty(G, K)_t$, it evidently satisfies (i) and (ii). Since $\sigma \in K^+$, it follows that in the compact realization, the operator $\pi^{Q^+}_{\varepsilon^+,\lambda}(\sigma)$ does not depend on λ. Therefore, condition (iii) can be proved by imitating the proof of Lemma 8.

Let us show that the conditions are sufficient. First, we wish to define $F_{\delta,r,H,\chi,\lambda}$ as in Theorem 2. The operator $S(\delta, r, H, \chi, \lambda)$ (see Section 6.1) intertwines $(\pi^P_{\delta,H,\chi,\lambda})^\sigma$ with $\pi^P_{\delta,H,\chi,\lambda}$. Therefore, when $\pi^P_{\delta,H,\chi,\lambda}$ is irreducible, the operators $S(\delta, r, H, \chi, \lambda)^n$ and $\pi^P_{\delta,H,\chi,\lambda}(\tau_\delta^n)$ are proportional (see Section 5.1). As $\tau_\delta^n \in K$, the second operator is independent of λ. By Lemma 6(ii), this is true also for the first operator restricted to the minimal K-types. Thus the proportionality factor is independent of λ when $\pi^P_{\delta,H,\chi,\lambda}$ is irreducible. Since these operators are analytic in λ, the proportionality factor is always independent of λ. Consequently, there exists $c \in \mathbb{C}$ such that $S'(\delta, r, H, \chi, \lambda) = cS(\delta, r, H, \chi, \lambda)$ verifies $(S'(\delta, H, \chi, \lambda))^n = \pi^P_{\delta,H,\chi,\lambda}(\tau_\delta^n)$. As a consequence, $S'(\delta, H, \chi, \lambda)$ determines a representation $(\pi^P_{\delta,H,\chi,\lambda})^+$ of G^+. It is clearly analytic in λ. We define

$$F_{H,\delta,r,\chi}(\lambda) = c^{-1}\phi\big((\pi^P_{\delta,H,\chi,\lambda})^+\big).$$

The relations between the $\Theta^\sigma_{\delta,r,H,\chi,\lambda}$ (see Propositions 2, 3) carry over to the twisted characters of the $(\pi^P_{\delta,H,\chi,\lambda})^+$ by Lemma 16. The fact that ϕ is $\mathbb{Z}$-linear on the set of tempered representations of G^+ therefore implies that the family of functions $F_{\delta,r,H,\chi}$ satisfies Theorem 3(iii) – (vi). Theorem 3(i) follows from condition (i) for ϕ. It remains to verify Theorem 3(ii). To achieve this, we show that the family of representations $(\pi^P_{\delta,H,\chi,\lambda})^+$ is equivalent to a family $\pi^{Q^+}_{\varepsilon^+,\lambda}$ for a suitable parabolic subgroup Q of G. In fact, Proposition 1 tells us that for $\lambda \in \mathfrak{a}^{*r\tau_\delta}$, the representation $\pi^P_{\delta,H,\chi,\lambda}$ is equivalent to $\pi^Q_{\varepsilon,\lambda}$ (with Q and ε as in Proposition 1). In particular, Q, ε, and λ are $\tilde r\tau_\delta$-stable. As a result, $\tilde r\tau_\delta \in Q'$, ε extends to an irreducible representation of M_Q^+, and $\lambda \in \mathfrak{a}^{*M_Q^+}$. The representation $\pi^{Q^+}_{\varepsilon^+,\lambda}$ is, like $(\pi^P_{\delta,H,\chi,\lambda})^+$, a representation of G^+ whose restriction to G is equivalent to $\pi^P_{\delta,H,\chi,\lambda}$. Thus ε^+ can be multiplied by a root of unity so that they are equivalent representations of G^+ for some λ_0 for which $\pi^P_{\delta,H,\chi,\lambda_0}$ is irreducible. By Lemma 16, one sees that the twisted characters of $(\pi^P_{\delta,H,\chi,\lambda})^+$ and $\pi^{Q^+}_{\varepsilon^+,\lambda}$ are identical for all λ. This means that the two representations are equivalent and that

$$F_{\delta,r,H,\chi}(\lambda) = c^{-1}\phi(\pi^{Q^+}_{\varepsilon^+,\lambda}), \quad \lambda \in i\mathfrak{a}^{*\langle r\tau_\delta, H\rangle}.$$

Theorem 2(ii) for $F_{\delta,r,H,\chi}$ follows from condition (iii) for ϕ. Thus Theorem 3 applies and furnishes a function $f \in C_c^\infty(G, K)_t$ such that

$$\phi\big((\pi^P_{\delta,H,\chi,\lambda})^+\big) = \mathrm{tr}\big(\pi^P_{\delta,H,\chi,\lambda}(f)\,\pi^+(\sigma)\big)$$

for all data (r, H, χ, λ). According to Theorem 1, every irreducible tempered σ-stable representation of G appears as such a $\pi^P_{\delta,H,\chi,\lambda}$. This implies that for every σ-stable irreducible representation π of G, there exists an extension π^+ to G^+ with

$$\phi(\pi^+) = \mathrm{tr}\big(\pi(f)\,\pi^+(\sigma)\big). \tag{9.9}$$

Condition (ii) implies that this is also true for all $\pi^+ \in \hat{G}'_{\mathrm{temp}}$.

Finally, if π^+ is an irreducible representation of G^+ whose restriction to G is reducible, then both sides of (9.9) are zero. This shows that f has the required properties. □

Appendix

THEOREM 5

Suppose that Δ is a root system of a subspace of a finite-dimensional real vector space E, suppose that W is its Weyl group, and suppose that β is a nontrivial automorphism of E, of finite order, preserving Δ and a subset of positive roots. Suppose further that $W_\beta = \{w_{|E^\beta} \mid w \in W,\ w(E^\beta) = E^\beta\}$. Let $S(E)$ denote the algebra of polynomial functions on E. Then, the restriction map from $S(E)$ to $S(E^\beta)$ induces a surjection from $S(E)^W$ to $S(E^\beta)^{W_\beta}$.

Proof

The theorem reduces easily to the case where Δ generates E. We first treat the case when the Dynkin diagram of Δ is connected. In this case, the automorphism β is either an involution or is an automorphism of order three of the Dynkin diagram of type D_4. As the theorem is known to hold when β is an involution (see [CD2, appendice A]), we assume that the latter holds. Following [Bo, Chapter VI, Section 4.8], $E = \mathbb{R}^4$ and $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is the canonical basis of $\mathbb{R}^4$. The roots system Δ is equal to

$$\{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 4\},$$

and the base for Δ is

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \qquad \alpha_2 = \varepsilon_2 - \varepsilon_3, \qquad \alpha_3 = \varepsilon_3 - \varepsilon_4, \qquad \alpha_4 = \varepsilon_3 + \varepsilon_4.$$

The elements of the Weyl group W are of the form $w = \mathrm{sgn} \circ s$, where sgn denotes a sign change on the ε_i of product 1 and s is a permutation of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$. Without loss of generality, we have

$$\beta(\alpha_1) = \alpha_3, \qquad \beta(\alpha_3) = \alpha_4, \qquad \beta(\alpha_4) = \alpha_1, \qquad \beta(\alpha_2) = \alpha_2.$$

The subspace E^β is generated by α_2 and $\alpha_1 + \alpha_3 + \alpha_4$. It is accordingly also generated by

$$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = \varepsilon_1 + \varepsilon_2 \qquad \text{and} \qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \varepsilon_1 + \varepsilon_3.$$

At this point, it is convenient to conjugate β by the element $x \in W$, which fixes $\varepsilon_1, \varepsilon_4$ and negates both ε_2 and ε_3. The fixed-point set $E^{\beta'}$ of the resulting automorphism $\beta' = x\beta x^{-1}$ is generated by

$$\alpha_1 = \varepsilon_1 - \varepsilon_2 = x(\varepsilon_1 + \varepsilon_2) \qquad \text{and} \qquad \alpha_2 = \varepsilon_2 - \varepsilon_3 = x(\varepsilon_1 + \varepsilon_3) - (\varepsilon_1 - \varepsilon_2).$$

If $E^{\beta'}$ is stable under $w = \mathrm{sgn} \circ s$, then s must fix ε_4, and $\mathrm{sgn} = \pm\mathrm{Id}$. Thus $W_{\beta'}$ is isomorphic to the direct product $S_3 \times \{\pm 1\}$, where S_3 is the permutation group of $\varepsilon_1, \varepsilon_2, \varepsilon_3$. Using the canonical coordinates, $S(E^{\beta'})^{S_3}$ is isomorphic to the polynomial algebra generated by the restriction of $X_1^2 + X_2^2 + X_3^2$ and $X_1X_2X_3$ to $E^{\beta'}$. Let u_1 and u_2 denote the respective restrictions. As u_1 is invariant under -1, the subalgebra $S(E^{\beta'})^{W_{\beta'}}$ is generated by u_1 and u_2^2. On the other hand, according to [Bo, Chapter VI, Section 4.8], $S(E)^W$ is generated by the symmetric polynomials

$$\begin{aligned}
t_1(X) &= (X_1)^2 + \cdots + (X_4)^2,\\
t_2(X) &= (X_1)^2(X_2)^2 + \cdots + (X_3)^2(X_4)^2,\\
t_3(X) &= (X_1)^2(X_2)^2(X_3)^2 + \cdots + (X_2)^2(X_3)^2(X_4)^2,\\
t_4(X) &= X_1X_2X_3X_4.
\end{aligned}$$

The desired surjectivity now follows from the fact that $(t_1)_{|E^{\beta'}} = u_1$ and $(t_3)_{|E^{\beta'}} = u_2^2$.

We now prove the theorem in the case of disconnected Dynkin diagrams. In this circumstance, the automorphism β may permute the connected components. The decomposition of a permutation into a product of disjoint cycles allows us to reduce the problem to the case where β permutes n isomorphic copies of a connected Dynkin diagram cyclically among themselves. The cyclic permutation given by β on these connected components allows us to identify each of them with a given D', generating a space E' and a root system R$'$ with Weyl group W'. The nth power of β induces an automorphism β' of the Dynkin diagram D'. One has

$$E^{\beta} = \{(x, \dots, x) \mid x \in E',\ \beta'(x) = x\}.$$

As a result, the theorem reduces to the connected case proved above. □

COROLLARY 1
The natural restriction map from $\mathcal{PW}(E)$ *to* $\mathcal{PW}(E^{\beta})$ *induces a surjection from* $\mathcal{PW}(E)^{W}$ *to* $\mathcal{PW}(E^{\beta})^{W_{\beta}}$.

Proof
The corollary follows from Theorem 5 by the argument given in [CD2, lemme C.1]. □

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