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GEOMETRIC SIDE OF A LOCAL RELATIVE TRACE FORMULA

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ABSTRACT. Following a scheme suggested by B. Feigon, we investigate a local relative trace formula in the situation of a reductive p-adic group G relative to a symmetric subgroup $H = \underline{H}(\mathrm{F})$ where $\underline{H}$ is split over the local field F of characteristic zero and $G = \underline{G}(\mathrm{F})$ is the restriction of scalars of $\underline{H}_{/\,\mathrm{E}}$ relative to a quadratic unramified extension E of F. We adapt techniques of the proof of the local trace formula by J. Arthur in order to get a geometric expansion of the integral over $H \times H$ of a truncated kernel associated to the regular representation of G.

CONTENTS

INTRODUCTION

In this article, we investigate a local relative trace formula in the situation of p-adic groups relative to a symmetric subgroup. This work is inspired by the recent results of B. Feigon (see [F]), where she investigated what she called a local relative trace formula on PGL(2) and a local Kuznetsov trace formula for $U(2)$.

Before we describe our setting and results, we would like to explain on the toy model of finite groups the framework of the formulas of Feigon. We even start with the more general framework of the relative trace formula initiated by H. Jacquet (cf. [Jac97]; see also [O] for an account of some applications of this relative trace formula).

Let G be a finite group and let H, H', Γ be subgroups of G. We endow any finite set with the counting measure. We denote by r the right regular representation of

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G on $L^2(\Gamma\backslash G)$ and we consider the H-fixed linear form ξ on $L^2(\Gamma\backslash G)$ defined by

$$\xi = \sum_{h\in H\cap\Gamma\backslash H} \delta_{\Gamma h}, \tag{0.1}$$

where $\delta_{\Gamma h}$ is the Dirac measure of the coset Γh or, in other words,

$$\xi(\psi) = \int_{H\cap\Gamma\backslash H} \psi(\Gamma h)dh, \quad \psi \in L^2(\Gamma\backslash G).$$

We define similarly ξ' relative to H'.

We view ξ, ξ' as elements of $L^2(\Gamma\backslash G)$ and we form the coefficient $c_{\xi,\xi'}(g) = (r(g)\xi, \xi')$. Integrating against functions on G, it defines a "distribution" Θ on G which is right invariant by H and left invariant by H'. The relative trace formula in this context gives two expressions of $\Theta(f)$ for f a function on G: the first one, called the geometric side, in terms of orbital integrals, and the second one, called the spectral side, in terms of irreducible representations of G.

First we deal with the geometric side. For this purpose we introduce suitable orbital integrals. For $\gamma \in \Gamma$, we set $[\gamma] := (H'\cap\Gamma)\gamma(H\cap\Gamma)$ and introduce two subgroups of $H'\times H$:

$$(H'\times H)_\gamma = \{(h',h) | h'\gamma h^{-1} = \gamma\}, (H'\cap\Gamma\times H\cap\Gamma)_\gamma = (H'\times H)_\gamma \cap (\Gamma\times\Gamma).$$

Then, we define the orbital integral of a function f on G by

$$I([\gamma], f) = \int_{(H'\times H)_\gamma\backslash(H'\times H)} f(h'\gamma h^{-1})dh'dh.$$

Let f be a function on G. Since $r(g)\delta_{\Gamma h} = \delta_{\Gamma hg^{-1}}$, the definition of ξ and ξ' gives

$$\Theta(f) = \sum_{g\in G} f(g)\Theta(g) = \sum_{g\in G} f(g)\frac{1}{vol(\Gamma\cap H)}\frac{1}{vol(\Gamma\cap H')}\sum_{h\in H}\sum_{h'\in H'}(\delta_{\Gamma hg^{-1}}, \delta_{\Gamma h'}).$$

Changing g in $g^{-1}h$ and using the fact that $(\delta_{\Gamma g}, \delta_{\Gamma h'})$ is equal to 1 for $g \in \Gamma h'$ and to zero otherwise, one gets

$$\Theta(f) = \frac{1}{vol(\Gamma\cap H)}\frac{1}{vol(\Gamma\cap H')}\sum_{h\in H}\sum_{h'\in H'}\sum_{\gamma\in\Gamma} f(h'\gamma h). \tag{0.2}$$

A simple computation of volumes leads to the geometric expression of Θ in terms of orbital integrals:

$$\Theta(f) = \sum_{[\gamma]\in H'\cap\Gamma\backslash\Gamma/\Gamma\cap H} vol((H'\cap\Gamma\times H\cap\Gamma)_\gamma\backslash(H'\times H)_\gamma)I([\gamma], f). \tag{0.3}$$

Let us shift to the spectral side. We decompose $L^2(\Gamma\backslash G)$ into isotypic components $\bigoplus_{\pi\in\widehat{G}}\mathcal{H}_\pi$, where $\widehat{G}$ is the unitary dual of G. The restriction of ξ and ξ' to $\mathcal{H}_\pi$ will be denoted ξ_π and ξ'_π respectively. The spectral formula for Θ is the simple equality

$$\Theta = \sum_{\pi\in\widehat{G}} c_{\xi_\pi,\xi'_{\pi'}}. \tag{0.4}$$

Notice that it might also be interesting to decompose further the representation into irreducible representations, and the restriction of ξ to each of them will be called a period.

There is a third interpretation of the distribution Θ. If f is a function on G, then the operator $r(f)$ on $L^2(\Gamma\backslash G)$ is an integral operator whose kernel K_f is the function on $\Gamma\backslash G \times \Gamma\backslash G$ given by

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

By (0.2), one gets easily the following expression of $\Theta(f)$:

$$\Theta(f) = \int_{(H'\cap\Gamma\backslash H')\times(H\cap\Gamma\backslash H)} K_f(h',h)dh'dh. \tag{0.5}$$

This point of view is probably the best one. But it is important to have the representation theoretic meaning of Θ.

The toy model for the local relative trace formula of Feigon appears as a particular case of the above relative trace formula. In that case, the groups G, H, and H' are products $G_1\times G_1$, $H_1\times H_1$, and $H_1'\times H_1'$ respectively, and Γ is the diagonal of $G_1\times G_1$. Then $\Gamma\backslash G$ identifies with G_1, and the right representation corresponds to the representation R of $G_1\times G_1$ on $L^2(G_1)$ given by $[R(x,y)\phi](g) = \phi(x^{-1}gy)$. Hence we have

$$\xi(\psi) = \int_{H_1} \psi(h)dh, \quad \psi\in L^2(G_1).$$

The spectral side is more concrete. If $(\pi_1,\mathcal{H}_{\pi_1})$ is an irreducible unitary representation of G_1, then $G_1\times G_1$ acts on $\mathrm{End}(\mathcal{H}_{\pi_1})$ by an irreducible representation denoted by π. It is unitary if we use the scalar product $(\cdot,\cdot)$ associated to the Hilbert-Schmidt norm. Moreover $L^2(G_1)$ is canonically isomorphic to the direct sum $\bigoplus_{\pi_1\in\widehat{G_1}}\mathrm{End}(\mathcal{H}_{\pi_1})$, where $\widehat{G_1}$ is the unitary dual of G_1. Let $P_\pi\in\mathcal{H}_{\pi_1}$ be the orthogonal projector onto the space of invariant vectors under H_1. Then the period map ξ_π, which is a linear form on $\mathrm{End}(\mathcal{H}_{\pi_1})$, is given by

$$\xi_\pi(T) = \int_{H_1} Tr(\pi_1(h)T)dh = (T,P_\pi), \quad T\in\mathrm{End}(\mathcal{H}_{\pi_1}).$$

One further decomposes ξ_π by using an orthonormal basis $(\eta_{\pi_1,i})$ of the space of H_1-invariant vectors. We will use the identification of $\mathrm{End}(\mathcal{H}_{\pi_1})$ with the tensor product of $\mathcal{H}_{\pi_1}$ with its conjugate complex vector space. Under this identification, one has

$$P_\pi = \sum_i \eta_{\pi_1,i}\otimes\eta_{\pi_1,i}.$$

We define similar notation for ξ' relative to H'. Then, for two functions f_1,f_2 on G_1, the spectral side of (0.4) can be written

$$\Theta(f_1\otimes f_2) = \sum_{\pi_1\in\widehat{G_1}}\sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1)c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

For the geometric side, we define the orbital integral of a function f on G_1 by

$$I(g,f) = \int_{(H_1'\times H_1)_g\backslash H_1'\times H_1} f(h'gh^{-1})dhdh',$$

which depends only on the double coset $H_1'gH_1$. Then one gets by (0.3) the equality

$$\Theta(f_1\otimes f_2) = \sum_{g\in H_1'\backslash G_1/H_1} v(g)I(g,f_1)I(g,f_2),$$

where the $v(g)$'s are positive constants depending on volumes. Hence the final form of the local relative trace formula is

$$\sum_{g\in H_1'\backslash G_1/H_1} v(g)I(g,f_1)I(g,f_2) = \sum_{\pi_1\in\hat{G}_1}\sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1)c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

This formula allows us to invert the orbital integrals $I(g,f_1)$ for any $g\in H_1'\backslash G_1/H_1$. For this purpose, one chooses $g_1\in G_1$ and takes for f_2 the Dirac measure at g_1. Then $I(g_1,f_2)=1$, and the other orbital integrals of f_2 are zero. Hence

$$v(g_1)I(g_1,f_1) = \sum_{\pi_1\in\hat{G}_1}\sum_{i,i'} c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_1)c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2).$$

In order to make the formula more precise, one needs to compute the constants $c_{\eta_{\pi_1,i},\eta'_{\pi_1,i'}}(f_2)$.

The inversion of orbital integrals is one of our motivations for investigating a local relative trace formula in the situation of p-adic groups relative to a symmetric subgroup H, and we will take $H=H'$.

In this article, we consider a reductive algebraic group $\underline{H}$ defined over a non-archimedean local field F of characteristic 0. We fix a quadratic unramified extension E of F and we consider the group $\underline{G}:=\mathrm{Res}_{\mathrm{E}\,/\,\mathrm{F}}\underline{H}$ obtained by restriction of scalars of $\underline{H}$. Here $\underline{H}$ is considered as a group defined over E. We denote by H and G the group of F-points of $\underline{H}$ and $\underline{G}$ respectively. Then G is isomorphic to $\underline{H}(\mathrm{E})$, and H appears as the fixed points of G under the involution of G induced by the nontrivial element of the Galois group of E / F. We assume that $\underline{H}$ is split over F and we fix a maximal split torus A_0 of H. The groups G and H correspond to G_1 and $H_1=H_1'$ respectively in our example of a local relative trace formula for finite groups.

The starting point of our study is the analogue to the expression (0.5). We consider the regular representation R of $G\times G$ on $L^2(G)$ given by $(R(g_1,g_2)\psi)(x)=\psi(g_1^{-1}xg_2)$. Then for $f=f_1\otimes f_2$ where f_1 and f_2 are two smooth compactly supported functions on G, the corresponding operator $R(f)$ is an integral operator on $L^2(G)$ with smooth kernel

$$K_f(x,y)=\int_G f_1(xg)f_2(gy)dg=\int_G f_1(g)f_2(x^{-1}gy)dg.$$

As H may not be compact, even modulo the split component A_H of the center of H, we shall truncate this kernel to integrate it. We multiply this kernel by a product of functions $u(x,T)u(y,T)$ where $u(\cdot,T)$ is the characteristic function of a large compact subset in $A_H\backslash H$ depending on a parameter $T\in \mathfrak{a}_0=\mathrm{Rat}(A_0)\otimes_{\mathbb{Z}}\mathbb{R}$ ($\mathrm{Rat}(A_0)$ is the group of F-rational characters of A_0) as in [Ar3] (cf. (2.7)). As H is split, we have $A_H=A_G$. Hence the kernel K_f is left invariant by the diagonal $\mathrm{diag}(A_H)$ of A_H, and we can integrate the truncated kernel over $\mathrm{diag}(A_H)\backslash H\times H$. We set

$$K^T(f):=\int_{\mathrm{diag}(A_H)\backslash(H\times H)} K_f(x_1,x_2)u(x_1,T)u(x_2,T)d\overline{(x_1,x_2)}.$$

In [Ar3], Arthur studies the integral of $K_f(x,x)u(x,T)$ over $A_G\backslash G$ to obtain its local trace formula on reductive groups.

We study the geometric expression of the distribution $K^T(f)$ and its dependence on the parameter T. Our main results (Theorem 2.3 and Corollary 2.11) assert that

$K^T(f)$ is asymptotic as T approaches infinity to another distribution $J^T(f)$ of the form

$$J^T(f) = \sum_{k=0}^{N} p_{\xi_k}(T,f)e^{\xi_k(T)}, \tag{0.6}$$

where $\xi_0 = 0, \dots, \xi_N$ are distinct points of the dual space $i\mathfrak{a}_0^*$ and each $p_{\xi_k}(T,f)$ is a polynomial function in T. Moreover, the constant term $\tilde{J}(f) := p_0(0,f)$ of $J^T(f)$ is well-defined and uniquely determined by $K^T(f)$. We give an explicit expression of this constant term in terms of weighted orbital integrals.

These results are analogous to those of [Ar3] for the group case. Our proof follows closely the study by Arthur of the geometric side of his local trace formula, which we were able to adapt under our assumptions to the case of double truncations.

In the first section, we introduce notation on groups and on symmetric spaces according to [RR]. The starting point of our study is the Weyl integration formula established in [RR], which takes into account the (H,H)-double classes of σ-regular elements of G (cf. (1.30) and (1.32)). These double classes are expressed in terms of σ-tori, which are tori whose elements are anti-invariant by σ. Under our assumptions, there is a bijective correspondence $S \to S_\sigma$ between maximal tori of H and maximal σ-tori of G which preserves H-conjugacy classes.

Then the Weyl integration formula can be written in terms of Levi subgroups $M \in \mathcal{L}(A_0)$ of H containing A_0 and M-conjugacy classes of maximal anisotropic tori of M (cf. (1.33)):

$$\begin{aligned}\int_G f(g)dg \\ = \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M}\sum_{x_m\in\kappa_S} c_{S,x_m}\int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{\mathrm{diag}(A_M)\backslash H\times H} f(h^{-1}x_m\gamma l) \\ \times\, d\overline{(h,l)}d\gamma,\end{aligned}$$

where κ_S is a finite subset of G, c_M and c_{S,x_m} are positive constants, $\mathcal{T}_M$ is a suitable set of anisotropic tori of M, and Δ_σ is a jacobian.

A fundamental result for our proofs concerns the orbital integral $\mathcal{M}(f)$ of a compactly supported smooth function f on G. It is defined on σ-regular points by

$$\mathcal{M}(f)(x_m\gamma) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\int_{\mathrm{diag}(A_S)\backslash H\times H} f(h^{-1}x_m\gamma l)d\overline{(h,l)},$$

where S is a maximal torus of H, $x_m \in \kappa_S$, and $\gamma \in S_\sigma$ such that $x_m\gamma$ is σ-regular. As in the group case using the exponential map and the property that each root of S_σ has multiciplity 2 in the Lie algebra of G, we prove that the orbital integral is bounded on the subset of σ-regular points of G (cf. Theorem 1.2).

In the second section, we explain the truncation process based on the notion of (H,M)-orthogonal sets and prove our main results. Using the Weyl integration formula, we can write

$$K^T(f) = \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M}\sum_{x_m\in\kappa_S} c_{S,x_m}\int_{S_\sigma} K^T(x_m,\gamma,f)d\gamma,$$

where

$$K^T(x_m,\gamma,f)=|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{\mathrm{diag}(A_M)\backslash H\times H}\int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2)$$

$$\times f_2(x_1^{-1}x_m\gamma x_2)u_M(x_1,y_1,x_2,y_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)}$$

and

$$u_M(x_1,y_1,x_2,y_2,T)=\int_{A_H\backslash A_M} u(y_1^{-1}ax_1,T)u(y_2^{-1}ax_2,T)da.$$

The function $J^T(f)$ is obtained in a similar way to $K^T(f)$, where we replace the weight function $u_M(x_1,y_1,x_2,y_2,T)$ by another weight function $v_M(x_1,y_1,x_2,y_2,T)$.

The weight function v_M is given by

$$v_M(x_1,y_1,x_2,y_2,T):=\int_{A_H\backslash A_M}\sigma_M(h_M(a),\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))da,$$

where $\sigma_M(\cdot,\mathcal{Y})$ is the function defined in [Ar3, equation (3.8)] depending on an (H,M)-orthogonal set $\mathcal{Y}$ and $\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)$ is an (H,M)-orthogonal set obtained as the "minimum" of two (H,M)-orthogonal sets $\mathcal{Y}_M(x_1,y_1,T)$ and $\mathcal{Y}_M(x_2,y_2,T)$ (cf. (2.4), Lemma 2.2, and (2.11)). If $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are two (H,M)-orthogonal positive sets, then the "minimum" $\mathcal{Z}$ of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ satisfies the property that the convex hull $\mathcal{S}_M(\mathcal{Z})$ in $a_H\backslash a_M$ of the points of $\mathcal{Z}$ is the intersection of the convex hulls $\mathcal{S}_M(\mathcal{Y}_1)$ and $\mathcal{S}_M(\mathcal{Y}_2)$ in $a_H\backslash a_M$ of the points of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ respectively.

If $\|T\|$ is large compared to $\|x_i\|,\|y_i\|,i=1,2$, then $\sigma_M(\cdot,\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))$ is just the characteristic function of $\mathcal{S}_M(\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))$. In that case, this function is equal to the product of $\sigma_M(\cdot,\mathcal{Y}_M(x_1,y_1,T))$ and $\sigma_M(\cdot,\mathcal{Y}_M(x_2,y_2,T))$.

A key step of our proof is a good estimate of

$$|u_M(x_1,y_1,x_2,y_2,T)-v_M(x_1,y_1,x_2,y_2,T)|$$

when $x_i,y_i,i=1,2$, satisfy $f_1(y_1^{-1}x_m\gamma y_2)f_2(x_1^{-1}x_m\gamma x_2)\neq 0$ for some $\gamma\in S_\sigma$ and $x_m\in\kappa_S$. Then, using that orbital integrals are bounded, we deduce our result on $|K^T(f)-J^T(f)|$.

This work is a first step towards a local relative trace formula. For the spectral side, we have to prove that $K^T(f)$ is asymptotic to a distribution $k^T(f)$ which is of general form (0.6) and constructed from spectral data. We hope that we can express the constant term of $k^T(f)$ in terms of regularized local period integrals introduced by Feigon in [F] in the same way as Jacquet-Lapid-Rogawski regularized period integrals for automorphic forms in [JLR]. In [DH], we have explicated the spectral side of such a local relative trace formula for PGL(2).

1. Preliminaries

1.1. **Reductive p-adic groups.** Let F be a non-archimedean local field of characteristic 0 and odd residual characteristic q. Let $|\cdot|_{\mathrm{F}}$ denote the normalized valuation on F.

For any algebraic variety $\underline{M}$ defined over F, we identify $\underline{M}$ with $\underline{M}(\overline{\mathrm{F}})$, where $\overline{\mathrm{F}}$ is an algebraic closure of F, and we set $M:=\underline{M}(\mathrm{F})$.

We will use the same convention as in [W2]. One considers various algebraic groups $\underline{J}$ defined over F, sentences such as

(1.1) "let M be an algebraic group" will mean "let M be the F-points of an algebraic group $\underline{M}$ defined over F", and "let A be a split torus" will mean "let A be the group of F-points of a torus, $\underline{A}$, defined and split over F".

If J is an algebraic group, one denotes by $\mathrm{Rat}(J)$ the group of its rational characters defined over F. If V is a vector space, V^* denotes its dual. If V is real, $V_{\mathbb{C}}$ refers to its complexification.

Let $\underline{G}$ be an algebraic reductive group defined over F. We fix a maximal split torus A_0 of G and we denote by M_0 its centralizer in G.

Let A_G be the maximal split torus of the center of G and let

$$a_G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Rat}(G), \mathbb{R}).$$

One has the canonical map $h_G : G \to a_G$, which is defined by

$$e^{\langle h_G(x), \chi\rangle} = |\chi(x)|_{\mathrm{F}}, \quad x \in G, \chi \in \mathrm{Rat}(G). \tag{1.2}$$

The restriction of rational characters from G to A_G induces an isomorphism

$$\mathrm{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathrm{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.3}$$

Notice that $\mathrm{Rat}(A_G)$ appears as a generating lattice in the dual space a_G^* of a_G and

$$a_G^* \simeq \mathrm{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.4}$$

The kernel G^1 of h_G is the intersection over all characters $\chi \in \mathrm{Rat}(G)$ of G of the kernels of $|\chi|_F$. The group G^1 is normal in G and contains the derived group G_{der} of G. Moreover, it is well-known that

(1.5) the group G^1 is generated by the compact subgroups of G.

G. Henniart has communicated to us an unpublished proof of this result by N. Abe, F. Herzig, G. Henniart, and M. F. Vigneras.

(1.6) One denotes by $a_{G,\mathrm{F}}$ (resp. $\tilde{a}_{G,\mathrm{F}}$) the image of G (resp., A_G) by h_G. Then G/G^1 is isomorphic to the lattice $a_{G,\mathrm{F}}$.

If P is a parabolic subgroup of G with Levi subgroup M, we keep the same notation with M instead of G.

The inclusions $A_G \subset A_M \subset M \subset G$ determine a surjective morphism $a_{M,\mathrm{F}} \to a_{G,\mathrm{F}}$ (resp. an injective morphism, $\tilde{a}_{G,\mathrm{F}} \to \tilde{a}_{M,\mathrm{F}}$) which extends uniquely to a surjective linear map h_{MG} from a_M to a_G (resp. injective linear map between a_G and a_M). The second map allows us to identify a_G with a subspace of a_M, and the kernel of the first one, a_M^G, satisfies

$$a_M = a_M^G \oplus a_G. \tag{1.7}$$

For $M = M_0$, we set $a_0 := a_{M_0}$ and $a_0^G := a_{M_0}^G$. We fix a scalar product $(\cdot,\cdot)$ on a_0 which is invariant under the Weyl group $W(G, A_0)$ of (G, A_0). Then a_G identifies with the fixed point set of a_0 by $W(G, A_0)$, and a_0^G is an invariant subspace of a_0 under $W(G, A_0)$. Hence it is the orthogonal subspace to a_G in a_0. The space

a_G^* might be viewed as a subspace of a_0^* by (1.7). Moreover, by definition of the surjective map $a_0 \to a_G$, one deduces that

(1.8) if $m_0 \in M_0$, then $h_G(m_0)$ is the orthogonal projection of $h_{M_0}(m_0)$ onto a_G.

From (1.7) applied to (M, M_0) instead of (G, M), one obtains a decomposition $a_0 = a_0^M \oplus a_M$. From the $W(G, A_0)$-invariance of the scalar product on a_0, one gets:

(1.9) The decomposition $a_0 = a_0^M \oplus a_M$ is an orthogonal decomposition. The space a_M^* appears as a subspace of a_0^*, and in the identification of a_0 with a_0^* given by the scalar product, a_M^* identifies with a_M.

The decomposition $a_M = a_M^G \oplus a_G$ is orthogonal with respect to the restriction to a_M of the $W(G, A_0)$-invariant scalar product on a_0, and the natural map h_{MG} is identified with the orthogonal projection of a_M onto a_G.

(1.10) In particular, $a_{G,\mathrm{F}}$ is the orthogonal projection of $a_{M,\mathrm{F}}$ onto a_G. Moreover, we have $\tilde{a}_{G,\mathrm{F}} = a_G \cap \tilde{a}_{M,\mathrm{F}}$ (cf. [Ar3, equation (1.4)]).

By a Levi subgroup of G, we mean a group M containing M_0 which is the Levi component of a parabolic subgroup of G. If P is a parabolic subgroup containing M_0, then it has a unique Levi subgroup denoted by M_P which contains M_0. We will denote by N_P the unipotent radical of P.

For a Levi subgroup M, we write $\mathcal{L}(M)$ for the finite set of Levi subgroups of G which contain M and we also let $\mathcal{P}(M)$ denote the finite set of parabolic subgroups P with $M_P = M$.

Let K be the fixator of a special point in the apartment of A_0 in the Bruhat-Tits building of G. We have the Cartan decomposition

$$G = KM_0K. \tag{1.11}$$

If $P = M_P N_P$ is a parabolic subgroup of G containing M_0, then

$$G = PK = M_P N_P K. \tag{1.12}$$

If $x \in G$, we can write

$$x = m_P(x) n_P(x) k_P(x), \quad m_P(x) \in M_P, \quad n_P(x) \in N_P, \quad k_P(x) \in K. \tag{1.13}$$

We set

$$h_P(x) := h_{M_P}(m_P(x)). \tag{1.14}$$

The point $m_P(x)$ is defined up to multiplication by an element of $K \cap M_P$, but $h_P(x)$ does not depend of this choice.

We introduce a norm $\|\cdot\|$ on G as in [W2, Section I.1] (called height function in [W2]). Let $\Lambda_0 : G \to \mathrm{GL}_n(\mathrm{F})$ be an algebraic embedding. For $g \in G$, we write

$$\Lambda_0(g) = (a_{i,j})_{i,j=1,\dots,n}, \quad \Lambda_0(g^{-1}) = (b_{i,j})_{i,j=1,\dots,n}.$$

We set

$$\|g\| := \sup_{i,j} \sup(|a_{i,j}|_\mathrm{F}, |b_{i,j}|_\mathrm{F}). \tag{1.15}$$

If $\Lambda : G \to \mathrm{GL}_d(\mathrm{F})$ is another algebraic embedding, then the norm $\|\cdot\|_\Lambda$ attached to Λ as above is equivalent to $\|\cdot\|$ in the following sense: there are a positive constant C_Λ and a positive integer d_Λ such that

$$\|g\|_\Lambda \leq C_\Lambda \|g\|^{d_\Lambda}.$$

This allows us to use results of [W2] for estimates on norms.

The following properties of the norm $\| \cdot \|$ are immediate consequences of its definition:

$$1 \leq \|x\| = \|x^{-1}\|, \quad x \in G, \tag{1.16}$$

$$\|xy\| \leq \|x\|\|y\|, \quad x, y \in G. \tag{1.17}$$

In order to have estimates, we introduce the following notation. Let r be a positive integer. Let f and g be two positive functions defined on a subset W of G^r.

(1.18) We write $f(x) \preccurlyeq g(x)$, $x \in W$, if and only if there are a positive constant c and a positive integer d such that $f(x) \leq cg(x)^d$ for all $x \in W$.

(1.19) We write $f(x) \approx g(x)$, $x \in W$, if $f(x) \preccurlyeq g(x), x \in W$, and $g(x) \preccurlyeq f(x)$, $x \in W$.

If f_1, f_2, and f_3 are positive functions on G^r, we clearly have:
if $f_1(x) \preccurlyeq f_2(x), x \in W$, and $f_2(x) \preccurlyeq f_3(x), x \in W$, then $f_1(x) \preccurlyeq f_3(x), x \in W$;
if $f_1(x) \approx f_2(x), x \in W$, and $f_2(x) \approx f_3(x), x \in W$, then $f_1(x) \approx f_3(x), x \in W$.
Moreover, if f_1, f_2, g_1 and g_2 are positive functions on G^r which take values greater than or equal to 1, we obtain easily the following properties:

(1.20)
(1) for all positive integers d, we have $f_1(x) \approx f_1(x)^d, x \in W$;
(2) if $f_1(x) \preccurlyeq g_1(x), x \in W$, and $f_2(x) \preccurlyeq g_2(x), x \in W$, then $(f_1 f_2)(x) \preccurlyeq (g_1 g_2)(x), x \in W$;
(3) if $f_1(x) \approx g_1(x), x \in W$, and $f_2(x) \approx g_2(x), x \in W$, then $(f_1 f_2)(x) \approx (g_1 g_2)(x), x \in W$.

Since $\|x\| = \|xyy^{-1}\| \leq \|xy\|\|y\|$ and $\|xy\| \leq \|x\|\|y\|$, we obtain

(1.21) If Ω is a compact subset of G, then $\|x\| \approx \|x\omega\|, \quad x \in G, \quad \omega \in \Omega$.

Let $P = M_P N_P$ be a parabolic subgroup of G containing M_0. Then each $x \in G$ can be written $x = m_P(x) n_P(x) k$, where $m_P(x) \in M_P, n_P(x) \in N_P$, and $k \in K$. By [Ar3, equation (4.5)], we then have

$$\|m_P(x)\| + \|n_P(x)\| \preccurlyeq \|x\|, \quad x \in G. \tag{1.22}$$

Recall that G^1 is the kernel of $h_G : G \to a_G$. Let us prove that

$$\|xa\| \approx \|x\|\|a\|, \quad x \in G^1, \quad a \in A_G. \tag{1.23}$$

According to the Cartan decomposition (1.11), if $g \in G$ we denote by $m_0(g)$ an element of M_0 such that there exist $k, k' \in K$ with $g = k m_0(g) k'$. Notice that $\|h_{M_0}(m_0(g))\|$ does not depend on our choice of $m_0(g)$. By (1.21), one has

$$\|g\| \approx \|m_0(g)\|, \quad g \in G, \tag{1.24}$$

and, by [W2, equation I.1(6)], we have

$$\|m_0\| \approx e^{\|h_{M_0}(m_0)\|}, \quad m_0 \in M_0. \tag{1.25}$$

Let $x \in G^1$ and $a \in A_G$. Then $m_0(x) \in G^1 \cap M_0$ and $m_0(xa) = m_0(x)a$. Thus, one has $h_G(m_0(x)) = 0$. We deduce from (1.7) and (1.8) that $h_{M_0}(m_0(x))$ belongs to a_0^G. Since $h_{M_0}(m_0(x)a) = h_{M_0}(m_0(x)) + h_{M_0}(a)$ and $h_{M_0}(a) \in a_G$, we obtain by orthogonality that

$$\frac{1}{2}(\|h_{M_0}(m_0(x))\| + \|h_{M_0}(a)\|) \leq \|h_{M_0}(m_0(x)a)\| \leq \|h_{M_0}(m_0(x))\| + \|h_{M_0}(a)\|.$$

Hence (1.23) follows from (1.24) and (1.25).

We denote by $C_c^\infty(G)$ the space of smooth functions on G with compact support. We normalize Haar measures according to [Ar3, Section 1]. Unless otherwise stated, the Haar measure on a compact group will be normalized to have total volume 1.

Let M be a Levi subgroup of G. We fix a Haar measure on a_M so that the volume of the quotient $a_M/\tilde{a}_{M,\mathrm{F}}$ equals 1.

Let $P = MN_P \in \mathcal{P}(M)$. We denote by δ_P the modular function of P given by

$$\delta_P(mn) = e^{2\rho_P(h_M(m))}, \quad m \in M, \quad n \in N_P,$$

where $2\rho_P$ is the sum of roots, with multiplicity, of (P, A_M). Let $\bar{P} = MN_{\bar{P}}$ be the parabolic subgroup which is opposite to P. If dn is a Haar measure on N_P, then the integral

$$\gamma(P) = \int_{N_P} e^{2\rho_{\bar{P}}(h_{\bar{P}}(n))} dn$$

is finite. Moreover, the measure $\gamma(P)^{-1} dn$ is independent of the choice of dn and thus defines a canonical Haar measure on N_P.

If dm is a Haar measure on M, then there exists a unique Haar measure dg on G, independent of the choice of the parabolic subgroup P, such that

$$\int_G f(g)dg = \frac{1}{\gamma(P)\gamma(\bar{P})} \int_{N_P} \int_M \int_{N_{\bar{P}}} f(nm\bar{n})\delta_P(m)^{-1} d\bar{n}\, dm\, dn,$$

for $f \in C_c^\infty(G)$. If so, we say that dm and dg are compatible. Compatibility has the obvious transitivity property with respect to Levi subgroups of M. Using the Iwasawa decomposition (1.12), these measures satisfy

$$\int_G f(g)dg = \frac{1}{\gamma(P)} \int_K \int_M \int_{N_P} f(mnk) dn\, dm\, dk.$$

1.2. **The symmetric space $H\backslash G$.** Let E be an unramified quadratic extension of F. Then $\mathrm{E} = \mathrm{F}[\tau]$ where τ^2 is not a square in F. We denote by σ the nontrivial element of the Galois group $\mathcal{G}al(\mathrm{E}/\mathrm{F})$ of E / F. The normalized valuation $|\cdot|_{\mathrm{E}}$ on E satisfies $|x|_{\mathrm{E}} = |x|_{\mathrm{F}}^2$ for $x \in \mathrm{F}$.

If $\underline{J}$ is an algebraic group defined over F, then J is as usual its group of points over F. Let $\underline{J} \times_{\mathrm{F}} \mathrm{E}$ be the group, defined over E, obtained from $\underline{J}$ by extension of scalars. We consider the group

$$\underline{\tilde{J}} := \mathrm{Res}_{\mathrm{E}/\mathrm{F}}(\underline{J} \times_{\mathrm{F}} \mathrm{E})$$

defined over F, obtained by restriction of scalars.

With our convention, one has $\tilde{J} = \underline{\tilde{J}}(\mathrm{F})$ and $\tilde{J}$ is isomorphic to $\underline{J}(\mathrm{E})$.

Let $\underline{H}$ be a reductive group defined over F. Throughout this article, we assume that $\underline{H}$ is split over F and we set $\underline{G} := \underline{\tilde{H}}$ and $G := \tilde{H}$. We fix a maximal split torus A_0 of H. Then A_0 is also a maximal split torus of G. We also have $A_H = A_G$.

The nontrivial element σ of $\mathcal{G}al(\mathrm{E}/\mathrm{F})$ induces an involution of $\underline{G}$ defined over F and denoted by the same letter. This automorphism σ extends to an E-automorphism σ_{E} on $\underline{G} \times_{\mathrm{F}} \mathrm{E}$.

We consider the canonical map φ defined over F from $\underline{G}$ to $(\underline{H} \times_{\mathrm{F}} \mathrm{E}) \times (\underline{H} \times_{\mathrm{F}} \mathrm{E})$ by $\varphi(g) = (g, \sigma(g))$.

(1.26) Then φ extends uniquely to an isomorphism Ψ defined over E from $\underline{G} \times_{\mathrm{F}} \mathrm{E}$ to $(\underline{H} \times_{\mathrm{F}} \mathrm{E}) \times (\underline{H} \times_{\mathrm{F}} \mathrm{E})$ such that $\Psi(g) = (g, \sigma(g))$ for all $g \in \underline{G}$. Moreover, if $\Psi(g) = (g_1, g_2)$, then $\Psi(\sigma_{\mathrm{E}}(g)) = (g_2, g_1)$.

Now we turn to the description of the geometric structure of the symmetric space $\mathcal{S} = H\backslash G$ according to [RR, Sections 2 and 3].

Let $\underline{\mathfrak{g}}$ be the Lie algebra of $\underline{G}$ and let $\mathfrak{g}$ be the Lie algebra of its F-points. We will say that $\mathfrak{g}$ is the Lie algebra of G and the Lie algebra $\mathfrak{h}$ of H consists of the elements of $\mathfrak{g}$ invariant by σ. We denote by $\mathfrak{q}$ the space of anti-invariant elements of $\mathfrak{g}$ by σ. Thus one has $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, and $\mathfrak{g}$ may be identified with $\mathfrak{h} \otimes_{\mathrm{F}} \mathrm{E}$.

As in [RR, Section 2], we say that a subspace $\mathfrak{c}$ of $\mathfrak{q}$ is a Cartan subspace of $\mathfrak{q}$ if $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{q}$ (or equivalently a maximal abelian subalgebra of $\mathfrak{q}$) made of semisimple elements. As $\mathrm{E} = \mathrm{F}[\tau]$, the multiplication by τ induces an isomorphism between the set of Cartan subspaces of $\mathfrak{q}$ and the set of Cartan subalgebras of $\mathfrak{h}$ which preserves H-conjugacy classes.

We denote by $\underline{\mathcal{P}}$ the connected component of 1 in the set of x in $\underline{G}$ such that $\sigma(x) = x^{-1}$. Then the map $\underline{p}$ from $\underline{G}$ to $\underline{\mathcal{P}}$ defined by $\underline{p}(x) = x^{-1}\sigma(x)$ induces an isomorphim of affine varieties, $\underline{p} : \underline{H}\backslash\underline{G} \to \underline{\mathcal{P}}$.

A torus $\underline{A}$ of $\underline{G}$ is called a σ-torus if $\underline{A}$ is a torus defined over F contained in $\underline{\mathcal{P}}$. Notice that such a torus is called a σ-split torus in [RR]. We would rather change the terminology, as σ-tori are not necessarily split over F. Each σ-torus is the centralizer in $\underline{\mathcal{P}}$ of a Cartan subspace of $\mathfrak{q}$ or equivalently of a Cartan subalgebra of $\mathfrak{h}$.

Let S be a maximal torus of H. We denote by $\underline{S}_\sigma$ the connected component of $\underline{\tilde{S}} \cap \underline{\mathcal{P}}$. Then $\underline{S}_\sigma$ is a σ-torus defined over F which identifies with the anti-diagonal $\{(s, s^{-1}); s \in \underline{S}\}$ of $\underline{S} \times \underline{S}$ by the isomorphism (1.26). Thus $\underline{S}_\sigma$ is a maximal σ-torus, and each maximal σ-torus arises in this way. The H-conjugacy classes of maximal tori of H are in a bijective correspondence with the H-conjugacy classes of maximal σ-tori of G by the map $S \mapsto S_\sigma$. The roots of $\underline{S}$ (resp. $\underline{S}_\sigma$) in $\underline{\mathfrak{h}} = Lie(\underline{H})$ (resp. $\underline{\mathfrak{q}} \otimes_{\mathrm{F}} \bar{\mathrm{F}}$) are the restrictions of the roots of $\underline{\tilde{S}}$ in $\underline{\mathfrak{g}} = Lie(\underline{G})$.

(1.27) Therefore, each root of $\underline{S}$ (resp. $\underline{S}_\sigma$) in $\underline{\mathfrak{g}}$ has multiplicity two. If $\underline{\tilde{S}}$ splits over a finite extension F' of F, we denote by $\Phi(S'_\sigma, \mathfrak{g}')$ (resp. $\Phi(S', \mathfrak{h}')$) the set of roots of $\underline{S}_\sigma(\mathrm{F}')$ in $\mathfrak{g} \otimes_{\mathrm{F}} \mathrm{F}'$ (resp. $\underline{S}(\mathrm{F}')$ in $\mathfrak{h} \otimes_{\mathrm{F}} \mathrm{F}'$). Let $\underline{\tilde{\mathfrak{s}}}$ be the Lie algebra of $\underline{\tilde{S}}$. Then the differential of each root α of $\Phi(\tilde{S}', \mathfrak{g}')$ defines a linear form on $\tilde{\mathfrak{s}} \otimes_{\mathrm{F}} \mathrm{F}'$ denoted by the same letter.

Let $\mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F})$ be the Galois group of $\overline{\mathrm{F}}/\mathrm{F}$. By [RR, Section 3], the set of (H, S_σ)-double cosets in $\underline{H}\underline{S}_\sigma \cap G$ are parametrized by the finite set I of cohomology classes in $H^1(\mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F}), \underline{H} \cap \underline{S}_\sigma)$ which split in both $\underline{H}$ and $\underline{S}_\sigma$. To each such class m, we attach an element $x_m \in G$ of the form $x_m = h_m a_m^{-1}$ with $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$ such that $m_\gamma = h_m^{-1}\gamma(h_m) = a_m^{-1}\gamma(a_m)$ for all $\gamma \in \mathcal{G}al(\overline{\mathrm{F}}/\mathrm{F})$.

Lemma 1.1. *Let $x \in G$ such that $x = hs$ with $h \in \underline{H}$ and $s \in \underline{\tilde{S}}$. Then xSx^{-1} is a maximal torus of H, and there exists $h' \in H$ such that $x' = h'x$ centralizes the split connected component A_S of S.*

Proof. By replacing S by an H-conjugate if necessary, we may assume that $A := A_S$ is contained in the fixed maximal split torus A_0 of H. Since H is split, A_0 is also a maximal split torus of G.

As $x = hs \in G$, the torus $\underline{S}' := x\underline{S}x^{-1}$ is equal to $h\underline{S}h^{-1} \subset \underline{H}$. Thus $\underline{S}'$ is defined over F and is contained in $\underline{H}$. Hence we get the first assertion.

Let $S' := \underline{S}'(\mathrm{F})$ and let A' be the split connected component of S'. There exists $h_1 \in H$ such that $h_1 A' h_1^{-1} \subset A_0$. We set $x_1 = h_1 x$. Then we have $A_1 := x_1 A x_1^{-1} \subset A_0$.

Let $M = Z_G(A)$ and $M_1 = Z_G(A_1) = x_1 M x_1^{-1}$. Then A_0 and $x_1 A_0 x_1^{-1}$ are maximal split tori of M_1. Therefore, there exists $y_1 \in M_1$ such that $y_1 x_1 A_0 x_1^{-1} y_1^{-1} = A_0$. As H is split, the Weyl group of A_0 in G coincides with the Weyl group of A_0 in H. Thus there exist $h_2 \in N_H(A_0)$ and $v \in Z_G(A_0)$ such that $z := y_1 x_1 = h_2 v$.

For $a \in A \subset A_0$, one has $zaz^{-1} = h_2 a h_2^{-1} = y_1 x_1 a x_1^{-1} y_1^{-1} = x_1 a x_1^{-1}$ since $x_1 a x_1^{-1} \in A_1$ and $y_1 \in M_1$. One deduces that $x' := h_2^{-1} h_1 x$ centralizes A. □

This lemma allows us to state the following result.

(1.28) For each maximal torus S of H, we can fix a finite set of representatives $\kappa_S = \{x_m\}_{m \in I}$ of the (H, S_σ)-double cosets in $\underline{H}\underline{S}_\sigma \cap G$ such that each element x_m may be written $x_m = h_m a_m^{-1}$ where $h_m \in \underline{H}$ centralizes A_S and $a_m \in \underline{S}_\sigma$. Hence x_m centralizes A_S.

1.3. **Weyl integration formula and orbital integrals.** We first recall basic notions on the symmetric space according to [RR, Section 3]. An element x in $\underline{G}$ is called σ-semisimple if the double coset $\underline{H}x\underline{H}$ is Zariski closed. This is equivalent to saying that $\underline{p}(x)$ is a semisimple point of $\underline{G}$. We say that a σ-semisimple element x is σ-regular if this closed double coset $\underline{H}x\underline{H}$ is of maximal dimension. This is equivalent to saying that the centralizer of $\underline{p}(x)$ in $\mathfrak{q}$ (resp. $\underline{\mathcal{P}}$) is a Cartan subspace of $\mathfrak{q}$ (resp. a maximal σ-torus of $\underline{G}$).

We denote by $G^{\sigma-reg}$ the set of σ-regular elements of G.

For $g \in G$, we denote by $D_G(g)$ the coefficient of the least power of t appearing nontrivially in $\det(t + 1 - \mathrm{Ad}(g))$. We define the H-bi-invariant function Δ_σ on G by $\Delta_\sigma(x) = D_G(\underline{p}(x))$. Then, by [RR, Lemmas 3.2 and 3.3], the set of $g \in G$ such that $\Delta_\sigma(g) \neq 0$ coincides with $G^{\sigma-reg}$.

Let S be a maximal torus of H with Lie algebra $\mathfrak{s}$. Then $\tilde{\mathfrak{s}} := \mathfrak{s} \otimes_{\mathrm{F}} \mathrm{E}$ identifies with the Lie algebra of $\tilde{S}$. For $g \in x_m S_\sigma$ with $x_m \in \kappa_S$, one has

$$\Delta_\sigma(g) = D_G(\underline{p}(g)) = \det(1 - \mathrm{Ad}(\underline{p}(g)))_{\mathfrak{g}/\tilde{\mathfrak{s}}}. \tag{1.29}$$

By [RR, Theorem 3.4(1)], the set $G^{\sigma-reg}$ is a disjoint union

$$G^{\sigma-reg} = \bigcup_{\{S\}_H} \bigcup_{x_m \in \kappa_S} H\big((x_m S_\sigma) \cap G^{\sigma-reg}\big)H, \tag{1.30}$$

where $\{S\}_H$ runs the H-conjugacy classes of maximal tori of H.

If $x_m \in \kappa_S$, then $x_m = h_m a_m$ for some $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$; hence $\underline{p}(x_m) = a_m^{-2}$ commutes with S and S_σ. Therefore for $\gamma \in S_\sigma$, we have

$$\underline{p}(x_m\gamma) = \underline{p}(x_m)\gamma^{-2} \quad \text{and} \quad H x_m \gamma S = H x_m \gamma.$$

We have the following Weyl integration formula (cf. [RR, Theorem 3.4(2)]).

Let f be a compactly supported smooth function on G. Then we have

$$\begin{aligned} &\int_G f(y)dy \\ &= \sum_{\{S\}_H} \sum_{x_m \in \kappa_S} c^0_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|^{1/2}_{\mathrm{F}} \int_{S\backslash H} \int_H f(hx_m\gamma l)dhd\bar{l}d\gamma, \end{aligned} \tag{1.31}$$

where the constants c^0_{S,x_m} are explicitly given in [RR, Theorem 3.4(1)].

For our purpose, we need another version of this Weyl integration formula. Let S be a maximal torus of H. We denote by A_S its split connected component. Since the quotient $A_S\backslash S$ is compact, by our choice of measure, the integration over $S\backslash H$ in the Weyl formula above can be replaced by an integration over $A_S\backslash H$. Moreover, it is convenient to change h into h^{-1}. As every $x_m \in \kappa_S$ commutes with A_S (cf. (1.28)), one can replace the integration over $(A_S\backslash H) \times H$ by an integration over $\mathrm{diag}(A_S)\backslash(H \times H)$, where $\mathrm{diag}(A_S)$ is the diagonal of A_S. This gives the following Weyl integration formula equivalent to (1.31):

(1.32)

$$\begin{aligned} &\int_G f(y)dy \\ &= \sum_{\{S\}_H} \sum_{x_m \in \kappa_S} c^0_{S,x_m} \int_{S_\sigma} |\Delta_\sigma(x_m\gamma)|^{1/2}_{\mathrm{F}} \int_{\mathrm{diag}(A_S)\backslash(H\times H)} f(h^{-1}x_m\gamma l)d\overline{(h,l)}d\gamma. \end{aligned}$$

We will now describe the H-conjugacy classes of maximal tori of H in terms of Levi subgroups M of H containing A_0 (i.e., $M \in \mathcal{L}(A_0)$) and M-conjugacy classes of some tori of M.

Let $M \in \mathcal{L}(A_0)$ and let $N_H(M)$ be its normalizer in H. If S is a maximal torus of M, we denote by $W(M,S)$ (resp. $W(H,S)$) its Weyl group in M (resp. H). We choose a set $\mathcal{T}_M$ of representatives for the M-conjugacy classes of maximal tori S in M such that $A_M\backslash S$ is compact. For $M, M' \in \mathcal{L}(A_0)$, we write $M \sim M'$ if M and M' are conjugate under H.

Let S be a maximal torus of H whose split connected component A_S is contained in A_0. Then the centralizer M of A_S belongs to $\mathcal{L}(A_0)$ and S is a maximal torus of M such that $A_M\backslash S$ is compact. If S' is a maximal torus H-conjugated to S such that $A_{S'}$ is contained in A_0, then the centralizer M' of $A_{S'}$ in H belongs to $\mathcal{L}(A_0)$ and $M' \sim M$.

Since each maximal torus of H is H-conjugated to a maximal torus S such that $A_S \subset A_0$, we obtain a surjective map $S \mapsto \{S\}_H$ from the set of S in $\mathcal{T}_M$, where M runs through a system of representatives of $\mathcal{L}(A_0)_{/\sim}$, to the set of H-conjugacy classes of maximal tori of H.

Let $M \in \mathcal{L}(A_0)$. By [Ko , equation (7.12.3)], the cardinal of the class of M in $\mathcal{L}(A_0)_{/\sim}$ is equal to

$$\frac{|W(H, A_0)|}{|W(M, A_0)||N_H(M)/M|},$$

where $N_H(M)$ is the normalizer of M in H.

According to [Ko , Lemma 7.1], if S is a maximal torus of M, then the number of M-conjugacy classes of maximal tori S' in M, such that S' is H-conjugated to

S, is equal to

$$\frac{|N_H(M)/M||W(M,S)|}{|W(H,S)|}.$$

Therefore, we can rewrite (1.32) as follows:

$$\begin{aligned}\int_G f(g)dg &= \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M}\sum_{x_m\in\kappa_S} c_{S,x_m}\int_{S_\sigma}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \\ &\quad\times\int_{\mathrm{diag}(A_M)\backslash H\times H} f(h^{-1}x_m\gamma l)d\overline{(h,l)}d\gamma,\end{aligned} \tag{1.33}$$

where

$$c_M=\frac{|W(M,A_0)|}{|W(H,A_0)|}\quad\text{and}\quad c_{S,x_m}=\frac{|W(H,S)|}{|W(M,S)|}c^0_{S,x_m}.$$

Let $f\in C_c^\infty(G)$. We define the orbital integral $\mathcal{M}(f)$ of f on $G^{\sigma-reg}$ as follows. Let S be a maximal torus of H. For $x_m\in\kappa_S$ and $\gamma\in S_\sigma$ such that $x_m\gamma\in G^{\sigma-reg}$, we set

$$\begin{aligned}\mathcal{M}(f)(x_m\gamma) &:= |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\int_{\mathrm{diag}(A_S)\backslash(H\times H)} f(h^{-1}x_m\gamma l)d\overline{(h,l)} \\ &= |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\int_{S\backslash H}\int_H f(hx_m\gamma l)dhd\bar{l}.\end{aligned} \tag{1.34}$$

Our definition corresponds, up to a positive constant factor, to [RR, Definition 3.8]. Indeed, by definition of Δ_σ, we have $\Delta_\sigma(x_m\gamma)=D_G(\underline{p}(x_m\gamma))$. Since we can write $x_m=h_ma_m$ with $h_m\in\underline{H}$ and $a_m\in\underline{S}_\sigma$, we have $\underline{p}(x_m\gamma)=\underline{p}(x_m)\gamma^{-2}=a_m^{-2}\gamma^{-2}$ for $\gamma\in S_\sigma$. Let F' be an extension of E such that $\tilde{S}$ splits over F' and $a_m\in\underline{S}_\sigma(\mathrm{F}')$. Since each root α of $\underline{S}_\sigma(\mathrm{F}')$ in $\mathfrak{g}\otimes_{\mathrm{F}}\mathrm{F}'$ has multiplicity $m(\alpha)=2$, using notation of (1.27), we obtain

$$\Delta_\sigma(x_m\gamma)=\prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')}(1-\underline{p}(x_m)^\alpha\gamma^{-2\alpha})^2=\prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')}(\gamma^\alpha-\underline{p}(x_m)^\alpha\gamma^{-\alpha})^2.$$

Hence

$$\begin{aligned}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}'}^{1/4} &= \prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')}|(\gamma^\alpha-\underline{p}(x_m)^\alpha\gamma^{-\alpha})^{m(\alpha)-1}|_{\mathrm{F}'}^{1/2} \\ &= \prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')}|(\gamma^\alpha-\underline{p}(x_m)^\alpha\gamma^{-\alpha})|_{\mathrm{F}'}^{1/2}.\end{aligned}$$

Then the Weyl integration formula (1.31) is given in terms of orbital integrals as in [RR, p. 126] by

$$\int_G f(y)dy=\sum_{\{S\}_H}\sum_{x_m\in\kappa_S}c^0_{S,x_m}\int_{S_\sigma}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4}\mathcal{M}(f)(x_m\gamma)d\gamma.$$

Theorem 1.2. *Let $f\in C_c^\infty(G)$ and S be a maximal torus of H. Let $x_m\in\kappa_S$.*

(1) *There exists a compact set Ω in S_σ such that, for any γ in the complementary of Ω in S_σ with $x_m\gamma\in G^{\sigma-reg}$, one has $\mathcal{M}(f)(x_m\gamma)=0$.*

(2) *One has*

$$\sup_{\gamma\in S_\sigma;\,x_m\gamma\in G^{\sigma-reg}}|\mathcal{M}(f)(x_m\gamma)|<+\infty.$$

Proof. The proof follows the one of the group case (see [HC3, proof of Theorem 14]). We write it here for the convenience of the reader.

Let us first show (1). Let ω be the support of f. We consider the set ω_S of elements γ in S_σ such that $x_m\gamma$ is in the closure of $H\omega H$. For $g \in G$, we consider the polynomial function

$$\det(1 - t - \mathrm{Ad}\,\underline{p}(g)) = (-1)^n t^n + q_{n-1}(g)t^{n-1} + \cdots + q_l(g)t^l, \tag{1.35}$$

where l is the rank of G and n is its dimension. Each q_j is an $H \times H$ bi-invariant regular function on G and thus is bounded on $x_m\omega_S$. Therefore, the roots of $\det(1 - t - \mathrm{Ad}\,\underline{p}(g))$ are bounded on $x_m\omega_S$.

For $\gamma \in S_\sigma$, we have $\underline{p}(x_m\gamma) = \underline{p}(x_m)\gamma^{-2}$. We choose a finite extension F' of F such that $\underline{\tilde{S}}$ splits over F' and $\underline{p}(x_m) \in \underline{S}_\sigma(\mathrm{F}')$. Using notation of (1.27), the roots of $\det(1-t-\mathrm{Ad}\,\underline{p}(x_m\gamma))$ are the numbers $(1-\underline{p}(x_m)^\alpha\gamma^{-2\alpha})$ for $\alpha \in \Phi(S'_\sigma, \mathfrak{g}')$. Since these roots are bounded on $x_m\omega_S$, we obtain that the maps $\gamma \to \gamma^\alpha$, $\alpha \in \Phi(S'_\sigma, \mathfrak{g}')$, are bounded on ω_S. This implies that ω_S is bounded, and hence the closure Ω of ω_S satisfies the first assertion.

It remains to show (2). According to (1), if $\gamma \notin \Omega$, then $\mathcal{M}(f)(x_m\gamma) = 0$. Thus it is enough to prove that, for each $\gamma_0 \in S_\sigma$, there exists a neighborhood V_{γ_0} of γ_0 in S_σ such that

$$\sup_{\gamma \in V_{\gamma_0}, x_m\gamma \in G^{\sigma-reg}} |\mathcal{M}(f)(x_m\gamma)| < +\infty. \tag{1.36}$$

Let $y_0 := \underline{p}(x_m\gamma_0)$. Let us first assume that y_0 is central in G. Then we have $\Delta_\sigma(x_m\gamma_0\gamma) = D_G(y_0\gamma^{-2}) = D_G(\gamma^{-2})$ for $\gamma \in S_\sigma$ and $x_m\gamma_0 h(x_m\gamma_0)^{-1} \in H$ for $h \in H$. We define the function f_0 on G by $f_0(g) := f(x_m\gamma_0 g)$. Then we have $\mathcal{M}(f_0)(\gamma) = \mathcal{M}(f)(x_m\gamma_0\gamma)$ for $\gamma \in S_\sigma \cap G^{\sigma-reg}$. Therefore we can restrict ourselves to the case $y_0 = 1$. As in the group case, we use the exponential map "exp", which is well-defined in a neighborhood of 0 in $\mathfrak{g}$, since the characteristic of F is equal to zero (cf. [HC4, Section 10]). As in [HC1, proof of Lemma 15], we can choose an H-invariant open neighborhood V_0 of 0 in $\mathfrak{h}$ such that the map $X \in V_0 \mapsto \exp(\tau X)$ is an isomorphism, and a homeomorphism onto its image, and such that there exists an H-invariant function $\varphi \in C_c^\infty(\mathfrak{h})$ such that $\varphi(X) = 1$ for $X \in V_0$. We define $\bar{f}$ in $C_c^\infty(\mathfrak{h})$ by $\bar{f}(X) = \varphi(X)\int_H f(h\exp(\tau X))dh$.

Let $\mathfrak{s}$ be the Lie algebra of S. For $X \in \mathfrak{s}$, we set $\eta(X) = |\det(\mathrm{ad}X)_{\mathfrak{h}/\mathfrak{s}}|_{\mathrm{F}}$. We consider a finite extension F' of F such that $\underline{\tilde{S}}$ splits over F' and $\underline{p}(x_m) \in \underline{S}_\sigma(\mathrm{F}')$. We use here notation introduced in (1.27). Since each root of S'_σ in $\mathfrak{g}'$ has multiplicity 2, we have for $X \in V_0$, regular in $\mathfrak{h}$,

$$\frac{|\Delta_\sigma(\exp\tau X)|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \frac{|D_{G'}(\exp(-2\tau X))|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \frac{\prod_{\alpha\in\Phi(S',\mathfrak{h}')}|1 - e^{-2\tau\alpha(X)}|_{\mathrm{F}'}}{\prod_{\alpha\in\Phi(S',\mathfrak{h}')}|\alpha(X)|_{\mathrm{F}'}}$$

$$= |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|} \prod_{\alpha\in\Phi(S',\mathfrak{h}')} |1 - \tau\alpha(X) + \frac{4\tau^2\alpha(X)^2}{3!} + \cdots|_{\mathrm{F}'}.$$

We can reduce V_0 in such way that each term of this product is equal to 1. Thus we obtain

$$\begin{aligned}\mathcal{M}(f)(\exp \tau X) &= |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|/2}\eta(X)^{1/2}\int_{H/S}\Big(\int_H f(h\exp\tau \mathrm{Ad}(l)X)dh\Big)d\bar{l}\\ &= |2\tau|_{\mathrm{F}'}^{|\Phi(S',\mathfrak{h}')|/2}\eta(X)^{1/2}\int_{H/S}\bar{f}(\mathrm{Ad}(l)X)d\bar{l},\end{aligned}$$

for $X \in V_0$, regular in $\mathfrak{h}$. Hence the estimate (1.36) follows from the result on the Lie algebra given in [HC3, Theorem 13].

Now, if $y_0 = \underline{p}(x_m\gamma_0)$ is not central in G, we consider the centralizer $\underline{\mathcal{Z}}$ of y_0 in $\underline{H}$. Let $\underline{\mathcal{Z}}^0$ be the identity component of $\underline{\mathcal{Z}}$. By [Bo, Section III.9], the group $\underline{\mathcal{Z}}^0$ is defined over F. As usual, we set $\underline{\tilde{\mathcal{Z}}}^0 := \mathrm{Res}_{\mathrm{E}\,/\,\mathrm{F}}(\underline{\mathcal{Z}}^0 \times_{\mathrm{F}} \mathrm{E})$ and we denote by $\underline{\tilde{\mathfrak{z}}}$ its Lie algebra. By definition of $\tilde{\mathfrak{z}}$, one has

$$|\det(1-\mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}} \neq 0.$$

Thus there exists a neighborhood V of 1 in S_σ such that, for all $\gamma \in V$,

$$|\det(1-\mathrm{Ad}(y_0\gamma^{-2}))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}} = |\det(1-\mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}} \neq 0. \tag{1.37}$$

Let ω be the support of f. From [HC3, Lemma 19], there exist a neighborhood V_1 of y_0 in $\tilde{S}$ and a compact subset $\overline{C_G}$ of $\tilde{\mathcal{Z}}^0\backslash G$ such that if $g \in G$ satisfies $g^{-1}V_1 g \cap \underline{p}(\omega) \neq \emptyset$, then its image $\bar{g}$ in $\tilde{Z}^0\backslash G$ belongs to $\overline{C_G}$.

We choose a neighborhood W of 1 in S_σ such that $W \subset V$ and $\underline{p}(x_m\gamma_0\gamma) = y_0\gamma^{-2} \in V_1$ for all $\gamma \in W$. By [Bo, Section III.9.1], the quotient $\mathcal{Z}^0\backslash H$ is a closed subset of $\tilde{\mathcal{Z}}^0\backslash G$. Hence

(1.38) the set $\overline{C} := \overline{C_G} \cap \mathcal{Z}^0\backslash H$ is a compact subset of $\mathcal{Z}^0\backslash H$ such that if $l \in H$ satisfies $l^{-1}y_0\gamma^{-2}l \in \underline{p}(\omega)$ for some $\gamma \in W$, then its image $\bar{l}$ in $\mathcal{Z}^0\backslash H$ belongs to $\overline{C}$.

Let $\gamma \in W$ such that $x_m\gamma_0\gamma \in G^{\sigma-reg}$. One has

$$\int_{S\backslash H}\int_H f(hx_m\gamma_0\gamma l)dhd\bar{l} = \int_{\mathcal{Z}^0\backslash H}\int_{S\backslash \mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi l)dhd\bar{\xi}d\bar{l}. \tag{1.39}$$

By our choice of W, the map

$$\bar{l} \in \mathcal{Z}^0\backslash H \mapsto \int_{S\backslash \mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi l)dhd\bar{\xi}$$

vanishes outside $\overline{C}$. We choose $u \in C_c^\infty(H)$ such that the map $\overline{u} \in C_c^\infty(\mathcal{Z}^0\backslash H)$, defined by $\overline{u}(\bar{l}) := \int_{\mathcal{Z}^0} u(\xi l)d\xi$, is equal to 1 on $\overline{C}$. As u and f are compactly supported, the map

$$\Phi : z \in \tilde{\mathcal{Z}}^0 \mapsto \int_H u(l)\int_H f(hx_m\gamma_0 zl)dhdl$$

is well-defined.

Since $y_0 = \underline{p}(x_m\gamma_0) = (x_m\gamma_0)^{-1}\sigma(x_m\gamma_0)$ and $\mathcal{Z}^0$ centralizes y_0, we have $\xi(x_m\gamma_0)^{-1}\sigma(x_m\gamma_0) = (x_m\gamma_0)^{-1}\sigma(x_m\gamma_0)\xi$ for $\xi \in \mathcal{Z}^0$. Thus $x_m\gamma_0\xi(x_m\gamma_0)^{-1} \in H$, and Φ is left invariant by $\mathcal{Z}^0$.

We claim that $\Phi \in C_c^\infty(\mathcal{Z}^0\backslash\tilde{\mathcal{Z}}^0)$. Indeed, fix l in the support of u. If $f(hx_m\gamma_0 zl)$ is nonzero for some $h \in H$ and $z \in \tilde{\mathcal{Z}}^0$, then $\underline{p}(hx_m\gamma_0 zl) = \underline{p}(x_m\gamma_0 zl)$ belongs to $\underline{p}(\omega)$. Since z commutes with $y_0 = \underline{p}(x_m\gamma_0)$, we have $\underline{p}(x_m\gamma_0 zl) = l^{-1}y_0\underline{p}(z)\sigma(l)$.

As u is compactly supported, we get that $\Phi(z) = 0$ when $\underline{p}(z)$ is outside a compact set. Hence the map Φ is a compactly supported function on $\mathcal{Z}^0\backslash\tilde{\mathcal{Z}}^0$.

By assumption, the function f is right invariant by a compact open subgroup of G. Thus f is right invariant by some compact open subgroup of H. We denote by $\tau_l f$ the right translate of f by an element $l \in G$. Since u is compactly supported, the vector space generated by $\tau_l f$, when $l \in H$ runs through the support of u, is finite dimensional. Hence one can find a compact open subgroup J_1 of $\tilde{Z}^0$ such that, for each l in the support of u, the function $\tau_l f$ is right invariant by J_1. This implies that Φ is smooth, and our claim follows.

Therefore, there exists $\varphi \in C_c^\infty(\tilde{\mathcal{Z}}^0)$ such that

$$\Phi(z) = \int_{\mathcal{Z}^0} \varphi(\xi z) d\xi = \int_H u(l) \int_H f(hx_m\gamma_0 zl) dh dl, \quad z \in \tilde{\mathcal{Z}}^0.$$

We obtain

$$\begin{aligned}\int_{S\backslash\mathcal{Z}^0}\int_{\mathcal{Z}^0} \varphi(\xi_1\gamma\xi_2) d\xi_1 d\bar{\xi_2} &= \int_H u(l) \big(\int_{S\backslash\mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi_2 l) dh d\bar{\xi_2}\big) dl \\ &= \int_{\mathcal{Z}^0\backslash H}\int_{\mathcal{Z}^0} u(\xi_1 l)\big(\int_{S\backslash\mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi_2\xi_1 l) dh d\bar{\xi_2}\big) d\xi_1 d\bar{l} \\ &= \int_{\mathcal{Z}^0\backslash H} \overline{u}(\bar{l}) \big(\int_{S\backslash\mathcal{Z}^0}\int_H f(hx_m\gamma_0\gamma\xi_2 l) dh d\bar{\xi_2}\big) d\bar{l}.\end{aligned}$$

The map $\bar{u}$ being equal to 1 on the compact set $\overline{C}$, we obtain, using (1.39) and the definition of $\overline{C}$ (cf. (1.38)),

$$\int_{S\backslash\mathcal{Z}^0}\int_{\mathcal{Z}^0} \varphi(\xi_1\gamma\xi_2) d\xi_1 d\bar{\xi_2} = \int_{S\backslash H}\int_H f(hx_m\gamma_0\gamma l) dh d\bar{l}.$$

By (1.37) and the choice of W, one has

$$|D_G(y_0\gamma^{-2})|_{\mathrm{F}} = |D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}} |\det(1 - \mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}}, \quad \gamma \in W.$$

Then we get, for $\gamma \in W$ satisfying $x_m\gamma_0\gamma \in G^{\sigma-reg}$,

$$\mathcal{M}(f)(x_m\gamma_0\gamma) = |\det(1 - \mathrm{Ad}(y_0))_{\mathfrak{g}/\tilde{\mathfrak{z}}}|_{\mathrm{F}}^{1/4} |D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}}^{1/4} \int_{S\backslash\mathcal{Z}^0}\int_{\mathcal{Z}^0} \varphi(\xi_1\gamma\xi_2) d\xi_1 d\bar{\xi_2}.$$

Since $|D_{\tilde{\mathcal{Z}}^0}(\gamma^{-2})|_{\mathrm{F}}$ coincides with the function $|\Delta_\sigma|_{\mathrm{F}}$ for the group $\tilde{Z}^0$ evaluated at γ (cf. (1.29)), the estimate (1.36) for f is obtained by applying the first case to φ defined on $\tilde{\mathcal{Z}}^0$. □

2. Geometric side of the local relative trace formula

2.1. **Truncation.** In this section, we will recall some needed results of [Ar3, Section 3]. We keep the notation of Section 1.1 for the group H. Since H is split, one has $M_0 = A_0$. We fix a Levi subgroup $M \in \mathcal{L}(A_0)$ of H. Let $P \in \mathcal{P}(M)$. We recall that A_M denotes the maximal split torus of the center of M.

Let Σ_P be the set of roots of A_M in the Lie algebra of P, let Σ_P^r be the subset of reduced roots, and let Δ_P be the subset of simple roots.

As usual, for $\beta \in \Delta_P$, the "co-root" $\check{\beta} \in a_M$ is defined as follows: if $P \in \mathcal{P}(A_0)$ is a minimal parabolic subgroup, then $\check{\beta} = 2\beta/(\beta,\beta)$, where a_0^* identifies with a_0 through the scalar product on a_0. In the general case, we choose $P_0 \in \mathcal{P}(A_0)$ contained in P. Then there exists a unique $\alpha \in \Delta_{P_0}$ such that $\beta = \alpha_{|a_M}$. The "co-root" $\check{\beta}$ is the projection of $\check{\alpha}$ onto a_M with respect to the decomposition $a_0 = a_M \oplus a_0^M$. This projection does not depend on the choice of P_0.

We denote by a_P^+ the positive Weyl chamber of elements $X \in a_M$ satisfying $\alpha(X) > 0$ for all $\alpha \in \Sigma_P$.

Let $M \in \mathcal{L}(A_0)$. A set of points in a_M indexed by $P \in \mathcal{P}(M)$,

$$\mathcal{Y} = \mathcal{Y}_M := \{Y_P \in a_M; P \in \mathcal{P}(M)\},$$

is called an (H, M)-orthogonal set if, for any pair of adjacent parabolic subgroups P, P' in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$, one has $Y_P - Y_{P'} = r_{P,P'}\check{\alpha}$ for some real number $r_{P,P'}$. The orthogonal set is called positive if every number $r_{P,P'}$ is nonnegative. For example, this is the case when the number

$$d(\mathcal{Y}) = \inf\{\alpha(Y_P); \alpha \in \Delta_P, Y_P \in \mathcal{Y}, P \in \mathcal{P}(M)\} \tag{2.1}$$

is nonnegative.

One example is the set

$$\{-h_P(x); P \in \mathcal{P}(M)\},$$

defined for any point $x \in H$ (see 1.14 and 1.2 for the definiton of h_P). Indeed, this is a positive (H, M)-orthogonal set according to [Ar1, Lemma 3.6].

(2.2) If L belongs to $\mathcal{L}(M)$ and Q is a group in $\mathcal{P}(L)$, we define Y_Q to be the projection onto a_L of any point Y_P, with $P \in \mathcal{P}(M)$ and $P \subset Q$. Then Y_Q is independent of P and $\mathcal{Y}_L := \{Y_Q; Q \in \mathcal{P}(L)\}$ is an (H, L)-orthogonal set.

We shall write $\mathcal{S}_M(\mathcal{Y})$ for the convex hull in a_M/a_H of an (H, M)-orthogonal set $\mathcal{Y}$. Notice that $\mathcal{S}_M(\mathcal{Y})$ depends only on the projection onto a_M^H of each $Y_P \in \mathcal{Y}$, $P \in \mathcal{P}(M)$.

If each Y_P, for $P \in \mathcal{P}(M)$, is in the positive Weyl chamber a_P^+ (this condition is equivalent to saying that $d(\mathcal{Y})$ is positive), we have a simple description of $\mathcal{S}_M(\mathcal{Y}) \cap a_P^+$ (cf. [Ar3, Lemma 3.1]). We denote by $(\omega_\gamma^P)_{\gamma \in \Delta_P}$ the set of weights, that is, the dual basis in $(a_M^H)^*$ of the set of co-roots $\{\check{\gamma}; \gamma \in \Delta_P\}$. Then we have

$$\mathcal{S}_M(\mathcal{Y}) \cap a_P^+ = \{X \in a_P^+; \omega_\gamma^P(X - Y_P) \leq 0, \gamma \in \Delta_P\}. \tag{2.3}$$

We now recall a decomposition of the characteristic function of $\mathcal{S}_M(\mathcal{Y})$ valid when $\mathcal{Y}$ is positive (cf. [Ar3, equation (3.8)]). Suppose that Λ is a point in $a_{M,\mathbb{C}}^*$ whose real part $\Lambda_R \in a_M^*$ is in general position. For $P \in \mathcal{P}(M)$, let Δ_P^Λ be the set of simple roots $\alpha \in \Delta_P$ such that $\Lambda_R(\check{\alpha}) < 0$. Let φ_P^Λ be the characteristic function of the set of $X \in a_M$ such that $\omega_\alpha^P(X) > 0$ for each $\alpha \in \Delta_P^\Lambda$ and $\omega_\alpha^P(X) \leq 0$ for each α in the complementary of Δ_P^Λ in Δ_P. We define

$$\sigma_M(X, \mathcal{Y}) := \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(X - Y_P). \tag{2.4}$$

Then:

(2.5) By [Ar3, Section 3, p. 22], the function $\sigma_M(\cdot, \mathcal{Y})$ vanishes on the complementary of $\mathcal{S}_M(\mathcal{Y})$ and is bounded. Moreover, if $\mathcal{Y}$ is positive, then $\sigma_M(\cdot, \mathcal{Y})$ is exactly the characteristic function of $\mathcal{S}_M(\mathcal{Y})$.

For $P \in \mathcal{P}(M)$, we denote by $(\tilde{\omega}_\gamma^P)_{\gamma \in \Delta_P}$ the set of co-weights, that is, the dual basis in a_M^H of Δ_P.

Lemma 2.1. *Let P and P' be two adjacent parabolic subgroups in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$. Then:*

(1) *For all β in $\Delta_P - \{\alpha\}$, there exists a unique β' in $\Delta_{P'} - \{-\alpha\}$ such that $\beta' = \beta + k_\beta \alpha$ where k_β is a nonnegative integer. Moreover, the map $\beta \mapsto \beta'$ is a bijection between $\Delta_P - \{\alpha\}$ and $\Delta_{P'} - \{-\alpha\}$.*

(2) *For all β in $\Delta_P - \{\alpha\}$, one has $\tilde{\omega}_{\beta'}^{P'} = \tilde{\omega}_\beta^P$.*

Proof. We denote by $\mathbb{N}$ the set of nonnegative integers and by $\mathbb{N}^*$ the subset of positive integers.

We will first show (1). As P and P' are adjacent, we have $\Sigma_{P'} = \big(\Sigma_P - \{\alpha\}\big) \cup \{-\alpha\}$. Let $\beta \in \Delta_P - \{\alpha\}$. If $\beta \in \Delta_{P'}$, then we set $\beta' := \beta$. Assume that β is not in $\Delta_{P'}$. Since $\beta \in \Sigma_{P'}$, there exists $\Theta \subset \Delta_{P'} - \{-\alpha\}$ such that $\beta = \sum_{\delta \in \Theta} n_\delta \delta - k_\beta \alpha$, where the n_δ's are positive integers and k_β is a nonnegative integer. Each δ in Θ belongs to Σ_P. Therefore, there are nonnegative integers $(r_{\delta,\eta})_{\eta \in \Delta_P}$ such that $\delta = \sum_{\eta \in \Delta_P} r_{\delta,\eta} \eta$. Set $\beta_1 := \sum_{\delta \in \Theta} n_\delta \delta = \beta + k_\beta \alpha$. Let $\gamma \in \Delta_P - \{\alpha\}$. If $\gamma \neq \beta$, one has $\beta_1(\tilde{\omega}_\gamma^P) = \beta(\tilde{\omega}_\gamma^P) = 0$. Thus, for each $\delta \in \Theta$, we have $r_{\delta,\gamma} = 0$. Hence $\delta = r_{\delta,\beta}\beta + r_{\delta,\alpha}\alpha$.

On the other hand, one has $\beta_1(\tilde{\omega}_\beta^P) = \beta(\tilde{\omega}_\beta^P) = 1$. Thus, for all $\delta \in \Theta$, one has $\sum_{\delta \in \Theta} n_\delta r_{\delta,\beta} = 1$. Since $n_\delta \in \mathbb{N}^*$ and $r_{\delta,\beta} \in \mathbb{N}$, one deduces that there exists a unique $\delta_0 \in \Theta$ such that $r_{\delta_0,\beta} \neq 0$ and one has $n_{\delta_0} = r_{\delta_0,\beta} = 1$. This implies that $\Theta = \{\delta_0\}$ and $\beta = \delta_0 - k_\beta \alpha$. We can take $\beta' := \delta_0$. Hence we obtain the existence of β' in all cases.

If $\beta_1' \in \Delta_{P'}$ satisfies $\beta_1' = \beta + k_\beta^1 \alpha$, then $\beta' = \beta_1' + (k_\beta - k_\beta^1)\alpha$. Since the roots β_1', β' and $-\alpha$ belong to the set $\Delta_{P'}$ of simple roots, we deduce that $\beta_1' = \beta'$. This gives the unicity of β'.

Let γ and β be in Δ_P such that $\gamma' = \beta'$. Then we have $\beta = \gamma + (k_\gamma - k_\beta)\alpha$. Since γ, β, and α belong to Δ_P, the same argument as above leads to $\beta = \gamma$. Hence, the map $\beta \mapsto \beta'$ is injective.

It now remains to show (2). Let $\beta \in \Delta_P - \{\alpha\}$. By definition, we have $\beta' = \beta + k_\beta \alpha \in \Delta_{P'} - \{-\alpha\}$ with $k_\beta \in \mathbb{N}$. Thus $\alpha(\tilde{\omega}_{\beta'}^{P'}) = \alpha(\tilde{\omega}_\beta^P) = 0$ and $\beta(\tilde{\omega}_{\beta'}^{P'}) = \beta'(\tilde{\omega}_{\beta'}^{P'}) = 1$. If $\gamma \in \Delta_P - \{\beta, \alpha\}$, then $\gamma' = \gamma + k_\gamma \alpha$ is different from β' by assertion (1). Thus we have $\gamma(\tilde{\omega}_{\beta'}^{P'}) = \gamma'(\tilde{\omega}_{\beta'}^{P'}) = 0$. One deduces that $\tilde{\omega}_{\beta'}^{P'} = \tilde{\omega}_\beta^P$. □

The above lemma allows us to define the minimum between two orthogonal sets.

(2.6) Let $P \in \mathcal{P}(M)$. For Y^1 and Y^2 in a_M, we denote by $\inf^P\{Y^1, Y^2\}$ the unique element Z in a_M^H such that, for all $\gamma \in \Delta_P$, one has $(\tilde{\omega}_\gamma^P, Z) = \inf\{(\tilde{\omega}_\gamma^P, Y^1), (\tilde{\omega}_\gamma^P, Y^2)\}$.

Lemma 2.2. *Let $\mathcal{Y}^1 = \{Y_P^1, P \in \mathcal{P}(M)\}$ and $\mathcal{Y}^2 = \{Y_P^2, P \in \mathcal{P}(M)\}$ be two (H,M)-orthogonal sets. Let $\mathcal{Z} := \inf(\mathcal{Y}^1, \mathcal{Y}^2)$ be the set of $Z_P := \inf^P\{Y_P^1, Y_P^2,\}$ when P runs $\mathcal{P}(M)$.*

(1) *The set $\mathcal{Z}$ is an (H,M)-orthogonal set.*

(2) *If $d(\mathcal{Y}^j) > 0$ for $j = 1, 2$, then $d(\mathcal{Z}) > 0$. In this case, the convex hull $\mathcal{S}_M(\mathcal{Z})$ is the intersection of $\mathcal{S}_M(\mathcal{Y}^1)$ and $\mathcal{S}_M(\mathcal{Y}^2)$.*

Proof. Let P and P' be two adjacent parabolic subgroups in $\mathcal{P}(M)$ whose chambers in a_M share the wall determined by the simple root $\alpha \in \Delta_P \cap (-\Delta_{P'})$. Let $\gamma \in \Delta_P - \{\alpha\}$. By definition of orthogonal sets, one has, for $j = 1$ or 2, $(\tilde{\omega}_\gamma^P, Y_P^j) = (\tilde{\omega}_\gamma^P, Y_{P'}^j)$.

By Lemma 2.1, we have $\tilde{\omega}^P_\gamma = \tilde{\omega}^{P'}_{\gamma'}$. Hence we obtain $(\tilde{\omega}^P_\gamma, Z_P) = (\tilde{\omega}^{P'}_{\gamma'}, Z_{P'})$ and $(\tilde{\omega}^{P'}_{\gamma'}, Z_{P'}) = (\tilde{\omega}^P_\gamma, Z_{P'})$. Since the scalar product on a_0 identifies a_M to a_M^*, one deduces that $Z_P - Z_{P'}$ is proportional to $\check{\alpha}$. The assertion (1) then follows.

Let us show (2). Let $j \in \{1,2\}$ and $P \in \mathcal{P}(M)$. By definition, we have $d(\mathcal{Y}^j) > 0$ if and only if $\alpha(Y^j_P) > 0$ for all $\alpha \in \Delta_P$. By [Ar1, Corollary 2.2], this implies that $(\tilde{\omega}^P_\alpha, Y^j_P) > 0$ for all $\alpha \in \Delta_P$. Let $\alpha \in \Delta_P$. Writing

$$Y^j_P = (\tilde{\omega}^P_\alpha, Y^j_P)\alpha + \sum_{\beta \in \Delta_P - \{\alpha\}} (\tilde{\omega}^P_\beta, Y^j_P)\beta + X^j,$$

with $X^j \in a_H$, the condition $\alpha(Y^j_P) > 0$ is equivalent to

$$\sum_{\beta \in \Delta_P - \{\alpha\}} (\tilde{\omega}^P_\beta, Y^j_P)[-(\beta,\alpha)] < (\tilde{\omega}^P_\alpha, Y^j_P)(\alpha,\alpha).$$

Since the real numbers $(\tilde{\omega}^P_\beta, Y^j_P)$, for $\beta \in \Delta_P$, and $-(\beta,\alpha)$, for $\alpha \neq \beta$ in Δ_P, are nonnegative, one deduces that

$$\begin{aligned}
& \sum_{\beta \in \Delta_P - \{\alpha\}} (\tilde{\omega}^P_\beta, Z_P)[-(\beta,\alpha)] \\
= & \sum_{\beta \in \Delta_P - \{\alpha\}} \inf\big((\tilde{\omega}^P_\beta, Y^1_P), (\tilde{\omega}^P_\beta, Y^2_P)\big)[-(\beta,\alpha)] \\
\leq & \ \inf\Big(\sum_{\beta \in \Delta_P - \{\alpha\}} (\tilde{\omega}^P_\beta, Y^1_P)[-(\beta,\alpha)], \sum_{\beta \in \Delta_P - \{\alpha\}} (\tilde{\omega}^P_\beta, Y^2_P)[-(\beta,\alpha)]\Big) \\
< & \ \inf\big((\tilde{\omega}^P_\alpha, Y^1_P), (\tilde{\omega}^P_\alpha, Y^2_P)\big)(\alpha,\alpha) = (\tilde{\omega}^P_\alpha, Z_P)(\alpha,\alpha).
\end{aligned}$$

This implies that $\alpha(Z_P) > 0$ for $\alpha \in \Delta_P$, and thus $d(\mathcal{Z}) > 0$.

To get the property of the convex hulls, it is enough to prove that, for all $P \in \mathcal{P}(M)$, $a_P^+ \cap \mathcal{S}_M(\mathcal{Y}^1) \cap \mathcal{S}_M(\mathcal{Y}^2) = a_P^+ \cap \mathcal{S}_M(\mathcal{Z})$. By [Ar3, Lemma 3.1], one has

$$a_P^+ \cap \mathcal{S}_M(\mathcal{Y}^j) = \{X \in a_P^+; \omega^P_\gamma(X - Y^j_P) \leq 0, \gamma \in \Delta_P\}.$$

Since $\tilde{\omega}^P_\gamma = c_\gamma \omega^P_\gamma$ for $\gamma \in \Delta_P$, where c_γ is a positive real number, the assertion follows easily. $\square$

2.2. **The truncated kernel.** We consider the regular representation R of $G \times G$ on $L^2(G)$ defined by

$$\big(R(y_1,y_2)\phi\big)(x) = \phi(y_1^{-1}xy_2), \quad \phi \in L^2(G), \quad y_1, y_2 \in G.$$

Consider $f \in C_c^\infty(G \times G)$ of the form $f(y_1,y_2) = f_1(y_1)f_2(y_2)$ with $f_1, f_2 \in C_c^\infty(G)$. Then

$$R(f) := \int_G \int_G f_1(y_1)f_2(y_2)R(y_1,y_2)dy_1dy_2$$

is an integral operator with smooth kernel

$$K_f(x,y) = \int_G f_1(xg)f_2(gy)dg = \int_G f_1(g)f_2(x^{-1}gy)dg.$$

In our case (i.e., H is split), one has $A_H = A_G$, and the kernel K_f is invariant by the diagonal $\mathrm{diag}(A_H)$ of A_H in $H \times H$. Since H is not compact, we introduce truncation to integrate this kernel on $\mathrm{diag}(A_H)\backslash(H \times H)$.

Recall that $a_{0,\mathrm{F}}$ is the image of M_0 by h_0. We fix a point T in $a_{0,\mathrm{F}}$. Let $P_0 \in \mathcal{P}(A_0)$. According to [Bou, Chapter 5, Section 3, no. 3.3, Theorem 2], the closure $\bar{a}^+_{P_0}$ of the positive Weyl chamber $a^+_{P_0}$ is a fundamental domain of the Weyl group

$W(H, A_0)$. We denote by T_{P_0} the unique translate by the Weyl group $W(H, A_0)$ of T in $\bar{a}^+_{P_0}$. Then

$$\mathcal{Y}_T := \{T_{P_0}; P_0 \in \mathcal{P}(A_0)\}$$

is an (H, A_0)-orthogonal set (see [Ar3, p. 20]). We shall assume that the number

$$d(T) := \inf_{\alpha \in \Delta_{P_0},\, P_0 \in \mathcal{P}(A_0)} \alpha(T_{P_0})$$

is suitably large. This means that the distance from T to any of the root hyperplanes in a_0 is large enough.

(2.7) We denote by $u(\cdot, T)$ the characteristic function in $A_H \backslash H$ of the set of points x such that

$$x = k_1 a k_2 \quad \text{with } a \in A_H \backslash A_0, \quad k_1, k_2 \in K \quad \text{and} \quad h_{A_0}(a) \in \mathcal{S}_{A_0}(\mathcal{Y}_T),$$

where $H = KA_0K$ is the Cartan decomposition of H.

We consider $u(\cdot, T)$ as an A_H-invariant function on H. Thus there is a compact set Ω_T of H such that if $u(x, T) \neq 0$, then $x \in A_H \Omega_T$. Let Ω be a compact subset of G containing the support of f_1 and f_2. We consider $g \in G$ and $x_1, x_2 \in H$ such that $f_1(g) f_2(x_1^{-1} g x_2) u(x_1, T) u(x_2, T) \neq 0$. Hence there are ω_1, ω_2 in Ω_T and a_1, a_2 in A_H such that $x_1 = \omega_1 a_1$, $x_2 = \omega_2 a_2$, and we have $g \in \Omega$ and $x_1^{-1} g x_2 = \omega_1^{-1} g \omega_2 a_1^{-1} a_2 \in \Omega$ since $A_H = A_G$. Therefore $a_1^{-1} a_2$ lies in a compact subset of A_H. Hence the map $(g, x_1, x_2) \mapsto f_1(g) f_2(x_1^{-1} g x_2) u(x_1, T) u(x_2, T)$ is a compactly supported function on $G \times \operatorname{diag}(A_H) \backslash (H \times H)$, and we can define

$$K^T(f) := \int_{\operatorname{diag}(A_H) \backslash H \times H} K_f(x_1, x_2) u(x_1, T) u(x_2, T) d\overline{(x_1, x_2)}.$$

By Fubini's Theorem, we have

$$K^T(f) = \int_G \int_{\operatorname{diag}(A_H) \backslash H \times H} f_1(g) f_2(x_1^{-1} g x_2) u(x_1, T) u(x_2, T) d\overline{(x_1, x_2)} dg.$$

By applying the Weyl integration formula (1.33), we get that

$$K^T(f) = \sum_{M \in \mathcal{L}(A_0)} c_M \sum_{S \in \mathcal{T}_M} \sum_{x_m \in \kappa_S} c_{S, x_m} \int_{S_\sigma} K^T(x_m, \gamma, f) d\gamma, \tag{2.8}$$

where, for $S \in \mathcal{T}_M$, $x_m \in \kappa_S$, and almost $\gamma \in S_\sigma$, $K^T(x_m, \gamma, f)$ is given by

$$K^T(x_m, \gamma, f) = |\Delta_\sigma(x_m \gamma)|_{\mathrm{F}}^{1/2} \int_{\operatorname{diag}(A_M) \backslash H \times H} \int_{\operatorname{diag}(A_H) \backslash H \times H} f_1(y_1^{-1} x_m \gamma y_2)$$

$$\times f_2(x_1^{-1} y_1^{-1} x_m \gamma y_2 x_2) u(x_1, T) u(x_2, T) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}.$$

Let us recall that, for any $S \in \mathcal{T}_M$, each x_m in κ_S and γ in S_σ commute with A_M. We first replace (x_1, x_2) by $(y_1 x_1, y_2 x_2)$ in the integral over $\overline{(x_1, x_2)}$. The resulting integral over $\operatorname{diag}(A_H) \backslash H \times H$ can be expressed as a double integral over $a \in A_H \backslash A_M$ and $(x_1, x_2) \in \operatorname{diag}(A_M) \backslash H \times H$, which depends on $\overline{(y_1, y_2)} \in$

$\mathrm{diag}(A_M)\backslash H \times H$. Since A_M commutes with $x_m \in \kappa_S$ and $\gamma \in S_\sigma$, we obtain that

$$K^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{\mathrm{diag}(A_M)\backslash H\times H} \int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2) \times f_2(x_1^{-1}x_m\gamma x_2) u_M(x_1, y_1, x_2, y_2, T) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}, \tag{2.9}$$

where
$$u_M(x_1, y_1, x_2, y_2, T) = \int_{A_H\backslash A_M} u(y_1^{-1}ax_1, T)u(y_2^{-1}ax_2, T)da.$$

Our goal is to prove that $K^T(f)$ is asymptotic to another integral $J^T(f)$, obtained similarly to $K^T(f)$, where the weight function $u_M(x_1, y_1, x_2, y_2, T)$ is replaced by another weight function $v_M(x_1, y_1, x_2, y_2, T)$ defined as follows.

We fix $M \in \mathcal{L}(A_0)$ and $P \in \mathcal{P}(M)$. Let $P_0 \in \mathcal{P}(A_0)$, contained in P, and let T_P be the projection of T_{P_0} on a_M with respect to the decomposition $a_0 = a_M \oplus a_0^M$. From (2.2) and (2.2), the set $\mathcal{Y}_M(T) := \{T_P; P \in \mathcal{P}(M)\}$ is an (H, M)-orthogonal set independent of the choice of P_0. Moreover, by [Ar3, equation (3.2)], we have $d(\mathcal{Y}_M(T)) \geq d(T) > 0$. Thus $\mathcal{Y}_M(T)$ is a positive (H, M)-orthogonal set.

For x, y in H, set

$$Y_P(x, y, T) := T_P + h_P(y) - h_{\bar{P}}(x).$$

By [Ar3, p. 30], $\mathcal{Y}_M(x, y, T) := \{Y_P(x, y, T); P \in \mathcal{P}(M)\}$ is an (H, M)-orthogonal set, which is positive when $d(T)$ is sufficiently large relative to x and y.

For x_1, x_2, y_1, and y_2 in H, let

$$Z_P(x_1, y_1, x_2, y_2, T) := \inf{}^P(Y_P(x_1, y_1, T), Y_P(x_2, y_2, T)), \tag{2.10}$$

where $\inf^P$ is defined in (2.6) and

$$\mathcal{Y}_M(x_1, y_1, x_2, y_2, T) := \{Z_P(x_1, y_1, x_2, y_2, T); P \in \mathcal{P}(M)\}. \tag{2.11}$$

By Lemma 2.6, the set $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ is an (H, M)-orthogonal set. Moreover, when $d(T)$ is large relative to x_i, y_i, for $i = 1, 2$, one has $d(\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) > 0$. Hence this set is a positive (H, M)-orthogonal set.

Let v_M be the weight function defined by

$$v_M(x_1, y_1, x_2, y_2, T) := \int_{A_H\backslash A_M} \sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T))da, \tag{2.12}$$

where σ_M is given by (2.4).

We set

$$J^T(f) := \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M} \sum_{x_m\in\kappa_S} c_{S,x_m} \int_{S_\sigma} J^T(x_m, \gamma, f)d\gamma, \tag{2.13}$$

where

$$J^T(x_m, \gamma, f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} \int_{\mathrm{diag}(A_M)\backslash H\times H} \int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2) \times f_2(x_1^{-1}x_m\gamma x_2) v_M(x_1, y_1, x_2, y_2, T) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}. \tag{2.14}$$

Our main result is the following. Its proof is postponed to Section 2.4.

Theorem 2.3. *Let $\delta > 0$. Then there are positive numbers C and ε such that, for all $T \in a_{0,\mathrm{F}}$ with $d(T) \geq \delta\|T\|$, one has*

$$|K^T(f) - J^T(f)| \leq Ce^{-\varepsilon\|T\|}. \tag{2.15}$$

2.3. **Preliminaries to estimates.** We fix a norm $\|\cdot\|$ on G as in (1.15). Let F' be a finite extension of F. We set $\underline{G}' := \underline{G} \times_{\mathrm{F}} \mathrm{F}'$ and $G' := \underline{G}'(\mathrm{F}')$. One can extend the absolute value $|\cdot|_{\mathrm{F}}$ to F' and the norm $\|\cdot\|$ to G'. For x, y in G', we set

$$\|(x,y)\| := \|x\|\|y\|.$$

To obtain our estimates, we will use $\preccurlyeq$ and $\approx$ defined respectively in (1.18) and (1.19). As the norm takes values greater than or equal to 1, we can freely apply the properties (1.20).

Lemma 2.4. *Let S be a maximal torus of H and let M be the centralizer of A_S in H. We fix $x_m \in G \cap \underline{MS}_\sigma = \tilde{M} \cap \underline{MS}_\sigma$. Then one has*

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \preccurlyeq \inf_{s'\in \underline{S}(\mathrm{F}')} \|(s'x_m^{-1}x_1, s'x_2)\|, \quad x_1, x_2 \in H. \tag{2.16}$$

Proof. Since H^1A_H is of finite index in H, we may assume, using (1.21), that x_1 and x_2 belong to H^1A_H. As $A_G = A_H$, using the invariance of the property (2.16) by the left action of $\mathrm{diag}(A_H)$ on (x_1, x_2), it is enough to prove the result for $x_1 \in H^1$ and $x_2 = a_2y_2$ with $a_2 \in A_H$ and $y_2 \in H^1$.

To establish (2.16), we first assume that $A_S = A_H$, which implies that the quotient $A_H\backslash S$ is compact. By (1.21), there is a positive constant C such that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \le C \inf_{a\in A_H} \|(ax_m^{-1}x_1, ax_2)\|.$$

We deduce from (1.17) that

$$\|(ax_m^{-1}x_1, ax_2)\| \le \|x_m^{-1}\|\|a\|^2\|a_2\|\|x_1\|\|y_2\|.$$

Taking the lower bound in $a \in A_H$, there is a positive constant C_1 such that

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \le C_1\|x_1\|\|a_2\|\|y_2\|. \tag{2.17}$$

In the following, we will need [Ar3, Lemma 4.1], which we recall here.

(2.18) If S_0 is a maximal torus of H with $A_H\backslash S_0$ compact, then there exists an element $s_0 \in S_0$ such that

$$\|y\| \preccurlyeq \|y^{-1}s_0y\|, \quad y \in H^1.$$

On one hand, we apply this result to $S_0 = S$. As $\underline{S}(\mathrm{F}')$ commutes with s_0, one deduces, using the property (1.17) of the norm, that

$$\|y_2\| \preccurlyeq \|s'y_2\|^2\|s_0\|, \quad y_2 \in H^1, \quad s' \in \underline{S}(\mathrm{F}'). \tag{2.19}$$

On the other hand, as $x_m \in G \cap \underline{MS}_\sigma$, $S_1 := x_mSx_m^{-1}$ is a maximal torus of H which satisfies $A_{S_1} = A_H$. Applying (2.18) to $S_0 = S_1$, there exists $s_1 \in S$ such that

$$\|x_1\| \preccurlyeq \|x_1^{-1}x_ms_1x_m^{-1}x_1\|, \quad x_1 \in H^1. \tag{2.20}$$

The same argument as above leads to

$$\|x_1\| \preccurlyeq \|s'x_m^{-1}x_1\|^2\|s_1\|, \quad x_1 \in H^1, \quad s' \in \underline{S}(\mathrm{F}'). \tag{2.21}$$

Then, by (2.17), (2.19), and (2.21), and applying the properties (1.20), we deduce that

$$\begin{aligned} \inf\nolimits_{s\in S} \|(sx_m^{-1}x_1, sa_2y_2)\| \quad &\preccurlyeq \quad \|s'x_m^{-1}x_1\|\|s'y_2\|\|a_2\|, \\ &\qquad s' \in \underline{S}(\mathrm{F}'), \quad x_1, y_2 \in H^1, \quad a_2 \in A_H. \end{aligned} \tag{2.22}$$

To obtain our result, we have to prove that
(2.23)
$$\|s'x_m^{-1}x_1\|\|s'y_2\|\|a_2\| \preccurlyeq \|(s'x_m^{-1}x_1, s'a_2y_2)\|, \quad s'\in\underline{S}(\mathrm{F}'), \quad x_1, y_2\in H^1, \quad a_2\in A_H.$$

We can write $\underline{S} = \underline{T}\underline{A}_H$ where $\underline{T}$ is a maximal torus of the derived group $\underline{H}_{der}$ of $\underline{H}$. We set $T' := \underline{T}(\mathrm{F}')$ and $A'_H := \underline{A}_H(\mathrm{F}')$. Then T' is contained in H'^1. Moreover, the intersection of $\underline{T}$ and $\underline{A}_H$ is finite. Hence, one has the exact sequence

$$1 \to \underline{T} \cap \underline{A}_H \to \underline{T} \times \underline{A}_H \to \underline{S} \to 1.$$

Going to F'-points, the long exact sequence in cohomology implies that $T'A'_H$ is of finite index in $\underline{S}(\mathrm{F}')$. Thus, by (1.21), it is enough to prove (2.23) for $s' = t'a' \in \underline{S}(\mathrm{F}')$ with $t' \in T'$ and $a' \in A'_H$. By (1.5), if $x_1 \in H^1$, then $x_1 \in H'^1 \subset G'^1$ and $x_m^{-1}x_1x_m \in G'^1$. As H is split, we have $A'_H = A'_G$. As $t' \in H'^1$, (1.23) gives

$$\|a't'x_m^{-1}x_1\| \approx \|a't'x_m^{-1}x_1x_m\| \approx \|a'\|\|t'x_m^{-1}x_1x_m\|, \quad a'\in A'_H, \quad t'\in T', \quad x_1\in H^1,$$

and

$$\|a't'y_2\| \approx \|a'\|\|t'y_2\|, \quad a' \in A'_H, \quad t' \in T', \quad y_2 \in H^1.$$

Applying (1.20), we deduce that
(2.24)
$$\begin{aligned}\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| &\approx \|a_2\|\|a'\|^2\|t'x_m^{-1}x_1x_m\|\|t'y_2\| \\ &\approx \|a_2\|\|a'\|\|t'x_m^{-1}x_1x_m\|\|t'y_2\|, \\ & \quad t' \in T', \quad a' \in A'_H, \quad x_1, y_2 \in H^1, \quad a_2 \in A_H.\end{aligned}$$

Let us prove that

$$\|a'\|\|a'a_2\| \approx \|a'\|\|a_2\|, \quad a' \in A'_H, \quad a_2 \in A_H. \tag{2.25}$$

According to (1.17), one has $\|a'a_2\| \leq \|a'\|\|a_2\|$. Then $\|a'\|\|a'a_2\| \leq (\|a'\|\|a_2\|)^2$, as $1 \leq \|a_2\|$. Since $\|a'\| = \|a'a_2a_2^{-1}\| \leq \|a'a_2\|\|a_2\|$, we have $\|a'\|\|a_2\| \leq (\|a'a_2\|\|a_2\|)^2$, and (2.25) follows. Applying (2.25) in (2.24), we deduce that
(2.26)
$$\begin{aligned}\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| &\approx \|a'\|\|t'x_m^{-1}x_1x_m\|\|a'a_2\|\|t'y_2\|, \\ & \quad t' \in T', \quad a' \in A'_H, \quad x_1, y_2 \in H^1, \quad a_2 \in A_H.\end{aligned}$$

As $x_m^{-1}H^1x_m \subset G'^1$ and $A'_H = A'_G$, we obtain from (1.23) that

$$\|a'\|\|t'x_m^{-1}x_1x_m\| \approx \|a't'x_m^{-1}x_1x_m\| \approx \|a't'x_m^{-1}x_1\|, \quad a' \in A'_H, \quad t'\in T', \quad x_1\in H^1,$$

and

$$\|a'a_2\|\|t'y_2\| \approx \|a'a_2t'y_2\|, \quad a' \in A'_H, \quad t' \in T', \quad a_2 \in A_H, \quad y_2 \in H^1.$$

Applying this in (2.26) and using (1.20), we deduce that

$$\|t'a'x_m^{-1}x_1\|\|a't'y_2\|\|a_2\| \preccurlyeq \|a't'x_m^{-1}x_1\|\|a't'a_2y_2\|, \quad a'\in A'_H, \quad t'\in T', \quad x_1, y_2\in H^1.$$

Then the property (2.23) follows. This finishes the proof of the lemma when $A_H\backslash S$ is compact.

We now prove (2.16) for any maximal torus S of H. Let A_S be the maximal split torus of S and let M be the centralizer of A_S in H. Thus we have $A_M = A_S$ and $A_M\backslash S$ is compact. Let $P = MN_P \in \mathcal{P}(M)$ and let K be a compact subgroup of H such that $H = PK$. Each $x \in H$ can be written $x = m_P(x)n_P(x)k(x)$ with

$m_P(x) \in M, n_P(x) \in N_P$, and $k(x) \in K$. Then there is a positive constant C such that

(2.27)
$$\begin{aligned} &\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \\ \leq\ & C\inf_{s\in S} \left(\|sx_m^{-1}m_P(x_1)\|\|sm_P(x_2)\|\right)\|n_P(x_1)\|\|n_P(x_2)\|, \quad x_1, x_2 \in H. \end{aligned}$$

By assumption on x_m, there exist $h_m \in \underline{M}$ and $a_m \in \underline{S}_\sigma$ such that $x_m = h_m a_m \in \tilde{M}$. Hence we can apply the first part of the proof to (M, S) instead of (H, S). Therefore, we obtain

$$\begin{aligned} &\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \\ &\quad \preccurlyeq \inf_{s'\in \underline{S}(\mathrm{F}')} \left(\|s'x_m^{-1}m_P(x_1)\|\|s'm_P(x_2)\|\right)\|n_P(x_1)\|\|n_P(x_2)\|, \qquad x_1, x_2 \in H. \end{aligned}$$

To compare the right-hand side of this inequality to the one of (2.16), we will use the Iwasawa decomposition (1.12) of H'. Let K' be a compact subgroup of H' such that $H' = \underline{P}(\mathrm{F}')K' = \underline{M}(\mathrm{F}')\underline{N}_P(\mathrm{F}')K'$. According to (1.13), each y in H' can be written $y = m'_P(y)n'_P(y)k'$ with $m'_P(y) \in \underline{M}(\mathrm{F}')$, $n'_P(y) \in \underline{N}_P(\mathrm{F}')$, and $k' \in K'$. Then, for $x \in H$ and $z \in \underline{M}(\mathrm{F}')$, we have $zx = zm_P(x)n_P(x)k = m'_P(zx)n'_P(zx)k'$ with $k \in K$ and $k' \in K'$. We have $m'_P(zx) \in zm_P(x)(K' \cap M')$ and $n'_P(zx) = n_P(x)$, hence

$$\|m'_P(zx)\| \approx \|zm_P(x)\| \text{ and } \|n'_P(zx)Vert = \|n_P(x)\|.$$

Using (1.22), it follows that

$$\|zm_P(x)\| \preccurlyeq \|zx\| \quad \text{and} \quad \|n_P(x)| \preccurlyeq \|zx\|, \quad z \in \underline{M}(\mathrm{F}'), \quad x \in H.$$

Hence, by (1.20),

$$\|zm_P(x)\|\|n_P(x)\| \preccurlyeq \|zx\|, \quad z \in \underline{M}(\mathrm{F}'), \quad x \in H. \tag{2.28}$$

We deduce that

$$\|s'm_P(x_2)\|\|n_P(x_2)\| \preccurlyeq \|s'x_2\|, \quad s' \in \underline{S}(\mathrm{F}'), \quad x_2 \in H. \tag{2.29}$$

Since $x_m = h_m a_m$ with $h_m \in \underline{M}$ and $a_m \in \underline{S}_\sigma$, one has $x_m s' x_m^{-1} \in \underline{M} \cap H' = \underline{M}(\mathrm{F}')$ for $s' \in \underline{S}(\mathrm{F}')$. Therefore, we deduce from (2.28) that

$$\|x_m s' x_m^{-1} m_P(x_1)\|\|n_P(x_1)\| \preccurlyeq \|x_m s' x_m^{-1} x_1\|, \quad s' \in \underline{S}(\mathrm{F}'), \quad x_1 \in H. \tag{2.30}$$

Since $\|s'x_m^{-1}m_P(x_1)\| \leq \|x_m^{-1}\|\|x_m s' x_m^{-1} m_P(x_1)\|$ and $\|x_m s' x_m^{-1} x_1\| \leq \|x_m\|\|s'x_m^{-1}x_1\|$, we deduce the estimate (2.16) from (2.27), (2.29), and (2.30). This finishes the proof of the lemma. ☐

The following lemma is the analogue of [Ar3, Lemma 4.2].

Lemma 2.5. *Let S be a maximal torus of H and let $x_m \in \kappa_S$. Then there is a positive integer k with the property that, for any given compact subset Ω of G, there exists a positive constant C_Ω such that, for all $\gamma \in S_\sigma$, with $x_m\gamma \in G^{\sigma-reg}$ and all x_1, x_2 in H satisfying $x_1^{-1}x_m\gamma x_2 \in \Omega$, one has*

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leq C_\Omega |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k}.$$

Proof. Let F' be a finite extension of E such that $\tilde{\underline{S}}$ splits over F'. Recall that we can write $x_m = h_m a_m$ with $h_m \in \underline{H}$ and $a_m \in \underline{S}_\sigma$. Thus we may and will choose F' such that $h_m \in \underline{H}(\mathrm{F}')$ and $a_m \in \underline{S}_\sigma(\mathrm{F}')$. For convenience, if $\underline{J}$ is an algebraic variety defined over F, we set $J' := \underline{J}(\mathrm{F}')$.

According to Lemma 2.4, it is enough to prove the existence of a positive integer k satisfying the property that, for any compact subset Ω' of $G'^{\sigma-reg}$, there exists $C_{\Omega'} > 0$ such that

$$\inf_{s'\in S'} \left(\|s'x_m^{-1}x_1\|\|s'x_2\|\right) \leq C_{\Omega'}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k} \tag{2.31}$$

for all $x_1, x_2 \in H'$ and $\gamma \in S_\sigma$ satisfying $x_m\gamma \in G^{\sigma-reg}$ and $x_1^{-1}x_m\gamma x_2 \in \Omega'$.

Let $B' = S'N'$ be a Borel subgroup of H' containing S' and K' be a compact subgroup of H' such that $H' = S'N'K' = N'S'K'$. We can also write $H' = (h_mS'h_m^{-1})(h_mN'h_m^{-1})(h_mK'h_m^{-1})$. By (1.21), one can reduce the proof to the statement for $x_1 \in (h_mS'h_m^{-1})(h_mN'h_m^{-1})$ and $x_2 \in S'N'$.

Let $x_1 = h_ms_1n_1h_m^{-1}$ and $x_2 = s_1s_2n_2$ with $s_1, s_2 \in S'$ and $n_1, n_2 \in N'$. Since $x_m = h_ma_m$, we have $x_ms_1x_m^{-1} = h_ms_1h_m^{-1}$. Hence, for any $s' \in S'$, we have $s'x_m^{-1}x_1 = s'x_m^{-1}x_ms_1x_m^{-1}h_mn_1h_m^{-1} = s's_1x_m^{-1}h_mn_1h_m^{-1}$. We thus obtain

$$\inf_{s'\in S'} \left(\|s'x_m^{-1}x_1\|\|s'x_2\|\right) = \inf_{s'\in S'} \left(\|s'x_m^{-1}h_mn_1h_m^{-1}\|\|s's_2n_2\|\right).$$

Notice that $x_1^{-1}x_m\gamma x_2 = h_mn_1^{-1}h_m^{-1}x_ms_1^{-1}x_m^{-1}x_m\gamma s_1s_2n_2 = h_mn_1^{-1}h_m^{-1}x_m\gamma s_2n_2$.

Therefore, we are reduced to proving (2.31) for $x_1 = h_mn_1h_m^{-1}$ with $n_1 \in N'$, $x_2 \in S'N' = N'S'$, and $\gamma \in S_\sigma$ such that $x_m\gamma$ is σ-regular and $x_1^{-1}x_m\gamma x_2 \in \Omega'$. We write now $x_2 = n_2s_2$ (notice the change of notation). By the properties of the norm, there is some positive constant C' such that

$$\inf_{s'\in S'} \left(\|s'x_m^{-1}x_1\|\|s'x_2\|\right) \leq C'\|n_1\|\|s_2\|\|n_2\|, \quad x_1 = h_mn_1h_m^{-1}, \quad x_2 = n_2s_2. \tag{2.32}$$

We want to estimate $\|n_1\|\|s_2\|\|n_2\|$ when $x_1 = h_mn_1h_m^{-1}$ and $x_2 = n_2s_2$ satisfy $x_1^{-1}x_m\gamma x_2 \in \Omega'$. For this, we use the isomorphism Ψ from G' to $H' \times H'$ defined in (1.26). If $x \in H'$, then $\Psi(x) = (x, x)$, and if $y \in G$ satisfies $y^{-1} = \sigma(y)$, then $\Psi(y) = (y, y^{-1})$. We set $(y_1, y_2) := \Psi(x_1^{-1}x_m\gamma x_2)$. Then we have

$$y_1 = h_mn_1^{-1}a_m\gamma n_2s_2 = h_m(n_1^{-1}a_m\gamma n_2(a_m\gamma)^{-1})(a_m\gamma s_2)$$

and

$$y_2 = h_mn_1^{-1}a_m^{-1}\gamma^{-1}n_2s_2 = h_m\left(n_1^{-1}a_m^{-1}\gamma^{-1}n_2\gamma a_m\right)(a_m\gamma)^{-1}s_2.$$

Since $H' = N'S'K'$, the condition $x_1^{-1}x_m\gamma x_2 \in \Omega'$ implies that there exist two compact subsets $\Omega_N \subset N'$ and $\Omega_S \subset S'$ depending only on Ω' such that

$$\begin{aligned} &n_1^{-1}a_m\gamma n_2(a_m\gamma)^{-1} \in \Omega_N, \quad n_1^{-1}a_m^{-1}\gamma^{-1}n_2\gamma a_m \in \Omega_N,\\ &a_m\gamma s_2 \in \Omega_S \quad \text{and} \quad (a_m\gamma)^{-1}s_2 \in \Omega_S. \end{aligned}$$

We deduce from the second property that s_2 and γ must lie in compact subsets of S'. We set

$$\nu_1(\gamma, n_1, n_2) := n_1^{-1}a_m\gamma n_2(a_m\gamma)^{-1} \quad \text{and} \quad \nu_2(\gamma, n_1, n_2) := n_1^{-1}(a_m\gamma)^{-1}n_2a_m\gamma.$$

We consider the map ψ from $N' \times N'$ into itself defined by $\psi(n_1, n_2) = (\nu_1, \nu_2)$. Recall that $\Phi(S', \mathfrak{h}')$ denotes the set of roots of S' in the Lie algebra $\mathfrak{h}'$ of H' (cf. (1.27)). Let $\mathfrak{n}'$ be the Lie algebra of N'. For $\alpha \in \Phi(S', \mathfrak{h}')$, we denote by $X_\alpha \in \mathfrak{n}'$ a root vector in $\mathfrak{h}'$ corresponding to α. Then $a_m\gamma$ acts on X_α by $a_\alpha := (a_m\gamma)^\alpha$. The differential $d_{(n_1,n_2)}\psi$ of ψ at $(n_1, n_2) \in N' \times N'$ is given by $d_{(n_1,n_2)}\psi(X_1, X_2) = (\mathrm{Ad}(a_m\gamma n_2^{-1}(a_m\gamma)^{-1})Y_1, \mathrm{Ad}((a_m\gamma)^{-1}n_2^{-1}a_m\gamma)Y_2)$, where

$$Y_1 = -\mathrm{Ad}(n_1)X_1 + \mathrm{Ad}(a_m\gamma)\mathrm{Ad}(n_2)X_2$$

and

$$Y_2 = -\mathrm{Ad}(n_1)X_1 + \mathrm{Ad}(a_m\gamma)^{-1}\mathrm{Ad}(n_2)X_2.$$

The map $(X_1, X_2) \mapsto (Y_1, Y_2)$ is the composition of the map

$$(X_1, X_2) \mapsto (\mathrm{Ad}(n_1)X_1, \mathrm{Ad}(n_2)X_2),$$

whose determinant is equal to 1, with $d_e\psi$, where e is the neutral point of $N' \times N'$. We deduce that the jacobian of ψ at (n_1, n_2) is independent of (n_1, n_2). At the neutral point $e \in N' \times N'$, we have $d_e\psi(X_\alpha, 0) = (-X_\alpha, -X_\alpha)$ and $d_e\psi(0, X_\alpha) = (a_\alpha X_\alpha, a_{-\alpha}X_\alpha)$. Hence, the jacobian of ψ is equal to

$$\begin{aligned}|\prod_{\alpha\in\Phi(S',\mathfrak{h}')} a_\alpha(1-a_{-2\alpha})|_{\mathrm{F}'} &= |\det(\mathrm{Ad}(a_m\gamma))_{\mathfrak{h}'/\mathfrak{s}'}|_{\mathrm{F}'}|\det(1-\mathrm{Ad}(a_m\gamma)^{-2})_{\mathfrak{h}'/\mathfrak{s}'}|_{\mathrm{F}'}\\ &= |D_{H'}((a_m\gamma)^{-2})|_{\mathrm{F}'}.\end{aligned}$$

Recall that $x_m\gamma$ is assumed to be σ-regular. Thus, by (1.29), one has $\Delta_\sigma(x_m\gamma) = D_{H'}(a_m^{-2}\gamma^{-2}) \neq 0$. Then, arguing as in [HC2, proof of Lemmas 10 and 11], we deduce that the map ψ is an F'-rational isomorphism of $\underline{N} \times \underline{N}$ onto itself whose inverse $(\nu_1, \nu_2) \mapsto (n_1, n_2) := (n_1(\gamma, \nu_1, \nu_2), n_2(\gamma, \nu_1, \nu_2))$ is rational. Moreover, there is a positive integer k such that the map

$$(y, \nu_1, \nu_2) \mapsto D_{\underline{H}}(y)^k(n_1(y, \nu_1, \nu_2), n_2(y, \nu_1, \nu_2))$$

is defined by an F'-rational morphism between the algebraic varieties $\underline{S} \times \underline{N} \times \underline{N}$ and $\underline{N} \times \underline{N}$. Since ν_1, ν_2, and γ lie in compact subsets depending only on Ω', one deduces that there exists a constant $C_{\Omega'} > 0$ such that

$$\|(n_1(\gamma, \nu_1, \nu_2), n_2(\gamma, \nu_1, \nu_2))\| \leq C_{\Omega'}|D_{H'}(a_m^{-2}\gamma^{-2})|_{\mathrm{F}'}^{-k} = C_{\Omega'}|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k}.$$

The lemma then follows from (2.32) and the fact that s_2 lies in a compact set. $\square$

2.4. **Proof of Theorem 2.3.** Our goal is to prove that $|K^T(f) - J^T(f)|$ is bounded by a function which approaches 0 as T approaches infinity. By definition, $K^T(f)$ and $J^T(f)$ are finite linear combinations of $\int_{S_\sigma} K^T(x_m, \gamma, f)d\gamma$ and $\int_{S_\sigma} J^T(x_m, \gamma, f)d\gamma$ respectively, where $M \in \mathcal{L}(A_0)$, S is a maximal torus of M satisfying $A_S = A_M$, and $x_m \in \kappa_S$ (cf. (2.8) and (2.13)).

We fix $M \in \mathcal{L}(A_0)$ and a maximal torus S of M such that $A_S = A_M$. Let $x_m \in \kappa_S$. To obtain our result, it is enough to establish the estimate (2.15) for $\int_{S_\sigma} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, T)|d\gamma$. This will be done in Corollary 2.9 below.

For $\varepsilon > 0$, we define

$$S_\sigma(\varepsilon, T) := \{\gamma \in S_\sigma; 0 < |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}} \leq e^{-\varepsilon\|T\|}\}. \tag{2.33}$$

Lemma 2.6.

(1) *There exists $\varepsilon_0 > 0$ such that the map $\gamma \mapsto |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0}$ is locally integrable on S_σ.*

(2) *Let $\varepsilon > 0$. Let B be a bounded subset of S_σ and let p be a nonnegative integer. Then there is a positive constant $C_{B,p}$ depending on B and p, such that*

$$\int_{S_\sigma(\varepsilon,T)\cap B} |\log|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1}|^p d\gamma \leq C_{B,p}e^{-\frac{\varepsilon\varepsilon_0\|T\|}{2}}.$$

Proof. The proof of (1) follows from the one of the group case (cf. [HC3, Lemma 43]). We use the similar statement on Lie algebras and the exponential map. We denote by $\mathfrak{s}$ the Lie algebra of S. For $X \in \mathfrak{s}$, we set $\eta(X) = |\det(\mathrm{ad}X)_{|\mathfrak{h}/\mathfrak{s}}|_{\mathrm{F}}$. By [HC3, Lemma 44], there exists $\varepsilon_0 > 0$ such that $X \mapsto \eta(X)^{-2\varepsilon_0}$ is locally integrable on $\mathfrak{s}$. To obtain the statement, it is sufficient to prove that

(2.34) for each $\gamma_0 \in S_\sigma$, there exists a compact neighborhood U_0 of 1 such that the integral $\int_{U_0} |\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma$ converges.

If $x_m\gamma_0$ is σ-regular, then there is a compact neighborhood U_0 of 1 in S_σ such that $|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}} = |\Delta_\sigma(x_m\gamma_0)|_{\mathrm{F}} \neq 0$ for all $\gamma \in U_0$. Hence (2.34) is clear.

Let us now assume that $x_m\gamma_0$ is not σ-regular. We choose an extension F' of E such that $\underline{\tilde{S}}$ splits over F' and $\underline{p}(x_m) \in \underline{\tilde{S}}_\sigma(\mathrm{F}')$. We use notation of (1.27). Let Φ_0 be the set of roots α in $\Phi(S'_\sigma, \mathfrak{g}')$ such that $\underline{p}(x_m\gamma_0)^\alpha = 1$. We set

$$\nu(\gamma) = \prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')-\Phi_0} |1 - \underline{p}(x_m\gamma_0)^\alpha\gamma^{-2\alpha}|_{\mathrm{F}'}^2.$$

We have $\Delta_\sigma(x_m\gamma_0\gamma) = D_{G'}(\underline{p}(x_m\gamma_0)\gamma^{-2}) = \det(1-\mathrm{Ad}(\underline{p}(x_m\gamma_0)\gamma^{-2}))_{|\mathfrak{g}/\tilde{\mathfrak{s}}}$, and each root of $\Phi(S'_\sigma, \mathfrak{g}')$ has multiplicity 2. Hence, we obtain

$$|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}'} = \nu(\gamma) \prod_{\alpha\in\Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2.$$

We choose a compact neighborhood W of 1 in S_σ such that $\nu(\gamma) = \nu(1) \neq 0$ for $\gamma \in W$. Let $\beta = \sup_{\gamma\in W} \prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')-\Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2$. Then, for $\gamma \in W$, we have

$$\beta|\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}'} = \beta\nu(1) \prod_{\alpha\in\Phi_0} |1 - \gamma^{-2\alpha}|_{\mathrm{F}'}^2 \geq \nu(1)|\Delta_\sigma(\gamma)|_{\mathrm{F}'}.$$

Consider the exponential map. There exist two open neighborhoods ω and U of 0 in $\mathfrak{s}$ and 1 in S_σ respectively such that the map $X \mapsto \exp(\tau X)$ is well-defined on ω and is an isomorphism and a homeomorphism onto U. For $X \in \omega$ regular in $\mathfrak{s}$, we have

$$\frac{|\Delta_\sigma(\exp(\tau X))|_{\mathrm{F}'}^{1/2}}{\eta(X)} = \prod_{\alpha\in\Phi(S'_\sigma,\mathfrak{g}')} \frac{|1 - e^{2\tau\alpha(X)}|_{\mathrm{F}'}}{|\alpha(X)|_{\mathrm{F}'}}.$$

We can choose a compact neighborhood $\omega_0 \subset \omega$ of 0 in $\mathfrak{s}$ such that the above product is a positive constant c and $U_0 := \exp(\tau\omega_0)$ is contained in W. Then

$$\begin{aligned}\int_{U_0} |\Delta_\sigma(x_m\gamma_0\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma &\leq \Big(\frac{\nu(1)}{\beta}\Big)^{-\varepsilon_0} \int_{U_0} |\Delta_\sigma(\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma \\ &= \Big(\frac{\nu(1)}{\beta}\Big)^{-\varepsilon_0} c \int_{\omega_0} \eta(X)^{-2\varepsilon_0} dX.\end{aligned}$$

The right-hand side of this inequality is finite by our choice of ε_0. The assertion (2.34) follows.

To show (2), let us pick $\varepsilon_0 > 0$ as in (1). We set

$$I_p = \int_{S_\sigma(\varepsilon,T)\cap B} |\log|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1}|^p d\gamma.$$

If p is a positive integer, then there is positive constant C' such that $|\log y|^p \leq C' y^{\varepsilon_0/2}$ for all $y \geq 1$. Since $|\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-1} \geq e^{\varepsilon\|T\|} \geq 1$ for all $\gamma \in S_\sigma(\varepsilon, T)$, we get

$$I_p \leq C' \int_{S_\sigma(\varepsilon,T)\cap B} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0/2} d\gamma \leq C' e^{-\frac{\varepsilon\varepsilon_0\|T\|}{2}} \int_B |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma.$$

If $p = 0$, then, by definition of $S_\sigma(\varepsilon, T)$, one has

$$I_0 = \int_{S_\sigma(\varepsilon,T)\cap B} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{\varepsilon_0} d\gamma \leq e^{-\varepsilon\varepsilon_0\|T\|} \int_B |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-\varepsilon_0} d\gamma.$$

In the two cases, the result follows from (1). □

Lemma 2.7. *Let $\varepsilon_0 > 0$ as in Lemma* 2.6*. Given $\varepsilon > 0$, we can choose a constant $c > 0$ such that, for any $T \in a_{0,\mathrm{F}}$, one has*

$$\int_{S_\sigma(\varepsilon,T)} \big(|K^T(x_m, \gamma, f)| + |J^T(x_m, \gamma, f)|\big) d\gamma \leq c e^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}.$$

Proof. We recall that for almost $\gamma \in S_\sigma$, we have

$$\begin{aligned} K^T(x_m, \gamma, f) \quad = \quad & |\Delta_\sigma(x_m\gamma)|^{1/2} \int_{\mathrm{diag}(A_M)\backslash H\times H} \int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1} x_m \gamma y_2) \\ & \times f_2(x_1^{-1} x_m \gamma x_2) u_M(x_1, y_1, x_2, y_2, T) \overline{d(x_1, x_2)}\, \overline{d(y_1, y_2)}, \end{aligned}$$

where

$$u_M(x_1, y_1, x_2, y_2, T) = \int_{A_H\backslash A_M} u(y_1^{-1} a x_1, T) u(y_2^{-1} a x_2, T) da.$$

We first establish an estimate of u_M. Let $x, y \in H$ and $a \in A_M$. According to (1.11) applied to H, we can write $y^{-1}ax = k_1 a_0 k_2$ with $k_1, k_2 \in K$ and $a_0 \in A_0$. By definition of the norm, there is a positive constant C_0 such that

$$\log \|y^{-1}ax\| \leq C_0(\|h_{A_0}(a_0)\| + 1).$$

If $u(y^{-1}ax, T) \neq 0$, then, by definition of $u(\cdot, T)$ (cf. (2.7)), the projection of $h_{A_0}(a_0)$ in $a_H\backslash a_M$ belongs to the convex hull in $a_H\backslash a_M$ of the $W(H, A_0)$-translates of T. Thus, there is a constant $C_1 > 0$ such that

$$\inf_{z\in A_H} \log \|y^{-1}zax\| \leq C_1(\|T\| + 1). \tag{2.35}$$

We assume that $\|T\| \geq 1$. Taking $C_2 = \max(2C_1, 1)$ and using the property (1.17) of the norm, we obtain

$$\inf_{z\in A_H} \log \|za\| \leq C_2(\|T\| + \log \|x\| + \log \|y\|). \tag{2.36}$$

Applying this inequality to (x_1, y_1) and (x_2, y_2) such that $u(y_1^{-1}ax_1, T)u(y_2^{-1}ax_2, T) \neq 0$, we get

$$\inf_{z\in A_H} \log \|za\| \leq C_2(\|T\| + \log \|x_1\| + \log \|y_1\| + \log \|x_2\| + \log \|y_2\|).$$

As $\|x\| \leq \|x_m\|\|x_m^{-1}x\|$ and $1 \leq \|T\|$, and taking the integral over $a \in A_H\backslash A_M$ on the above inequality, we deduce the following inequality:

(2.37)

$$u_M(x_1, y_1, x_2, y_2, T) \preccurlyeq (\|T\| + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|), \qquad x_1, y_1, x_2, y_2 \in H.$$

The function $u_M(x_1, y_1, x_2, y_2, T)$ is invariant by the diagonal (left) action of A_M on (x_1, x_2) and (y_1, y_2). As x_m commutes with $A_S = A_M$ (cf. Lemma 1.1), we can replace $\log \|x_m^{-1}x_1\| + \log \|x_2\|$ and $\log \|x_m^{-1}y_1\| + \log \|y_2\|$ by $\inf_{a\in A_M} \log \|(ax_m^{-1}x_1, ax_2)\|$ and $\inf_{a\in A_M} \log \|(ax_m^{-1}y_1, ay_2)\|$ respectively. By assumption, the quotient $A_M\backslash S$ is compact. Then, using (1.21), one has

$$\inf_{a\in A_M} \|(ax_m^{-1}x, ax')\| \approx \inf_{s\in S} \|(sx_m^{-1}x, sx')\|, \quad x, x' \in H.$$

Therefore, as $\|T\| \geq 1$, the inequality (2.37) gives

$$u_M(x_1, y_1, x_2, y_2, T) \preccurlyeq \|T\| + \log \inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| + \log \inf_{s\in S} \|(sx_m^{-1}y_1, sy_2)\|, \qquad x_1, y_1, x_2, y_2 \in H.$$

In other words, this means that there are a positive constant C_3 and a positive integer d such that, for all x_1, y_1, x_2, and $y_2 \in H$, one has

$$u_M(x_1, y_1, x_2, y_2, T) \leq C_3(\|T\| + \log \inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| + \log \inf_{s\in S} \|(sx_m^{-1}y_1, sy_2)\|)^d.$$

Let Ω be a compact set containing the support of f_1 and f_2. By Lemma 2.5, there is a positive integer k (independent of Ω) and a positive constant C_Ω such that if $x_m\gamma \in x_m S_\sigma$ is a σ-regular point with $f_1(y_1^{-1}x_m\gamma y_2)f_2(x_1^{-1}x_m\gamma x_2) \neq 0$ for some x_1, x_2, y_1, and y_2 in H, then

$$u_M(x_1, y_1, x_2, y_2, T) \leq C_\Omega(\|T\| + \log |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k})^d.$$

This inequality and the expression of $K^T(x_m, \gamma, f)$ thus give that for $\gamma \in S_\sigma$ with $x_m\gamma \in G^{\sigma-reg}$, we have

$$|K^T(x_m, \gamma, f)| \leq C_\Omega(\|T\| + \log |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k})^d |\mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma)|, \tag{2.38}$$

where $\mathcal{M}(f_j)$ is the orbital integral of f_j defined in (1.34). By Theorem 1.2, these orbital integrals are bounded by a positive constant C_4 on $(x_m S_\sigma) \cap G^{\sigma-reg}$. Hence, we obtain

$$|K^T(x_m, \gamma, f)| \leq C_\Omega C_4^2(\|T\| + \log |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{-k})^d.$$

Let B be the set of γ in S_σ such that $x_m\gamma$ is σ-regular and $K^T(x_m, \gamma, f) \neq 0$. Then B is bounded by Theorem 1.2 and (2.38). Using Lemma 2.6, we can find a constant $C > 0$ such that

$$\int_{S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f)| d\gamma \leq Ce^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}. \tag{2.39}$$

If $\|T\| \leq 1$, then (2.35) implies that if $u(x^{-1}ay, T) \neq 0$, then

$$\inf_{z\in A_H} \log \|za\| \leq 2C_1 + \log \|x\| + \log \|y\|.$$

The same arguments used to get (2.37) thus imply that there is a positive constant $C_1' \geq 1$ such that

$$u_M(x_1, y_1, x_2, y_2, T) \preccurlyeq (C_1' + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|), \tag{2.40}$$

for x_1, y_1, x_2, and y_2 in H. Replacing $\|T\|$ by C_1' in the argument after (2.37), we deduce that $\int_{S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f)| d\gamma$ is bounded. Hence, one obtains (2.39) for $\|T\| \leq 1$.

We will now establish a similar estimate when K^T is replaced by J^T. For this, it is enough to prove that the weight function v_M has an estimate like (2.37). We will see that this follows easily from the definition of v_M. Indeed, for x_1, y_1, x_2 and y_2 in H, one has by definition

$$v_M(x_1, y_1, x_2, y_2, T) := \int_{A_H\backslash A_M} \sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T))da,$$

where $\sigma_M(\cdot, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T))$ is a bounded function which vanishes in the complement of the convex hull $\mathcal{S}_M(\mathcal{Y}_M(x_1, y_1, x_2, y_2, T))$ of the (H, M)-orthogonal set $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ (cf. (2.5)). As $\mathcal{Y}_M(x_1, y_1, x_2, y_2, T)$ is the set of points $Z_P = \inf^P(T_P + h_P(y_1) - h_{\bar{P}}(x_1), T_P + h_P(y_2) - h_{\bar{P}}(x_2))$ for $P \in \mathcal{P}(M)$ (cf. (2.11)), if $\sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) \neq 0$, then $\|X\| \leq \|Z_P\|$ for $P \in \mathcal{P}(M)$. By definition of T_P, one has $\|T_P\| \leq \|T\|$. Let us prove that, for any $P \in \mathcal{P}(M)$, one has

$$\|h_P(x)\| \preccurlyeq 1 + \log \|x\|, \quad x \in H. \tag{2.41}$$

Let us first compare $\|m\|$ and $\|h_M(m)\|$ for any $m \in M$. Let $M = K_M A_0 K_M$ be the Cartan decomposition of M where K_M is a suitable compact subgroup of M. Then each $m \in M$ can be written $m = ka(m)k'$, with $k, k' \in K_M$ and $a(m) \in A_0$. As K_M is compact, (1.21) gives the property $\|m\| \approx \|a(m)\|$, $m \in M$, and this property does not depend on our choice of $a(m)$. By (1.25), we have $\|a\| \approx e^{\|h_{A_0}(a)\|}$, $a \in A_0$. Hence, there are a positive constant C and a nonnegative integer d such that $e^{\|h_{A_0}(a(m))\|} \leq C\|m\|^d$, $m \in M$. Applying (1.8) to (M, A_0), one has, for any $a \in A_0$, that $h_M(a)$ is the orthogonal projection of $h_{A_0}(a)$ onto $\mathfrak{a}_M$. Thus $\|h_M(a)\| \leq \|h_{A_0}(a)\|$. As $h_M(m) = h_M(a(m))$ for any $m \in M$, we then obtain that there is a positive constant C' such that

$$\|h_M(m)\| \leq \|h_{A_0}(a(m))\| \leq C'(1 + \log \|m\|), \quad m \in M. \tag{2.42}$$

By definition of m_P and h_P (cf. (1.13) and (1.14)), we have $h_P(x) = h_M(m_P(x))$ for any $x \in H$. Moreover, according to (1.22), we have $\|m_P(x)\| \preccurlyeq \|x\|, x \in H$. Thus our claim (2.41) follows from (2.42).

Therefore, there are a positive constant C_1 and a positive integer d such that if $\sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) \neq 0$, then

$$\|h_M(a)\| \leq \|Z_P\| \leq C_1(\|T\| + \log \|x_1\| + \log \|y_1\| + \log \|x_2\| + \log \|y_2\|)^d.$$

As $\|x\| \leq \|x_m\|\|x_m^{-1}x\|$ for any $x \in H$, this gives the following estimate of v_M analogous to (2.37) and (2.40):

(2.43)

if $\|T\| > 1$, then

$$v_M(x_1, y_1, x_2, y_2, T) \preccurlyeq \|T\| + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|, \quad x_1, y_1, x_2, y_2 \in H,$$

and

(2.44)

there is a positive constant C_2' such that, for any $\|T\| \leq 1$, one has

$$v_M(x_1, y_1, x_2, y_2, T) \preccurlyeq C_2' + \log \|x_m^{-1}x_1\| + \log \|x_m^{-1}y_1\| + \log \|x_2\| + \log \|y_2\|, \quad x_1, y_1, x_2, y_2 \in H.$$

Arguing exactly as we did above for K^T, we deduce that there is a positive constant C' such that

$$\int_{S_\sigma(\varepsilon, T)} |J^T(x_m, \gamma, f)|d\gamma \leq C' e^{-\frac{\varepsilon\varepsilon_0\|T\|}{4}}.$$

This finishes the proof of the lemma. □

Lemma 2.8. *Fix $\delta > 0$. Then there exist positive numbers C, ε_1, and ε_2 such that, for all $T \in a_{0,\mathrm{F}}$ with $d(T) \geq \delta\|T\|$ and for all x_1, y_1, x_2 and y_2 in the set $H_{\varepsilon_2} := \{x \in H; \|x\| \leq e^{\varepsilon_2\|T\|}\}$, one has*

$$|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)| \leq Ce^{-\varepsilon_1\|T\|}. \tag{2.45}$$

Proof. If $\|T\|$ remains bounded, then, by (2.37), (2.40), (2.43) and (2.44), the functions u_M and v_M are bounded and the result (2.45) is trivial. Thus we have only to prove the lemma for $\|T\|$ sufficiently large and $d(T) \geq \delta\|T\|$.

By [Ar3, equation (5.8)], we can choose ε_2 such that $d(\mathcal{Y}_M(x, y, T)) > 0$ for all $x, y \in H_{\varepsilon_2}$. By the discussion of [Ar3, bottom of page 38 and top of page 39], there is a constant $C_0 > 0$ such that, for T with $d(T) \geq \delta\|T\|$ and $\|T\| > C_0$, $x, y \in H_{\varepsilon_2}$, and $a \in A_H\backslash A_M$, one has

$$u(y^{-1}ax, T) = \sigma_M(h_M(a), \mathcal{Y}_M(x, y, T)).$$

By Lemma 2.2, we have, for $X \in a_M$,

$$\sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) = \sigma_M(X, \mathcal{Y}_M(x_1, y_1, T))\sigma_M(X, \mathcal{Y}_M(x_2, y_2, T)).$$

Thus, one deduces that

$$\sigma_M(h_M(a), \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) = u(y_1^{-1}ax_1, T)u(y_2^{-1}ax_2, T), \quad a \in A_H\backslash A_M.$$

Hence, for T such that $d(T) \geq \delta\|T\| \geq \delta C_0$ and x_i, y_i in H_{ε_2}, we have

$$u_M(x_1, y_1, x_2, y_2, T) = v_M(x_1, y_1, x_2, y_2, T).$$

This finishes the proof of the lemma. □

Theorem 2.3 then follows from the corollary below.

Corollary 2.9. *Fix $\delta > 0$. There exist two positive numbers ε and c such that, for all T with $d(T) \geq \delta\|T\|$, one has*

$$\int_{S_\sigma} |K^T(x_m, \gamma, f) - J^T(x_m\gamma, f)|\, d\gamma \leq ce^{-\varepsilon\|T\|}. \tag{2.46}$$

Proof. By Lemma 2.7, it is enough to prove that we can find positive numbers ε, ε', and C_0 such that

$$\int_{S_\sigma - S_\sigma(\varepsilon, T)} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, f)| d\gamma \leq C_0 e^{-\varepsilon'\|T\|}, \tag{2.47}$$

where $S_\sigma(\varepsilon, T)$ is defined in (2.33).

Let $\varepsilon > 0$. Let Ω be a compact subset of G which contains the supports of f_1 and f_2. We will estimate $|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)|$ for x_1, x_2, y_1 and y_2 in H satisfying $x_1^{-1}x_m\gamma x_2 \in \Omega$ and $y_1^{-1}x_m\gamma y_2 \in \Omega$ for some $\gamma \in S_\sigma - S_\sigma(\varepsilon, T)$ with $x_m\gamma \in G^{\sigma-reg}$. For this, we will use the invariance of the functions u_M and v_M by the diagonal left action of A_M on (x_1, x_2) and (y_1, y_2) respectively.

By Lemma 2.5, there are a positive integer k and a positive constant C_Ω (depending only on Ω) such that, for all $\gamma \in S_\sigma - S_\sigma(\varepsilon, T)$ with $x_m\gamma \in G^{\sigma-reg}$ and for all x_i, y_i in H, $i = 1, 2$, with $x_1^{-1}x_m\gamma x_2$ and $y_1^{-1}x_m\gamma y_2$ in Ω, one has

$$\inf_{s\in S} \|(sx_m^{-1}x_1, sx_2)\| \leq C_\Omega \Delta_\sigma(x_m\gamma)^{-k} \leq C_\Omega e^{k\varepsilon\|T\|} \tag{2.48}$$

and

$$\inf_{s\in S} \|(sx_m^{-1}y_1, sy_2)\| \leq C_\Omega \Delta_\sigma(x_m\gamma)^{-k} \leq C_\Omega e^{k\varepsilon\|T\|}.$$

As $A_M\backslash S$ is compact, we deduce from (1.21) and (2.48) that there is a constant $C'_\Omega > 0$ such that

$$\inf_{a\in A_M} \|(ax_m^{-1}x_1, ax_2)\| \leq C'_\Omega e^{k\varepsilon\|T\|}.$$

Thus, for $\eta > 0$, there exists $a_0 \in A_M$ such that

$$\|a_0x_m^{-1}x_1\|\|a_0x_2\| \leq C_\Omega e^{k\varepsilon\|T\|} + \eta. \tag{2.49}$$

Since $A_M = A_S$, the point a_0 commutes with x_m by (1.28), and we have $\|a_0x_1\| \leq \|x_m\|\|x_m^{-1}a_0x_1\|$.

If $\|T\|$ remains bounded, then $\|a_0x_i\|$, $i = 1, 2$, are bounded by a constant independent of $\|T\|$. By the same arguments, there exists $a_1 \in A_M$ such that $\|a_1y_i\|$, $i = 1, 2$, are bounded by a constant independent of $\|T\|$. Using the invariance of u_M and v_M by the left action of $\mathrm{diag}(A_M)$ on (x_1, x_2) and (y_1, y_2) respectively and the estimates (2.37), (2.40), (2.43), and (2.44) for u_M and v_M, we deduce that $|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)|$ is bounded by a constant independent of T and of x_i, y_i, $i = 1, 2$. Recall that, by Theorem 1.2, the constant

$$C_1 := \int_{S_\sigma} \mathcal{M}(|f_1|)(x_m\gamma)\mathcal{M}(|f_2|)(x_m\gamma)d\gamma$$

is finite. We deduce that $\int_{S_\sigma - S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, f)|d\gamma$ is bounded; hence we obtain (2.47).

We assume that $\|T\|$ is not bounded. Let $\varepsilon_1, \varepsilon_2$, and C be as in Lemma 2.8. Taking $\|T\|$ to be sufficiently large and ε such that $k\varepsilon$ is smaller than the constant ε_2, we can assume by (2.49) that

$$\|a_0x_i\| \leq e^{\varepsilon_2\|T\|}, \quad i = 1, 2.$$

The same arguments are valid for $\|y_i\|$, $i = 1, 2$. Thus there is $a_1 \in A_M$ such that

$$\|a_1y_i\| \leq e^{\varepsilon_2\|T\|}, \quad i = 1, 2.$$

Using Lemma 2.8 and the invariance of u_M and v_M by the left action of the diagonal of A_M on (x_1, x_2) and (y_1, y_2) respectively, we deduce that, for all T with $d(T) \geq \delta\|T\|$, one has

$$|u_M(x_1, y_1, x_2, y_2, T) - v_M(x_1, y_1, x_2, y_2, T)| \leq Ce^{-\varepsilon_1\|T\|}.$$

Hence, we obtain

$$\int_{S-S_\sigma(\varepsilon,T)} |K^T(x_m, \gamma, f) - J^T(x_m, \gamma, T)| \leq CC_1e^{-\varepsilon_1\|T|},$$

where $C_1 := \int_{S_\sigma} \mathcal{M}(|f_1|)(x_m\gamma)\mathcal{M}(|f_2|)(x_m\gamma)d\gamma$. This finishes the proof of the corollary. □

2.5. **The function $J^T(f)$.** The goal of this section is to prove that $J^T(f)$ is of the form

$$\sum_{k=0}^{N} p_k(T, f)e^{\xi_k(T)}, \tag{2.50}$$

where $\xi_0 = 0, \xi_1, \ldots, \xi_N$ are distinct points in $i\mathfrak{a}_0^*$ and each $p_k(T, f)$ is a polynomial function of T. Moreover, the constant term $\tilde{J}(f) := p_0(0, f)$ is well-defined and is uniquely determined by $K^T(f)$. Except for one detail, our arguments and calculations are the same as those of [Ar3, Section 6]. We give the details of the proof for the convenience of the reader.

Recall that $J^T(f)$ is a finite sum of the distributions

$$J^T(x_m,\gamma,f) = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2}\int_{\mathrm{diag}(A_M)\backslash H\times H}\int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2) \times f_2(x_1^{-1}x_m\gamma x_2)v_M(x_1,y_1,x_2,y_2,T)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)},$$

where $M\in\mathcal{L}(A_0)$, S is a maximal torus of M such that $A_S=A_M$, $x_m\in\kappa_S$, and $v_M(x_1,y_1,x_2,y_2,T):=\int_{A_H\backslash A_M}\sigma_M(h_M(a),\mathcal{Y}_M(x_1,y_1,x_2,y_2,T))da$, where $\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)$ is defined in (2.11).

We first study the weight function v_M as a function of T. We fix $M\in\mathcal{L}(A_0)$ and x_1,y_1,x_2 and y_2 in H.

Let $\mathscr{L}_M:=(a_{M,\mathrm{F}}+a_H)/a_H$ and $\widetilde{\mathscr{L}_M}:=(\tilde{a}_{M,\mathrm{F}}+a_H)/a_H$ be the projection in a_M/a_H of the lattices $a_{M,\mathrm{F}}$ and $\tilde{a}_{M,\mathrm{F}}$ respectively. According to (1.10), one has

$$\tilde{a}_{M,\mathrm{F}}/\tilde{a}_{H,\mathrm{F}}=\tilde{a}_{M,\mathrm{F}}/\tilde{a}_{M,\mathrm{F}}\cap a_H\simeq\widetilde{\mathscr{L}_M}. \tag{2.51}$$

For $M=A_0$, we replace the subscript A_0 by 0. We denote by $\mathscr{L}^\vee:=\mathrm{Hom}(\mathscr{L},2\pi i\mathbb{Z})$ the dual lattice of a lattice $\mathscr{L}$.

Let $P\in\mathcal{P}(M)$. We introduce the following sublattice of $\mathscr{L}_M$. For $k\in\mathbb{N}$, we set

$$\mu_{\alpha,k}:=k\log(q)\check{\alpha},\quad \alpha\in\Delta_P,$$

where q is the order of the residual field of F and

$$\mathscr{L}_{M,k}:=\sum_{\alpha\in\Delta_P}\mathbb{Z}\mu_{\alpha,k}.$$

Then $\mathscr{L}_{M,k}$ is a lattice in $a_M^H\simeq a_M/a_H$ independent of P, and, according to [Ar2, Section 4], one can find $k\in\mathbb{N}^*$ such that, for all $M\in\mathcal{L}(A_0)$, one has

$$\mathscr{L}_{M,k}\subset\widetilde{\mathscr{L}_M}.$$

The set of points $\sum_{\alpha\in\Delta_P}y_\alpha\mu_{\alpha,k}$ with $y_\alpha\in]-1,0]$ is a fundamental domain of $\mathscr{L}_{M,k}$, which we denote by $\mathcal{D}_{M,k}$.

(2.52) For $X\in\mathscr{L}_M/\mathscr{L}_{M,k}$ and $Y\in a_M/a_H$, we denote by $\bar{X}_P(Y)$ the representative of X in $\mathscr{L}_M$ such that $\bar{X}_P(Y)-Y\in\mathcal{D}_{M,k}$.

For $\lambda\in a_{M,\mathbb{C}}^*$, we set

$$\theta_{P,k}(\lambda)=vol(a_M^H/\mathscr{L}_{M,k})^{-1}\prod_{\alpha\in\Delta_P}(1-e^{-\lambda(\mu_{\alpha,k})}). \tag{2.53}$$

We fix $T\in a_{0,\mathrm{F}}$. By definition of σ_M (cf. (2.4)), the function v_M depends only on the image of T_P in $\mathscr{L}_M$. Hence we can assume that T lies in the lattice $\mathscr{L}_0$. For $P\in\mathcal{P}(M)$, the map $T\mapsto T_P$ sends surjectively $\mathscr{L}_0$ onto the intersection of $\mathscr{L}_M$ with the closure $\overline{a_P^+}$ of the chamber associated to P. Thus, we may restrict T to lie at the intersection of $\mathscr{L}_0$ with suitable regular points in some positive chamber a_0^+ of $a_H\backslash a_0$. Then the points T_P range over suitable regular points in $\mathscr{L}_M\cap a_P^+$.

We recall that $\mathcal{Y}_M(x_1,y_1,x_2,y_2,T)$ is the set of points $Z_P:=Z_P(x_1,y_1,x_2,y_2,T)$ defined in (2.10). Thus, we can write

$$Z_P=T_P+Z_P^0\quad\text{with } Z_P^0:=\inf{}^P(h_P(y_1)-h_{\overline{P}}(x_1),h_P(y_2)-h_{\overline{P}}(x_2)). \tag{2.54}$$

Notice that the points Z_P^0 do not necessarily belong to the lattice $\mathscr{L}_M$. It is the only difference from [Ar3, Section 6] in what follows.

Lemma 2.10. *There are a positive integer N independent of M and polynomial functions $q_\xi(T)$ for $\xi \in (\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee$ (depending on x_1, y_1, x_2 and y_2) such that*

$$v_M(x_1, y_1, x_2, y_2, T) = \sum_{\xi \in (\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} q_\xi(T) e^{\xi(T)}.$$

Moreover, the constant term $\tilde{v}_M(x_1, y_1, x_2, y_2) := q_0(0)$ of $v_M(x_1, y_1, x_2, y_2, T)$ is given by

$$\tilde{v}_M(x_1, y_1, x_2, y_2) = \lim_{\Lambda \to 0} \Big(\sum_{P \in \mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X \in \mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle \Lambda, \bar{X}_P(Z_P^0)\rangle} \theta_{P,k}(\Lambda)^{-1} \Big).$$

Proof. The kernel of the surjective map $h_M : A_H \backslash A_M \to \tilde{a}_{M,\mathrm{F}}/\tilde{a}_{H,\mathrm{F}}$ is a compact group which has volume 1 by our convention of choice of measure. Thus, using (2.51), we can write

$$v_M(x_1, y_1, x_2, y_2, T) = \sum_{X \in \widetilde{\mathscr{L}_M}} \sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)).$$

For our study, it is convenient to take a sum over $\mathscr{L}_M$. The finite quotient $\widetilde{\mathscr{L}_M}^\vee/\mathscr{L}_M^\vee$ can be identified with the character group of $\mathscr{L}_M/\widetilde{\mathscr{L}_M}$ under the pairing

$$(\nu, X) \in \widetilde{\mathscr{L}_M}^\vee/\mathscr{L}_M^\vee \times \mathscr{L}_M/\widetilde{\mathscr{L}_M} \mapsto e^{\nu(X)}.$$

Hence, by the inversion formula on finite abelian groups, we obtain

$$\begin{aligned} &v_M(x_1, y_1, x_2, y_2, T) \\ &\quad = |\mathscr{L}_M/\widetilde{\mathscr{L}_M}|^{-1} \sum_{\nu \in \widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee} \sum_{X \in \mathscr{L}_M} \sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) e^{\nu(X)}. \end{aligned}$$

Coming back to the definition of σ_M (cf. (2.4)), we fix a small point $\Lambda \in (a_M/a_H)^*_\mathbb{C}$ whose real part Λ_R is in general position. One then has

$$\begin{aligned} \sigma_M(X, \mathcal{Y}_M(x_1, y_1, x_2, y_2, T)) &= \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(X - Z_P) \\ &= \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(X - Z_P) e^{\Lambda(X)}. \end{aligned}$$

By definition of φ_P^Λ, the function $X \mapsto e^{\Lambda(X)}$ is rapidly decreasing on the support of $X \mapsto \varphi_P^\Lambda(X - Z_P)$. Hence the product of these two functions is summable over $X \in \mathscr{L}_M$. Therefore, we can write

$$v_M(x_1, y_1, x_2, y_2, T) = \sum_{\nu \in \widetilde{\mathscr{L}_M}^\vee/\mathscr{L}_M^\vee} \lim_{\Lambda \to 0} \sum_{P \in \mathcal{P}(M)} F_P^T(\Lambda, \nu), \tag{2.55}$$

where

$$F_P^T(\Lambda, \nu) := |\mathscr{L}_M/\widetilde{\mathscr{L}_M}|^{-1} \sum_{X \in \mathscr{L}_M} (-1)^{|\Delta_P^\Lambda|} \varphi_P^\Lambda(X - Z_P) e^{(\Lambda + \nu)(X)}.$$

The above discussion implies that

$$\text{the map } \Lambda \mapsto \sum_{P \in \mathcal{P}(M)} F_P^T(\Lambda, \nu) \text{ is analytic at } \Lambda = 0. \tag{2.56}$$

We fix $P \in \mathcal{P}(M)$. We want to express $F_P^T(\Lambda,\nu)$ in terms of a product of geometric series. For this, we write

(2.57)
$$F_P^T(\Lambda,\nu) = |\mathscr{L}_M/\widetilde{\mathscr{L}_M}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} \sum_{X'\in\mathscr{L}_{M,k}} (-1)^{|\Delta_P^\Lambda|}\varphi_P^\Lambda(X+X'-Z_P) \times e^{(\Lambda+\nu)(X+X')}.$$

Let $X \in \mathscr{L}_M/\mathscr{L}_{M,k}$. Recall that $\bar{X}_P(Y)$ is the representative of X in $\mathscr{L}_M$ such that $\bar{X}_P(Y) - Y \in \mathcal{D}_{M,k}$. We set

$$\bar{X}_P^\Lambda(Y) := \bar{X}_P(Y) + \sum_{\alpha\in\Delta_P^\Lambda} \mu_{\alpha,k}.$$

Thus $\bar{X}_P^\Lambda(Y)$ is also a representative of X in $\mathscr{L}_M$. Taking $Y := Z_P$, we can set

$$\varphi_P^\Lambda(X+X'-Z_P) = \varphi_P^\Lambda(\bar{X}_P^\Lambda(Z_P)+X'-Z_P)$$

in (2.57). The set of points $X' \in \mathscr{L}_{M,k}$ such that this characteristic function equals 1 is exactly the set

$$\{\sum_{\alpha\in\Delta_P^\Lambda} n_\alpha\mu_{\alpha,k} - \sum_{\alpha\in\Delta_P-\Delta_P^\Lambda} n_\alpha\mu_{\alpha,k}; n_\alpha\in\mathbb{N}\}.$$

Therefore, a simple calculation as in [Ar3, top of p. 45] gives

$$(2.58)\qquad \begin{aligned} &(-1)^{|\Delta_P^\Lambda|} \sum_{X'\in\mathscr{L}_{M,k}} \varphi_P^\Lambda(X+X'-Z_P)e^{(\Lambda+\nu)(X+X')} \\ = \; & e^{(\Lambda+\nu)(\bar{X}_P(Z_P))} \prod_{\alpha\in\Delta_P} (1-e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1}. \end{aligned}$$

We have fixed the Haar measure on $a_M^H \simeq a_M/a_G$ with the property that the quotient of a_M/a_H by the lattice $\widetilde{\mathscr{L}_M}$ has volume 1. Thus we have

$$|\mathscr{L}_M/\widetilde{\mathscr{L}_M}|^{-1} \prod_{\alpha\in\Delta_P} (1-e^{-(\Lambda+\nu)(\mu_{\alpha,k})})^{-1} = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1}\theta_{P,k}(\Lambda+\nu)^{-1}.$$

By the above equality, (2.57) and (2.58), we obtain

$$(2.59)\qquad F_P^T(\Lambda,\nu) = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle\Lambda+\nu,\bar{X}_P(Z_P)\rangle}\theta_{P,k}(\Lambda+\nu)^{-1}.$$

Let $X \in \mathscr{L}_M/\mathscr{L}_{M,k}$. We recall that T_P belongs to $\mathscr{L}_M$ for $P \in \mathcal{P}(M)$ and $Z_P = T_P + Z_P^0$ (cf. (2.54)). By definition (cf. (2.52)), the point $\bar{X}_P(Z_P)$ is the unique representative of X in $\mathcal{L}_M$ such that $\bar{X}_P(Z_P) - T_P - Z_P^0 \in \mathcal{D}_{M,k}$ and $\overline{(X-T_P)}_P(Z_P^0)$ is the unique representative of $X - T_P$ in $\mathscr{L}_M$ such that $\overline{(X-T_P)}_P(Z_P^0) - Z_P^0 \in \mathcal{D}_{M,k}$. Hence we deduce that

$$(2.60)\qquad \bar{X}_P(Z_P) = \overline{(X-T_P)}_P(Z_P^0) + T_P.$$

Replacing X by $X - T_P$ in (2.59), we obtain

$$(2.61)\quad F_P(\Lambda,\nu)^T = |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle\Lambda+\nu,T_P+\bar{X}_P(Z_P^0)\rangle}\theta_{P,k}(\Lambda+\nu)^{-1},$$

where $\bar{X}_P(Z_P^0)$ is independent of T. Thus, by (2.55), we have established that $v_M(x_1, y_1, x_2, y_2, T)$ is equal to

(2.62)

$$\sum_{\nu\in\widetilde{\mathscr{L}}_M^\vee/\mathscr{L}_M^\vee} \lim_{\Lambda\to 0}\big(\sum_{P\in\mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle\Lambda+\nu, T_P+\bar{X}_P(Z_P^0)\rangle} \times\theta_{P,k}(\Lambda+\nu)^{-1}\big).$$

Recall that the expression in brackets is analytic at $\Lambda = 0$ (cf. (2.56)). To analyze this expression as a function of T, we argue as in [W1, p. 315]. We give the details for the convenience of the reader. We replace Λ by $z\Lambda$. The map $z \mapsto \theta_{P,k}(z\Lambda+\nu)^{-1}$ may have a pole at $z = 0$. Let r denotes the biggest order of this pole when P runs over $\mathcal{P}(M)$. Then, using Taylor expansions, one deduces that

$$\begin{aligned}
&\lim_{\Lambda\to 0}\big(\sum_{P\in\mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle\Lambda+\nu, T_P+\bar{X}_P(Z_P^0)\rangle}\theta_{P,k}(\Lambda+\nu)^{-1}\big)\\
=\ &\sum_{m=0}^{r}\sum_{P\in\mathcal{P}(M)} C_m \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} \frac{\partial^m}{\partial z^m}(e^{\langle z\Lambda+\nu, T_P+\bar{X}_P(Z_P^0)\rangle})_{[z=0]}\\
&\qquad\qquad\times\frac{\partial^{r-m}}{\partial z^{r-m}}(z^r\theta_{P,k}(z\Lambda+\nu)^{-1})_{[z=0]},
\end{aligned}$$

where $C_m = \frac{1}{m!(r-m)!}|\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1}$. But we have

$$\frac{\partial^m}{\partial z^m}(e^{\langle z\Lambda+\nu, T_P+\bar{X}_P(Z_P^0)\rangle})_{[z=0]} = (\langle\Lambda, T_P + \bar{X}_P(Z_P^0)\rangle)^m e^{\langle\nu, T_P+\bar{X}_P(Z_P^0)\rangle}$$

and $\frac{\partial^{r-m}}{\partial z^{r-m}}(z^r\theta_{P,k}(z\Lambda+\nu)^{-1})_{[z=0]}$ is independent of T_P. Therefore, we deduce that $v_M(x_1, y_1, x_2, y_2, T)$ is a finite sum of functions

$$q_{P,\nu}(T_P)e^{\nu(T_P)}, \quad \nu\in\widetilde{\mathscr{L}_M}^\vee/\mathscr{L}_M^\vee, \quad P\in\mathcal{P}(M),$$

where $q_{P,\nu}$ is a polynomial function on a_M.

Since $\mathscr{L}_0^\vee \subset \widetilde{\mathscr{L}_0}^\vee$ are lattices of the same rank, one can find a positive integer N such that $N\widetilde{\mathscr{L}_0}^\vee \subset \mathscr{L}_0^\vee$. Therefore, by our choice of T and the above expression, we can write

$$v_M(x_1, y_1, x_2, y_2, T) = \sum_{\xi\in(\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} q_\xi(T)e^{\xi(T)},$$

where $q_\xi(T)$ is a polynomial function of T. This gives the first part of the lemma.

Since the polynomials $q_\xi(T)$ are obviously uniquely determined, the constant term $\tilde{v}_M(x_1, y_1, x_2, y_2) := q_0(0)$ is well-defined. To calculate it, we take the summand corresponding to $\nu = 0$ in (2.62) and then set $T = 0$. We obtain

$$\tilde{v}_M(x_1, y_1, x_2, y_2) = \lim_{\Lambda\to 0}\big(\sum_{P\in\mathcal{P}(M)} |\mathscr{L}_M/\mathscr{L}_{M,k}|^{-1} \sum_{X\in\mathscr{L}_M/\mathscr{L}_{M,k}} e^{\langle\Lambda,\bar{X}_P(Z_P^0)\rangle}\theta_{P,k}(\Lambda)^{-1}\big).$$

This finishes the proof of the lemma. □

We substitute the expression we have obtained for v_M in Lemma 2.10 into the expression (2.14) for $J^T(x_m, \gamma, f)$. Hence we obtain the following similar decomposition for $J^T(f)$.

Corollary 2.11. *There is a decomposition*

$$J^T(f) = \sum_{\xi\in(\frac{1}{N}\mathscr{L}_0^\vee)/\mathscr{L}_0^\vee} p_\xi(T, f)e^{\xi(T)}, \quad T\in\mathscr{L}_0\cap a_0^+,$$

where N is a positive integer and each $p_\xi(T,f)$ is a polynomial function of T. Moreover, the constant term $\tilde{J}(f) := p_0(0,f)$ of $J^T(f)$ is given by

$$\tilde{J}(f) := \sum_{M\in\mathcal{L}(A_0)} c_M \sum_{S\in\mathcal{T}_M} \sum_{x_m\in\kappa_S} c_{S,x_m} \int_{S_\sigma} \tilde{J}(x_m,\gamma,f)d\gamma,$$

where

$$\begin{aligned}\tilde{J}(x_m,\gamma,f)& \\ = |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/2} &\int_{\mathrm{diag}(A_M)\backslash H\times H}\int_{\mathrm{diag}(A_M)\backslash H\times H} f_1(y_1^{-1}x_m\gamma y_2)f_2(x_1^{-1}x_m\gamma x_2)\\ &\times\tilde{v}_M(x_1,y_1,x_2,y_2)d\overline{(x_1,x_2)}d\overline{(y_1,y_2)}.\end{aligned}$$

Appendix A. Spherical character of a supercuspidal representation as weighted orbital integral

Let (π,V) be a unitary irreducible admissible representation of G. We say that π is H-distinguished if the space $V^{*H} = \mathrm{Hom}_H(\pi,\mathbb{C})$ of H-invariant linear forms on V is nonzero. In that case, a distribution $m_{\xi,\xi'}$, called a spherical character, can be associated to two H-invariant linear forms ξ,ξ' on V (cf. definition below). By [Ha, Theorem 1], spherical characters are locally integrable functions on G, which are smooth on the set of σ-regular points of G.

From now on, we assume that $A_H=\{1\}$. We fix an H-distinguished supercuspidal representation (τ,V) of G. We denote by $d(\tau)$ its formal degree.

The aim of this appendix is to deduce from our main results the value $m_{\xi,\xi'}(g)$ when $g\in G$ is σ-regular and $\xi,\xi'\in V^{*H}$, in terms of weighted orbital integrals of a matrix coefficient of τ (cf. Theorem A.2). This result is analogous to that of Arthur in the group case (see [Ar2]). Notice that this result of Arthur can be deduced from his local trace formula given in [Ar3], which was obtained later.

Let $(\cdot,\cdot)$ be a G-invariant hermitian inner product on V. Since τ is unitary, it induces an isomorphism $\iota : v\mapsto(\cdot,v)$ from the conjugate complex vector space $\overline{V}$ of V and the smooth dual $\check{V}$ of V, which intertwines the complex conjugate of τ and its contragredient $\check{\tau}$. If ξ is a linear form on V, we define the linear form $\overline{\xi}$ on $\overline{V}$ by $\overline{\xi}(u) := \overline{\xi(u)}$.

For ξ_1 and ξ_2 two nonzero H-invariant linear forms on V, we associate the spherical character m_{ξ_1,ξ_2} defined to be the distribution on G given by

$$m_{\xi_1,\xi_2}(f) := \sum_{u\in\mathcal{B}} \xi_1\big(\tau(f)u\big)\overline{\xi_2(u)},$$

where $\mathcal{B}$ is an orthonormal basis of V. Since $\tau(f)$ is of finite rank, this sum is finite. Moreover, this sum does not depend on the choice of $\mathcal{B}$. Indeed, let (τ^*,V^*) be the dual representation of τ. For $f\in C_c^\infty(G)$, we set $\check{f}(g) := f(g^{-1})$. By [R, Theorems III.3.4 and I.1.2], the linear form $\tau^*(\check{f})\xi$ belongs to $\check{V}$. Hence we can write $\iota^{-1}(\tau^*(\check{f})\xi) = \sum_{v\in\mathcal{B}}\big(\tau^*(\check{f})\xi\big)(v)\cdot v$, where $(\lambda,v)\mapsto\lambda\cdot v$ is the action of $\mathbb{C}$ on $\overline{V}$. Therefore, we deduce easily that one has

$$m_{\xi_1,\xi_2}(f) = \overline{\xi}_2\big(\iota^{-1}(\tau^*(\check{f})\xi_1)\big). \tag{A.1}$$

Since τ is a supercuspidal representation, we can define the $H\times H$-invariant pairing $\mathcal{L}$ on $V\times\overline{V}$ by

$$\mathcal{L}(u,v):=\int_H(\tau(h)u,v)dh.$$

According to [Z, Theorem 1.5],

(A.2) the map $v\mapsto\xi_v : u\mapsto\mathcal{L}(u,v)$ is a surjective linear map from $\overline{V}$ onto V^{*H}.

For $v,w\in V$, we denote by $c_{v,w}$ the corresponding matrix coefficient defined by $c_{v,w}(g):=(\tau(g)v,w)$, $g\in G$.

Lemma A.1. *Let $\xi_1,\xi_2\in V^{*H}$ and $v,w\in V$. Then we have*

$$m_{\xi_1,\xi_2}(\check{c}_{v,w})=d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}.$$

Proof. By (A.2), there exist v_1 and v_2 in V such that $\xi_j=\xi_{v_j}$ for $j=1,2$. By definition of the spherical character, for $f\in C_c^\infty(G)$ and $\mathcal{B}$ an orthonormal basis of V, one has

$$\begin{aligned}m_{\xi_1,\xi_2}(f) &= \sum_{u\in\mathcal{B}}\int_H(\tau(h)\tau(f)u,v_1)dh\int_H\overline{(\tau(h)u,v_2)}dh\\ &= \sum_{u\in\mathcal{B}}\int_{H\times H}(u,\tau(\check{f})\tau(h_1)v_1)(\tau(h_2)v_2,u)dh_1dh_2\\ &= \int_{H\times H}(\tau(h_2)v_2,\tau(\check{f})\tau(h_1)v_1)dh_1dh_2.\end{aligned}$$

Hence we obtain

$$m_{\xi_1,\xi_2}(f)=\int_{H\times H}\int_G f(g)(\tau(h_1gh_2)v_2,v_1)dgdh_1dh_2. \tag{A.3}$$

Let $f(g):=\check{c}_{v,w}(g)=\overline{(\tau(g)w,v)}$. By the orthogonality relation of Schur, for $h_1,h_2\in H$, one has

$$\int_G(\tau(g)\tau(h_2)v_2,\tau(h_1)v_1)\overline{(\tau(g)w,v)}dg=d(\tau)^{-1}(\tau(h_2)v_2,w)(v,\tau(h_1)v_1).$$

Thus we deduce that

$$m_{\xi_1,\xi_2}(f)=d(\tau)^{-1}\xi_w(v_2)\xi_{v_1}(v)=d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}.$$

□

For $M\in\mathcal{L}(A_0)$, we define the weight function w_M on $H\times H$ by

$$w_M(y_1,y_2):=\tilde{v}_M(1,y_1,1,y_2),$$

where $\tilde{v}_M$ is defined in Lemma 2.10 and 1 is the neutral element of H. For $f\in C_c^\infty(G)$, we define the weighted orbital integral of f by

$$\mathcal{WM}(f)(g):=|\Delta_\sigma(g)|_{\mathrm{F}}^{1/2}\int_{H\times H}f(y_1gy_2)w_M(y_1,y_2)dy_1dy_2,\quad g\in G^{\sigma-reg}\cap\tilde{M}.$$

Theorem A.2. *Let $M\in\mathcal{L}(A_0)$ and $S\in\mathcal{T}_M$. Let $x_m\in\kappa_S$ and $\gamma\in S_\sigma$ be such that $x_m\gamma$ is σ-regular. Then, for $v,w\in V$, we have*

$$c_Mc_{S,x_m}\mathcal{WM}(c_{v,w})(x_m\gamma)=m_{\xi_w,\xi_v}(x_m\gamma).$$

Proof. Let f_1 be a matrix coefficient of τ and let $f_2 \in C_c^\infty(G)$. We set $f := f_1 \otimes f_2$. For $x \in G$, we define

$$F(g) := \int_G f_1(xu) f_2(ugx) du, \quad g \in G,$$

so that

$$K_f(x,y) = \big[\rho(yx^{-1})F\big](e), \text{ where } \rho \text{ is the right regular representation.}$$

If π is a unitary irreducible admissible representation of G, one has

$$\begin{aligned}\pi\big(\rho(yx^{-1})F\big) &= \int_{G\times G} f_1(xu) f_2(ugy)\pi(g) du dg \\ &= \int_{G\times G} f_1(xu) f_2(u_2)\pi(u^{-1}u_2 y^{-1}) du du_2 \\ &= \int_{G\times G} f_1(u_1^{-1}) f_2(u_2)\pi(u_1 x u_2 y^{-1}) du_1 du_2 = \pi(\check{f}_1)\pi(x)\pi(f_2)\pi(y^{-1}).\end{aligned}$$

Since τ is supercuspidal and f_1 is a matrix coefficient of τ, we deduce that $\pi\big(\rho(yx^{-1})F\big)$ is equal to 0 if π is not equivalent to τ. Therefore, applying the Plancherel formula [W2, Theorem VIII.1.1] to $\big[\rho(yx^{-1})\check{F}\big]$, we obtain

$$K_f(x,y) = d(\tau)\mathrm{tr}\big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y^{-1})\big).$$

We identify $\check{V} \otimes V$ with a subspace of Hilbert-Schmidt operators on V. Taking an orthonormal basis $\mathcal{B}_{HS}(V)$ of $\check{V} \otimes V$ for the scalar product $(S, S') := \mathrm{tr}(SS'^*)$, one obtains

$$\begin{aligned}K_f(x,y) &= d(\tau)\mathrm{tr}\Big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y)^*\Big) = d(\tau)(\tau(\check{f}_1)\tau(x)\tau(f_2), \tau(y)) \\ &= d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} (\tau(\check{f}_1)\tau(x)\tau(f_2), S^*)\overline{(\tau(y), S^*)} \\ &= d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} \mathrm{tr}\big(\tau(x)\tau(f_2)S\tau(\check{f}_1)\big)\mathrm{tr}\overline{\big(\tau(y)S\big)},\end{aligned}$$

where the sums over S are finite since $\tau(f_2)$ and $\tau(\check{f}_1)$ are of finite rank. Therefore, the truncated kernel $K^T(f)$ is equal to

$$d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} P_\tau^T(\check{\tau}\otimes\tau(f)S)\overline{P_\tau^T(S)},$$

where

$$P_\tau^T(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)u(h,T)dh, \quad S \in \check{V}\otimes V.$$

For $\check{v}\otimes v \in \check{V}\otimes V$, one has $\mathrm{tr}\big(\tau(h)(\check{v}\otimes v)\big) = c_{\check{v},v}(h)$. Since $c_{\check{v},v}$ is compactly supported, the truncated local period $P_\tau^T(S)$ converges, when $\|T\|$ approaches infinity, to

$$P_\tau(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)dh.$$

Therefore, we obtain

$$\lim_{\|T\|\to+\infty} K^T(f) = d(\tau) m_{P_\tau,P_\tau}(f), \tag{A.4}$$

where m_{P_τ,P_τ} is the spherical character of the representation $\check{\tau}\otimes\tau$ associated to the $H\times H$-invariant linear form P_τ on $\check{V}\otimes V$.

Recall that $\tilde{J}(f)$ is the constant term of $J^T(f)$. We deduce from Theorem 2.15 that

$$d(\tau)m_{P_\tau,P_\tau}(f) = \tilde{J}(f). \tag{A.5}$$

We now express m_{P_τ,P_τ} in terms of H-invariant linear forms on V. Let V_H be the orthogonal of V^{*H} in V. Since $\xi_u(v) = \overline{\xi_v(u)}$ for $u, v \in V$, the space $\overline{V_H}$ is the kernel of $v \mapsto \xi_v$. Let W be a complementary subspace of V_H in V. Then, the map $v \mapsto \xi_v$ is an isomorphism from $\overline{W}$ to V^{*H} and $(u, v) \mapsto \xi_v(u)$ is a nondegenerate hermitian form on W. Let $(e_1, \dots, e_n)$ be an orthogonal basis of W for this hermitian form. We set $\xi_i := \xi_{e_i}$ for $i = 1, \dots, n$. Thus we have $\xi_i(e_i) \neq 0$.

We identify $\overline{V}$ and $\check{V}$ by the isomorphism ι. We claim that

$$P_\tau = \sum_{i=1}^{n} \frac{1}{\xi_i(e_i)} \overline{\xi_i} \otimes \xi_i. \tag{A.6}$$

Indeed, we have $P_\tau(v \otimes u) = \xi_v(u) = \overline{\xi_u(v)}$. Hence, the two sides are equal to 0 on $\overline{V} \otimes V_H + \overline{V_H} \otimes V + \overline{V_H} \otimes V_H$ and take the same value $\xi_k(e_l)$ on $e_k \otimes e_l$ for $k, l \in \{1, \dots, n\}$. Hence, by definition of spherical characters, we deduce that

$$\begin{aligned} m_{P_\tau,P_\tau}(f_1 \otimes f_2) &= \sum_{u\otimes v \in\ o.b.(\bar{V}\otimes V)} P_\tau\Big(\bar{\tau}(f_1) \otimes \tau(f_2)(u \otimes v)\Big)\overline{P_\tau(u \otimes v)} \\ &= \sum_{u\otimes v \in\ o.b.(\bar{V}\otimes V)} \sum_{i,j=1}^{n} \frac{1}{\xi_i(e_i)\xi_j(e_j)} \overline{\xi_i}(\bar{\tau}(f_1)u)\xi_i(\tau(f_2)v)\overline{\overline{\xi_j}(u)\xi_j(v)}, \end{aligned}$$

where $o.b.(\bar{V} \otimes V)$ is an orthonormal basis of $\overline{V} \otimes V$. By definition of $\overline{\xi}$ for $\xi \in V^{*H}$, one has $\overline{\xi}(\overline{\tau}(f_1)u) = \overline{\xi(\tau(\overline{f_1})\bar{u})}$. Therefore, we obtain

$$m_{P_\tau,P_\tau}(f_1 \otimes f_2) = \sum_{i,j=1}^{n} \frac{1}{\xi_i(e_i)\xi_j(e_j)} \overline{m_{\xi_i,\xi_j}(\overline{f_1})} m_{\xi_i,\xi_j}(f_2). \tag{A.7}$$

Let v and w be in V. Let $f_1 := c_{v,w}$ so that $\overline{f_1} = \check{c}_{v,w}$. If $v \in V_H$ or $w \in V_H$, it follows from Lemma A.1 that $m_{\xi_i,\xi_j}(\overline{f_1}) = 0$ for $i, j \in \{1, \dots, n\}$. Hence $m_{P_\tau,P_\tau}(f_1 \otimes f_2) = 0$. Thus we deduce from (A.5) that

$$\tilde{J}(c_{v,w} \otimes f_2) = 0, \quad v \in V_H \quad \text{or} \quad w \in V_H. \tag{A.8}$$

Let $k, l \in \{1, \dots, n\}$. Let us take $f_1 := c_{e_k,e_l}$. Then $\overline{f_1} = \check{c}_{e_l,e_k}$, and, by Lemma A.1, one has $m_{\xi_i,\xi_j}(\overline{f_1}) = d(\tau)^{-1}\xi_i(e_l)\xi_j(e_k)$. Therefore, by (A.5) and (A.7), we obtain

$$\tilde{J}(c_{e_k,e_l} \otimes f_2) = m_{\xi_l,\xi_k}(f_2). \tag{A.9}$$

By sesquilinearity, one deduces from (A.8) and (A.9) that

$$\tilde{J}(c_{v,w} \otimes f_2) = m_{\xi_w,\xi_v}(f_2) \quad v, w \in V. \tag{A.10}$$

Let $(J_n)_n$ be a sequence of compact open subgroups whose intersection is equal to the neutral element of G. The characteristic function g_n of $J_n x_m \gamma J_n$ approaches the Dirac measure at $x_m\gamma$ as n approaches $+\infty$. Thus, if $v, w \in V$, then $m_{\xi_w,\xi_v}(g_n)$ converges to $m_{\xi_w,\xi_v}(x_m\gamma)$. Then, by Corollary 2.11, the constant term $\tilde{J}(c_{v,w} \otimes g_n)$ converges to $c_M c_{S,x_m} \mathcal{WM}(c_{v,w})(x_m\gamma)$. We thus deduce the theorem from (A.10). □

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