P. Delorme *et al.* (2021) "The Constant Term of Tempered Functions
on a Real Spherical Space,"
International Mathematics Research Notices, Vol. 00, No. 0, pp. 1–86
doi:10.1093/imrn/rnaa395

The Constant Term of Tempered Functions on a Real Spherical Space

Patrick Delorme[1], Bernhard Krötz[2], Sofiane Souaifi[3], and Raphaël Beuzart-Plessis[4]

[1]Université d'Aix-Marseille, I2M, UMR 7373, 13453 Marseille, France, [2]Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany, [3]Université de Strasbourg, IRMA, UMR 7501, 7 rue René Descartes, 67084 Strasbourg Cedex, France, and [4]Aix Marseille University, CNRS, Centrale Marseille, I2M, Marseille, France

Let Z be a unimodular real spherical space. We develop a theory of constant terms for tempered functions on Z, which parallels the work of Harish-Chandra. The constant terms f_I of an eigenfunction f are parametrized by subsets I of the set S of spherical roots that determine the fine geometry of Z at infinity. Constant terms are transitive i.e., $(f_J)_I = f_I$ for $I \subset J$, and our main result is a quantitative bound of the difference $f - f_I$, which is uniform in the parameter of the eigenfunction.

1 Introduction

Real spherical spaces are the natural algebraic homogeneous structures $Z = G/H$ attached to a real reductive group G. Formally, one defines *real spherical* by the existence of a minimal parabolic subgroup $P \subset G$ with PH open in G. On a more informal level, one could define real spherical spaces as the class of algebraic homogeneous spaces $Z = G/H$, which allow a uniform treatment of spectral theory, i.e., admit explicit Fourier analysis for $L^2(Z)$.

Real spherical spaces provide a wide class of algebraic homogeneous spaces. Important examples are the group G itself, viewed as a homogeneous space under its

Received August 27, 2019; Revised July 6, 2020; Accepted December 21, 2020
Communicated by Prof. Assaf Naor

© The Author(s) 2021. Published by Oxford University Press. All rights reserved. For permissions, please e-mail: journals.permission@oup.com

both sided symmetries $G \simeq G \times G/\operatorname{diag}(G)$, and, more generally, all symmetric spaces. In case H is reductive, a classification of all infinitesimal real spherical pairs $(\operatorname{Lie} G, \operatorname{Lie} H)$ was recently obtained and we refer to [26, 27] for the tables.

Harmonic analysis on spherical spaces was initiated by Sakellaridis and Venkatesh in the context of p-adic absolutely spherical varieties of wavefront type [39]. In particular, they developed a theory of asymptotics for smooth functions generalizing Harish-Chandra's concept of constant term for real reductive groups.

Harish-Chandra's approach to the Plancherel formula for $L^2(G)$, a cornerstone of 20th century mathematics (cf. [19]), was based on his theory of the constant term [17, 18] and his epochal work on the determination of the discrete series [15, 16]. In more precision, constant terms were introduced in [17] and then made uniform in the representation parameter in [18] by using the strong results on the discrete series in [15, 16]. Also in Harish-Chandra's approach towards the Plancherel formula for p-adic reductive groups, the constant term concept played an important role and we refer to Waldspurger's work [40] for a complete account (the constant term in [40] is called weak constant term). Likewise, the Plancherel theory of Sakellaridis and Venkatesh for p-adic spherical spaces is founded on their more general theory of asymptotics.

Carmona introduced a theory of constant term for real symmetric spaces [7] parallel to [17, 18], with the uniformity in the representation parameter relying on the description of the discrete series by Oshima-Matsuki [37]. This concept of constant term then crucially entered the proofs of Delorme [11] and van den Ban–Schlichtkrull [2] of the Plancherel formula for real symmetric spaces.

Motivated by [39], we develop in this paper a complete theory of constant term for real spherical spaces generalizing the works of Harish-Chandra [17, 18] and Carmona [7]. Let us point out that our results hold in full generality for all real spherical spaces, i.e., in contrast to [39], we are not required to make any limiting geometric assumptions on Z such as absolutely spherical or of wavefront type. Further, we do not need to make any assumptions on the discrete spectrum as in [39]. This is because of the recently obtained spectral gap theorem for the discrete series on a real spherical space [33], which then implies the uniformity of the constant term approximation in the representation parameter. The results of this paper then will be applied in the forthcoming paper [12], where we derive the Bernstein decomposition of $L^2(Z)$, which is a major step towards the Plancherel formula for Z.

Let us describe the results more precisely. In this introduction, G is the group of real points of a connected reductive algebraic group $\underline{G}$ defined over $\mathbb{R}$, and $H = \underline{H}(\mathbb{R})$

for an algebraic subgroup $\underline{H}$ of $\underline{G}$ defined over $\mathbb{R}$. Furthermore, we assume that Z is unimodular, i.e., Z carries a positive G-invariant Radon measure. We will say that A is a split torus of G if $A = \underline{A}(\mathbb{R})$, where $\underline{A}$ is a split $\mathbb{R}$-torus of $\underline{G}$.

Central to the geometric theory of real spherical spaces $Z = G/H$ is the local structure theorem (cf. [30, Theorem 2.3] and Section 2.1), which associates a parabolic subgroup $Q \supset P$, said Z-adapted to P, with Levi decomposition $Q = LU$.

Let A be a maximal split torus of G, which we choose to be contained in L and set $A_H := A \cap H$. We define A_Z to be the identity component of A/A_H and recall the spherical roots S as defined in e.g., Section 3.2[31] or Section 2.2. Suitably normalized the spherical roots are the simple roots for a certain root system on $\mathfrak{a}_Z = \operatorname{Lie} A_Z$ and give rise to the co-simplicial compression cone $\mathfrak{a}_Z^- := \{X \in \mathfrak{a}_Z \mid \alpha(X) \leq 0, \alpha \in S\}$, see [25]. Set $A_Z^- := \exp(\mathfrak{a}_Z^-) \subset A_Z$.

We move on to boundary degenerations $\mathfrak{h}_I$ of $\mathfrak{h}$, which are parametrized by subsets $I \subset S$. These geometric objects show up naturally in the compactification theory of Z (see [28], [25] and Section 3), which in turn is closely related to the polar decomposition (see (1.1) below). In more detail, let $\mathfrak{a}_I = \bigcap_{\alpha\in I} \operatorname{Ker}\alpha \subset \mathfrak{a}_Z$ and pick $X \in \mathfrak{a}_I^{--} = \{X \in \mathfrak{a}_I \mid \alpha(X) < 0, \alpha \in S \setminus I\}$. Let H_I be the analytic subgroup of G with Lie algebra

$$\operatorname{Lie} H_I = \lim_{t\to+\infty} e^{t\,\operatorname{ad} X} \operatorname{Lie} H\,,$$

where the limit is taken in the Grassmannian $\operatorname{Gr}(\mathfrak{g})$ of $\mathfrak{g} = \operatorname{Lie} G$ and does not depend on X. If we denote by $z_0 = H$ the standard base point of Z, then one can view H_I (up to components) as the invariance group of the asymptotic directions $\gamma_X(t) := \exp(tX) \cdot z_0$ for $t \to \infty$ and $X \in \mathfrak{a}_I^{--}$. Phrased more geometrically, $Z_I := G/H_I$ is (up to cover) asymptotically tangent to Z in direction of the curves γ_X, $X \in \mathfrak{a}_I^{--}$.

As a deformation of Z, the space Z_I is real spherical. Further, one has $A_{Z_I} = A_Z$ naturally, but the compression cone $\mathfrak{a}_{Z_I}^- = \{X \in \mathfrak{a}_Z \mid \alpha(X) \leq 0,\ \alpha \in I\}$ becomes larger. In particular, $\mathfrak{a}_I$ is the edge of the cone $\mathfrak{a}_{Z_I}^-$, which translates into the fact that $A_I = \exp(\mathfrak{a}_I)$ acts on Z_I from the right, commuting with the left action of G.

The general concept of "constant term" is to approximate functions f on Z in directions γ_X, $X \in \mathfrak{a}_I^{--}$, by functions f_I, called constant terms, on Z_I. The notion "constant" then refers to the fact that f_I should transform finitely under the right action of A_I.

The appropriate class of functions for which this works are tempered eigenfunctions on Z. In order to define them, we need to recall the polar decomposition, which asserts

$$Z = \Omega A_Z^- \mathcal{W} \cdot z_0 \,, \tag{1.1}$$

for a compact subset $\Omega \subset G$ and a certain finite subset $\mathcal{W}$ of G, which parametrizes the open P-orbits in Z (see Lemma 2.7 and Remark 2.11 for more explicit expressions of the elements of $\mathcal{W}$).

Let ρ_Q be the half sum of the roots of $\mathfrak{a}$ in Lie U. Actually, as Z is unimodular, $\rho_Q \in \mathfrak{a}_Z^*$.

For $f \in C^\infty(Z)$ and $N \in \mathbb{N}$, we set

$$q_N(f) = \sup_{g \in \Omega, w \in \mathcal{W}, a \in A_Z^-} a^{-\rho_Q}(1 + \|\log a\|)^{-N} |f(gaw \cdot z_0)|$$

and define $C^\infty_{temp,N}(Z)$ as the space of all $f \in C^\infty(Z)$ such that, for all u in the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the complexification $\mathfrak{g}_\mathbb{C}$ of $\mathfrak{g}$,

$$q_{N,u}(f) := q_N(L_u f)$$

is finite. The semi-norms $q_{N,u}$ induce a Fréchet structure on $C^\infty_{temp,N}(Z)$ for which the G-action is smooth and of moderate growth (in [4], these are called SF-representations). We define the space of *tempered functions* $C^\infty_{temp}(Z) = \bigcup_{N \in \mathbb{N}} C^\infty_{temp,N}(Z)$ and endow it with the inductive limit topology.

We denote by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$ and define $\mathcal{A}_{temp}(Z)$, resp. $\mathcal{A}_{temp,N}(Z)$, as the subspace of $C^\infty_{temp}(Z)$, resp. $C^\infty_{temp,N}(Z)$, consisting of $\mathcal{Z}(\mathfrak{g})$-finite functions.

Remark 1.1. Functions $f \in \mathcal{A}_{temp}(Z)$ can be described suitably in terms of representation theory. A variant of Frobenius reciprocity implies that elements $f \in \mathcal{A}_{temp}(Z)$ can be expressed as generalized matrix coefficients

$$f(gH) = m_{\eta,v}(gH) := \eta(\pi(g)^{-1}v) \,,$$

for $v \in V^\infty$, where (π, V^∞) is a $\mathcal{Z}(\mathfrak{g})$-finite SF-representation of G and $\eta : V^\infty \to \mathbb{C}$ a H-invariant continuous functional. If $V^{-\infty}$ denotes the continuous dual of V^∞, then an element $\eta \in (V^{-\infty})^H$ is called *Z-tempered* provided $m_{\eta,v} \in \mathcal{A}_{temp}(Z)$ for all $v \in V^\infty$. We denote the corresponding subspace by $(V^{-\infty})^H_{temp}$.

For $I \subset S$, we choose a set $\mathcal{W}_I \subset G$ parametrizing the open P-orbits on Z_I. Then it is the content of Section 2 that there is a map $\mathbf{m} : \mathcal{W}_I \to \mathcal{W}$ obtained from a natural matching of open P-orbits on Z_I with open P-orbits on Z. As Z_I is a real spherical space, we can define $C^{\infty}_{temp,N}(Z_I)$ and $\mathcal{A}_{temp,N}(Z_I)$ as before.

For $\mathcal{J}$ a finite codimensional ideal of the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ we denote by $\mathcal{A}_{temp,N}(Z : \mathcal{J})$ the space of elements of $\mathcal{A}_{temp,N}(Z)$, which are annihilated by $\mathcal{J}$. Note that, by definition, there exists for each $f \in \mathcal{A}_{temp,N}(Z)$ a co-finite ideal $\mathcal{J}$ such that $\mathcal{A}_{temp,N}(Z : \mathcal{J})$.

The main result of this paper is the following (cf. Remark 6.6 and Theorem 6.9 for (i)–(iii) and Theorem 8.10 for (iv)).

Theorem 1.2. Let $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g})$ be an ideal of finite co-dimension. Let $I \subset S$. Then there exists an $N_{\mathcal{J}} \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists a continuous G-equivariant linear map

$$\mathrm{CT}_{I,N} : \mathcal{A}_{temp,N}(Z : \mathcal{J}) \to \mathcal{A}_{temp,N+N_{\mathcal{J}}}(Z_I : \mathcal{J}), \ \ f \mapsto f_I$$

with the following properties for all $g \in G$ and $X_I \in \mathfrak{a}_I^{--}$:

(i) If we interpret f, resp. f_I, as functions on G that are right invariant under H, resp. H_I, then

$$\lim_{t\to+\infty} e^{-t\rho_Q(X_I)} \big(f(g\exp(tX_I)) - f_I(g\exp(tX_I))\big) = 0\,.$$

(ii) The assignment

$$\mathbb{R} \ni t \mapsto e^{-t\rho_Q(X_I)} f_I(g\exp(tX_I))$$

defines an exponential polynomial with unitary characters, i.e., it is of the form $\sum_{j=1}^{n} p_j(t)e^{i\nu_j t}$, where the p_j's are polynomials and the ν_j's are real numbers.

Conditions (i) and (ii) determine the constant term morphism $\mathrm{CT}_{I,N}$ uniquely. Moreover,

(iii) For each $w_I \in \mathcal{W}_I$ with $w = \mathbf{m}(w_I) \in \mathcal{W}$, and any compact subsets $\mathcal{C}_I \subset \mathfrak{a}_I^{--}$, $\Omega \subset G$, there exists $\varepsilon > 0$ and a continuous semi-norm p on $\mathcal{A}_{temp,N}(Z)$ such that for all $f \in \mathcal{A}_{temp,N}(Z:\mathcal{J})$ one has:

$$|(a_Z \exp(tX_I))^{-\rho_Q} \left(f(ga_Z \exp(tX_I)w \cdot z_0) - f_I(ga_Z \exp(tX_I)w_I \cdot z_{0,I})\right)| \\ \leq e^{-\varepsilon t}(1 + \|\log a_Z\|)^N p(f), \quad a_Z \in A_Z^-, X_I \in \mathcal{C}_I, g \in \Omega, t \geq 0\,.$$

(iv) The bound in (iii) is uniform for all $\mathcal{J}$ of codimension 1, i.e., $\mathcal{J} = \ker\chi$ for a character χ of $\mathcal{Z}(\mathfrak{g})$.

Given $f \in \mathcal{A}_{temp}(Z)$ and $I \subset S$, we call $f_I \in \mathcal{A}_{temp}(Z_I)$ the *constant term* of f associated to I. Note that properties (i) and (ii) in Theorem 1.2 determine f_I uniquely as a smooth function on Z_I. Furthermore, we may interpret Theorem 1.2(iii) in such a way that f_I controls the normal asymptotics of f in direction of $\mathfrak{a}_I^{--}$ emanating from the base points $w \cdot z_0$ for certain $w \in \mathcal{W}$.

Remark 1.3. (a) Theorem 1.2 can be phrased differently in the language of representation theory and it is worthwhile to mention this reformulation. Let V be a Harish-Chandra module and V^∞ its unique *SF*-completion. The subgroups H, H_I being real spherical implies that the invariant spaces $(V^{-\infty})^H$ and $(V^{-\infty})^{H_I}$ are both finite dimensional (cf. [35]). Inside, we find the subspaces of tempered functionals $(V^{-\infty})^H_{temp}$ and $(V^{-\infty})^{H_I}_{temp}$. Then Theorem 1.2 defines a linear map

$$(V^{-\infty})^H_{temp} \to (V^{-\infty})^{H_I}_{temp}, \quad \eta \mapsto \eta_I$$

such that, for all $v \in V^\infty$, the matrix coefficient $f = m_{\eta,v}$ is approximated by $f_I = m_{\eta_I,v}$ in the sense of Theorem 1.2(iii). As A_I normalizes H_I, we obtain an action of $\mathfrak{a}_I$ on the finite dimensional space $(V^{-\infty})^{H_I}_{temp}$. It is easy to see that temperedness implies that

$$\mathrm{Spec}_{\mathfrak{a}_I}(V^{-\infty})^{H_I}_{temp} \subset \rho_Q\big|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*\,,$$

which in turn translates into the behavior of f_I as exponential polynomial as recorded in Theorem 1.2(ii).

(b) It is possible to develop a constant term theory for all $\mathcal{Z}(\mathfrak{g})$-finite eigenfunctions on Z, i.e., the assumption of temperedness is not really necessary. This was carried out in [34, Th. 7.1] in case where Z is wavefront.

Parts (i)–(iii) of Theorem 1.2 generalize the work of Harish-Chandra in the group case (see [17, Sections 21 to 25], also the work of Wallach [42, Chapter 12], where the constant term is called leading term) and the one of Carmona for symmetric spaces (see [7]). The uniformity in (iv) generalizes the uniform results of Harish-Chandra in the group case (cf. [18, Section 10]) and Carmona for symmetric spaces (cf. [7, Section 5]).

As a corollary of Theorem 1.2, we obtain a characterization of tempered eigenfunctions f in the discrete series by the vanishing of their constant terms f_I, $I \subsetneq S$ (see Theorem 6.12). Again it is analogous to a result of Harish-Chandra. For this, we use in a crucial manner some results on discrete series from [31, Section 8].

The proof of Theorem 1.2 is inspired by the work of Harish-Chandra for real reductive groups G, [17, 18], who associates to a tempered eigenfunction f on G certain systems of linear differential equations. The main technical difficulty here is to set up the correct first order system (5.26) of differential equations on A_I associated to a function $f \in \mathcal{A}_{temp}(Z)$. This is based on novel insights on the algebra of invariant differential operators on Z. With the solution formula for the first order differential system (5.26), one then obtains, as in [17], for each $f \in \mathcal{A}_{temp}(Z)$, a unique smooth function $f_I \in C^\infty(Z_I)$ with properties (i), (ii) in Theorem 1.2 and also (iii) for $w_I = w = 1$. A main difficulty in this paper was to show that f_I is in fact tempered, which translates into the assertions in (iii) for all $w_I \in \mathcal{W}_I$. This, we deduce from Proposition 3.1 on geometric asymptotics related to the natural matching map $\mathbf{m} : \mathcal{W}_I \to \mathcal{W}$. Let us point out further that our treatment in Section 8 of the uniformity in Theorem 1.2(iv) constitutes a simplification to the so far existing state of the art in [42, Chapter 12].

Earlier versions of this article needed the assumption that Z is of wavefront type. This was mainly due to the lack of a better understanding of the algebra $\mathbb{D}(Z)$ of G-invariant differential operators on Z and their behavior under boundary degenerations, i.e., overlooking that there is a natural map $\mathbb{D}(Z) \to \mathbb{D}(Z_I)$ originating from Knop's work [24]. This was observed by Raphaël Beuzart-Plessis and is now recorded in Appendix C. With this insight, we could remove the wavefront assumption and make the paper valid in the full generality of real spherical spaces.

2 Notation

In this paper, we will denote (real) Lie groups by upper case Latin letters and their Lie algebras by lower case German letters. If R is a real Lie group, then R_0 will denote its identity component. Furthermore, if $\underline{Z}$ is an algebraic variety defined over $\mathbb{R}$ and k is any field containing $\mathbb{R}$, then we denote by $\underline{Z}(k)$ the k-points of $\underline{Z}$.

Let $\underline{G}$ be a connected reductive algebraic group defined over $\mathbb{R}$ and let $G := \underline{G}(\mathbb{R})$ be its group of real points.

Remark 2.1. More generally, we could define G as an open subgroup of the real Lie group $\underline{G}(\mathbb{R})$. The main analytic result of this paper (i.e., Theorem 1.2) is not affected by this more general assumption but we do not supply a complete proof here.

For an $\mathbb{R}$-algebraic subgroup $\underline{R}$ of $\underline{G}$, we set $R := \underline{R}(\mathbb{R})$ and note that $R \subset G$ is a closed subgroup.

Let now $\underline{H} \subset \underline{G}$ be an $\mathbb{R}$-algebraic subgroup. Having $\underline{G}$ and $\underline{H}$, we form the homogeneous variety $\underline{Z} = \underline{G}/\underline{H}$. We note that $\underline{Z}(\mathbb{C}) = \underline{G}(\mathbb{C})/\underline{H}(\mathbb{C})$ and denote by $z_0 = \underline{H}(\mathbb{C})$ the standard base point of $\underline{Z}(\mathbb{C})$. Set $Z = G/H$ and record the G-equivariant embedding

$$Z \to \underline{Z}(\mathbb{C}), \quad gH \mapsto g \cdot z_0 \,.$$

In the sequel, we consider Z as a submanifold of $\underline{Z}(\mathbb{C})$ and, in particular, identify z_0 with the standard base point H of $Z = G/H$ as well.

Remark 2.2. Note that Z is typically strictly smaller than $\underline{Z}(\mathbb{R})$, which is a finite union of G-orbits. An instructive example is the space of invertible symmetric matrices $\underline{Z} = \underline{GL_n}/\underline{O_n}$, which features $\underline{Z}(\mathbb{R}) = \bigcup_{p+q=n} \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(p,q)$. In particular, $\underline{Z}(\mathbb{R}) \supsetneq Z = G/H = \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$.

As a further piece of notation, we use, for an algebraic subgroup $\underline{R} \subset \underline{G}$ defined over $\mathbb{R}$, the notation $\underline{R}_{\underline{H}} := \underline{R} \cap \underline{H}$ and, likewise, $R_H := R \cap H$. In the sequel, we use the letter $\underline{P}$ to denote a minimal $\mathbb{R}$-parabolic subgroup of $\underline{G}$. The unipotent radical of $\underline{P}$ is denoted by $\underline{N}$.

2.1 The local structure theorem

From now on, we assume that Z is real spherical, that is, there is a choice of $\underline{P}$ such that $P \cdot z_0$ is open in Z.

Remark 2.3. Notice that $\underline{P}(\mathbb{C})\underline{H}(\mathbb{C})$ is Zariski open and hence dense in $\underline{G}(\mathbb{C})$ as $\underline{G}$ was assumed to be connected. Thus, any other choice $\underline{P}'$ of $\underline{P}$, with $P' \cdot z_0$ open, is conjugate to $\underline{P}$ under $\underline{H}$.

We now recall the local structure theorem for real spherical varieties (cf. [30, Theorem 2.3] or [25, Corollary 4.12]; see also [6, 23] for preceding versions in the complex case), which asserts that there is a unique parabolic subgroup $\underline{Q} \supset \underline{P}$ endowed with a

Levi decomposition $\underline{Q} = \underline{L} \ltimes \underline{U}$, defined over $\mathbb{R}$, such that

$$\underline{QH} = \underline{PH}, \tag{2.1a}$$

$$\underline{Q}_{\underline{H}} = \underline{L}_{\underline{H}}, \tag{2.1b}$$

$$\underline{L}_{\underline{H}} \supset \underline{L}_{\mathrm{n}}, \tag{2.1c}$$

where $\underline{L}_{\mathrm{n}}$ denotes the connected normal subgroup of $\underline{L}$ generated by all unipotent elements defined over $\mathbb{R}$.

Remark 2.4. (a) The Lie algebra $\mathfrak{l}_{\mathrm{n}}$ is the sum of all non-compact simple non-abelian ideals of $\mathfrak{l}$.

(b) As mentioned above, $\underline{Q}$ is the unique parabolic subgroup containing $\underline{P}$ with properties (2.1a)–(2.1c). Slightly differently, we could have defined $\underline{Q}$ via [28, Lemma 3.7], which asserts

$$\underline{Q}(\mathbb{C}) = \{g \in \underline{G}(\mathbb{C}) \mid g\underline{P}(\mathbb{C}) \cdot z_0 = \underline{P}(\mathbb{C}) \cdot z_0\}\,.$$

The group $\underline{L}_{\underline{H}}$ is uniquely determined by $\underline{Q}$ and we recall from [25, Lemma 13.5] that $\underline{L}_{\underline{H}}$ is an invariant of $\underline{Z}$, i.e., its $\underline{H}$-conjugacy class is defined over $\mathbb{R}$. In contrast to L_H, the Levi subgroup L is only unique up to conjugation with elements from U, which stabilize L_H. In this regard, we note that it is quite frequent that L_H is trivial and then L could be an arbitrary Levi of Q. For later purposes of compactifications, we will only use those choices of L that are obtained from the constructive proof of the local structure theorem (cf. [30, Sect. 2.1] or [25, Sect. 4]). In case $\underline{Z}$ is quasi-affine, this means that $\mathfrak{l}$ is defined as the centralizer of a generic hyperbolic element of $\mathfrak{g}^*$, which is contained in $(\mathfrak{h} + \mathfrak{n})^\perp$ (see the constructive proof of the local structure theorem in [30] or [25]).

Example 2.5. (The triple space) Let $\underline{G}_o := \mathrm{SL}(2)$ with $G_o = \mathrm{SL}(2, \mathbb{R})$ and form $\underline{G} := \underline{G}_o \times \underline{G}_o \times \underline{G}_o$. With $\underline{H} := \operatorname{diag} \underline{G}_o$ we obtain a real spherical space $Z = G/H$, the so-called triple space. It features $Z = \underline{Z}(\mathbb{R})$ as one deduces from the standard identification

$$\underline{Z} = \underline{G}/\underline{H} \to \underline{G}_o \times \underline{G}_o, \quad (g_1, g_2, g_3)H \mapsto (g_1 g_3^{-1}, g_2 g_3^{-1}),$$

which a $\underline{G}$-isomorphism of varieties where $\underline{G}$ acts on $\underline{G}_o \times \underline{G}_o$ as

$$(g_1, g_2, g_3) \cdot (x, y) = (g_1 x g_3^{-1}, g_2 y g_3^{-1}) \qquad (g_1, g_2, g_3, x, y \in \underline{G}_o)\,.$$

Let $\mathfrak{g}_o = \mathfrak{k}_o + \mathfrak{s}_o$ be the standard Cartan decomposition with $\mathfrak{k}_o = \mathfrak{so}(2,\mathbb{R})$. Let further $X_1, X_2 \in \mathfrak{s}_o$ be linearly independent elements, set $X_3 := -(X_1 + X_2)$ and define

$$\mathfrak{a}_1 := \mathbb{R}X_1, \quad \mathfrak{a}_2 = \mathbb{R}X_2, \quad \mathfrak{a}_3 = \mathbb{R}X_3 .$$

Then $\mathfrak{a} := \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ defines a Cartan subspace of $\mathfrak{g} = \mathfrak{g}_o \times \mathfrak{g}_o \times \mathfrak{g}_o$. We set $A_i = \exp(\mathfrak{a}_i)$ and then $A = A_1 \times A_2 \times A_3$. Observe that

$$\mathfrak{h}^\perp := \{(Y_1, Y_2, Y_3) \in \mathfrak{g} \mid Y_1 + Y_2 + Y_3 = 0\}$$

and thus

$$X_0 := (X_1, X_2, X_3) \in \mathfrak{a} \cap \mathfrak{h}^\perp .$$

Let now $P_i \subset G_o$ be any minimal parabolic subgroup of G_o above A_i. With $P = P_1 \times P_2 \times P_3$ we then obtain a minimal parabolic of G such that $PH \subset G$ is open. For later purpose we record $P_1 \cap P_2 \cap P_3 = \{\pm 1\}$ equals the center $Z(H)$ of H. This entails that $\underline{PH}/\underline{H} \simeq \underline{P}/Z(\underline{H})$ and thus $\underline{Q} = \underline{P}$.

If we write $P_i = M_i A_i N_i$, then $M_i = Z(G_o)$ with $N = N_1 \times N_2 \times N_3$ and therefore $X_0 \in (\mathfrak{h} + \mathfrak{n})^\perp$ is in accordance with the constructive proof of the local structure theorem.

A parabolic subgroup Q as above in (1.1) will be called Z-adapted to P.

Let $\underline{A_L}$ be the maximal split torus of the center of $\underline{L}$ and $\underline{A}$ be a maximal split torus of $\underline{P} \cap \underline{L}$. Note that $\underline{A_L} \subset \underline{A}$. Define $\underline{A_Z} := \underline{A}/\underline{A_H}$ and let (by slight abuse of notation) $A_Z := (A/A_H)_0 \simeq A_0/(A_H)_0$. From the fact that $\underline{L}_\mathrm{n} \subset \underline{L_H}$ and $\underline{A} = \underline{A_L}(\underline{A} \cap \underline{L}_\mathrm{n})$, we obtain $A_Z \simeq (A_L)_0/(A_L)_0 \cap H$ with $\mathfrak{a}_Z \simeq \mathfrak{a}_L/\mathfrak{a}_L \cap \mathfrak{h}$.

We choose a section $\mathbf{s} : A_Z \to (A_L)_0$ of the projection $(A_L)_0 \to A_Z$, which is a morphism of Lie groups. We will often use $\tilde{a}$ instead of $\mathbf{s}(a)$, $\tilde{\mathfrak{a}}_Z$ instead of $\mathbf{s}(\mathfrak{a}_Z)$ etc. (2.2)

Note that $Z_{\underline{G}}(\underline{A}) = \underline{MA}$ where $\underline{M} \subset \underline{P}$ is a maximal anisotropic subgroup. Moreover, $\underline{MA}$, as a Levi of $\underline{P}$, is connected (recall that Levi subgroups of connected algebraic groups are connected). Notice that $\underline{M}$ commutes with $\underline{A}$ and $P = MAN$. Observe that $M \cap A$ equals the 2-torsion points A_2 of A.

From (2.1a)–(2.1b), we obtain $\underline{PH}/\underline{H} = \underline{QH}/\underline{H} \simeq \underline{U} \times \underline{L}/\underline{L_H}$, and, taking real points, we get

$$[\underline{P} \cdot z_0](\mathbb{R}) \simeq U \times (\underline{L}/\underline{L_H})(\mathbb{R})\,.$$

Next, we collect some elementary facts about $(\underline{L}/\underline{L_H})(\mathbb{R})$. To begin with, we define

$$\widehat{M}_H := \{m \in M \mid m \cdot z_0 \subset \underline{A_Z}(\mathbb{R})\}$$

and note that M_H is a cofinite normal 2-subgroup of $\widehat{M}_H$, see Proposition B.2. We denote by $F_M := \widehat{M}_H/M_H$ this finite 2-group. Since the action of the $\underline{P}$-Levi $\underline{MA} \subset \underline{L}$ on $\underline{L}/\underline{L_H}$ is transitive, we obtain for the real points, by Proposition B.2,

$$(\underline{L}/\underline{L_H})(\mathbb{R}) = [M/M_H] \times^{F_M} \underline{A_Z}(\mathbb{R})\,. \tag{2.3}$$

From that, we derive the local structure theorem in the form

$$[\underline{P} \cdot z_0](\mathbb{R}) = U \times \big[[M/M_H] \times^{F_M} \underline{A_Z}(\mathbb{R})\big]\,, \tag{2.4}$$

which we will use later. Let us denote by $\underline{A_Z}(\mathbb{R})_2 \simeq \{-1,1\}^r$ the 2-torsion elements in $\underline{A_Z}(\mathbb{R}) \simeq (\mathbb{R}^\times)^r$ and note that $\underline{A_Z}(\mathbb{R})_2$ naturally parametrizes the connected components of $\underline{A_Z}(\mathbb{R})$, that is, the A_Z-orbits in $\underline{A_Z}(\mathbb{R})$. In particular, we record the natural isomorphism of Lie groups

$$\underline{A_Z}(\mathbb{R}) \simeq A_Z \times \underline{A_Z}(\mathbb{R})_2\,.$$

Observe that F_M naturally acts on $\underline{A_Z}(\mathbb{R})_2$. Hence, if we denote by $(P\backslash\underline{Z}(\mathbb{R}))_{\text{open}}$ the set of open P-orbits in $\underline{Z}(\mathbb{R})$, then we obtain from (2.4) and Corollary B.3 that:

Lemma 2.6. The map

$$\underline{A_Z}(\mathbb{R})_2/F_M \to (P\backslash\underline{Z}(\mathbb{R}))_{\text{open}}, \quad F_M a_Z \mapsto P a_Z$$

is a bijection.

If we intersect (2.4) with Z, we obtain

$$[\underline{P} \cdot z_0](\mathbb{R}) \cap Z = U \times \big[[M/M_H] \times^{F_M} A_{Z,\mathbb{R}}\big] \tag{2.5}$$

with $A_{Z,\mathbb{R}} := \underline{A_Z}(\mathbb{R}) \cap Z$. Observe that $A_{Z,\mathbb{R}}$ might not be a group and is in general only a A_Z-set. With $A_{Z,2} := \underline{A_Z}(\mathbb{R})_2 \cap A_{Z,\mathbb{R}}$, we then obtain

$$A_{Z,\mathbb{R}} = A_Z A_{Z,2} \simeq A_Z \times A_{Z,2} .$$

Note that F_M acts on $A_{Z,2}$ and thus we obtain, in analogy to Lemma 2.6, that the map

$$A_{Z,2}/F_M \to (P\backslash Z)_{\text{open}}, \quad F_M a_Z \mapsto Pa_Z$$

is a bijection. Next, we wish to find suitable representatives of the open P-orbits of Z in G, i.e., find, for each $F_M a_Z$ with $a_Z \in A_{Z,2}$, an element $w \in G$ such that $Pa_Z = Pw \cdot z_0$. For that, we consider the exact sequence

$$1 \to \underline{A_H} \to \underline{A} \to \underline{A_Z} = \underline{A}/\underline{A_H} \to 1 .$$

Now, note that this sequence stays, in general, only left exact when taking real points

$$1 \to \underline{A_H}(\mathbb{R}) \to \underline{A}(\mathbb{R}) \to \underline{A_Z}(\mathbb{R}) .$$

In particular, we typically do not find a preimage of a torsion element $t \in A_{Z,2} \subset \underline{A_Z}(\mathbb{R})$ in $A = \underline{A}(\mathbb{R})$. However, if we set $T := \exp(i\mathfrak{a}) \subset \underline{A}(\mathbb{C})$ and $T_Z := \exp(i\mathfrak{a}_Z) \subset \underline{A_Z}(\mathbb{C})$, then $T \to T_Z$ is surjective. In particular, each $t \in A_{Z,2}$ has a lift $\tilde{t} \in T$, which can even be chosen in $\exp(i\tilde{\mathfrak{a}}_Z) \subset T$. In this way we extend the lift $\mathbf{s} : A_Z \to (A_L)_0$ from (2.2) to a map $\underline{A_Z}(\mathbb{R}) \to A_L \exp(i\tilde{\mathfrak{a}}_Z)$, also denoted by $\mathbf{s}$ in the sequel. Thus, we have shown that

Lemma 2.7. There exists a set $\mathcal{W} \subset G$ of representatives of $(P\backslash Z)_{\text{open}}$ such that any element $w \in \mathcal{W}$ has a factorization in $\underline{G}(\mathbb{C})$ of the form

$w = \tilde{t}h$, where $\tilde{t} \in \exp(i\tilde{\mathfrak{a}}_Z)$ and $h \in \underline{H}(\mathbb{C})$ such that $t := \tilde{t} \cdot z_0 \in A_{Z,2}$.

In particular, if $a \in A_H$, $aw \cdot z_0 = w \cdot z_0$.

In the sequel, $\mathcal{W} \subset G$ is a choice of representatives of $(P\backslash Z)_{\text{open}}$ as in Lemma 2.7, assumed to contain 1 as a representative of $P \cdot z_0$.

Example 2.8. (a) (Group case) Let $G = G' \times G'$. In the group case, i.e.,

$$Z = G' \times G' / \operatorname{diag} G' \simeq G' ,$$

one has only one open $P = P' \times P'$-orbit in Z by the Bruhat decomposition of G'. Hence $\mathcal{W} = \{1\}$ in this case.

(b) (Triple case) We recall the triple space $Z = G_o \times G_o \times G_o / \operatorname{diag} G_o$ from Example 2.5. We recall that $\underline{P} \cdot z_0 \simeq \underline{P}/Z(\underline{H})$ with $Z(\underline{H}) = \{\pm 1\}$ the center of $\underline{H}$. Further we have $F_M = M/M_H$ in this case so that $\underline{A}_Z(\mathbb{R})_2/F_M$ consists in fact of two elements. Hence $\mathcal{W} = \{1, w\}$ has two elements in this case. If we were to consider disconnected $\mathrm{PGL}(2,\mathbb{R})$ instead of $G_o = \mathrm{SL}(2,\mathbb{R})$, then $\mathcal{W} = \{1\}$ would be trivial.

2.2 Spherical roots and polar decomposition

Let $K \subset G$ be a maximal compact subgroup associated to a Cartan involution θ of $\mathfrak{g}$ with $\theta(X) = -X$ for all $X \in \mathfrak{a}$. Furthermore, let κ be an $\operatorname{Ad} G$ and θ-invariant bilinear form on $\mathfrak{g}$ such that the quadratic form $X \mapsto \|X\|^2 = -\kappa(X, \theta X)$ is positive definite. We will denote by $(\cdot, \cdot)$ the corresponding scalar product on $\mathfrak{g}$. It defines a quotient scalar product and a quotient norm on $\mathfrak{a}_Z$ that we still denote by $\|\cdot\|$.

For later reference, we record that K is algebraic, i.e., $K = \underline{K}(\mathbb{R})$, and further, $M \subset K$ as we requested $\theta\big|_{\mathfrak{a}} = -\mathrm{id}_{\mathfrak{a}}$.

Let Σ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. If $\alpha \in \Sigma$, let $\mathfrak{g}^\alpha$ be the corresponding weight space for $\mathfrak{a}$. We write $\Sigma_{\mathfrak{u}}$ (resp. $\Sigma_{\mathfrak{n}}$) $\subset \Sigma$ for the set of $\mathfrak{a}$-roots in $\mathfrak{u}$ (resp. $\mathfrak{n}$) and set $\mathfrak{u}^- = \sum_{\alpha \in \Sigma_{\mathfrak{u}}} \mathfrak{g}^{-\alpha}$, i.e., the nilradical of the parabolic subalgebra $\mathfrak{q}^-$ opposite to $\mathfrak{q}$ with respect to $\mathfrak{a}$.

Let $(\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$ be the orthogonal complement of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{l}$ with respect to the scalar product $(\cdot, \cdot)$. One has:

$$\mathfrak{g} = \mathfrak{h} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \oplus \mathfrak{u}\,. \tag{2.6}$$

Let T be the restriction to $\mathfrak{u}^-$ of minus the projection from $\mathfrak{g}$ onto $(\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \oplus \mathfrak{u}$ parallel to $\mathfrak{h}$. Let $\alpha \in \Sigma_{\mathfrak{u}}$ and $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Then (cf. [31,equation (3.3)])

$$T(X_{-\alpha}) = \sum_{\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}} X_{\alpha,\beta}\,, \tag{2.7}$$

with $X_{\alpha,\beta} \in \mathfrak{g}^\beta \subset \mathfrak{u}$ if $\beta \in \Sigma_{\mathfrak{u}}$ and $X_{\alpha,0} \in (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$.

Let $\mathcal{M} \subset \mathbb{N}_0[\Sigma_{\mathfrak{u}}]$ be the monoid generated by:

$$\{\alpha + \beta \mid \alpha \in \Sigma_{\mathfrak{u}}, \beta \in \Sigma_{\mathfrak{u}} \cup \{0\} \text{ such that there exists } X_{-\alpha} \in \mathfrak{g}^{-\alpha} \text{ with } X_{\alpha,\beta} \neq 0\}. \tag{2.8}$$

The elements of $\mathcal{M}$ vanish on $\mathfrak{a}_H$ so $\mathcal{M}$ is identified with a subset of $\mathfrak{a}_Z^*$. We define

$$\begin{aligned} \mathfrak{a}_Z^{--} &= \{X \in \mathfrak{a}_Z \mid \alpha(X) < 0,\, \alpha \in \mathcal{M}\} \\ \text{and} \quad \mathfrak{a}_Z^{-} &= \{X \in \mathfrak{a}_Z \mid \alpha(X) \leq 0,\, \alpha \in \mathcal{M}\}. \end{aligned}$$

Following e.g., [31,Section 3.2], we recall that $\mathfrak{a}_Z^-$ is a co-simplicial cone, and our choice of spherical roots S consists of the irreducible elements of $\mathcal{M}$, which are extremal in $\mathbb{R}_{\geq 0}\mathcal{M}$. Here, an element of $\mathcal{M}$ is called irreducible if it cannot be expressed as a sum of two nonzero elements in $\mathcal{M}$.

Example 2.9. (Triple case continued) In the triple case of Example 2.5 we had $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2 \times \mathfrak{a}_3$ and accordingly $\Sigma_{\mathfrak{a}} = \Sigma_1 \amalg \Sigma_2 \amalg \Sigma_3$ with $\Sigma_i = \{\pm\alpha_i\}$ and α_i corresponding to $N_i \subset P_i \subset G_o$. Let now $0 \neq Y \in \mathfrak{g}_o^{-\alpha_1}$ and expand it as $Y = Y_0 + Y^+ + Y^-$ for $Y_0 \in \mathfrak{a}_i$ and $Y^\pm \in \mathfrak{g}_o^{\pm\alpha_i}$ for $i = 2, 3$. Then a simple computation shows that $Y_0 \neq 0$ and thus $\alpha_1 \in \mathcal{M}$. Likewise we obtain $\alpha_2, \alpha_3 \in \mathcal{M}$ and thus $S = \{\alpha_1, \alpha_2, \alpha_3\}$.

Later, we will also need the edge of $\mathfrak{a}_Z$

$$\mathfrak{a}_{Z,E} := \mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-) = \{X \in \mathfrak{a}_Z \mid \alpha(X) = 0,\, \alpha \in S\}\,.$$

Note that $\mathfrak{a}_{Z,E}$ (more precisely $\mathbf{s}(\mathfrak{a}_{Z,E})$) normalizes $\mathfrak{h}$ and, likewise, $A_{Z,E} := \exp(\mathfrak{a}_{Z,E}) \subset A_Z$.

We turn to the polar decomposition for Z. Set $A_Z^- := \exp(\mathfrak{a}_Z^-)$ and $A_{Z,\mathbb{R}}^- = A_{Z,2}A_Z^- \subset A_{Z,\mathbb{R}}$. By the definition of $\mathcal{W}$, we then record that

$$A_{Z,\mathbb{R}}^- = A_Z^- \mathcal{W} \cdot z_0\,.$$

Lemma 2.10 (Polar decomposition). There exists a compact subset $\Omega \subset G$ such that

$$Z = \Omega A_{Z,\mathbb{R}}^-\,. \tag{2.9}$$

Proof. Recall the group of 2-torsion points $\underline{A_Z}(\mathbb{R})_2$ of $\underline{A_Z}(\mathbb{R})$. According to [25,Theorem 13.2 with Remark 13.3(ii)] (building up on the earlier work [28,Theorem 5.13]), we have $\underline{Z}(\mathbb{R}) = \Omega \cdot \underline{A_Z}(\mathbb{R})_2 A_Z^-$, for some compact subset Ω of G. Note that $A_Z^- \underline{A_Z}(\mathbb{R})_2 \cap Z = A_{Z,\mathbb{R}}^-$ and the assertion follows. ■

Remark 2.11 (Passage to H connected). An analytically more general setup would be to work with connected H, i.e., with $Z_0 = G/H_0$ instead of $Z = G/H$. For that, only some adjustments are needed. In detail, by right-enlarging $\mathcal{W}$ with a set F_H of representatives for $H/H_0(H \cap M)$, we obtain with $\mathcal{W}_0 := \mathcal{W}F_H$ a set, which is in bijection with the set of the open $P \times H_0$-double cosets in G. Similarly, one obtains a polar decomposition for Z_0 as $Z_0 = \Omega A_Z^- \mathcal{W}_0 \cdot z_0$ with $z_0 = H_0$, now denoting the base point of Z_0.

In order not to introduce further notation and maintain readability, the main text is kept in the algebraic framework. At various places, we will comment on the necessary adjustments needed for H connected.

The polar decomposition is closely related to compactification theory of Z, which we summarize in the next section.

Example 2.12. In some cases it is possible to have a very specific choice of Ω, for example in the group case $Z = G' \times G' / \operatorname{diag} G' \simeq G'$ one can take $\Omega = K' \times K'$. Interesting is also the triple case. Here one has in fact $Z = KA_Z$ for $K = K_o \times K_o \times K_o$ by [10,Th. 3.2], but $Z \supsetneq KA^-_{Z,\mathbb{R}}$ as a consequence of [10,Th. 4.1].

3 Boundary degenerations and quantitative geometry at infinity

For a real spherical subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and any subset $I \subset S$, there is natural deformation of $\mathfrak{h}_I$ of $\mathfrak{h}$, see (3.1) below for the straightforward definition. We define $H_I = \langle \exp(\mathfrak{h}_I) \rangle$ as the analytic subgroup of G with Lie algebra $\mathfrak{h}_I$ and define the boundary degenerations of Z as $Z_I := G/H_I$. Let us mention that Z_I identifies (up to cover) with an open cone-subbundle in the normal bundle of a certain G-boundary orbit in a smooth compactification of Z. This more elaborate point of view will be taken in the forthcoming work [12], but is not the topic of this paper.

The compactification theory is reviewed here shortly in Section 3.3, but only as a tool to give a short proof of Proposition 3.1, which is the main result of this section. In more detail, let $\mathcal{W}_I$ be the set of open P-orbits in the deformed space Z_I. We introduce a natural matching map $\mathbf{m} : \mathcal{W}_I \to \mathcal{W}$ for open P-orbits. The definition of $\mathbf{m}$ involves certain sequences and the contents of Proposition 3.1 is about the rapid (i.e., exponentially fast) convergence of these sequences.

3.1 Boundary degenerations of Z

Let I be a subset of S and set:

$$\begin{aligned} \mathfrak{a}_I &= \{X \in \mathfrak{a}_Z \mid \alpha(X) = 0,\, \alpha \in I\}, & A_I &= \exp(\mathfrak{a}_I) \subset A_Z\,, \\ \mathfrak{a}_I^{--} &= \{X \in \mathfrak{a}_I \mid \alpha(X) < 0,\, \alpha \in S \setminus I\}, & A_I^{--} &= \exp(\mathfrak{a}_I^{--})\,. \end{aligned}$$

Then there exists an algebraic Lie subalgebra $\mathfrak{h}_I$ of $\mathfrak{g}$ such that, for all $X \in \mathfrak{a}_I^{--}$, one has:

$$\mathfrak{h}_I = \lim_{t \to +\infty} e^{\operatorname{ad} tX} \mathfrak{h} \tag{3.1}$$

in the Grassmannian $\mathrm{Gr}_d(\mathfrak{g})$ of $\mathfrak{g}$, where $d := \dim(\mathfrak{h})$ (cf. [31,equation (3.9)]).

Notice that $\tilde{\mathfrak{a}}_I$ normalizes $\mathfrak{h}_I$, and hence

$$\widehat{\mathfrak{h}}_I := \mathfrak{h}_I + \tilde{\mathfrak{a}}_I$$

defines a subalgebra of $\mathfrak{g}$ that does not depend on the section $\mathbf{s}$.

Let H_I be the analytic subgroup of G with Lie algebra $\mathfrak{h}_I$ and set $Z_I = G/H_I$. Then Z_I is a real spherical space for which:

(i) PH_I is open,
(ii) Q is Z_I-adapted to P,
(iii) $\mathfrak{a}_{Z_I} = \mathfrak{a}_Z$ and $\mathfrak{a}_{Z_I}^- = \{X \in \mathfrak{a}_Z \mid \alpha(X) \le 0,\ \alpha \in I\}$ contains $\mathfrak{a}_Z^-$

(cf. [31,Proposition 3.2]). Similarly to (2.6), one has:

$$\mathfrak{g} = \mathfrak{h}_I \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_\mathfrak{l}} \oplus \mathfrak{u}\,.$$

Let $T_I : \mathfrak{u}^- \to (\mathfrak{l} \cap \mathfrak{h})^{\perp_\mathfrak{l}} \oplus \mathfrak{u}$ be the restriction to $\mathfrak{u}^-$ of minus the projection of $\mathfrak{g}$ onto $(\mathfrak{l} \cap \mathfrak{h})^{\perp_\mathfrak{l}} \oplus \mathfrak{u}$ parallel to $\mathfrak{h}_I$. Furthermore, let $\langle I \rangle \subset \mathbb{N}_0[S]$ be the monoid generated by I. Let $X_{\alpha,\beta}^I = X_{\alpha,\beta}$ if $\alpha + \beta \in \langle I \rangle$ and zero otherwise. It follows from [31,equation (3.12)] that $X_{-\alpha} + \sum_{\beta \in \Sigma_\mathfrak{u} \cup \{0\}} X_{\alpha,\beta}^I \in \mathfrak{h}_I$. This implies that, for $\alpha \in \Sigma_\mathfrak{u}$,

$$T_I(X_{-\alpha}) = \sum_{\beta \in \Sigma_\mathfrak{u} \cup \{0\}} X_{\alpha,\beta}^I\,.$$

Let $A_{Z_I}^- = \exp \mathfrak{a}_{Z_I}^-$. Similarly to Z, the real spherical space Z_I has a polar decomposition:

$$Z_I = \Omega_I A_{Z_I}^- \mathcal{W}_I \cdot z_{0,I}\,, \tag{3.2}$$

where $z_{0,I} = H_I$, $\Omega_I \subset G$ compact and $\mathcal{W}_I \subset G$ finite (cf. Lemma 3 and Remark 5 for the choice of $\mathcal{W}_I$ as H_I is defined to be connected). In more detail, the Lie algebra $\mathfrak{h}_I$ is algebraic and we let $\underline{H}_I$ be the corresponding connected algebraic subgroup of $\underline{G}$. Using

Lemma 2 applied to the real spherical space $\underline{G}(\mathbb{R})/\underline{H}_I(\mathbb{R})$, we can make, using Remark 5, a choice for $\mathcal{W}_I$ such that elements $w_I \in \mathcal{W}_I$ are of the form

$$w_I = \tilde{t}_I h_I, \quad \text{for some } \tilde{t}_I \in \exp(i\tilde{\mathfrak{a}}_Z) \text{ and } h_I \in \underline{H}_I(\mathbb{C})\,. \tag{3.3}$$

We fix such a choice in the following, requesting in addition that $1 \in \mathcal{W}_I$.

3.2 Quantitative escape to infinity

Let $I \subset S$. Let us pick $X_I \in \mathfrak{a}_I^{--}$, i.e., $X_I \in \mathfrak{a}_I$ and $\alpha(X_I) < 0$ for all $\alpha \in S \setminus I$. For $s \in \mathbb{R}$, let

$$a_s := \exp(sX_I)\,. \tag{3.4}$$

Fix $w_I = \tilde{t}_I h_I \in \mathcal{W}_I$. According to [31,Lemma 3.9], there exists $w \in \mathcal{W}$ (uniquely determined by X_I) and $s_0 > 0$ with

$$Pw_I\tilde{a}_s H = PwH, \quad s \geq s_0\,. \tag{3.5}$$

Note that (cf. Lemma 2):

$$w = \tilde{t}h, \quad \text{for some } \tilde{t} \in \exp(i\tilde{\mathfrak{a}}_Z) \text{ and } h \in \underline{H}(\mathbb{C})\,. \tag{3.6}$$

A priori, w might depend on X_I, say $w(X_I)$. On the other hand, the limit (3.1) is locally uniform in compact subsets of $\mathfrak{a}_I^{--}$. In particular, the set of $Y \in \mathfrak{a}_I^{--}$ such that $w(Y) = w(X_I)$ is open and closed. Hence, w is independent of X_I. Given $w_I \in \mathcal{W}_I$ and $w \in \mathcal{W}$ such that (3.5) holds, we say that w *corresponds to* w_I and note that this correspondence sets up a natural map $\mathbf{m} : \mathcal{W}_I \to \mathcal{W}$.

According to [31,Lemma 3.10], there exist elements $u_s \in U$, $b_s \in A_Z$ and $m_s \in M$ such that:

$$\begin{array}{rcl} w_I\tilde{a}_s \cdot z_0 & = & u_s m_s \tilde{b}_s w \cdot z_0 \quad s \geq s_0\,, \\ \lim\limits_{s\to+\infty} (a_s b_s^{-1}) & = & 1, \\ \lim\limits_{s\to+\infty} u_s & = & 1\,, \\ \lim\limits_{s\to+\infty} m_s & = & m_{w_I}, \text{for some} m_{w_I} \in M\,. \end{array} \tag{3.7}$$

Notice that in case $w_I = 1$ we have $w = 1$ by (3.5) and our request that $1 \in \mathcal{W}$; also note $m_{w_I} = 1$ for $w_I = 1$.

The goal of this section is to give a quantitative version of the convergence in (3.7). For that, we first refer to Appendix A for the definition and basic properties of rapid convergence.

Recall the finite 2-group $F_M = \widehat{M}_H/M_H$ defined before (2.4) and fix with $\widetilde{F_M} \subset M$ a set of its representatives containing 1. Then we have the following result:

Proposition 3.1. The families $(a_s b : s^{-1})$ and (u_s) converge rapidly to 1 and one can choose the family (m_s) such that (m_s) converges rapidly to $m_{w_I} \in \widetilde{F_M}$.

Remark 3.2. (a) Proposition 3.1 allows us to change the representatives w_I to $m_{w_I}^{-1} w_I$ without losing the special form $w_I = \tilde{t}_I h_I$ with $\tilde{t}_I \in \exp i\tilde{\mathfrak{a}}_Z$. This is because of $F_M A_Z \subset A_{Z,\mathbb{R}} \subset \exp(i\tilde{\mathfrak{a}}_Z)A \cdot z_0$ Hence, we may and will assume in the sequel that $m_{w_I} = 1$ for all $w_I \in \mathcal{W}_I$.

(b) For H replaced by connected H_0, Proposition 3.1 stays valid with $\widetilde{F_M}$ right-enlarged by representatives of the component group $M_H/(M \cap H_0)$. However, this causes that we possibly cannot take $m_{w_I} = 1$ as in (a).

(c) In order to give a shorter proof of Proposition 3.1, we use the compactification theory of $\underline{Z}(\mathbb{R})$, which we review in the next paragraph. In particular, it yields the framework to consider $\widehat{z}_{0,I} := \lim_{s\to\infty} \tilde{a}_s \cdot z_0$ as an appropriate rapid limit in a suitable smooth compactification of Z.

Geometrically, compactification theory provides (up to cover) a first order approximation of Z_I to Z at the vertex $\widehat{z}_{0,I}$ at infinity. This first order approximation then yields readily $u_s \to 1$ rapidly and $m_s \to m_{w_I} \in \widetilde{F_M}$ rapidly. However, first order approximation can only give $a_s b : s^{-1} \to 1$ and to show that $a_s b_s^{-1} \to 1$ indeed rapidly, we need to use finer tools from finite dimensional representation theory.

3.3 Smooth equivariant compactifications

By an equivariant compactification of $\underline{Z}(\mathbb{R})$, we understand here a $\underline{G}$-variety $\widehat{\underline{Z}}$, defined over $\mathbb{R}$, such that $\widehat{\underline{Z}}(\mathbb{R})$ is compact and contains $\underline{Z}(\mathbb{R})$ as an open dense subset. In this context, we denote by ∂Z the boundary of Z in $\widehat{\underline{Z}}(\mathbb{R})$.

Suitable (i.e., smooth and equivariant) compactifications exist:

Proposition 3.3. Let $\underline{Z} = \underline{G}/\underline{H}$ be an algebraic real spherical space. Then there exists a smooth equivariant compactification $\widehat{\underline{Z}}(\mathbb{R})$ of $\underline{Z}(\mathbb{R})$ with the following property: for all $I \subset S$ and $X \in \mathfrak{a}_I^{--}$, the limit $z_X := \lim_{s\to\infty}(\exp(sX) \cdot z_0)$ exists in ∂Z and the convergence is rapid. If $\mathfrak{h}_X$ is the stabilizer Lie subalgebra of z_X in $\mathfrak{g}$, then $\mathfrak{h}_I \subset \mathfrak{h}_X \subset \widehat{\mathfrak{h}}_I$.

The proof of this result is implicit in the proof of [25,Theorem 13.2]. Since the constructive proof is of relevance for us, we allow ourselves to repeat the fairly short proof.

Proof. The starting point is the local structure theorem for the open $\underline{P}$-orbit on $\underline{Z}$ as in (2.4)

$$(\underline{P} \cdot z_0)(\mathbb{R}) = U \times \left[[M/M_H] \times^{F_M} \underline{A_Z}(\mathbb{R}) \right]. \tag{3.8}$$

One of the main results in [25], see loc.cit., Theorem 7.1, was that the compactification theory of $\underline{Z}$ can be reduced, via the local structure theorem, to the partial toric compactification theory of $\underline{A_Z}$. Let us be more precise and denote by Ξ the character group of $\underline{A_Z}$. Note that $\Xi \simeq \mathbb{Z}^n$ with $n = \dim \underline{A_Z}$. If we denote by $\mathcal{N}$ the co-character group of $\underline{A_Z}$, then there is a natural identification of $\mathfrak{a}_Z$ with $\mathcal{N} \otimes_{\mathbb{Z}} \mathbb{R}$. Further, the compression cone $\mathfrak{a}_Z^-$ identifies as a co-simplicial cone (in [25], one uses the rational valuation cone, denoted $\mathcal{Z}_k(X)$: take $k = \mathbb{R}$ and $X = \underline{Z}$. Then $\mathfrak{a}_Z^- = \mathbb{R} \otimes_{\mathbb{Q}} \mathcal{Z}_k(X)$). The set of spherical roots $S \subset \Xi$ are then the primitive (in Ξ) extremal elements, co-spanning $\mathfrak{a}_Z^-$. Best possible compactifications (a.k.a. wonderful compactifications) exist when $\#S = \dim \mathfrak{a}_Z$ and S is a basis of the lattice Ξ. In general, this is not satisfied and we proceed as follows: we choose a complete fan $\mathcal{F} \subset \mathfrak{a}_Z$, supported in $\mathfrak{a}_Z^-$, which is generated by simple simplicial cones $C_1, \ldots, C_N$, i.e.,

- $\bigcup C_i = \mathfrak{a}_Z^-$,
- $C_i \cap C_j$ is a face of both C_i and C_j for all $1 \leq i, j \leq N$,
- $C_i = \{X \in \mathfrak{a}_Z \mid d\psi_{ij}(X) \leq 0, 1 \leq j \leq n\}$ for $(\psi_{ij})_{1 \leq j \leq n}$ a basis of the lattice Ξ.

For the existence of such a subdivision, we refer to [9,Th. 11.1.9]. Now, attached to the fan $\mathcal{F}$, we construct the toric variety $\underline{A_Z}(\mathcal{F})$ expanding $\underline{A_Z}$ along $\mathcal{F}$. Note that the toric variety $\underline{A_Z}(\mathcal{F})$ is smooth, as the fan consists of simple cones (third bulleted property). Thus, we obtain a smooth variety

$$\underline{Z}_0(\mathcal{F}) := \underline{U} \times \left[[\underline{M}/\underline{M_H}] \times^{F_{\underline{M}}} \underline{A_Z}(\mathcal{F}) \right], \tag{3.9}$$

which can be enlarged to a smooth $\underline{G}$-variety $\underline{Z}(\mathcal{F}) := \underline{G} \cdot \underline{Z}_0(\mathcal{F})$, containing $\underline{Z}_0(\mathcal{F})$ as an open subset. This is the content of [25,Theorem 7.1]. Now, set $\widehat{\underline{Z}}(\mathbb{R}) := \underline{Z}(\mathcal{F})(\mathbb{R})$ and note that $\widehat{\underline{Z}}(\mathbb{R})$ is compact by [25,Corollary 7.12] as $\mathcal{F}$ was assumed to be complete.

We now claim that the limit $\lim_{s \to \infty}(\exp(sX) \cdot z_0)$ exists in $\underline{A_Z}(\mathcal{F})(\mathbb{R})$ and that the convergence is rapid. For that, we pick a cone C_i, which contains X, and let $\mathcal{F}_i$ be the complete fan supported in C_i, which is generated by C_i. Notice that $\underline{A}_{\underline{Z}}(\mathcal{F}_i)(\mathbb{C}) \simeq \mathbb{C}^n$ is open in $\underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{C})$. More specifically, the embedding of $\underline{A}_{\underline{Z}}(\mathbb{C}) \hookrightarrow \mathbb{C}^n$ is obtained by

$$\underline{A}_{\underline{Z}}(\mathbb{C}) \ni a \mapsto (\psi_{ij}(a))_{1\leq j\leq n} \in (\mathbb{C}^*)^n \subset \mathbb{C}^n\,. \tag{3.10}$$

Given the definition of C_i as the negative dual cone to the ψ_{ij}'s, $j = 1, \ldots, n$, the claim now follows.

Note that the stabilizer of $z_s := \exp(sX)\cdot z_0$ in G is given by $H_s := \exp(sX)H\exp(-sX)$ with Lie algebra $\mathfrak{h}_s := e^{s\,\mathrm{ad}\,X}\mathfrak{h}$. Since $z_s \to z_X$ in the smooth manifold $\widehat{\underline{Z}}(\mathbb{R})$, we obtain that the vector fields corresponding to elements of $\lim_{s\to\infty}\mathfrak{h}_s = \mathfrak{h}_I$ vanish at z_X. This shows that $\mathfrak{h}_I \subset \mathfrak{h}_X$. Finally the property $\mathfrak{h}_X \subset \widehat{\mathfrak{h}_I}$ is derived from [25,Theorem 7.3]. ■

We end this subsection with further remarks and explanations of the construction in the proof above.

Remark 3.4. (a) It is quite instructive to consider the special case of $\underline{Z} = \underline{G} = \underline{A}$. Here $A_Z^- = A_Z = A = A_{Z,E}$ with $S = \emptyset$. Upon identifying $\mathfrak{a}_Z$ with $\mathbb{R}^n$ via the character lattice Ξ, there are two standard choices for the complete fan $\mathcal{F}$ generated by the cones $C_1, \ldots, C_N$. The first one is for $N = 2^n$ and the cones given by the orthants: $C_\sigma = \sigma(\mathbb{R}_{\geq 0})^n$ for $\sigma \in \{-1, 1\}^n$. This fan leads to $\underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{R})^n$, the n-fold copy of the projective line. The other standard choice is obtained via the identification $\mathbb{R}^n \simeq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{e}$ with $\mathbf{e} = e_1 + \ldots + e_{n+1}$, where $(e_1, \ldots, e_{n+1})$ is the canonical basis of $\mathbb{R}^{n+1}$, and has $N = n+1$ cones given by:

$$C_i = [(\bigoplus_{j=1\,\mathrm{s.t.}\,j\neq i}^{n+1} \mathbb{R}_{\geq 0}e_j) + \mathbb{R}\mathbf{e}]/\mathbb{R}\mathbf{e} \qquad 1 \leq i \leq n+1\,.$$

This fan leads to the projective space $\underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R}) \simeq \mathbb{P}^n(\mathbb{R})$.

(b) In the previous example, we have seen that there are exactly N fixed points for G in the compactification $\widehat{\underline{Z}}(\mathbb{R})$, paramatrized by the cones C_i and explicitly given by limits $\widehat{z}_{\emptyset,i} := \lim_{t\to\infty}(\exp(tX)\cdot z_0)$, for some $X \in \operatorname{int} C_i$. This feature is not limited to this specific example but general: the compactification $\widehat{Z}(\mathbb{R})$ features exactly N closed $\underline{G}(\mathbb{R})$-orbits through the various $\widehat{z}_{\emptyset,i}$'s. This is in contrast to wonderful compactifications [25,Def. 11.4], where one has exactly one closed orbit [25,Th. 11.1]. For wonderful compactifications, one has $\mathfrak{a}_{Z,E} = \{0\}$ and S is a basis of the lattice Ξ. If one of these two conditions fails, one is in need of a further subdivision of $\mathfrak{a}_Z^-$ into simple simplicial cones C_i.

(c) Let $X \in \mathfrak{a}_I^{--}$ and $F \in \mathcal{F}$ be the smallest face in the fan that contains X. Then $\operatorname{span}_{\mathbb{R}} F \subset \mathfrak{a}_I$ and $\mathfrak{h}_X = \mathfrak{h}_I + \operatorname{span}_{\mathbb{R}} F$. In particular, for $X \in \mathfrak{a}_I^{--}$ generic, we have $\mathfrak{h}_X = \widehat{\mathfrak{h}}_I$.

(d) (cf. [25,Section 11]) In case $\underline{H} = N_{\underline{G}}(\underline{H})$ is self-normalizing, one obtains a wonderful compactification $\widehat{\underline{Z}}(\mathbb{R})$ by closing up $\underline{Z}(\mathbb{R})$ in the Grassmannian $\mathrm{Gr}_d(\mathfrak{g})$ of $d := \dim \mathfrak{h}$-dimensional subspaces of $\mathfrak{g}$. The embedding is given by $g \cdot z_0 \mapsto \mathrm{Ad}(g)\mathfrak{h}$ and, given the definition of $\mathfrak{h}_I$ as a limit (cf. (3.1)), one derives easily that the stabilizer $\widehat{H}_I$ of $\widehat{z}_{0,I}$ in G has Lie algebra $\widehat{\mathfrak{h}}_I$.

3.4 Proof of Proposition 3.1

We choose a smooth compactification $\widehat{\underline{Z}}(\mathbb{R}) = \underline{Z}(\mathcal{F})(\mathbb{R})$ as constructed in the previous section. To begin with, we note that the limit

$$\widehat{z}_{0,I} := \lim_{s\to\infty} \tilde{a}_s \cdot z_0 \tag{3.11}$$

exists. Moreover, $\widehat{z}_{0,I} \in \underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R})$ and the convergence is rapid. Further, we deduce from the fact that $\widehat{z}_{0,I}$ is fixed by $\underline{H}_I(\mathbb{C})$ and $w_I = \tilde{t}_I h_I$ that $\lim_{s\to\infty} w_I \tilde{a}_s \cdot z_0 = \tilde{t}_I \cdot \widehat{z}_{0,I} \in \underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R})$ is rapid. On the other hand, $w_I \tilde{a}_s \cdot z_0 = u_s m_s \tilde{b}_s w \cdot z_0 = u_s m_s \tilde{b}_s \tilde{t} \cdot z_0$, which, in local coordinates as given by (3.9), translates into:

$$w_I \tilde{a}_s \cdot z_0 = (u_s, [m_s, \tilde{t}\tilde{b}_s \cdot z_0]) \in U \times \left[[M/M_H] \times^{F_M} \underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R}) \right]. \tag{3.12}$$

Since $\lim_{s\to\infty} w_I \tilde{a}_s \cdot z_0 = (1, [1, \tilde{t} \cdot \widehat{z}_{0,I}])$ is rapid, we thus deduce that $\lim_{s\to\infty} u_s = 1$ is rapid as well. Next, we use the smooth projection $[M/M_H] \times^{F_M} \underline{A}_{\underline{Z}}(\mathcal{F})(\mathbb{R}) \to M/M_H F_M$ and obtain that $m_s(M_H F_M) \xrightarrow[s\to\infty]{rapid} 1(M_H F_M) \in M/M_H F_M$. In particular, we may assume that $m_s \xrightarrow[s\to\infty]{rapid} m_{w_I} \in M_H F_M$. Notice that we are free to replace m_s by elements of the form $m_s m_H$ with $m_H \in M_H$ as we have

$$m_s m_H \tilde{b}_s w \cdot z_0 = m_s m_H \tilde{b}_s \tilde{t} \ \cdot z_0 = m_s \tilde{b}_s \tilde{t} \cdot z_0 = m_s \tilde{b}_s w \cdot z_0 \,.$$

Thus, we may even assume that $m := m_{w_I} \in \widetilde{F_M}$ (which was defined just before Proposition 3.1).

We remain with showing $b_s a_s^{-1} \xrightarrow[s\to\infty]{rapid} 1$. Using the techniques from above, it is immediate that $d(a_s, b_s) \to 0$ rapidly for any Riemannian metric d on $\widehat{\underline{Z}}(\mathbb{R})$. However, the statement $a_s^{-1} b_s \to 1$ rapidly is a considerably finer assertion and difficult to obtain working with only one compactification. Thus, we change the strategy of proof and work with (varying) finite dimensional spherical representations instead. The representations give us various morphisms of $\underline{Z}$ into affine spaces.

We assume first that $\underline{Z}$ is quasi-affine. The representations we work with are finite dimensional irreducible representations (π, V) of $\underline{G}(\mathbb{C})$ featuring two properties:

- The representation is $\underline{H}(\mathbb{C})$-spherical, that is, there exists a vector $v_H \neq 0$ such that $\pi(h)v_H = v_H$ for all $h \in \underline{H}(\mathbb{C})$.
- Each $\underline{N}(\mathbb{C})$-fixed vector is fixed by $\underline{M}(\mathbb{C})$.

We remark that the second property is equivalent to the representation being $\underline{K}(\mathbb{C})$-spherical (Cartan–Helgason theorem). In particular, each of these representations is self-dual, its highest weight λ is an element of $\mathfrak{a}^*$ and its lowest weight is given by $-\lambda$. We write Λ_Z for the set of highest weights of all $\underline{H}(\mathbb{C})$ and $\underline{K}(\mathbb{C})$-spherical irreducible representations.

Given $\lambda \in \Lambda_Z$, we let (π, V) be such an irreducible representation of $\underline{G}(\mathbb{C})$ of highest weight λ. Furthermore, we fix a highest weight vector v^* in the dual representation V^* of V. From the fact that PH is open in G, we then deduce $v^*(v_H) \neq 0$ and, in particular, $V^H = \mathbb{C}v_H$ is one-dimensional. Moreover, it follows that $\Lambda_Z \subset \mathfrak{a}_Z^*$.

We expand v_H into $\mathfrak{a}$-weight vectors

$$v_H = \sum_{\mu \in \Lambda_\pi} v_{-\lambda+\mu},$$

with $\Lambda_\pi := \{\mu \in \mathfrak{a}^* \mid v_{-\lambda+\mu} \neq 0\}$. As v_H is $\mathfrak{a}_H$-fixed, we have $\Lambda_\pi \subset \mathfrak{a}_Z^*$ and, by [28,Lemma 5.3], we obtain:

$$\mu\big|_{\mathfrak{a}_Z^{--}} < 0, \qquad \mu \in \Lambda_\pi \backslash \{0\}. \tag{3.13}$$

Set

$$v_{H,s} := a_s^\lambda \pi(\tilde{a}_s) v_H \qquad s \geq 0$$

and note, as v_H is H-invariant, that this expression is independent of the choice of the particular section $\mathbf{s}$. From the definition, we then get

$$v_{H,s} = \sum_{\mu \in \Lambda_\pi} a_s^\mu v_{-\lambda+\mu} \,. \tag{3.14}$$

If we define

$$v_{H,I} := \sum_{\mu \in \Lambda_\pi \, \text{s.t.} \, \mu(X_I)=0} v_{-\lambda+\mu} \,,$$

then it is immediate from (3.13) and (3.14) that

$$v_{H,s} \to v_{H,I} \qquad \text{rapidly for } s \to \infty \,. \tag{3.15}$$

Recall $v^* \in V^*$, a highest weight vector in the dual representation. Then we obtain from $w_I \tilde{a}_s \cdot z_0 = u_s m_s \tilde{b}_s \tilde{t} \cdot z_0$ with $t = \tilde{t} \cdot z_0 \in A_{Z,2}$ (cf. Lemma 2) that:

$$v^*(\pi(w_I) v_{H,s}) = a_s^\lambda \left(v^*(\pi(u_s m_s \tilde{b}_s \tilde{t}) v_H) \right) = (a_s b_s^{-1})^\lambda t^{-\lambda} \left(v^*(v_H) \right) \,. \tag{3.16}$$

By (3.15), we thus obtain from (3.16) that:

$$(a_s b_s^{-1})^\lambda = t^\lambda \frac{v^*(\pi(w_I) v_{H,s})}{v^*(v_H)} \to t^\lambda \frac{v^*(\pi(w_I) v_{H,I})}{v^*(v_H)} \qquad \text{rapidly for } s \to \infty \,. \tag{3.17}$$

We now employ [31,Lemma 3.10] for the simple convergence $a_s b_s^{-1} \to 1$. Thus, (3.17) implies $t^\lambda \frac{v^*(\pi(w_I) v_{H,I})}{v^*(v_H)} = 1$ with

$$(a_s b_s^{-1})^\lambda \to 1 \qquad \text{rapidly for} s \to \infty, \, \lambda \in \Lambda_Z \,. \tag{3.18}$$

Assume for the moment that $\underline{Z}$ is quasi-affine. We claim that the set Λ_Z spans $\mathfrak{a}_Z^*$. In fact, this is as a consequence of [30,Lemma 3.4 and (3.2)] as in the notation of op. cit. each $f \in \mathbb{C}(Z)_\chi$ is a quotient $f = f_1/f_2$ for some $f_i \in \mathbb{C}[Z]_{\chi_i}$ corresponding to characters Λ_Z in the notation of this article. From the claim and (3.18) we then get $a_s b_s^{-1} \to 1$ rapidly.

If $\underline{Z}$ is not quasi-affine, then matters are reduced to the quasi-affine case via the so-called cone construction from algebraic geometry: we extend $\underline{G}(\mathbb{C})$ to $\underline{G}'(\mathbb{C}) := \underline{G}(\mathbb{C}) \times \mathbb{C}^*$ and, for a character $\psi : \underline{H}(\mathbb{C}) \to \mathbb{C}^*$ defined over $\mathbb{R}$, we set $\underline{H}'(\mathbb{C}) := \{(h, \psi(h)) \mid h \in \underline{H}(\mathbb{C})\}$.

In this way, we obtain a real spherical space $\underline{Z}' := \underline{G}'/\underline{H}'$, which projects $\underline{G}'$-equivariantly onto $\underline{Z}$. According to [25,Corollary 6.10], there is compatibility of compression cones:

$$\mathfrak{a}_Z'^{-} = \mathfrak{a}_Z^{-} \oplus \mathbb{R}\,. \tag{3.19}$$

Furthermore, according to Chevalley's quasiprojective embedding theorem for homogeneous spaces [21,Sect. 11.2], we find such a ψ such that $\underline{Z}'$ is quasi-affine and we complete the reduction to the quasi-affine case as follows: we lift the identity (3.6) to Z' and obtain

$$w_I \tilde{a}_s' \cdot z_0' = u_s m_s \tilde{b}_s' w \cdot z_0' \quad s \geq s_0\,,$$

with $\tilde{a}_s' \in \tilde{a}_s(1 \times \mathbb{R}^\times) \in G'$ and likewise for $\tilde{b}_s' \in \tilde{b}_s(1 \times \mathbb{R}^\times) \in G'$. Because of (3.19), we obtain the rapid convergence $b_s'(a_s')^{-1} \to 1$ in the quasi-affine environment of Z'. Projecting to Z then completes this final reduction step.

4 Z-tempered H-fixed continuous linear forms and the space $\mathcal{A}_{temp}(Z)$

In this section, we introduce the function space $\mathcal{A}_{temp}(Z)$ of tempered $\mathcal{Z}(\mathfrak{g})$-eigenfunctions on Z. Via Frobenius reciprocity, these functions can naturally be interpreted as matrix coefficients of smooth representations of G, which are of moderate growth (*SF*-representations for short). This section starts with a brief digression on *SF*-representations and then provides the definition of $\mathcal{A}_{temp}(Z)$.

4.1 *SF*-representations of G

Let us recall some definitions and results of [4].

A continuous representation (π, E) of a Lie group G on a locally convex complex topological vector space E is a representation such that the map:

$$G \times E \to E,\ (g, v) \mapsto \pi(g)v, \text{is continuous.}$$

If R is a compact subgroup of G and $v \in E$, we say that v is R-finite if $\pi(R)v$ generates a finite dimensional subspace of E. Let $V_{(R)}$ denote the vector space of R-finite vectors in

E. Let η be a continuous linear form on E and $v \in E$. Let us define the generalized matrix coefficient associated to η and v by:

$$m_{\eta,v}(g) := \langle \eta, \pi(g^{-1})v \rangle, \quad g \in G. \tag{4.1}$$

Let G be a real reductive group and $\|\cdot\|$ be a norm on G (cf. [41,Section 2.A.2] or [4,Section 2.1.2]). A continuous representation (π, E) of G is called a *Fréchet representation with moderate growth* if E is a Fréchet space and if, for any continuous semi-norm p on E, there exist a continuous semi-norm q on E and $N \in \mathbb{N}$ such that:

$$p(\pi(g)v) \leq q(v)\|g\|^N, \quad v \in E, g \in G. \tag{4.2}$$

This notion coincides with the notion of F-representations given in [4,Definition 2.6] for the large scale structure corresponding to the norm $\|\cdot\|$. We will adopt the terminology of F-representations.

Let (π, E) be an F-representation. A smooth vector in E is a vector v such that $g \mapsto \pi(g)v$ is smooth from G to E. The space E^∞ of smooth vectors in E is endowed with the Sobolev semi-norms that we define now. Fix a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$ and $k \in \mathbb{N}$. Let p be a continuous semi-norm on E and set

$$p_k(v) = \left(\sum_{m_1+\cdots+m_n \leq k} p(\pi(X_1^{m_1} \cdots X_n^{m_n})v)^2 \right)^{1/2}, \quad v \in E^\infty. \tag{4.3}$$

We endow E^∞ with the topology defined by the semi-norms p_k, $k \in \mathbb{N}$, when p varies in the set of continuous semi-norms of E, and denote by (π^∞, E^∞) the corresponding sub-representation of (π, E).

An SF-representation is an F-representation (π, E), which is smooth, i.e., such that $E = E^\infty$ as topological vector spaces. Let us remark that if (π, E) is an F-representation, then (π^∞, E^∞) is an SF-representation (cf. [4,Corollary 2.16]).

Recall our fixed maximal compact subgroup $K \subset G$.

Following [4], we call an SF-representation E *admissible* provided that $E_{(K)}$ is a Harish-Chandra module with respect to the pair $(\mathfrak{g}, K)$, that is, a $(\mathfrak{g}, K)$-module with finite K-multiplicities, which is finitely generated as a $\mathcal{U}(\mathfrak{g})$-module.

An admissible SF-representation will be called an SAF-representation of G.

It is a fundamental theorem of Casselman–Wallach (cf. [8], [42,Chapter 11] or [4]) that every Harish-Chandra module V admits a unique SF-completion V^∞, i.e., an

SF-representation V^∞ of G, unique up to isomorphism in the SF-category, such that:

$$V^\infty_{(K)} \simeq_{(\mathfrak{g},K)} V\,.$$

In particular, all SAF-representations of G are of the form V^∞ for a Harish-Chandra module V.

4.2 The spaces $C^\infty_{temp,N}(Z)$ and $\mathcal{A}_{temp,N}(Z)$

From now on and for the remainder of this paper, we will assume that Z is unimodular. Let ρ_Q be the half sum of the roots of $\mathfrak{a}$ in $\mathfrak{u}$. Let us show that

$$\rho_Q \text{ is trivial on } \mathfrak{a}_H\,.$$

As $\mathfrak{l}\cap\mathfrak{h}$-modules,

$$\mathfrak{g}/\mathfrak{h} = \mathfrak{u}\oplus(\mathfrak{l}/\mathfrak{l}\cap\mathfrak{h})\,.$$

But the action of $\mathfrak{a}_H = \mathfrak{a}\cap\mathfrak{h}$ on $(\mathfrak{l}/\mathfrak{l}\cap\mathfrak{h})$ is trivial. Since Z is unimodular, the action of $\mathfrak{a}_H$ has to be unimodular. Our claim follows.

Hence ρ_Q can be defined as a linear form on $\mathfrak{a}_Z$.

We have the notion of weight functions on a homogeneous space X of a locally compact group G (cf. [3,Section 3.1]). This is a function $w : X \to \mathbb{R}_{>0}$ such that, for every ball B of G (i.e., a compact symmetric neighborhood of 1 in G), there exists a constant $c = c(w,B)$ such that

$$w(g\cdot x) \le cw(x), \quad g\in B, x\in X\,. \tag{4.4}$$

One sees easily that, if w is a weight function, then $1/w$ is also a weight function.

Let $\Omega\subset G$ be a compact subset in accordance with the polar decomposition in Lemma 3. Then weight function $\mathbf{v}$ and $\mathbf{w}$ on Z are defined by

$$\mathbf{v}(z) := \mathrm{vol}_Z(Bz) \quad\text{and}\quad \mathbf{w}(z) := 1 + \sup_{a\in A_Z^- \text{ s.t. } z\in\Omega a\mathcal{W}\cdot z_0} \|\log(a)\|\,,$$

where B is some ball of G and $\|\cdot\|$ refers to the quotient norm on $\mathfrak{a}_Z = \mathfrak{a}/\mathfrak{a}_H$. It is then clear that $\mathbf{v}$ is a weight function and $\mathbf{w}$ is a weight function by [29,Proposition 3.4].

Recall that the equivalence class of $\mathbf{v}$ does not depend on B (see op. cit. Lemma 4.1 and beginning of Section 3 for the definition of the equivalence relation).

For any $N \in \mathbb{N}$, we define a norm p_N on $C_c(Z)$ by

$$p_N(f) := \sup_{z \in Z} \left(\mathbf{w}(z)^{-N} \mathbf{v}(z)^{1/2} |f(z)| \right) . \tag{4.5}$$

From the polar decomposition of Z (cf. (9)), one has

$$p_N(f) = \sup_{g \in \Omega, a \in A_Z^-, w \in \mathcal{W}} \left(\mathbf{w}(gaw \cdot z_0)^{-N} \mathbf{v}(gaw \cdot z_0)^{1/2} |f(gaw \cdot z_0)| \right) .$$

From the fact that $\mathbf{v}$ and $\mathbf{w}$ are weight functions on Z and from [29,Propositions 3.4(2) and 4.3], one then sees that:

The norm p_N is equivalent to the norm

$$f \mapsto q_N(f) := \sup_{g \in \Omega, a \in A_Z^-, w \in \mathcal{W}} \left(a^{-\rho_Q} (1 + \| \log a \|)^{-N} |f(gaw \cdot z_0)| \right) . \tag{4.6}$$

Moreover, due to the fact that $\mathbf{v}$ and $1/\mathbf{w}$ are weight functions on Z, one gets that G acts by left translations on $(C_c(Z), p_N)$ and, for any compact subset C of G, by changing z into $z' = g^{-1} \cdot z$ in (4.5), one sees that:

There exists $c > 0$ such that

$$p_N(L_g f) \leq c p_N(f), \quad g \in C, f \in C_c(Z) . \tag{4.7}$$

This is in essence what is needed to identify

$$C^\infty_{temp,N}(Z) := \{f \in C^\infty(Z) \mid p_{N,k}(f) < \infty, \, k \in \mathbb{N}\} \tag{4.8}$$

as an *SF*-module for G. Here, the $p_{N,k}$, $k \in \mathbb{N}$, are as in (4.3), with p replaced by p_N and (π, E) by the *SF*-representation $(L, C^\infty_{temp,N}(Z))$. Further, we endow the increasing union $C^\infty_{temp}(Z) := \bigcup_{N \in \mathbb{N}} C^\infty_{temp,N}(Z)$, with the inductive limit topology. We call $C^\infty_{temp}(Z)$ the space of *smooth tempered functions* on Z.

Inside of $C^\infty_{temp}(Z)$, we define $\mathcal{A}_{temp}(Z)$ as the subspace of $\mathcal{Z}(\mathfrak{g})$-finite functions. Likewise we define $\mathcal{A}_{temp,N}(Z)$.

4.3 Z-Tempered functionals

Let (π, E) be an SF-representation and E' its strong dual. An element $\eta \in (E')^H$ will be called Z*-tempered* provided

$$\text{There exists } N \in \mathbb{N} \text{ such that, for all } v \in E, \text{ one has } m_{\eta,v} \in C^\infty_{temp,N}(Z). \tag{4.9}$$

The Z-tempered functionals then define a subspace $(E')^H_{temp}$ of $(E')^H$. Frobenius reciprocity then asserts for an SF-representation (π, E) the following isomorphism of vector spaces:

$$\operatorname{Hom}(E, C^\infty_{temp}(Z)) \simeq (E')^H_{temp}, \tag{4.10}$$

which can be established as in [36,Lemma 6.5] via the Grothendieck factorization theorem for topological vector spaces.

In case $E = V^\infty$ is an SAF-representation, we adopt the more common terminology $V^{-\infty} := (V^\infty)'$ and recall the finiteness result for real spherical spaces (cf. [35,Theorem 3.2]):

$$\dim (V^{-\infty})^H < \infty. \tag{4.11}$$

For a finite codimensional ideal $\mathcal{J}$ of $\mathcal{Z}(\mathfrak{g})$, let

$$\mathcal{A}_{temp,N}(Z : \mathcal{J}) := \{f \in \mathcal{A}_{temp,N}(Z) \mid f \text{ is annihilated by } \mathcal{J}\} \tag{4.12}$$

and denote by $\mathcal{A}_{temp}(Z : \mathcal{J})$ the subspace of $\mathcal{A}_{temp}(Z)$ annihilated by $\mathcal{J}$.

Proposition 4.1. There exists an $N_0 \in \mathbb{N}$ such that $\mathcal{A}_{temp}(Z : \mathcal{J}) = \mathcal{A}_{temp,N_0}(Z : \mathcal{J})$. In particular, $\mathcal{A}_{temp}(Z : \mathcal{J})$ is an SAF-representation of G.

The proof of Proposition 4.1 is preceded by two lemmas.

Lemma 4.2. There exists a Harish-Chandra module $V_{\mathcal{J}}$ annihilated by $\mathcal{J}$ such that any Harish-Chandra module annihilated by $\mathcal{J}$ is a quotient of a finite direct sum of copies of $V_{\mathcal{J}}$.

Proof. According to Harish-Chandra [14,Thm. 7], there exist only finitely many isomorphism classes $V_1, \ldots, V_n$ of irreducible Harish-Chandra modules that are annihilated by $\mathcal{J}$. We can find a finite set $F \subset \widehat{K}$ of isomorphism classes of irreducible K-representations such that, for each $1 \leqslant i \leqslant n$, the $(\mathfrak{g}, K)$-module V_i is generated by its δ-isotypic component for some $\delta \in F$. Then every Harish-Chandra module, which is annihilated by $\mathcal{J}$ is generated by the sum of its δ-isotypic components for every $\delta \in F$. Let $R(K)$ be the algebra (for convolution) of K-finite functions on K and $I_F \subset R(K)$ the ideal of elements, which acts by zero in δ for any $\delta \in F$. Let $R(\mathfrak{g}, K)$ be the "Hecke algebra" of Knapp–Vogan [22,Section I.4], i.e., the algebra of K-finite distributions on G, which are supported in K. Then $R(\mathfrak{g}, K)$ is generated as a $\mathcal{U}(\mathfrak{g})$-module (either on the left or on the right) by $R(K)$ and moreover the category of $(\mathfrak{g}, K)$-module is naturally equivalent to the category of *non-degenerate* (also called *approximately unital* by Knapp–Vogan) $R(\mathfrak{g}, K)$-modules. Setting $V_{\mathcal{J}} = R(\mathfrak{g}, K)/(R(\mathfrak{g}, K)I_F + R(\mathfrak{g}, K)\mathcal{J})$ we see that $V_{\mathcal{J}}$ is a $(\mathfrak{g}, K)$-module, which is generated by any supplement subspace of I_F in $R(K)$ and annihilated by $\mathcal{J}$. Hence, by another result of Harish-Chandra, $V_{\mathcal{J}}$ is in fact a Harish-Chandra module, see [4,Th. 4.3] for a short proof. Moreover, it is clear that any Harish-Chandra module annihilated by $\mathcal{J}$ is a quotient of a finite sum of copies of $V_{\mathcal{J}}$. ■

Lemma 4.3. Let $f \in \mathcal{A}_{temp,N}(Z)$ be a K-finite element. Set $E^f := \overline{\mathrm{span}_{\mathbb{C}} L(G)f}$, with the closure taken in $C^{\infty}_{temp,N}(Z)$. Then E^f is an SAF-representation, i.e., $E^f_{(K)}$ is a Harish-Chandra module.

Proof. We consider the $(\mathfrak{g}, K)$-module $V^f := \mathcal{U}(\mathfrak{g})f$. Since f is $\mathcal{Z}(\mathfrak{g})$-finite, the same holds for V^f. Now, as a finitely generated and $\mathcal{Z}(\mathfrak{g})$-finite module, V^f is a Harish-Chandra module by a theorem of Harish-Chandra, see [4,Th. 4.3] for a short proof. Note that E^f is an F-representation of G containing the Harish-Chandra module V^f. Hence the closure $\overline{V^f}$ in E^f is a continuous G-representation. On the other hand E^f was generated by the G-translates of f. Hence $\overline{V^f} = E^f$ and thus $V^f \simeq_{(\mathfrak{g},K)} E^f_{(K)}$. ■

Proof of Proposition 4.1 Let $E_{\mathcal{J}} = V^{\infty}_{\mathcal{J}}$ be the SAF-globalization of $V_{\mathcal{J}}$ where $V_{\mathcal{J}}$ is as in Lemma 4.2. We will actually show that $\mathcal{A}_{temp}(Z : \mathcal{J})$ is precisely the image of

$$(E'_{\mathcal{J}})^H_{temp} \otimes E_{\mathcal{J}} \to \mathcal{A}_{temp}(Z) \tag{4.13}$$

$$\eta \otimes v \mapsto m_{\eta,v}\,.$$

Indeed, since $(E'_{\mathcal{J}})^H_{temp}$ is of finite dimension (cf. (4.11)), the image of (4.13) is contained in $\mathcal{A}_{temp,N_0}(Z : \mathcal{J})$ for some $N_0 \geqslant 0$ and, by unicity of the SAF-globalization, this image is closed in $\mathcal{A}_{temp,N}(Z : \mathcal{J})$ for every $N \geqslant N_0$. Hence, it suffices to show that it is also dense in $\mathcal{A}_{temp,N}(Z : \mathcal{J})$ for every $N \geqslant N_0$. Let f be any K-finite function $f \in \mathcal{A}_{temp,N}(Z : \mathcal{J})$. By Lemma 4.3 E^f is an SAF-representation annihilated by $\mathcal{J}$ and as such a quotient of finitely many copies of $E_{\mathcal{J}}$ by Lemma 4, i.e., there exists a surjective morphism $\bigoplus_{j=1}^n E_{\mathcal{J}} \to E^f$. Hence $((E^f)')^H_{temp} \to \bigoplus_{j=1}^n (E'_{\mathcal{J}})^H_{temp}$ injects and thus every K-finite function $f \in \mathcal{A}_{temp,N}(Z : \mathcal{J})$ is in the image of (4.13). Since K-finite functions are dense in $\mathcal{A}_{temp,N}(Z : \mathcal{J})$, this completes the proof. ■

We conclude this section with an illustration of invariant functionals for our guiding example.

Example 4.4. (Triple space continued) In this case $G = G_o \times G_o \times G_o$ and Harish-Chandra modules for $(\mathfrak{g}, K)$ are of the form $V = V_1 \otimes V_2 \otimes V_3$ with V_i Harish-Chandra modules for $(\mathfrak{g}_o, K_o)$. Likewise one has $V^\infty = V_1^\infty \widehat{\otimes} V_2^\infty \widehat{\otimes} V_3^\infty$. We denote by $\widetilde{V}_3$ the Harish-Chandra module dual to V_3. Note that

$$((V^\infty)')^H = \mathrm{Hom}_{G_o}(V_1^\infty \widehat{\otimes} V_2^\infty \widehat{\otimes} V_3^\infty, \mathbb{C})$$

and thus

$$((V^\infty)')^H \simeq \mathrm{Hom}_{G_o}(V_1^\infty \widehat{\otimes} V_2^\infty, \widetilde{V}_3^\infty)$$

by a standard argument: First $\mathrm{Hom}_{G_o}(V_1^\infty \widehat{\otimes} V_2^\infty \widehat{\otimes} V_3^\infty, \mathbb{C}) \simeq \mathrm{Hom}_{G_o}(V_1^\infty \widehat{\otimes} V_2^\infty, (V_3^\infty)')$ by Grothendieck's theory of tensor products for nuclear Fréchet spaces, and then, by the use of the Grothendieck factorization theorem, deduce that the image of some $T \in \mathrm{Hom}_{G_o}(V_1^\infty \widehat{\otimes} V_2^\infty, (V_3^\infty)')$ lies in fact in some Banach completion of $\widetilde{V}_3$. Thus we see that H-invariant functionals are related to branching problems of tensor product representation $V_1^\infty \otimes V_2^\infty$ for G_o. There is a vast literature on this subject, see for instance [38] or [5].

5 Ordinary differential equation for $\mathcal{Z}(\mathfrak{g})$-eigenfunctions on Z

Let $f \in \mathcal{A}_{temp}(Z)$. The goal of this section is to show that $f\big|_{A_I}$ gives a certain system of ordinary differential equations on A_I. In more precision, f is by definition annihilated by an ideal $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g})$ of finite codimension. We construct out of f a certain vector valued function Φ_f on A_I with values in a finite dimensional vector space U_f with dimension bounded by $\dim \mathcal{Z}(\mathfrak{g})/\mathcal{J}$. The function $f\big|_{A_I}$ is then recovered by contracting Φ_f with a vector in U_f. The function Φ_f in turn satisfies a first order linear differential system recorded in (5.30).

This section starts with a basic estimate for functions $f \in C^\infty_{temp,N}(Z)$, which will be crucial in the sequel: in a nutshell, we show that derivatives in direction of $\mathfrak{h}_I$ have decreasing decay in direction of A_I^-. After that, we have a short algebraic subsection on invariant differential operators on Z, where we review in particular the contents of Appendix C. With these preparatory subsections, we then derive the differential equation (5.30) for Φ_f. From the solution formula for Φ_f in Lemma 5.7, we then derive a variety of basic growth estimates for Φ_f.

5.1 Differentiating tempered functions in direction of $\mathfrak{h}_I$

Recall the basic notions about boundary degenerations related to subsets $I \subset S$ of spherical roots. Let us fix $I \subset S$ throughout this section. We define a piecewise linear functional on $\mathfrak{a}_I$ by

$$\tilde{\beta}_I(X) = \max_{\alpha \in S \setminus I} \alpha(X), \qquad X \in \mathfrak{a}_I\,, \tag{5.1}$$

and note that $\tilde{\beta}_I(X) < 0$ if $X \in \mathfrak{a}_I^{--}$. If $a \in A_I$ with $a = \exp X$, we set $a^{\tilde{\beta}_I} = e^{\tilde{\beta}_I(X)}$.

We begin this section with a crucial estimate:

Lemma 5.1. Let $Y \in \mathfrak{h}_I$ and $N \in \mathbb{N}$. There exists a continuous semi-norm on $C^\infty_{temp,N}(Z)$, p, such that

$$|(L_Y f)(a)| \leq a^{\rho_Q + \tilde{\beta}_I}(1 + \|\log a\|)^N p(f), \quad a \in A_I^{--}, f \in C^\infty_{temp,N}(Z)\,.$$

Proof. On one hand, if $Y \in \mathfrak{l} \cap \mathfrak{h}$,

$$(L_Y f)(a) = 0, \quad a \in A_I\,.$$

Hence, the conclusion of the Lemma holds for $Y \in \mathfrak{l} \cap \mathfrak{h}$.

On the other hand, from the definition of T_I (cf. beginning of Section 3.1), $\mathfrak{l} \cap \mathfrak{h}$ and the elements

$$Y_{-\alpha} = X_{-\alpha} + T_I(X_{-\alpha}) \in \mathfrak{h}_I\,,$$

for α varying in $\Sigma_{\mathfrak{u}}$ and $X_{-\alpha}$ in $\mathfrak{g}^{-\alpha}$, generate $\mathfrak{h}_I$ as a vector space. By linearity, it then remains to get the result for $Y = Y_{-\alpha}$.

Let $a \in A_I$ and $\tilde{a} = \mathbf{s}(a)$ (cf. (2.2) for the definition of $\mathbf{s}$). Then let us show that

$$\mathrm{Ad}(\tilde{a})Y_{-\alpha} = \tilde{a}^{-\alpha}Y_{-\alpha}\,.$$

One has $\mathrm{Ad}(\tilde{a})X_{-\alpha} = \tilde{a}^{-\alpha}X_{-\alpha}$ and $\mathrm{Ad}(\tilde{a})X_{\alpha,\beta} = \tilde{a}^{\beta}X_{\alpha,\beta}$. But $\alpha+\beta \in \langle I\rangle$. Hence, $\tilde{a}^{\alpha+\beta} = 1$ as $a \in A_I$. Our claim follows.

Let us get the statement for $(L_{Y_{-\alpha}}f)(a)$, $a \in A_I^{--}$ and $f \in C_{temp,N}(Z)$. One has:

$$(L_{Y_{-\alpha}}f)(a) = (L_{\tilde{a}^{-1}}(L_{Y_{-\alpha}}f))(z_0) = \tilde{a}^{\alpha}(L_{Y_{-\alpha}}L_{\tilde{a}^{-1}}f)(z_0)\,.$$

Recall that $\mathcal{M}$ is the monoid in $\mathbb{N}_0[\Sigma_{\mathfrak{u}}]$ defined in (2.8) and $\langle I\rangle$ denotes the monoid in $\mathbb{N}_0[S]$ generated by I. Let us notice that:

$$Y_{-\alpha} + \sum_{\beta\in\Sigma_{\mathfrak{u}}\cup\{0\}\,\mathrm{s.t.}\,\alpha+\beta\notin\langle I\rangle} X_{\alpha,\beta} \in \mathfrak{h}\,.$$

Hence one has:

$$\begin{aligned}(L_{Y_{-\alpha}}f)(a) &= -\tilde{a}^{\alpha}\textstyle\sum_{\beta\in\Sigma_{\mathfrak{u}}\cup\{0\}\,\mathrm{s.t.}\,\alpha+\beta\notin\langle I\rangle}(L_{X_{\alpha,\beta}}L_{\tilde{a}^{-1}}f)(z_0)\\ &= -\textstyle\sum_{\beta\in\Sigma_{\mathfrak{u}}\cup\{0\}\,\mathrm{s.t.}\,\alpha+\beta\notin\langle I\rangle}\tilde{a}^{\alpha+\beta}(L_{\tilde{a}^{-1}}L_{X_{\alpha,\beta}}f)(z_0)\,.\end{aligned}$$

But $\tilde{a}^{\alpha+\beta} = a^{\alpha+\beta}$ as $a \in A_I \subset A_Z$ and $\alpha+\beta \in S$. Then, as $(L_{\tilde{a}^{-1}}L_{X_{\alpha,\beta}}f)(z_0) = L_{X_{\alpha,\beta}}f(a)$, one has:

$$(L_{Y_{-\alpha}}f)(a) = -\sum_{\beta\in\Sigma_{\mathfrak{u}}\cup\{0\}\,\mathrm{s.t.}\,\alpha+\beta\notin\langle I\rangle} a^{\alpha+\beta}(L_{X_{\alpha,\beta}}f)(a)\,. \tag{5.2}$$

If $\alpha+\beta \notin \langle I\rangle$ as above and $L_{X_{\alpha,\beta}}f \neq 0$, one has $\alpha+\beta \in \mathcal{M}\setminus\langle I\rangle$ and, from the definition of β_I (cf. (5.1)):

$$a^{\alpha+\beta} \le a^{\tilde{\beta}_I}, \quad a \in A_I^{--}\,.$$

Then

$$|(L_{Y_{-\alpha}}f)(a)| \le a^{\tilde{\beta}_I}\sum_{\beta\in\Sigma_{\mathfrak{u}}\cup\{0\}\,\mathrm{s.t.}\,\alpha+\beta\notin\langle I\rangle}|(L_{X_{\alpha,\beta}}f)(a)|\,.$$

Hence, we get the inequality of the Lemma for $Y = Y_{-\alpha}$ by taking

$$p = \sum_{\beta \in \Sigma_{\mathfrak{u}} \cup \{0\} \text{s.t.} \, \alpha+\beta \notin \langle I \rangle} q_{N, X_{\alpha,\beta}} ,$$

with $q_{N,X}(f) := q_N(L_X f)$. ■

5.2 Algebraic preliminaries

For a real spherical space $Z = G/H$, we denote by $\mathbb{D}(Z)$ the algebra of G-invariant differential operators. We recall the deformations $Z_I = G/H_I$ of Z, which were defined with H_I to be connected. In particular, we point out that $H_S = H_0$ and that $Z_S \to Z$ is possibly a proper covering. However, we have $\mathbb{D}(Z) \subset \mathbb{D}(Z_S)$ naturally by Remark C.2. Next we describe $\mathbb{D}(Z_I)$ as in Appendix C.

Let R denote the right regular representation of G on $C^\infty(G)$. Differentiating R yields an algebra representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}_\mathbb{C}$:

$$R : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(C^\infty(G)), \quad u \mapsto R(u) .$$

Set $\mathfrak{b} := \mathfrak{a} + \mathfrak{m} + \mathfrak{u}$ and note that $\mathfrak{b} \subset \mathfrak{g}$ is a subalgebra with $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}_I$ for all $I \subset S$. Note that $\mathfrak{b} \cap \mathfrak{h}_I = \mathfrak{a}_H + \mathfrak{m}_H$ for all $I \subset S$, where $\mathfrak{m}_H = \mathfrak{m} \cap \mathfrak{h}$. Let $\mathfrak{b}_H := \mathfrak{a}_H + \mathfrak{m}_H$. With

$$\mathcal{U}_I(\mathfrak{b}) := \{u \in \mathcal{U}(\mathfrak{b}) \mid Xu \in \mathcal{U}(\mathfrak{g})\mathfrak{h}_I, \;\; X \in \mathfrak{h}_I\} , \tag{5.3}$$

we obtain a subalgebra of $\mathcal{U}(\mathfrak{b})$, which features $\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ as a two-sided ideal. Next, we explain the natural isomorphism

$$\mathbb{D}(Z_I) \simeq \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H \tag{5.4}$$

from (C1). For that, we denote for $f_I \in C^\infty(Z_I)$ by $\tilde{f}_I \in C^\infty(G)$ its natural lift to a right H_I-invariant function in G. Then, with regard to the quotient map $\pi : \mathcal{U}(\mathfrak{b}) \to \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, we take $\tilde{u} \in \mathcal{U}(\mathfrak{b})$ to be any lift of $u \in \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H \subset \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. Then we can define

$$(R_I(u)f_I)(gH_I) := (R(\tilde{u})\tilde{f}_I)(g), \qquad g \in G ,$$

as the right hand side is independent of the particular choice of the lift $\tilde{u}$ of u and the representative g of the coset gH_I. With this notion of R_I, the isomorphism in (5.4) is implemented by the assignment

$$\mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H \ni u \to R_I(u) \in \mathbb{D}(Z_I)\,. \tag{5.5}$$

For $f \in C^\infty(Z) \subset C^\infty(Z_S)$ and $u \in \mathbb{D}(Z) \subset \mathbb{D}(Z_S)$, we use the abbreviated notation $R(u)f$ without specifying any further index.

In the sequel, we consider $\mathbb{D}(Z_I)$ as a subspace of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ for any $I \subset S$. Notice that $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ is naturally a module for A_Z under the adjoint action, which yields us a notion of $\mathfrak{a}_Z$-weights of elements $u \in \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$.

Recall the center $\mathfrak{a}_{Z,E} = \mathfrak{a}_S$ of Z, which has the property that $A_{Z,E}$ normalizes H and as such acts on Z from the right, commuting with the left G-action on Z. In particular, we obtain a natural embedding $S(\mathfrak{a}_{Z,E}) \hookrightarrow \mathbb{D}(Z)$. When applied to the real spherical space $Z_I = G/H_I$, $I \subset S$, we note that $\mathfrak{a}_I = \mathfrak{a}_{Z_I,E}$ and record the inclusion $S(\mathfrak{a}_I) \hookrightarrow \mathbb{D}(Z_I)$.

We rephrase Theorem C.5 from Appendix C as:

Lemma 5.2. For $I \subset S$, the following assertions hold:

1. For any $u \in \mathbb{D}(Z) \subset \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ and $X \in \mathfrak{a}_I^{--}$, the limit

$$\mu_I(u) := \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X} u$$

exists in the vector space $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, lies in the subspace $\mathcal{U}(\mathfrak{b})_I/\mathcal{U}(\mathfrak{b})_I\mathfrak{b}_H$ and defines via R_I (see (5.5)) and defines an injective algebra morphism $\mu_I : \mathbb{D}(Z) \to \mathbb{D}(Z_I)$, which does not depend on X.

2. For any non-zero $u \in \mathbb{D}(Z)$, the $\mathfrak{a}_Z$-weights of $\mu_I(u)$ and u are non-positive on $\mathfrak{a}_Z^-$ and the $\mathfrak{a}_Z$-weights of $\mu_I(u) - u$ are negative on $\mathfrak{a}_I^{--}$.

This Lemma shows that we can view $\mathbb{D}(Z_I)$ as a subalgebra of $\mathbb{D}(Z_\emptyset)$. Since $\mathfrak{h}_\emptyset = \mathfrak{l} \cap \mathfrak{h} + \bar{\mathfrak{u}}$ is of a particular simple shape, i.e., close to a parabolic, the algebra $\mathbb{D}(Z_\emptyset)$ can be described easily. For that, let $M_H := \exp(\mathfrak{m}_H) < M$ and keep in mind the standard isomorphim

$$\mathbb{D}(M/M_H) \simeq \mathcal{U}(\mathfrak{m})^{M_H}/(\mathcal{U}(\mathfrak{m})\mathfrak{m}_H \cap \mathcal{U}(\mathfrak{m})^{M_H})\,. \tag{5.6}$$

Lemma 5.3. The natural map

$$\Phi : S(\mathfrak{a}_Z) \otimes \left[\mathcal{U}(\mathfrak{m})^{M_H}/(\mathcal{U}(\mathfrak{m})\mathfrak{m}_H \cap \mathcal{U}(\mathfrak{m})^{M_H})\right] \to \mathcal{U}_\emptyset(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H, \quad u \otimes v \mapsto uv + \mathcal{U}(\mathfrak{b})\mathfrak{b}_H$$

is an isomorphism. In particular, via (5.4) and (5.6), we obtain a natural isomorphim of algebras

$$\mathbb{D}(Z_\emptyset) \simeq S(\mathfrak{a}_Z) \otimes \mathbb{D}(M/M_H)\,. \tag{5.7}$$

Proof. In the absolutely spherical case, this is found in [24,Section 6] (what is called X_h, the horospherical deformation of a G-variety X, would correspond to our $Z_\emptyset$). The slightly more general case is an easy adaptation. In the following proof, we replace from (5.8) onwards $H_\emptyset$ by its algebraic closure, which is legitimate by Remark C.2(b).

Recall that

$$\mathcal{U}_\emptyset(\mathfrak{b}) = \{u \in \mathcal{U}(\mathfrak{b}) \mid [X, u] \in \mathcal{U}(\mathfrak{g})\mathfrak{h}_\emptyset,\ X \in \mathfrak{h}_\emptyset\}\,.$$

In particular, $\mathcal{U}_\emptyset(\mathfrak{b})$ is ad $\mathfrak{a}$-invariant and we obtain a spectral decomposition

$$\mathcal{U}_\emptyset(\mathfrak{b}) = \sum_{\lambda \in \mathfrak{a}^*} \mathcal{U}_\emptyset(\mathfrak{b})_\lambda\,.$$

For $\lambda = 0$, we further have

$$\mathcal{U}_\emptyset(\mathfrak{b})_0 = \mathcal{U}_\emptyset(\mathfrak{b}) \cap \mathcal{U}(\mathfrak{a}+\mathfrak{m}) = \mathcal{U}(\mathfrak{a})(\mathcal{U}(\mathfrak{m})^{M_H} + \mathcal{U}(\mathfrak{m})\mathfrak{m}_H)\,,$$

from which we easily derive that Φ is injective.

It remains to be seen that Φ is surjective. For that, it suffices to show that $[\mathcal{U}_\emptyset(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H]_\lambda \simeq \mathbb{D}(Z_\emptyset)_\lambda = 0$ for $\lambda \neq 0$. To verify that, we pass to the graded level and first note that the symbol map gives an embedding

$$\operatorname{gr} \mathbb{D}(Z_\emptyset) \hookrightarrow \operatorname{Pol}(T^* Z_\emptyset)^G\,, \tag{5.8}$$

with $\operatorname{Pol}(T^*Z_\emptyset) := \mathbb{C}[T^*Z_\emptyset]$ the regular (polynomial) functions on the quasi-affine variety $T^*Z_\emptyset$. We identify the cotangent bundle $T^*Z_\emptyset$ with $G \times^{H_\emptyset} (\mathfrak{g}/\mathfrak{h}_\emptyset)^*$ and obtain $\operatorname{Pol}(T^*Z_\emptyset)^G \simeq \operatorname{Pol}((\mathfrak{g}/\mathfrak{h}_\emptyset)^*)^{H_\emptyset}$. Thus we have $\operatorname{gr}\mathbb{D}(Z_\emptyset) \subset \operatorname{Pol}((\mathfrak{g}/\mathfrak{h}_\emptyset)^*)^{H_\emptyset}$ naturally. Recall the invariant non-degenerate bilinear form κ on $\mathfrak{g}$. This form yields a G-equivariant identification of $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ and induces an $H_\emptyset$-equivariant identification of $(\mathfrak{g}/\mathfrak{h}_\emptyset)^*$ with $\mathfrak{h}_\emptyset^\perp := \{X \in$

$\mathfrak{g} \mid \kappa(X,Y)=0, Y\in\mathfrak{h}_\emptyset\}$. The proof of the Lemma will then be completed by showing that the restriction map

$$\mathrm{Pol}(\mathfrak{h}_\emptyset^\perp)^{H_\emptyset} \to \mathrm{Pol}(\mathfrak{b}_H^{\perp_{\mathfrak{m}+\mathfrak{a}}})$$

is injective, where $\mathfrak{b}_H^{\perp_{\mathfrak{m}+\mathfrak{a}}} := \{X\in\mathfrak{m}+\mathfrak{a} \mid \kappa(X,Y)=0, Y\in\mathfrak{b}_H\}$. This is now fairly standard. Note that $\mathfrak{h}_\emptyset^\perp = \mathfrak{b}_H^{\perp_{\mathfrak{m}+\mathfrak{a}}} + \bar{\mathfrak{u}}$. Next let $X = X_\mathfrak{a} + X_\mathfrak{m} \in \mathfrak{a}+\mathfrak{m}$ with $X_\mathfrak{a}\in\mathfrak{a}$ and $X_\mathfrak{m}\in\mathfrak{m}$. Suppose further that $\alpha(X_\mathfrak{a})>0$ for $\alpha\in\Sigma(\mathfrak{a},\mathfrak{u})$. Then, by a slight modification of [30,Lemma 2.5], we have

$$\mathrm{Ad}(\bar{U})X = X + [X,\bar{\mathfrak{u}}] = X + \bar{\mathfrak{u}}. \tag{5.9}$$

Now $\bar{U}\subset H_\emptyset$ and the fact that Z (and hence $Z_\emptyset$) is unimodular implies that there exists an element $X_\mathfrak{a}$ as above which lies in $\mathfrak{a}_H^{\perp_\mathfrak{a}}$ (see Lemma 5.4 below). It follows then from (5.9) that any $f\in\mathrm{Pol}(\mathfrak{h}_\emptyset^\perp)^{H_\emptyset}$ is constant in the $\bar{\mathfrak{u}}$-variable of $\mathfrak{h}_\emptyset^\perp$, i.e., the restriction map above is injective. This completes the proof of the Lemma. ■

Lemma 5.4. Let Z be a unimodular real spherical space. Then the following assertions hold:

1. Z is quasi-affine, i.e., $\underline{Z}(\mathbb{C}) = \underline{G}(\mathbb{C})/\underline{H}(\mathbb{C})$ is a quasi-affine algebraic variety.
2. There exists an $X\in\mathfrak{a}_H^{\perp_\mathfrak{a}} \simeq \mathfrak{a}_Z$ such that $\alpha(X)>0$ for all $\alpha\in\Sigma_\mathfrak{u}$.

Proof. [12,Example 12.6 and Lemma 12.7]. ■

Let us denote by $\mathfrak{Z}(Z_\emptyset)$ the center of $\mathbb{D}(Z_\emptyset)$. We then obtain from (5.7) that

$$\mathfrak{Z}(Z_\emptyset) \simeq S(\mathfrak{a}_Z)\otimes\mathfrak{Z}(M/M_H)\,. \tag{5.10}$$

We wish to describe the image of the natural map $\mathcal{Z}(\mathfrak{g}) \to \mathfrak{Z}(Z_\emptyset)\subset\mathbb{D}(Z_\emptyset)$ more closely, i.e., derive a slight extension of [24,Lemma 6.4].

In order to do so, we have to recall first the construction of the Harish-Chandra homomorphism and then relate it to the Knop homomorphism for $\mathfrak{Z}(M/M_H)$.

We begin with a short summary on the Harish-Chandra homomorphims. The natural inclusion $\mathcal{Z}(\mathfrak{g})\subset\mathcal{U}(\mathfrak{a})\mathcal{Z}(\mathfrak{m})\oplus\mathcal{U}(\mathfrak{g})\bar{\mathfrak{n}}$ yields, via projection to the first summand, an injective algebra morphism

$$\gamma_{0,\mathfrak{a}+\mathfrak{m}} : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{a})\otimes\mathcal{Z}(\mathfrak{m})\,.$$

With $\mathfrak{t} \subset \mathfrak{m}$ a maximal torus (which will be specified more closely below), we obtain with $\mathfrak{j} := \mathfrak{a} + \mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. We choose a positive system of roots $\Sigma^+(\mathfrak{j}_\mathbb{C})$ of the the root system of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{j}_\mathbb{C}$ such that the nonzero restrictions to $\mathfrak{a}$ yield the root spaces of $\mathfrak{n}$. Note that all roots are real-valued on $\mathfrak{j}_\mathbb{R} := \mathfrak{a} + i\mathfrak{t}$ and we denote by $\rho_\mathfrak{j} \in \mathfrak{j}_\mathbb{R}^*$ the corresponding half sum. Then, similar to what was just explained, we obtain, by projection along the negative $\mathfrak{m}_\mathbb{C}$-root spaces with respect to $\mathfrak{t}_\mathbb{C}$, an injective algebra morphism $\gamma_{0,\mathfrak{m}} : \mathcal{Z}(\mathfrak{m}) \to \mathcal{U}(\mathfrak{t})$. Putting matters together, we obtain with

$$\gamma_0 := (\mathrm{Id}_{S(\mathfrak{a})} \otimes \gamma_{0,\mathfrak{m}}) \circ \gamma_{0,\mathfrak{a}+\mathfrak{m}}$$

an injective algebra morphism $\gamma_0 : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{j})$. If we identify $\mathcal{U}(\mathfrak{j}) = S(\mathfrak{j})$ with the polynomials $\mathbb{C}[\mathfrak{j}_\mathbb{C}^*]$ on $\mathfrak{j}_\mathbb{C}^*$, the Harish-Chandra isomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{j})^{W_\mathfrak{j}}$ is then obtained by twisting γ_0 with the $\rho_\mathfrak{j}$-shift, i.e., $\gamma(z)(\cdot) = \gamma_0(z)(\cdot + \rho_\mathfrak{j})$ as polynomials on $\mathfrak{j}_\mathbb{C}^*$. For our purpose we are in fact more interested in the unnormalized Harish-Chandra morphism $\gamma_0 : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{j}) \simeq \mathcal{U}(\mathfrak{j})$.

Next we recall the Knop homomorphism for $\mathfrak{Z}(M/M_H)$. Set $\mathfrak{t}_H := \mathfrak{t} \cap \mathfrak{h}$ and $\mathfrak{t}_Z := \mathfrak{t}/\mathfrak{t}_H$. Note that M/M_H is affine, i.e., the complexification $M_\mathbb{C}/(M_H)_\mathbb{C}$ is an affine homogeneous space. We will request, from our choice of $\mathfrak{t}$, that the complexification of $\mathfrak{t}_Z$ is a flat for $M_\mathbb{C}/(M_H)_\mathbb{C}$, i.e., compatible with the local structure theorem (cf. [25,Theorem 4.2] applied to $Y = X = M_\mathbb{C}/(M_H)_\mathbb{C}$ and $k = \mathbb{C}$). Set $\rho_\mathfrak{m} := \rho_\mathfrak{j}\big|_{i\mathfrak{t}}$ and let W_M be the little Weyl group of the affine space $M_\mathbb{C}/(M_H)_\mathbb{C}$. Then [24,Theorem in the Introduction part(a)] yields the Knop isomorphism

$$k : \mathfrak{Z}(M/M_H) \to \mathbb{C}[\mathfrak{t}_{Z,\mathbb{C}}^* + \rho_\mathfrak{m}]^{W_M} .$$

For our purpose it is easier to work with the unnormalized Knop homomorphism, which yields us an algebra monomorphism:

$$k_0 : \mathfrak{Z}(M/M_H) \to S(\mathfrak{t})/S(\mathfrak{t})\mathfrak{t}_H$$

The important thing to notice here is that the Knop homomorphism k_0 is compatible with the unnormalized Harish-Chandra homomorphism $\gamma_{0,\mathfrak{m}} : \mathcal{Z}(\mathfrak{m}) \to S(\mathfrak{t})$ in the sense that the diagram

$$\begin{tikzcd} \mathcal{Z}(\mathfrak{m}) \arrow[r] \arrow[d, "\gamma_{0,\mathfrak{m}}"'] & \mathfrak{Z}(M/M_H) \arrow[d, "k_0"'] \\ S(\mathfrak{t}) \arrow[r] & S(\mathfrak{t})/S(\mathfrak{t})\mathfrak{t}_H\,. \end{tikzcd} \tag{5.11}$$

is commutative, see [24,Lemma 6.4]. To summarize, we obtain from (5.7), the just explained construction of the Harish-Chandra homomorphism and (5.11) an injective algebra morphism

$$j_0 : \mathfrak{Z}(Z_\emptyset) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H) \tag{5.12}$$

together with the following commutative diagram

$$\begin{tikzcd} \mathcal{Z}(\mathfrak{g}) \arrow[r] \arrow[d, "\gamma_0"'] & \mathfrak{Z}(Z_\emptyset) \arrow[d, "j_0"'] \\ S(\mathfrak{j}) \arrow[r] & S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)\,. \end{tikzcd} \tag{5.13}$$

In this diagram, the upper lower horizontal arrow is obtained from the natural $\mathcal{Z}(\mathfrak{g})$-module structure of $\mathfrak{Z}(Z_\emptyset)$ and the lower horizontal arrow is the natural projection $S(\mathfrak{j}) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)$.

Example 5.5. (Triple space continued) For the triple space $Z = G_o \times G_o \times G_o/\operatorname{diag} G_o$ we have $\mathfrak{g} = \mathfrak{g}_o \times \mathfrak{g}_o \times \mathfrak{g}_o$ and thus $\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3]$ with $\mathcal{C}_i$ the Casimir operator of the i-th $\mathfrak{g}_o$-factor in $\mathfrak{g}$. Also we have $\mathfrak{j} = \mathfrak{a}$ and the Brion-Knop little Weyl group W_Z coincides with the Weyl group $W_\mathfrak{a} \simeq (\mathbb{Z}/2\mathbb{Z})^3$. Thus, by the Knop isomomorphism, we have $\mathbb{D}(Z) \simeq \mathcal{Z}(\mathfrak{g})$. Now $\mathfrak{Z}(Z_\emptyset) = S(\mathfrak{a}) \simeq \mathbb{C}[z_1, z_2, z_3]$ with z_i the co-root coordinates and the above algebra inclusion

$$\mathcal{Z}(\mathfrak{g}) = \mathbb{C}[\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3] \hookrightarrow \mathfrak{Z}(Z_\emptyset) = \mathbb{C}[z_1, z_2, z_3]$$

is given by the assignment

$$\mathcal{C}_i \mapsto \frac{1}{4} z_i^2 - \frac{1}{2} z_i\,.$$

Note that $\mathbb{D}(Z_I)$ is naturally a module for $\mathcal{Z}(\mathfrak{g})$, the center of $\mathcal{U}(\mathfrak{g})$. We define by $\mathbf{D}(Z_I)$ the commutative subalgebra of $\mathbb{D}(Z)$, which is generated by $S(\mathfrak{a}_I)$ and the image of $\mathcal{Z}(\mathfrak{g})$ in $\mathbb{D}(Z_I)$.

Lemma 5.6. The $\mathcal{Z}(\mathfrak{g})$-module $\mathfrak{Z}(Z_\emptyset)$ is finitely generated. In particular, $\mathbf{D}(Z_I)$ is a finitely generated $\mathcal{Z}(\mathfrak{g})$-module for all $I \subset S$.

Proof. Since $S(\mathfrak{j})$ is a module of finite rank over $S(\mathfrak{j})^{W_\mathfrak{j}}$ (Chevalley's theorem), we obtain from (5.13) and $\operatorname{im} \gamma = S(\mathfrak{j})^{W_\mathfrak{j}}$ that $\mathfrak{Z}(Z_\emptyset)$ is a finitely generated $\mathcal{Z}(\mathfrak{g})$-module. Since $\mathbf{D}(Z_I)$ is naturally a submodule of $\mathbf{D}(Z_\emptyset)$ via the injective algebra morphism μ_I of Lemma 5.2, the second assertion follows from the fact that $\mathcal{Z}(\mathfrak{g}) \simeq S(\mathfrak{j})^{W_\mathfrak{j}}$ is a polynomial ring (again by Chevalley) and hence noetherian. ■

Let us denote by $\mathbf{D}_0(Z)$ the image of $\mathcal{Z}(\mathfrak{g})$ in $\mathbf{D}(Z_\emptyset) \subset \mathbb{D}(Z)$. As we will see later, some aspects become simpler if we work with the slightly smaller algebra $\mathbf{D}_0(Z)$. It follows from Lemma 5.6 that $\mathbf{D}(Z_I)$ is a finitely generated $\mu_I(\mathbf{D}_0(Z))$-module.

Fix now $I \subset S$. Since $\mathbf{D}(Z_I)$ is finitely generated over $\mu_I(\mathbf{D}_0(Z))$, there exists a finite dimensional vector subspace U of $\mathbf{D}(Z_I)$ containing 1 such that the map

$$\begin{array}{rcl} \mu_I(\mathbf{D}_0(Z)) \otimes U & \longrightarrow & \mathbf{D}(Z_I) \\ v \otimes u & \longmapsto & vu \end{array} \tag{5.14}$$

is a linear surjective map.

Let $\mathcal{I}$ be a finite codimensional ideal of $\mathbf{D}_0(Z)$ and let $\mathcal{I}' := \mu_I(\mathcal{I})$. Let $C = C(\mathcal{I})$ be a finite dimensional vector subspace of $\mu_I(\mathbf{D}_0(Z))$ containing 1 such that $\mu_I(\mathbf{D}_0(Z)) = C + \mathcal{I}'$. Hence:

$$\mathbf{D}(Z_I) = (C + \mathcal{I}')U = CU + \mathcal{I}'U\,, \tag{5.15}$$

where $\mathcal{I}'U$ (resp. CU) is the linear span of $\{vu \mid v \in \mathcal{I}', u \in U\}$ (resp. $\{vu \mid v \in C, u \in U\}$).

Since $\mathcal{I}'$ is an ideal on $\mathbf{D}_0(Z)$, we obtain that:

$$\mathcal{I}'U = \mathcal{I}'\mu_I(\mathbf{D}_0(Z))U = \mathcal{I}'\mathbf{D}(Z_I) = \mathbf{D}(Z_I)\mathcal{I}'\,. \tag{5.16}$$

Hence, (5.15) implies that:

$$\mathbf{D}(Z_I) = CU + \mathbf{D}(Z_I)\mathcal{I}'\,. \tag{5.17}$$

In case $\mathcal{I}$ is a one codimensional ideal of $\mathbf{D}_0(Z)$, one may and will take $C = \mathbb{C}1$, and then $CU = U$.

In general, we choose a finite dimensional subspace $U_{\mathcal{I}} \subset CU$, possibly depending on $\mathcal{I}$, such that the sum in (5.17) becomes direct:

$$\mathbf{D}(Z_I) = U_{\mathcal{I}} \oplus \mathbf{D}(Z_I)\mathcal{I}' . \tag{5.18}$$

Let $s_{\mathcal{I}}$, resp. $q_{\mathcal{I}}$, be the linear map from $\mathbf{D}(Z_I)$ to $U_{\mathcal{I}}$, resp. $\mathbf{D}(Z_I)\mathcal{I}'$, deduced from this direct sum decomposition. The algebra $\mathbf{D}(Z_I)$ acts on $U_{\mathcal{I}}$ by a representation $\rho_{\mathcal{I}}$ defined by:

$$\rho_{\mathcal{I}}(v)u = s_{\mathcal{I}}(vu), \quad v \in \mathbf{D}(Z_I), u \in U_{\mathcal{I}} . \tag{5.19}$$

In fact:

The representation $(\rho_{\mathcal{I}}, U_{\mathcal{I}})$ is isomorphic to the natural representation of $\mathbf{D}(Z_I)$ on $\mathbf{D}(Z_I)/\mathbf{D}(Z_I)\mathcal{I}'$.

We notice that, for $v \in \mathbf{D}(Z_I)$ and $u \in U_{\mathcal{I}}$,

$$vu = \rho_{\mathcal{I}}(v)u + q_{\mathcal{I}}(vu). \tag{5.20}$$

If $(u_i)_{i=1,\dots,n}$ is a basis of U, then we obtain, from $\mathbf{D}(Z_I)\mathcal{I}' = \mu_I(\mathcal{I})U = U\mu_I(\mathcal{I})$ (see (5.16)), elements $z_i = z_i(v, u, \mathcal{I}) \in \mathcal{I}$, not necessarily unique, such that:

$$q_{\mathcal{I}}(vu) = \sum_{i=1}^{n} u_i \mu_I(z_i) . \tag{5.21}$$

Moreover, we record from Lemma 5.2(ii) that:

$$\mu_I(z_i) - z_i \text{ has } \mathfrak{a}_Z\text{-weights non-positive on } \mathfrak{a}_Z^- \text{ and negative on } \mathfrak{a}_I^{--} . \tag{5.22}$$

In order to use it later, we denote by $\mathcal{F} = \mathcal{F}(\mathcal{I})$ the (finite) set of all these $\mathfrak{a}_Z$-weights that occur when v describes $\mathfrak{a}_I \subset \mathbf{D}(Z_I)$ and u describes $U_{\mathcal{I}}$. Let us define a piecewise linear functional on $\mathfrak{a}_Z$ by:

$$\beta_I(X) := \max_{\lambda \in \mathcal{F} \cup (S \setminus I)} \lambda(X), \quad X \in \mathfrak{a}_Z . \tag{5.23}$$

Note that $\beta_I\big|_{\mathfrak{a}_Z^-} \leq 0$ and $\beta_I\big|_{\mathfrak{a}_I^{--}} < 0$.

5.3 The function Φ_f on A_Z and related differential equations

Fix $N \in \mathbb{N}$ and $\mathcal{I}$ a finite codimensional ideal in $\mathbf{D}_0(Z)$. Recall the surjective morphism $\mathcal{Z}(\mathfrak{g}) \to \mathbf{D}_0(Z)$ and let $\mathcal{J}$ be the corresponding preimage of $\mathcal{I}$. Set

$$\mathcal{A}_{temp}(Z : \mathcal{I}) := \mathcal{A}_{temp}(Z : \mathcal{J}),$$

with $\mathcal{A}_{temp}(Z : \mathcal{J})$ defined in (4.12).

Recall that we identified for any $I \subset S$ the algebra $\mathbf{D}(Z_I)$ as a subspace of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. Now given $f \in C^\infty(Z)$, we denote by $\tilde{f} \in C^\infty(G)$ its lift to a right H-invariant smooth function on G. For $u \in \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ we let further $\tilde{u} \in \mathcal{U}(\mathfrak{b})$ be any lift. Then, for all $a_Z \in A_Z$ the notion

$$(R_u f)(a_Z) := (R(\tilde{u})\tilde{f})(\tilde{a}_Z)$$

is defined, i.e., independent of the lift $\tilde{u}$ and the section $\mathbf{s}$.

Recall that $(\rho_{\mathcal{I}}, U_{\mathcal{I}})$ is the finite dimensional $\mathbf{D}(Z_I)$-module defined in (5.19) and in particular $U_{\mathcal{I}} \subset \mathbf{D}(Z_I) \subset \mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. For any $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$, let us define a function $\Phi_f : A_Z \to U_{\mathcal{I}}^*$ by:

$$\langle \Phi_f(a_Z), u\rangle := (R_u f)(a_Z), \quad u \in U_{\mathcal{I}}, a_Z \in A_Z. \tag{5.24}$$

Hence, for $X \in \mathfrak{a}_I \subset \mathbf{D}(Z_I)$,

$$\langle (R_X\Phi_f)(a_Z), u\rangle = (R_{Xu} f)(a_Z), \quad a_Z \in A_Z, u \in U_{\mathcal{I}}. \tag{5.25}$$

Hence, by using (5.20) and (5.21) for Xu, one gets

$$R_X\Phi_f = {}^t\rho_{\mathcal{I}}(X)\Phi_f + \Psi_{f,X}, \quad X \in A_I, \tag{5.26}$$

where $\Psi_{f,X} : A_Z \to U_{\mathcal{I}}^*$ is given by:

$$\langle \Psi_{f,X}(a_Z), u\rangle := \sum_i (R_{u_i\mu_I(z_i)} f)(a_Z), \quad a_Z \in A_Z, u \in U_{\mathcal{I}}, \tag{5.27}$$

with $z_i = z_i(X, u, \mathcal{I})$ given by (5.21).

Since $R_{z_i} f = 0$ as $z_i \in \mathcal{I}$ and f is annihilated by $\mathcal{I}$, one then has:

$$\langle \Psi_{f,X}(a_Z), u \rangle = \sum_i (R_{u_i(\mu_I(z_i)-z_i)} f)(a_Z), \quad a_Z \in A_Z, u \in U_{\mathcal{I}} . \tag{5.28}$$

One sets:

$$\Gamma_{\mathcal{I}}(X) = {}^t\rho_{\mathcal{I}}(X), \quad X \in \mathfrak{a}_I . \tag{5.29}$$

Hence, we arrive at the fundamental first order ordinary differential equation:

$$R_X \Phi_f = \Gamma_{\mathcal{I}}(X)\Phi_f + \Psi_{f,X}, \quad X \in \mathfrak{a}_I . \tag{5.30}$$

Notice that $\Gamma_{\mathcal{I}}$ is a representation of the abelian Lie algebra $\mathfrak{a}_I$ on $U_{\mathcal{I}}^*$.

For $\lambda \in \mathfrak{a}_{I,\mathbb{C}}^*$, one denotes by $U_{\mathcal{I},\lambda}^*$ the space of joint generalized eigenvectors of $U_{\mathcal{I}}^*$ by the endomorphisms $\Gamma_{\mathcal{I}}(X)$, $X \in \mathfrak{a}_I$, for the eigenvalue λ. Let $\mathcal{Q}_{\mathcal{I}}$ be the (finite) subset of $\lambda \in \mathfrak{a}_{I,\mathbb{C}}^*$ such that $U_{\mathcal{I},\lambda}^* \neq \{0\}$. One has:

$$U_{\mathcal{I}}^* = \bigoplus_{\lambda \in \mathcal{Q}_{\mathcal{I}}} U_{\mathcal{I},\lambda}^* . \tag{5.31}$$

If $\lambda \in \mathcal{Q}_{\mathcal{I}}$, let E_λ be the projector of $U_{\mathcal{I}}^*$ onto $U_{\mathcal{I},\lambda}^*$ parallel to the sum of the other $U_{\mathcal{I},\mu}^*$'s.

Define, for $\lambda \in \mathcal{Q}_{\mathcal{I}}$,

$$\Phi_{f,\lambda} := E_\lambda \circ \Phi_f .$$

We conclude this subsection with the solution formula for the system (5.30) (see the next Lemma 5.7) and with two elementary estimates for Φ_f and $\Psi_{f,X}$ in Lemma 5.8 below.

Lemma 5.7. Let $f \in \mathcal{A}_{temp}(Z : \mathcal{I})$. One has,

(i) for all $a_Z \in A_Z$, $t \in \mathbb{R}$, $X \in \mathfrak{a}_I$,

$$\Phi_f(a_Z \exp(tX)) = e^{t\Gamma_{\mathcal{I}}(X)} \Phi_f(a_Z) + \int_0^t e^{(t-s)\Gamma_{\mathcal{I}}(X)} \Psi_{f,X}(a_Z \exp(sX))\, ds ,$$

(ii) for all $a_Z \in A_Z$, $t \in \mathbb{R}$, $X \in \mathfrak{a}_I$, $\lambda \in \mathcal{Q}_{\mathcal{I}}$,

$$\Phi_{f,\lambda}(a_Z \exp(tX)) = e^{t\Gamma_{\mathcal{I}}(X)} \Phi_{f,\lambda}(a_Z) + \int_0^t E_\lambda e^{(t-s)\Gamma_{\mathcal{I}}(X)} \Psi_{f,X}(a_Z \exp(sX))\, ds .$$

Proof. The equality (i) is an immediate consequence of (5.30). Indeed, we apply the elementary result on first order linear differential equation to the function $s \mapsto F(s) = \Phi_f(a_Z \exp(sX))$, whose derivative $F'(s) = (R_X \Phi_f)(a_Z \exp(sX))$ satisfies

$$F'(s) = \Gamma_{\mathcal{I}}(X)F(s) + \Psi_{f,X}(a_Z \exp(sX)) .$$

The equality (ii) follows by applying E_λ to both sides of the equality of (i). ■

We recall the definition of β_I from (5.23).

Lemma 5.8. Let $N \in \mathbb{N}$.

(i) There exists a continuous semi-norm on $C^\infty_{temp,N}(Z)$, p, such that

$$\|L_v \Phi_f(a_Z)\| \leq a_Z^{\rho_Q}(1 + \|\log a_Z\|)^N p(L_v f)$$

for all $v \in \mathcal{U}(\mathfrak{a})$, $a_Z \in A_Z^-$ and $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$.

(ii) There exists a continuous semi-norm q on $C^\infty_{temp,N}(Z)$ such that, for all compact subset $\Omega_A \subset A_Z$, there exists a constant $C = C(\Omega_A) > 0$ with:

$$\|L_v \Psi_{f,X}(a_Z)\| \leq C a_Z^{\rho_Q + \beta_I}(1 + \|\log a_Z\|)^N \|X\| q(L_v f)$$

for $a_Z \in \Omega_A A_Z^-$, $X \in \mathfrak{a}_I$ and $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$.

Proof. (i) We first consider the case of $v = 1$. Let $u \mapsto u^t$ denote the principal anti-automorphism of $U(\mathfrak{g})$.

Let $u \in \mathbf{D}(Z_I) \subset \mathcal{U}(\mathfrak{a}_Z + \mathfrak{m}_Z + \mathfrak{u})$. One has:

$$(R_u f)(a_Z) = (L_{(\mathrm{Ad}(a_Z)u)^t} f)(a_Z) .$$

Since $\mathrm{Ad}(\mathfrak{a}_Z^-)$ contracts the $\mathfrak{a}_Z$-weights of u (see Lemma 5.2(ii)), the assertion for $v = 1$ follows from the continuity of the left regular action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty_{temp,N}(Z)$. The more general case is obtained by the fact that the assignment $f \mapsto \Phi_f$ is A-equivariant for the left regular representation of A on functions on Z, resp. A_Z.

(ii) We recall from (5.28) that:

$$\langle \Psi_{f,X}(a_Z), u \rangle = \sum_i (R_{u_i(\mu_I(z_i) - z_i)} f)(a_Z), \quad a_Z \in A_Z, u \in U_{\mathcal{I}} ,$$

with $z_i = z_i(X, u, \mathcal{I})$. In particular, this identity readily reduces to the case of $v = 1$ as left and right regular representation commute.

Since the $\mathfrak{a}_Z$-weights of u_i are non-positive on $\mathfrak{a}_Z^-$ (see Lemma 5.2(ii)), we obtain that $u_i(\mu_I(z_i) - z_i)$ decomposes into a finite sum over $\mathcal{F} - (\mathfrak{a}_Z^-)^\star$ of $\mathfrak{a}_Z$-weight vectors:

$$u_i(\mu_I(z_i) - z_i) = \sum_\lambda v_{i,\lambda} .$$

Here, $(\mathfrak{a}_Z^-)^\star$ denotes the dual cone of $\mathfrak{a}_Z^-$. Then:

$$\begin{aligned} \langle \Psi_{f,X}(a_Z), u\rangle &= \textstyle\sum_i \sum_\lambda (L_{(\mathrm{Ad}(a_Z)(v_{i,\lambda}))^t} f)(a_Z) \\ &= \textstyle\sum_i \sum_\lambda a_Z^\lambda (L_{v_{i,\lambda}^t} f)(a_Z) . \end{aligned}$$

Let $k := \max_{u \in U_\mathcal{I}, X \in \mathfrak{a}_I}(\deg(v_{i,\lambda}))$. Assume first that $\Omega_A = \{1\}$ and $\|X\| = 1$. Then it follows from the continuity of the left action of $\mathcal{U}(\mathfrak{g})$ on $C^\infty_{temp,N}(Z)$ and the definition of β_I that there is an appropriate Sobolev norm $q = p_{N,k}$ such that the bound in (ii) holds for $C = 1$. In general, if $u \in \mathcal{U}(\mathfrak{g})$, $a \in \Omega_A$ and $a_Z \in A_Z^-$, one has:

$$(L_u f)(a a_Z) = L_{a^{-1}}(L_{\mathrm{Ad}(a^{-1})u} f)(a_Z)$$

and the assertion follows from:

$$q(L_{a^{-1}} f) \le C q(f), \quad f \in C^\infty_{temp,N}(Z), a \in \Omega_A .$$

■

5.4 The decomposition of Φ_f into eigenspaces

We recall the representation $\Gamma_\mathcal{I} : \mathfrak{a}_I \to \mathrm{End}(U_\mathcal{I}^*)$ of the abelian Lie algebra $\mathfrak{a}_I$ from (5.29) and $\mathcal{Q}_\mathcal{I}$ the set of its generalized $\mathfrak{a}_I$-eigenvalues.

We endow $U_\mathcal{I}^*$ with a scalar product and, if $T \in \mathrm{End}(U_\mathcal{I}^*)$, we denote by $\|T\|$ its Hilbert–Schmidt norm. It is clear that, for any $\lambda \in \mathcal{Q}_\mathcal{I}$, the projector E_λ defined just after (5.31) commutes with the operators $\Gamma_\mathcal{I}(X)$, $X \in \mathfrak{a}_I$. For $\lambda \in \mathcal{Q}_\mathcal{I}$, we set

$$E_\lambda(X) := e^{-\lambda(X)} \left(E_\lambda \circ e^{\Gamma_\mathcal{I}(X)}\right), \quad X \in \mathfrak{a}_I .$$

As $E_\lambda \circ [\Gamma_\mathcal{I}(X) - \lambda(X)\mathrm{Id}_{U_\mathcal{I}^*}]$ is nilpotent, one readily obtains that:

Lemma 5.9. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}$. We can choose $c \geq 0$ such that:

$$\|E_{\lambda}(X)\| \leq c(1 + \|X\|)^{N_{\mathcal{I}}}, \quad X \in \mathfrak{a}_I,$$

where $N_{\mathcal{I}}$ is the dimension of $U_{\mathcal{I}}$.

Next, we decompose $\mathcal{Q}_{\mathcal{I}}$ into three disjoints subsets $\mathcal{Q}_{\mathcal{I}}^{+}$, $\mathcal{Q}_{\mathcal{I}}^{0}$ and $\mathcal{Q}_{\mathcal{I}}^{-}$ as follows:

(1) $\lambda \in \mathcal{Q}_{\mathcal{I}}^{+}$ if $\operatorname{Re}\lambda(X_I) > \rho_Q(X_I)$ for some $X_I \in \mathfrak{a}_I^{--}$,
(2) $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0}$ if $\operatorname{Re}\lambda(X_I) = \rho_Q(X_I)$ for all $X_I \in \mathfrak{a}_I^{--}$,
(3) $\lambda \in \mathcal{Q}_{\mathcal{I}}^{-}$ if $\lambda \notin \mathcal{Q}_{\mathcal{I}}^{+} \cup \mathcal{Q}_{\mathcal{I}}^{0}$, i.e., for all $X_I \in \mathfrak{a}_I^{--}$, $\operatorname{Re}\lambda(X_I) \leq \rho_Q(X_I)$ and there exists $X_I \in \mathfrak{a}_I^{--}$ such that $\operatorname{Re}\lambda(X_I) < \rho_Q(X_I)$.

The next two propositions will be central for the definition of the constant term in the next section. We first state the results and then provide the proofs in a sequence of lemmas. The proofs of these results follow closely the work of Harish-Chandra (cf. [17,Section 22]): to see the analogy replace M_1^{+} in [17] by A_Z^{-} and M_1 by $A_{Z_I}^{-}$.

Proposition 5.10. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0}$ and $f \in \mathcal{A}_{temp}(Z:\mathcal{I})$. Then, for $X_I \in \mathfrak{a}_I^{--}$, the following limit

$$\lim_{t \to +\infty} e^{-t\Gamma_{\mathcal{I}}(X_I)} \Phi_{f,\lambda}(a_Z \exp(tX_I)), \quad a_Z \in A_Z,$$

exists and is independent of $X_I \in \mathfrak{a}_I^{--}$.

For $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0}$ and $f \in \mathcal{A}_{temp}(Z:\mathcal{I})$, we now set

$$\Phi_{f,\lambda,\infty}(a_Z) := \lim_{t \to +\infty} e^{-t\Gamma_{\mathcal{I}}(X_I)} \Phi_{f,\lambda}(a_Z \exp(tX_I)), \quad a_Z \in A_Z. \tag{5.32}$$

Further we define

$$\Phi_{f,\lambda,\infty}(a_Z) := 0, \quad a_Z \in A_Z, \lambda \in \mathcal{Q}_{\mathcal{I}}^{+} \cup \mathcal{Q}_{\mathcal{I}}^{-}, f \in \mathcal{A}_{temp}(Z:\mathcal{I}). \tag{5.33}$$

Proposition 5.11. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}$ and $f \in \mathcal{A}_{temp}(Z : \mathcal{I})$. Then there exists $\delta > 0$ such that for all $a_Z \in A_Z$, $X_I \in \mathfrak{a}_I^{--}$ and $t \geq 0$:

$$\begin{aligned}
&\|\Phi_{f,\lambda}(a_Z \exp(tX_I)) - \Phi_{f,\lambda,\infty}(a_Z \exp(tX_I))\| \\
\leq & e^{t(\rho_Q+\delta\beta_I)(X_I)} \Big(\|E_\lambda(tX_I)\| \|\Phi_f(a_Z)\| \\
&+ \int_0^\infty e^{-s(\rho_Q+\beta_I/2)(X_I)} \|E_\lambda((t-s)X_I)\| \|\Psi_{f,X_I}(a_Z \exp(sX_I))\| \, ds \Big) .
\end{aligned}$$

5.4.1 *Proof of Proposition 5.10*

We say that an integral depending on a parameter converges uniformly if the absolute value of the integrand is bounded by an integrable function independently of the parameter.

Lemma 5.12. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}$ and $X_I \in \mathfrak{a}_I^{--}$ be such that $\operatorname{Re}\lambda(X_I) > (\rho_Q + \beta_I)(X_I)$. Then

(i) The integral

$$\int_0^\infty E_\lambda e^{-s\Gamma_{\mathcal{I}}(X_I)} \Psi_{f,X_I}(a_Z \exp(sX_I)) \, ds$$

converges uniformly on any compact subset of A_Z.

(ii) The assignment

$$a_Z \mapsto \int_0^\infty E_\lambda e^{-s\Gamma_{\mathcal{I}}(X_I)} \Psi_{f,X_I}(a_Z \exp(sX_I)) \, ds$$

is a well-defined map on A_Z. Its derivative along $u \in S(\mathfrak{a}_Z)$ is given by derivation under the integral sign.

Proof. One has

$$E_\lambda e^{-s\Gamma_{\mathcal{I}}(X_I)} = e^{-s\lambda(X_I)} E_\lambda e^{s(\lambda(X_I) - \Gamma_{\mathcal{I}}(X_I))} = e^{-s\lambda(X_I)} E_\lambda(-sX_I) .$$

Hence, from Lemma 5.9, one has:

$$\|E_\lambda e^{-s\Gamma_{\mathcal{I}}(X_I)}\| \leq c(1 + \|sX_I\|)^{N_{\mathcal{I}}} e^{-s\operatorname{Re}\lambda(X_I)} . \tag{5.34}$$

Using Lemma 5.8(ii), (5.34) and the assumption $\operatorname{Re}\lambda(X_I) > (\rho_Q + \beta_I)(X_I)$, we obtain that the integral in (i) converges uniformly on compact subsets of A_Z.

The assertion from (ii) follows in the same way and using the theorem on derivatives of integral depending of a parameter. ■

Fix $N \in \mathbb{N}$ such that $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$ and $\lambda \in \mathcal{Q}_{\mathcal{I}}$ and put, for X_I as in Lemma 5.12, i.e., $X_I \in \mathfrak{a}_I^{--}$ such that $\operatorname{Re} \lambda(X_I) > (\rho_Q + \beta_I)(X_I)$:

$$\Phi_{f,\lambda,\infty}(a_Z, X_I) := \lim_{t \to +\infty} e^{-t\Gamma_{\mathcal{I}}(X_I)} \Phi_{f,\lambda}(a_Z \exp(tX_I)), \quad a_Z \in A_Z . \tag{5.35}$$

It follows from Lemmas 5.7(ii) and 5.12 that this limit exists and is C^∞ on A_Z. Moreover

$$\Phi_{f,\lambda,\infty}(a_Z, X_I) = \Phi_{f,\lambda}(a_Z) + \int_0^\infty E_\lambda e^{-s\Gamma_{\mathcal{I}}(X_I)} \Psi_{f,X_I}(a_Z \exp(sX_I))\, ds, \quad a_Z \in A_Z . \tag{5.36}$$

Lemma 5.13. Let $X_1, X_2 \in \mathfrak{a}_I^{--}$ and suppose that

$$\operatorname{Re} \lambda(X_i) > (\rho_Q + \beta_I)(X_i), \quad i = 1, 2 .$$

Then

$$\Phi_{f,\lambda,\infty}(a_Z, X_1) = \Phi_{f,\lambda,\infty}(a_Z, X_2), \quad a_Z \in A_Z .$$

Proof. Same as the proof of [17,Lemma 22.8]. We give it for sake of completeness. Let $a_Z \in A_Z$. Applying Lemma 5.7(ii) to $a_Z \exp(t_1 X_1)$ instead of a_Z, X_2 instead of X and t_2 instead of t, one gets:

$$\begin{aligned} & e^{-\Gamma_{\mathcal{I}}(t_1 X_1 + t_2 X_2)} \Phi_{f,\lambda}(a_Z \exp(t_1 X_1) \exp(t_2 X_2)) \\ =& e^{-t_1 \Gamma_{\mathcal{I}}(X_1)} \Phi_{f,\lambda}(a_Z \exp(t_1 X_1)) \\ & + \int_0^{t_2} E_\lambda e^{-\Gamma_{\mathcal{I}}(t_1 X_1 + s_2 X_2)} \Psi_{f,X_2}(a_Z \exp(t_1 X_1 + s_2 X_2))\, ds_2 , \end{aligned}$$

for $t_1, t_2 > 0$. From Lemmas 5.9 and 5.8(ii) applied to $X = t_1 X_1 + s_2 X_2$ and $(X, a_Z) = (X_2, a_Z \exp(t_1 X_1 + s_2 X_2))$ respectively, one sees that:

$$\int_0^\infty \| E_\lambda e^{-\Gamma_{\mathcal{I}}(t_1 X_1 + s_2 X_2)} \| \| \Psi_{f,X_2}(a_Z \exp(t_1 X_1 + s_2 X_2)) \| \, ds_2$$

tends to 0 when $t_1 \to +\infty$. Hence:

$$\begin{aligned}&\lim_{t_1,t_2\to+\infty} e^{-\Gamma_{\mathcal{I}}(t_1X_1+t_2X_2)}\Phi_{f,\lambda}(a_Z\exp(t_1X_1+t_2X_2))\\ =&\lim_{t_1\to+\infty} e^{-\Gamma_{\mathcal{I}}(t_1X_1)}\Phi_{f,\lambda}(a_Z\exp(t_1X_1))\\ =&\Phi_{f,\lambda,\infty}(a_Z,X_1)\,.\end{aligned}$$

Since the first limit on the above equality is symmetrical in X_1 and X_2, one then deduces that:

$$\Phi_{f,\lambda,\infty}(a_Z,X_1)=\Phi_{f,\lambda,\infty}(a_Z,X_2).$$

■

Proof of Proposition 5.10 If $\lambda\in\mathcal{Q}^0_{\mathcal{I}}$, the hypothesis of (5.35) is satisfied. Together with the preceeding Lemma, it shows the proposition. ■

5.4.2 *Proof of Proposition 5.11*

Lemma 5.14. For $X_I\in\mathfrak{a}_I^{--}$ such that $\operatorname{Re}\lambda(X_I)>\rho_Q(X_I)$, one has:

$$\Phi_{f,\lambda,\infty}(a_Z,X_I)=0,\quad a_Z\in A_Z\,.$$

Proof. One has

$$\|e^{-t\Gamma_{\mathcal{I}}(X_I)}\Phi_{f,\lambda}(a_Z\exp(tX_I))\|\le e^{-t\operatorname{Re}\lambda(X_I)}\|E_\lambda(-tX_I)\|\|\Phi_f(a_Z\exp(tX_I))\|\,.$$

From Lemmas 5.9 and 5.8(i), one then has

$$\|e^{-t\Gamma_{\mathcal{I}}(X_I)}\Phi_{f,\lambda}(a_Z\exp(tX_I))\|\le Ca_Z^{\rho_Q}(1+\|\log a_Z\|)^N(1+\|tX_I\|)^{N+N_{\mathcal{I}}}e^{t(\rho_Q-\operatorname{Re}\lambda)(X_I)}\,.$$

The right hand side of the inequality tends to zero as $t\to+\infty$. Hence, the Lemma follows from the definition (5.35) of $\Phi_{f,\lambda,\infty}(a_Z,X_I)$. ■

Lemma 5.15. Assume $\lambda\in\mathcal{Q}^+_{\mathcal{I}}$ and $X_I\in\mathfrak{a}_I^{--}$ such that $\operatorname{Re}\lambda(X_I)>(\rho_Q+\beta_I)(X_I)$. Then, for any $a_Z\in A_Z$,

$$\Phi_{f,\lambda,\infty}(a_Z,X_I)=0$$

and

$$\Phi_{f,\lambda}(a_Z \exp(tX_I)) = -\int_t^{\infty} E_\lambda e^{(t-s)\Gamma_{\mathcal{I}}(X_I)} \Psi_{f,X_I}(a_Z \exp(sX_I))\, ds\,, \quad t \in \mathbb{R}\,.$$

Proof. Since $\lambda \in \mathcal{Q}_{\mathcal{I}}^{+}$, there exists $X_0 \in \mathfrak{a}_I^{--}$ such that $\operatorname{Re}\lambda(X_0) > \rho_Q(X_0)$. Then, from Lemma 5.14, $\Phi_{f,\lambda,\infty}(a_Z, X_0) = 0$, and, from Lemma 5.13, as $\operatorname{Re}\lambda(X_0) > \rho_Q(X_0) > (\rho_Q + \beta_I)(X_0)$, one has $\Phi_{f,\lambda,\infty}(a_Z, X_I) = \Phi_{f,\lambda,\infty}(a_Z, X_0)$ for any $X_I \in \mathfrak{a}_I^{--}$ such that $\operatorname{Re}\lambda(X_I) > (\rho_Q + \beta_I)(X_I)$. This proves the first part of the Lemma. The second part follows from (5.36) by change of variables and when we replace a_Z by $a_Z \exp(tX_I)$. ∎

Corollary 5.16. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}^{+}$ and $X_I \in \mathfrak{a}_I^{--}$ be such that $\operatorname{Re}\lambda(X_I) \geq (\rho_Q + \beta_I/2)(X_I)$. Then, for $a_Z \in A_Z$ and $t \geq 0$,

$$\|\Phi_{f,\lambda}(a_Z \exp(tX_I))\| \leq \int_t^{\infty} e^{(t-s)(\rho_Q+\beta_I/2)(X_I)} \|E_\lambda((t-s)X_I)\| \|\Psi_{f,X_I}(a_Z \exp(sX_I))\|\, ds\,.$$

Proof. Since $\beta_I(X_I) < 0$ and $\operatorname{Re}\lambda(X_I) \geq (\rho_Q+\beta_I/2)(X_I)$, one has, in particular, $\operatorname{Re}\lambda(X_I) > (\rho_Q + \beta_I)(X_I)$. Then one can see, from Lemmas 5.15 and 5.12, that:

$$\|\Phi_{f,\lambda}(a_Z \exp(tX_I))\| \leq \int_t^{\infty} e^{(t-s)\operatorname{Re}\lambda(X_I)} \|E_\lambda((t-s)X_I)\| \|\Psi_{f,X_I}(a_Z \exp(sX_I))\|\, ds\,.$$

Our assertion follows, since $\operatorname{Re}\lambda(X_I) \geq (\rho_Q + \beta_I/2)(X_I)$ implies that $(t-s)\operatorname{Re}\lambda(X_I) \leq (t-s)(\rho_Q + \beta_I/2)(X_I)$ for $s \geq t$. ∎

Lemma 5.17. Let $X_I \in \mathfrak{a}_I^{--}$ be such that $\operatorname{Re}\lambda(X_I) \leq (\rho_Q + \beta_I/2)(X_I)$. Then

$$\begin{aligned}\|\Phi_{f,\lambda}(a_Z \exp(tX_I))\| \leq\ & e^{t(\rho_Q+\beta_I/2)(X_I)}\Big(\|E_\lambda(tX_I)\| \|\Phi_f(a_Z)\| \\ & + \int_0^{\infty} e^{-s(\rho_Q+\beta_I/2)(X_I)} \|E_\lambda((t-s)X_I)\| \|\Psi_{f,X_I}(a_Z \exp(sX_I))\|\, ds\Big), \\ & \qquad t \geq 0, a_Z \in A_Z\,.\end{aligned}$$

Proof. We use Lemma 5.7(ii) and the inequality $(t-s)\operatorname{Re}\lambda(X_I) \leq (t-s)(\rho_Q + \beta_I/2)(X_I)$ for $s \leq t$ to get an analogue of the inequality of the Lemma, where $\int_0^\infty$ is replaced by $\int_0^t$. The Lemma follows. ∎

Like in [17,after the proof of Lemma 22.8], one sees that one can choose $0 < \delta \leq 1/2$ such that:

$$\operatorname{Re}\lambda(X_I) \leq (\rho_Q + \delta\beta_I)(X_I), \quad X_I \in \mathfrak{a}_I^{--}, \lambda \in \mathcal{Q}_{\mathcal{I}}^{-}. \tag{5.37}$$

Lemma 5.18. Let $\lambda \in \mathcal{Q}_{\mathcal{I}}^{-}$ and $X_I \in \mathfrak{a}_I^{--}$. Then, for $a_Z \in A_Z$, $t \geq 0$,

$$\begin{aligned} \|\Phi_{f,\lambda}(a_Z \exp(tX_I))\| \quad &\leq \quad e^{t(\rho_Q+\delta\beta_I)(X_I)}\Big(\|E_\lambda(tX_I)\|\|\Phi_f(a_Z)\| \\ &\quad + \int_0^\infty e^{-s(\rho_Q+\beta_I/2)(X_I)}\|E_\lambda((t-s)X_I)\|\|\Psi_{f,X_I}(a_Z \exp(sX_I))\|\,ds\Big). \end{aligned}$$

Proof. This is proved like Lemma 5.17, using that $\operatorname{Re}\lambda(X_I) \leq (\rho_Q + \delta\beta_I)(X_I)$ and $0 < \delta \leq 1/2$. ■

Notice now that, if $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0}$, it follows from Lemma 5.13 and the definition of β_I (cf. (5.23)) that:

For $a_Z \in A_Z$, $\Phi_{f,\lambda,\infty}(a_Z, X_I)$ is independent of $X_I \in \mathfrak{a}_I^{--}$.

We will denote it by $\Phi_{f,\lambda,\infty}(a_Z)$.

Lemma 5.19. Assume $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0}$ and let $X_I \in \mathfrak{a}_I^{--}$. Then one has, for $t \geq 0$ and $a_Z \in A_Z$,

$$\begin{aligned} &\|\Phi_{f,\lambda}(a_Z \exp(tX_I)) - \Phi_{f,\lambda,\infty}(a_Z \exp(tX_I))\| \\ \leq \quad &e^{t(\rho_Q+\delta\beta_I)(X_I)} \int_0^\infty e^{-s(\rho_Q+\beta_I/2)(X_I)}\|E_\lambda((t-s)X_I)\|\|\Psi_{f,X_I}(a_Z \exp(sX_I))\|\,ds. \end{aligned}$$

Proof. From (5.36), one deduces:

$$\Phi_{f,\lambda,\infty}(a_Z \exp(tX_I)) = \Phi_{f,\lambda}(a_Z \exp(tX_I)) + \int_t^\infty E_\lambda e^{(t-s)\Gamma_{\mathcal{I}}(X_I)}\Psi_{f,X_I}(a_Z \exp(sX_I))\,ds.$$

The Lemma now follows from the fact that $(t-s)\beta_I(X_I) \geq 0$ whenever $s \geq t$. ■

We recall that we have defined:

$$\Phi_{f,\lambda,\infty}(a_Z) := 0, \quad a_Z \in A_Z, \lambda \in \mathcal{Q}_{\mathcal{I}}^{+} \cup \mathcal{Q}_{\mathcal{I}}^{-}.$$

Proof of Proposition 5.11 If $\lambda \in \mathcal{Q}_{\mathcal{I}}^{0} \cup \mathcal{Q}_{\mathcal{I}}^{-}$, our assertion follows from Lemmas 5.18 and 5.19. On the other hand, if $\lambda \in \mathcal{Q}_{\mathcal{I}}^{+}$, we can apply Lemmas 5.15 and 5.17, and Corollary 5.16. ■

6 Definition and properties of the constant term

In this section, we define the constant term f_I of a function $f \in \mathcal{A}_{temp}(Z)$ in terms of the $\Phi_{f,\lambda,\infty}$ from the previous section. At first, f_I is defined as a smooth function on A_Z but then will be extended to a smooth function on $Z_I = G/H_I$. The main difficulty then is to show that the function $f_I \in C^\infty(Z_I)$ is indeed tempered. For that, we need to show certain consistency relations of f_I with respect to the matching map $\mathbf{m} : \mathcal{W}_I \to \mathcal{W}$, see Proposition 6.7. The consistency relations are immediate from our strong results of rapid convergence in Proposition 3.1. As an application, we characterize the functions of the discrete series as those with all constant terms vanishing, see Theorem 6.12.

Throughout this section, we fix a subset I of S and a finite codimensional ideal $\mathcal{I}$ in $\mathbf{D}_0(Z)$.

6.1 Definition of the constant term

For $f \in \mathcal{A}_{temp}(Z : \mathcal{I})$ let us define f_I as the function on A_Z by:

$$f_I(a_Z) := \sum_{\lambda \in \mathcal{Q}^0_{\mathcal{I}}} \langle \Phi_{f,\lambda,\infty}(a_Z), 1 \rangle, \quad a_Z \in A_Z\,, \tag{6.1}$$

where $\Phi_{f,\lambda,\infty}$ has been defined in (5.32) and (5.33). From Lemma 6.2 and since the eigenvalues of $E_\lambda(\Gamma_{\mathcal{I}}(X))$, for any $X \in \mathfrak{a}_I$, are contained in $\rho_Q(X) + i\mathbb{R}$ if $\lambda \in \mathcal{Q}^0_{\mathcal{I}}$, one has that:

$$\begin{aligned}&\text{For any } X \in \mathfrak{a}_I \text{, the map } t \mapsto e^{-t\rho_Q(X)} f_I(\exp(tX)) \text{ is an}\\ &\text{exponential polynomial with unitary characters.}\end{aligned} \tag{6.2}$$

We will soon extend f_I to a smooth function on G, which is right invariant under H_I, i.e., f_I descends to a smooth function on Z_I. This will be prepared with a few estimates in the next subsection.

6.2 Some estimates

In this subsection, we establish some estimates analogous to the ones given in [17,Section 23].

Lemma 6.1. Let $N \in \mathbb{N}$. There exists a continuous semi-norm q on $C^\infty_{temp,N}(Z)$ such that, for all $\lambda \in \mathcal{Q}_\mathcal{I}$, $a_Z \in A_Z^-$, $X_I \in \mathfrak{a}_I^{--}$, $t \geq 0$ and $f \in \mathcal{A}_{temp,N}(Z:\mathcal{I})$,

$$\|\Phi_{f,\lambda}(a_Z \exp(tX_I)) - \Phi_{f,\lambda,\infty}(a_Z \exp(tX_I))\|$$

$$\leq (a_Z \exp(tX_I))^{\rho_Q} e^{t\delta\beta_I(X_I)}(1 + \|\log a_Z\|)^N(1 + t\|X_I\|)^{\dim U_\mathcal{I}} q(f)\,.$$

Proof. The assertion of the Lemma follows from Proposition 5.11, Lemmas 5.8 and 5.9, and the fact that $a_Z^{\beta_I} \leq 1$ for $a_Z \in A_Z^-$. ■

Lemma 6.2. For all $X \in \mathfrak{a}_I, a_Z \in A_Z, \lambda \in \mathcal{Q}_\mathcal{I}$ and $f \in \mathcal{A}_{temp}(Z:\mathcal{I})$ one has

$$\Phi_{f,\lambda,\infty}(a_Z \exp X) = e^{\Gamma_\mathcal{I}(X)} \Phi_{f,\lambda,\infty}(a_Z),\,.$$

Proof. According to (5.33), one may assume $\lambda \in \mathcal{Q}^0_\mathcal{I}$. From Lemma 5.7(ii) applied with $t = 1$, one has, for $a_Z \in A_Z$, $X \in \mathfrak{a}_I$,

$$e^{-\Gamma_\mathcal{I}(X)}\Phi_\lambda(a_Z \exp X) = \Phi_\lambda(a_Z) + \int_0^1 E_\lambda e^{-s\Gamma_\mathcal{I}(X)}\Psi_X(a_Z \exp(sX))\,ds\,.$$

Let $Y \in \mathfrak{a}_I^{--}$. Replacing a_Z by $a_Z \exp(tY)$ and multiplying by $e^{-t\Gamma_\mathcal{I}(Y)}$, one gets:

$$\begin{aligned} e^{-\Gamma_\mathcal{I}(X+tY)}\Phi_\lambda(a_Z \exp(X + tY)) &= e^{-\Gamma_\mathcal{I}(tY)}\Phi_\lambda(a_Z \exp(tY)) \\ &\quad + \int_0^1 E_\lambda e^{-\Gamma_\mathcal{I}(sX+tY)}\Psi_X(a_Z \exp(sX + tY))\,ds\,. \end{aligned}$$

Since $\lambda \in \mathcal{Q}^0_\mathcal{I}$, we obtain, from (5.34) and Lemma 5.8(ii), that the integral in this equality tends to 0 for $t \to \infty$. Recalling the definition of $\Phi_{f,\lambda,\infty}$ (cf. (5.35)), one gets

$$e^{-\Gamma_\mathcal{I}(X)}\Phi_{f,\lambda,\infty}(a_Z \exp X) = \Phi_{f,\lambda,\infty}(a_Z), \quad X \in \mathfrak{a}_I, a_Z \in A_Z\,.$$

■

Lemma 6.3. Let $N \in \mathbb{N}$. There exists a continuous semi-norm p on $C^\infty_{temp,N}(Z)$ such that, for all $f \in \mathcal{A}_{temp,N}(Z:\mathcal{I})$, $\lambda \in \mathcal{Q}^0_\mathcal{I}$,

$$\|\Phi_{f,\lambda,\infty}(a_{Z_I})\| \leq a_{Z_I}^{\rho_Q}(1 + \|\log a_{Z_I}\|)^{N+\dim U_\mathcal{I}} p(f), \quad a_{Z_I} \in A_{Z_I}^-\,.$$

Proof. We fix $X \in \mathfrak{a}_I^{--}$. Let $a_{Z_I} \in A_{Z_I}^-$. If t is large enough, $a_{Z_I}\exp(tX) \in A_Z^-$. More precisely, if $a_{Z_I} = \exp Y$ with $Y \in \mathfrak{a}_{Z_I}^-$, t has to be such that $\alpha(Y + tX) \leq 0$ for all $\alpha \in S \setminus I$. For this, it is enough that $t \geq |\frac{\alpha(Y)}{\alpha(X)}|$ for all $\alpha \in S \setminus I$. But $|\frac{\alpha(Y)}{\alpha(X)}|$ is bounded above by $C\|Y\|$ for some constant $C > 0$. We will take:

$$t = C\|Y\| \tag{6.3}$$

and write $a_{Z_I} = a_Z \exp(-tX)$ with $a_Z = a_{Z_I}\exp(tX) \in A_Z^-$. Since $\lambda \in \mathcal{Q}_{\mathcal{I}}^0$ and $\exp(-tX) = a_Z^{-1} a_{Z_I}$, one has, from Lemma 6.2,

$$\|\Phi_{f,\lambda,\infty}(a_{Z_I})\| = \|E_\lambda e^{-t\Gamma_{\mathcal{I}}(X)}\Phi_{f,\lambda,\infty}(a_Z)\| = a_{Z_I}^{\rho_Q} a_Z^{-\rho_Q}\|E_\lambda(-tX)\Phi_{f,\lambda,\infty}(a_Z)\|\,. \tag{6.4}$$

We know from Lemma 5.9 that $\|E_\lambda(-tX)\|$ is bounded by a constant times $(1 + t\|X\|)^{N_{\mathcal{I}}}$, where $N_{\mathcal{I}}$ is the dimension of $U_{\mathcal{I}}$. Using (6.3) and as X is fixed, one concludes that there exists $C_1 > 0$ such that:

$$\|E_\lambda(-tX)\| \leq C_1(1 + \|\log a_{Z_I}\|)^{N_{\mathcal{I}}}\,.$$

We remark that $\|\log a_Z\| \leq \|\log a_{Z_I}\| + \|tX\|$ is bounded by some constant times $\|\log a_{Z_I}\|$ because $t = C\|Y\|$ and $\|X\|$ is fixed. Then, using (6.4), the Lemma follows from Lemma 6.1 (applied with $t = 0$) and Lemma 5.8(i). ∎

We recall that $\Phi_{f,\lambda,\infty} = 0$ for $\lambda \in \mathcal{Q}_{\mathcal{I}}^+ \cup \mathcal{Q}_{\mathcal{I}}^-$ (cf. (5.33)). We obtain then, from Lemma 6.1, that:

Lemma 6.4. Let $N \in \mathbb{N}$. There exists a continuous semi-norm q on $\mathcal{A}_{temp,N}(Z)$ such that, for any $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$, $a_Z \in A_Z^-$, $X_I \in \mathfrak{a}_I^{--}$ and $t \geq 0$,

$$|(a_Z\exp(tX_I))^{-\rho_Q}\left[f(a_Z\exp(tX_I)) - f_I(a_Z\exp(tX_I))\right]|$$

$$\leq e^{t\delta\beta_I(X_I)}(1 + \|\log a_Z\|)^N(1 + t\|X_I\|)^{\dim U_{\mathcal{I}}}q(f)\,.$$

Note that the Lemma implies that:

$$\lim_{t\to\infty}(a_Z\exp(tX_I))^{-\rho_Q}[f(a_Z\exp(tX_I)) - f_I(a_Z\exp(tX_I))] = 0, \quad a_Z \in A_Z^-, X_I \in \mathfrak{a}_I^{--}\,. \tag{6.5}$$

6.3 The constant term as a smooth function on Z_I

Let us first start by the following general remark:

$$\begin{aligned}&\text{If an exponential polynomial function of one variable, } P(t)\text{, with}\\ &\text{unitary characters, satisfies: } \lim_{t\to+\infty} P(t)=0\,,\\ &\text{then } P\equiv 0.\end{aligned} \tag{6.6}$$

We define some linear forms η and η_I on $\mathcal{A}_{temp}(Z:\mathcal{I})$ by:

$$\begin{aligned}\langle \eta, f\rangle &= f(z_0),\\ \langle \eta_I, f\rangle &= f_I(z_{0,I}), \quad f\in\mathcal{A}_{temp}(Z:\mathcal{I})\,.\end{aligned}$$

Let us remark that η is a continuous linear form on $\mathcal{A}_{temp,N}(Z:\mathcal{I})$ for any $N\in\mathbb{N}$.

Note that we obtain from the definition (4.1) that:

$$m_{\eta_I,f}(a_Z)=f_I(a_Z), \quad a_Z\in A_Z\,.$$

Lemma 6.5. Let $N\in\mathbb{N}$. The linear form η_I is the unique linear form on $\mathcal{A}_{temp,N}(Z:\mathcal{I})$ such that:

(i) For any $f\in\mathcal{A}_{temp,N}(Z:\mathcal{I})$ and $X_I\in\mathfrak{a}_I^{--}$,

$$\lim_{t\to\infty} e^{-t\rho_Q(X_I)}[m_{\eta,f}(\exp(tX_I))-m_{\eta_I,f}(\exp(tX_I))]=0\,.$$

(ii) For any $f\in\mathcal{A}_{temp,N}(Z:\mathcal{I})$ and $X\in\mathfrak{a}_I$, $t\mapsto e^{-t\rho_Q(X)}m_{\eta_I,f}(\exp(tX))$ is an exponential polynomial with unitary characters.

(iii) Moreover, η_I is continuous on $\mathcal{A}_{temp,N}(Z:\mathcal{I})$ and H_I-invariant.

Proof. The assertion (i) is (6.5) and (ii) is (6.2).

To prove the unicity of such an η_I satisfying (i) and (ii), we use (6.6). If η_I' is another linear form satisfying (i) and (ii), then, for any $f\in\mathcal{A}_{temp,N}(Z:\mathcal{I})$,

$$m_{\eta_I,f}(\exp(tX_I))-m_{\eta_I',f}(\exp(tX_I))=0, \quad X_I\in\mathfrak{a}_I^{--}, t\in\mathbb{R}\,.$$

This equality applied to $t=0$ implies that $\eta_I=\eta_I'$.

Let us show the continuity of η_I. By taking $a_Z = 1$ in the inequality of Lemma 6.4, one gets:

$$|f(z_0) - f_I(z_{0,I})| \leq Cq(f), \text{i.e.}, |\langle \eta, f \rangle - \langle \eta_I, f \rangle| \leq Cq(f)\,.$$

Moreover η is a continuous map on $\mathcal{A}_{temp,N}(Z : \mathcal{I})$. This implies that η_I is continuous on $\mathcal{A}_{temp,N}(Z : \mathcal{I})$.

It remains to get that η_I is H_I-invariant. From (6.5), for any $X_I \in \mathfrak{a}_I^{--}$,

$$\lim_{t \to \infty} e^{-t\rho_Q(X_I)}[f(\exp(tX_I)) - f_I(\exp(tX_I))] = 0\,.$$

One applies this to $L_Y f$, $Y \in \mathfrak{h}_I$ and gets:

$$\lim_{t \to \infty} e^{-t\rho_Q(X_I)} \left[(L_Y f)(\exp(tX_I)) - (L_Y f)_I(\exp(tX_I))\right] = 0\,. \tag{6.7}$$

On the other hand, from Lemma 5.1, one has:

$$\lim_{t \to \infty} e^{-t\rho_Q(X_I)}(L_Y f)(\exp(tX_I)) = 0\,. \tag{6.8}$$

Hence, one gets, from (6.7) and (6.8), that:

$$\lim_{t \to \infty} e^{-t\rho_Q(X_I)}(L_Y f)_I(\exp(tX_I)) = 0\,.$$

But $t \mapsto e^{-t\rho_Q(X_I)}(L_Y f)_I(\exp(tX_I))$ is an exponential polynomial with unitary characters (cf. (6.2)). Hence, from (6.6), it is identically equal to 0. This implies that:

$$\eta_I(L_Y f) = 0\,.$$

Then η_I is continuous and $\mathfrak{h}_I$-invariant, and hence H_I-invariant. This completes the proof of (iii). ■

Let $N \in \mathbb{N}$ be fixed. For $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$, since η_I is continuous, we obtain with

$$g \mapsto f_I(g) := m_{\eta_I, f}(g), \quad g \in G\,, \tag{6.9}$$

a smooth extension of f_I previously defined on A_Z. Note that, as η_I is H_I-invariant, f_I defines a smooth function on Z_I denoted by the same symbol. Further, note that the assignment $f \mapsto f_I$ is G-equivariant, in symbols:

$$(L_g f)_I = L_g f_I, \quad g \in G. \tag{6.10}$$

Remark 6.6. As a consequence of Lemma 6.5 and the above equivariance relation (6.10), for all $g \in G$ and $X_I \in \mathfrak{a}_I^{--}$,

$$\lim_{t\to\infty} e^{-t\rho_Q(X_I)}[f(g\exp(tX_I)) - f_I(g\exp(tX_I))] = 0.$$

and $X \mapsto e^{-\rho_Q(X)} f_I(g\exp X)$ is an exponential polynomial on $\mathfrak{a}_I$ with unitary characters. Moreover, f_I is the unique smooth function on G with these two properties.

6.4 Consistency relations for the constant term

Let $w_I \in \mathcal{W}_I$ and $w \in \mathcal{W}$. Set $H_{I,w_I} = w_I H_I w_I^{-1}$ and $H_w = wHw^{-1}$. Consider the real spherical spaces $Z_w = G/H_w$ and $Z_{I,w_I} = G/H_{I,w_I}$, and put $z_0^w = H_w \in Z_w$ and $z_{0,I}^{w_I} = H_{I,w_I} \in Z_{I,w_I} = G/H_{I,w_I}$. Then (cf. [31,Corollary 3.8]) Q is Z_w-adapted to P and $A_{Z_w} = A_Z$ with $A_{Z_w}^- = A_Z^-$.

For $f \in C^\infty(Z)$, let us define f^w by:

$$f^w(g \cdot z_0^w) = f(gw \cdot z_0), \quad g \in G.$$

In the same way, one defines ϕ^{w_I} for $\phi \in C^\infty(Z_I)$. Then $f^w \in C^\infty(Z_w)$ and $\phi^{w_I} \in C^\infty(Z_{I,w_I})$.

Proposition 6.7 (Consistency relations for the constant term). Let $w_I \in \mathcal{W}_I$ and $w = \mathbf{m}(w) \in \mathcal{W}$. Let $f \in \mathcal{A}_{temp,N}(Z:\mathcal{I})$. Then $f^w \in \mathcal{A}_{temp,N}(Z_w:\mathcal{I})$ and

$$(f_I)^{w_I}(a_Z) = (f^w)_I(a_Z), \quad a_Z \in A_Z.$$

Here, $f^w \in \mathcal{A}_{temp}(Z_w:\mathcal{I})$, $(f^w)_I \in C^\infty(Z_{w,I})$, $f_I \in C^\infty(Z_I)$, $f_I^{w_I} \in C^\infty(Z_{I,w_I})$, and, from [31,Proposition 3.2(5) and Corollary 3.8], one has:

$$A_{Z_{w,I}} = A_{Z_w} = A_Z,$$
$$A_{Z_{I,w_I}} = A_{Z_I} = A_Z.$$

Hence, both sides of the equality are well-defined on A_Z.

The proof of Proposition 6 is prepared by a simple technical lemma. Recall the elements $a_s = \exp(sX_I)$ for $X_I \in \mathfrak{a}_I^{--}$.

Lemma 6.8. Let (g'_s) be a family in G, which converges rapidly to $g \in G$. Let $f \in \mathcal{A}_{temp,N}(Z)$. Then there exist $C > 0$ and $\varepsilon > 0$ such that:

$$|(L_{(g'_s)^{-1}} f)(a_s) - (L_{g^{-1}} f)(a_s)| \leq C a_s^{\rho_Q} e^{-\varepsilon s}, \quad s \geq s_0 .$$

Proof. As (g'_s) converges rapidly to g when s tends to $+\infty$, there exists s'_0, C', ε' strictly positive and $(X_s) \subset \mathfrak{g}$ such that, for all $s \geq s'_0$,

$$g'_s = g \exp X_s \text{and} \|X_s\| \leq C' e^{-\varepsilon' s} . \tag{6.11}$$

As $L_{g^{-1}}$ preserves $\mathcal{A}_{temp,N}(Z)$, one is reduced to prove, for all $f \in \mathcal{A}_{temp,N}(Z)$, that there exist $C, \varepsilon, s_0 > 0$ such that:

$$|f(\exp(X_s)a_s) - f(a_s)| \leq C a_s^{\rho_Q} e^{-\varepsilon s} .$$

But, by the mean value theorem, if $a \in A_Z$ and $X \in \mathfrak{g}$,

$$|f(\exp(X)a) - f(a)| \leq \sup_{t \in [0,1]} |(L_{-X} f)(\exp(tX)a)| \|X\| .$$

From (6.11), one then sees that it is enough to prove that, if $\|X\|$ is bounded by a constant $C'' > 0$, there exists a constant $C''' > 0$ such that:

$$\sup_{t \in [0,1]} |(L_{-X} f)(\exp(tX)a)| \leq C''' a^{\rho_Q} (1 + \| \log a \|)^N, \quad a \in A_Z^- . \tag{6.12}$$

Decomposing $-X$ in a basis (X_i) of $\mathfrak{g}$ and using the continuity of the endomorphisms L_{X_i} of $\mathcal{A}_{temp,N}(Z)$, one sees that there exists a continuous semi-norm such that:

$$|(L_{-X} f)(a)| \leq a^{\rho_Q} (1 + \| \log a \|)^N q(f), \quad a \in A_Z^- .$$

But $f \mapsto \sup_{\|X\| \leq C''} q(L_{\exp(-tX)} f)$ is a continuous semi-norm on $\mathcal{A}_{temp,N}(Z)$. Hence, as L_{-X} and $L_{\exp(-tX)}$ commute, (6.12) follows. This achieves to prove the Lemma. ■

Proof of Proposition 6.7 If $a \in A$, one has:

$$[(L_a f)^W]_I = [L_a (f^W)]_I \text{as} (L_a f)^W = L_a f^W .$$

Hence, it is enough to prove the identity of the Proposition for $a_Z = z_0$. Then, using (6.6) and Remark 6.6, it is enough to prove that $s \mapsto (f_I)^{w_I}(a_s)$ is an exponential polynomial with unitary characters satisfying:

$$\lim_{s\to+\infty} a_s^{-\rho_Q}[f^w(a_s) - (f_I)^{w_I}(a_s)] = 0\,. \tag{6.13}$$

But, from (3.7),

$$\tilde{a}_s w \cdot z_0 = (\tilde{a}_s \tilde{b}_s^{-1} m_s^{-1} u_s^{-1})(u_s m_s \tilde{b}_s w) \cdot z_0 = g_s w_I \tilde{a}_s \cdot z_0\,,$$

for $s \geq s_0$, where $g_s = \tilde{a}_s \tilde{b}_s^{-1} m_s^{-1} u_s^{-1}$. Then one has:

$$f^w(a_s) = L_{w_I^{-1} g_s^{-1}} f(a_s)\,.$$

On the other hand, from [31,Lemma 3.5] for $Z = Z_I$, as $A_{Z_I,E} = A_I$ (cf. loc.cit., equation (3.13)), one has:

$$\tilde{a}_s w_I \cdot z_{0,I} = w_I \tilde{a}_s \cdot z_{0,I}\,, \tag{6.14}$$

which implies that:

$$(L_{w_I^{-1}} f_I)(\tilde{a}_s \cdot z_{0,I}) = (f_I)^{w_I}(a_s)\,. \tag{6.15}$$

Now, according to Proposition 3.1 – this is the key ingredient! –, the sequence $(g_s w_I)$ converges rapidly to w_I. Hence, we can apply Lemma 6.8 with $g'_s = g_s w_I$ and find $C', \varepsilon', s'_0 > 0$ such that:

$$a_s^{-\rho_Q} |(L_{w_I^{-1} g_s^{-1}} f)(a_s) - (L_{w_I^{-1}} f)(a_s)| \leq C' e^{-\varepsilon' s}, \quad s \geq s'_0\,. \tag{6.16}$$

Using Lemma 6.4, one has, for some $C'', \varepsilon' > 0$,

$$a_s^{-\rho_Q} |(L_{w_I^{-1}} f)(a_s) - (L_{w_I^{-1}} f_I)(a_s)| \leq C'' e^{-\varepsilon' s}, \quad s \geq s'_0\,.$$

Hence, from (6.15) and (6.16), one deduces (6.13). It remains to prove that:

$$s \mapsto (f_I)^{w_I}(a_s) = f_I(a_s w_I \cdot z_{0,I})$$

is an exponential polynomial with unitary characters. But, from [31,Lemma 3.5] applied to Z_I,

$$(f_I)^{w_I}(a_s) = f_I(w_I a_s)\,.$$

Hence, our claim follows from (6.14). This achieves the proof of the Proposition. ■

6.5 Constant term approximation

Now we turn to the main Theorem of this section.

Theorem 6.9 (Constant term approximation). Let $I \subset S$ and $\mathcal{I}$ be a finite codimensional ideal of $\mathbf{D}_0(Z)$.

(i) For all $N \in \mathbb{N}$, the map $f \mapsto f_I$ is a continuous linear map from $\mathcal{A}_{temp,N}(Z : \mathcal{I})$ to $\mathcal{A}_{temp,N+\dim\, U_{\mathcal{I}}}(Z_I : \mu_I(\mathcal{I}))$.

(ii) Let $N \in \mathbb{N}$ and $\mathcal{C}_I$ be a compact subset of $\mathfrak{a}_I^{--}$. For $w_I \in \mathcal{W}_I$ let $w = \mathbf{m}(w_I) \in \mathcal{W}$. Then there exist $\varepsilon > 0$ and a continuous semi-norm p on $C^\infty_{temp,N}(Z)$ such that, for all $f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})$,

$$|(a_Z \exp(tX))^{-\rho_Q} \left(f(ga_Z \exp(tX_I)w \cdot z_0) - f_I(ga_Z \exp(tX_I)w_I \cdot z_{0,I})\right)|$$

$$\leq e^{-\varepsilon t}(1 + \|\log a_Z\|)^N p(f), \qquad a_Z \in A_Z^-, X_I \in \mathcal{C}_I, g \in \Omega, w_I \in \mathcal{W}_I, t \geq 0\,.$$

Proof. We first show (i). In view of (4.6), it suffices to prove that, for any $w_I \in \mathcal{W}_I$, there exists a continuous semi-norm p on $\mathcal{A}_{temp,N}(Z : \mathcal{I})$ such that:

$$\sup_{g\in\Omega, a_{Z_I}\in A^-_{Z_I}} |a_{Z_I}^{-\rho_Q}(1 + \|\log a_{Z_I}\|)^{-(N+\dim\, U_{\mathcal{I}})} f_I(ga_{Z_I}w_I)| \leq p(f), \quad f \in \mathcal{A}_{temp,N}(Z : \mathcal{I})\,.$$

For $w_I = 1$, one has $w = \mathbf{m}(w_I) = 1$. Our claim then follows from (6.10), (6.1) and Lemma 6.3 and the continuity of the left regular representation of G on $C^\infty_{temp,N}(Z)$ (see (4.7)).

For general w_I, one uses Proposition 6 to get $f_I(a_{Z_I}w_I) = (f^w)_I(a_{Z_I})$ and the above inequality for H^w instead of H. This shows (i).

Using Proposition 6, one is reduced to prove (ii) for $w_I = w = 1$, by changing H into H_w. Moreover, from (6.10) and (4.7), one is reduced to show (ii) for $g = 1$. In that case, (ii) follows from (5.24) (applied with $u = 1$), (6.1) and Lemma 6.1 by choosing $\varepsilon > 0$ and p in the following way.

Set $N_{\mathcal{I}} := \dim U_{\mathcal{I}}$. Let us consider the continuous function $\varphi : (X_I, t) \mapsto e^{t\delta\beta_I(X_I)/2}(1 + t\|X_I\|)^{N_{\mathcal{I}}}$ on $\mathfrak{a}_I \times \mathbb{R}$, which is smooth on the second variable and positive on $\mathfrak{a}_I \times \mathbb{R}_{\geq 0}$. Recall that $\delta\beta_I(X_I) < 0$ for any $X_I \in \mathfrak{a}_I^{--}$. Since $\mathcal{C}_I$ is a compact subset of $\mathfrak{a}_I^{--}$, by continuity, $C := \max_{X_I \in \mathcal{C}_I} \varphi(X_I, -2N_{\mathcal{I}}/\delta\beta_I(X_I) - 1/\|X_I\|)$ and $\varepsilon := -\delta/2[\max_{X_I \in \mathcal{C}_I}(\beta_I(X_I))]$ exist and $\varepsilon > 0$. Moreover, φ has values $\leq C$ on $\mathcal{C}_I \times \mathbb{R}$. Hence $C > 0$ and, by Lemma 6.1, ε yields the inequality in (ii) for $t \geq 0$ by setting $p := Cq$. ∎

Remark 6.10 (Statement for H_0 connected). Theorem 6.1 remains valid for H replaced by H_0: exchange the expression $f_I(ga_Z \exp(tX)w_I \cdot z_{0,I})$ by $f_I(gm_{w_I}^{-1}a_Z \exp(tX)w_I \cdot z_{0,I})$ for certain $m_{w_I} \in M$, see Remark 5(b). Likewise, this will hold for Theorem 8.1 below, which generalizes Theorem 6.1.

Remark 6.11. Reformulation of Theorem 6.1 in terms of representation theory

Let (π, V^∞) be an *SAF*-representation of G, for example $V^\infty = \mathcal{A}_{temp}(Z : \mathcal{I})$ (see Proposition 4.1). Then Theorem 6.1(i) gives rise to a linear map

$$(V^{-\infty})^H_{temp} \longrightarrow (V^{-\infty})^{H_I}_{temp}, \quad \eta \mapsto \eta_I$$

and correspondingly, for every $v \in V^\infty$, an approximation of the matrix coefficient $g \mapsto f(g \cdot z_0) = m_{\eta,v}(g)$ by $g \mapsto f_I(g \cdot z_{0,I}) = m_{\eta_I,v}(g)$ as in Theorem 6.1(ii). In this language, the consistency relations from Proposition 6 then translate into

$$(w \cdot \eta)_I = w_I \cdot \eta_I \qquad w_I \in \mathcal{W}_I, w = \mathbf{m}(w_I),$$

where, for an element $\xi \in V^{-\infty}$ and $g \in G$, we use the notation $g \cdot \xi = \xi(g^{-1}\cdot)$ for the dual action.

6.6 Application to the relative discrete series for Z

Let χ be a normalized unitary character of $A_{Z,E} = \exp(\mathfrak{a}_{Z,E})$, i.e., $d\chi_{|\mathfrak{a}_{Z,E}} = \rho_Q|_{\mathfrak{a}_{Z,E}}$.

We recall that, if $a \in A_{Z,E}$ and $w \in \mathcal{W}$,

$$\tilde{a}wH = waH \tag{6.17}$$

(cf. [31,Lemma 3.5]).

As $\widetilde{A}_{Z,E}$ normalizes H, there is a right action $(a, z) \mapsto z \cdot a$ of $A_{Z,E}$ on Z. Let $C^\infty(Z, \chi)$ be the space of C^∞ functions on Z such that:

$$f(z \cdot a) = \chi(a) f(z), \quad a \in A_{Z,E}, z \in Z$$

and observe that

$$|a^{-\rho_Q} f(z \cdot a)| = |f(z)|, \quad a \in A_{Z,E}, z \in Z, \tag{6.18}$$

as χ was requested to be normalized unitary.

If $f \in C^\infty(Z, \chi)$, $u \in \mathcal{U}(\mathfrak{g})$ and $N \in \mathbb{N}$, then (6.17) and (6.18) allow us to define

$$r_{N,u}(f) = \sup_{g \in \Omega, a \in A_Z^-/A_{Z,E}, w \in \mathcal{W}} |a^{-\rho_Q}(1 + \|\log a\|)^N (L_u f)(gaw \cdot z_0)|,$$

with $\| \cdot \|$ refering to the quotient norm on $\mathfrak{a}_Z/\mathfrak{a}_{Z,E}$. Moreover, we set

$$\mathcal{C}(Z, \chi) = \{f \in C^\infty(Z, \chi) \mid r_{N,u}(f) < \infty, N \in \mathbb{N}, u \in \mathcal{U}(\mathfrak{g})\}.$$

Since $\widetilde{A}_{Z,E}$ normalizes H, we obtain a closed subgroup $\widehat{H} := H\widetilde{A}_{Z,E}$ (not depending on the section **s**) and a real spherical space $\widehat{Z} = G/\widehat{H}$. We extend χ trivially to H and then define a character of $\widehat{H}$ still denoted χ. Let us define $L^2(\widehat{Z}; \chi)$ as in [31,Section 8.1].

Let $w \in \mathcal{W}$. We recall that $H_w = wHw^{-1}$ and $Z_w = G/H_w$. Let f be in $C^\infty(Z, \chi)$. Recall that f_w defined by $f_w(g) = f(gwH)$, $g \in G$, is right H_w-invariant and defines an element of $C^\infty(Z_w)$ and even of $C^\infty(Z_w, \chi)$ by using the relation (6.17). This element will still be denoted f_w. Moreover, by "transport of structure", if f is Z-tempered, f_w is Z_w-tempered.

Let η be a Z-tempered H-fixed linear form on V^∞. Let $w \in \mathcal{W}$. Then $\mathfrak{a}_{Z_w} = \mathfrak{a}_Z$ and $w \cdot \eta$ is H_w-invariant and Z_w-tempered by "transport of structure". By [31,Corollary 3.8], Q is Z_w-adapted to P. Moreover, the set of spherical roots for Z_w is equal to S (see [31,equation (3.2), definition of S in the beginning of Section 3.2 and Lemma 3.7]). Hence, one can define $(w \cdot \eta)_I$, $w \in \mathcal{W}$.

Theorem 6.12. Let (π, V^∞) be an *SAF*-representation of G, with V its associated Harish-Chandra module, and η be a Z-tempered continuous linear form on V^∞, which transforms under a unitary character χ of $A_{Z,E}$. Then the following assertions are equivalent:

(i) For all $v \in V$, $m_{\eta,v} \in L^2(\widehat{Z};\chi)$.
(ii) For all proper subsets I of S and $w \in \mathcal{W}$, $(w\cdot\eta)_I = 0$.
(iii) For all $v \in V^\infty$, $m_{\eta,v} \in \mathcal{C}(Z,\chi)$.

Proof. Let us assume (i). We may assume that $V \subset L^2(\widehat{Z};\chi)$ via the embedding $v \mapsto m_{\eta,v}$. Let $\mathcal{H}$ be the unitary completion of V in $L^2(\widehat{Z};\chi)$. Then $V^\infty = \mathcal{H}^\infty$ and this implies that the statement is independent of the particular choice of the maximal compact subgroup K. We choose now K as in [31,Sect. 6]. In particular we obtain an open neighborhood U_A of 1 in A such that all $m_{\eta,v}$, $v \in V$ admit absolutely convergent power series expansions [31,(6.2)]

$$m_{\eta,v}(aw) = \sum_{j=1}^{l}\sum_{\alpha\in\mathbb{N}_0[S]} a^{\Lambda_j+\alpha} q_{\alpha,j,w}(\log a) \qquad (a \in U_A\cdot A_Z, w \in \mathcal{W}), \tag{6.19}$$

where $\Lambda_j \in \mathfrak{a}_{Z,\mathbb{C}}^*$, $1 \le j \le l$, and $q_{\alpha,j,w}$ are polynomials on $\mathfrak{a}_Z$. Let $S = \{\sigma_1,\dots,\sigma_s\}$ and $\omega_1,\dots,\omega_s \in \mathfrak{a}_Z$ be such that:

$$\begin{aligned} &\sigma_i(\omega_j) = \delta_{i,j}, && i,j = 1,\dots,s \\ &\omega_i \perp \mathfrak{a}_{Z,E}, && i = 1,\dots,s\,. \end{aligned}$$

Here we use the scalar product on $\mathfrak{a}_Z$ defined before (2.7). According to [31,Theorem 8.5] the condition that all $m_{\eta,v} \in L^2(\widehat{Z};\chi)$ implies that

$$\mathrm{Re}(\Lambda_k - \rho_Q)(\omega_j) > 0, \quad j = 1,\dots,s,\ k = 1,\dots,l. \tag{6.20}$$

Now (6.19) in combination with (6.20) imply the existence of an $\varepsilon > 0$ such that for all $v \in V$ there exists a constant $C_v > 0$ such that

$$|m_{\eta,v}(aw)| \le C_v a^{(1+\varepsilon)\rho_Q} \qquad (a \in A_Z^-)\,. \tag{6.21}$$

On the other hand by the constant term approximation (Lemma 6.5(i) applied to Z_w and η replaced by $w\cdot\eta$) we obtain that

$$\lim_{t\to\infty} e^{t\rho_Q(X_I)}\Big(m_{w\cdot\eta,v}(\exp(tX_I)) - m_{(w\cdot\eta)_I,v}(\exp(tX_I))\Big) = 0$$

for all $X \in \mathfrak{a}_I^{--} \subset \mathfrak{a}_Z^-$. Moreover $t \mapsto m_{(w\cdot\eta)_I,v}(\exp(tX))$ is the unique exponential polynomial with unitary characters having this approximation property by Lemma 6.5(ii). Hence (6.21) implies that this exponential polynomial is zero. In particular, $\langle (w\cdot\eta)_I, v\rangle = m_{(w\cdot\eta)_I,v}(1) = 0$ for all $v \in V$, and hence, by density of V in V^∞, $(w\cdot\eta)_I = 0$, that is (ii).

Let us assume that (ii) holds. Let $\mathcal{I}$ be an ideal of $\mathcal{Z}(\mathfrak{g})$, which annihilates V or V^∞. It is of finite codimension. Since η is Z-tempered, there exists $N_0 \in \mathbb{N}$ such that, for all $v \in V^\infty$, $m_{\eta,v} \in \mathcal{A}_{temp,N_0}(Z:\mathcal{I})$ (cf. (4.9)). Let $v \in V^\infty$ and set $f = m_{\eta,v}$. Then one can apply Theorem 6.1 to Z_w and f_w for w_I equal to 1: Let $I \subsetneq S$, $\mathcal{C}$ be a compact subset of $\mathfrak{a}_I^{--}$, Ω_1 be a compact subset of G and $u \in \mathcal{U}(\mathfrak{g})$. Then there exists a continuous semi-norm p on $C^\infty_{temp,N_0}(Z)$, $\varepsilon > 0$ such that:

$$\begin{aligned} &|(a_Z \exp(tX))^{-\rho_Q}(L_u f)(g a_Z \exp(tX) w\cdot z_0)| \\ &\le e^{-\varepsilon t}(1+\|\log a_Z\|)^{N_0} p(f), \quad a_Z \in A_Z^-/A_{Z,E}, X \in \mathcal{C}, g \in \Omega_1, w \in \mathcal{W}, t \ge 0\,. \end{aligned} \tag{6.22}$$

Note, as η transforms under a unitary character for $A_{Z,E}$ (see (6.17)), the left hand side in (6.22) depends only on $a_Z \exp(tX)$ mod $A_{Z,E}$.

Let $\mathbf{S}_1$ be the unit sphere in $\mathfrak{a}_Z/\mathfrak{a}_{Z,E}$ and let $\widetilde{X}_0 \in \mathbf{S}_1 \cap \mathfrak{a}_Z^-/\mathfrak{a}_{Z,E}$. Let Ω_0 be an open neighborhood of $\widetilde{X}_0$ in $S_1 \cap \mathfrak{a}_Z^-/\mathfrak{a}_{Z,E}$ such that, for all $\widetilde{X} \in \Omega_0$, $\alpha(\widetilde{X}) \le \alpha(\widetilde{X}_0)/2$, $\alpha \in S$. Let I be the set of $\alpha \in S$ such that $\alpha(\widetilde{X}_0) = 0$. One has $I \neq S$ as we may assume that $S \neq \emptyset$. Let $X_0 \in \mathfrak{a}_I$ be a lift of $\widetilde{X}_0$ and note that $X_0 \in \mathfrak{a}_I^{--}$. Let $Y \in \Omega_0$ and $t \ge 0$. Then $t(Y - X_0/2) \in \mathfrak{a}_Z^-/\mathfrak{a}_{Z,E}$ and $\exp(tY) = \exp t(Y - \widetilde{X}_0/2)\exp(t\widetilde{X}_0/2) \in A_Z/A_{Z,E}$. Using (6.22) for $X = X_0/2$ and $a_Z = \exp t(Y - \widetilde{X}_0/2) \in A_Z^-/A_{Z.E}$ one gets: For any $N \in \mathbb{N}$ there exists a $c > 0$, depending on $N, \varepsilon, f, \Omega_0$ and Ω_1 such that

$$\begin{aligned} &|(\exp(tY))^{-\rho_Q}(L_u f)(g \exp(tY) w\cdot z_0)| \\ &\le e^{-\varepsilon t}(1 + t\|Y - \widetilde{X}_0/2\|)^{N_0} p(f) \le c(1+t)^{-N}, \quad Y \in \Omega_0, g \in \Omega_1, w \in \mathcal{W}, t \ge 0\,, \end{aligned}$$

One deduces easily from this that, for any $u \in \mathcal{U}(\mathfrak{g})$ and $N \in \mathbb{N}$:

$$\sup_{g\in\Omega_1, w\in\mathcal{W}, a\in\exp(\mathbb{R}^+\Omega_0)} a^{-\rho_Q}(1+\|\log a\|)^N |(L_u f)(gaw\cdot z_0)| < +\infty\,.$$

Using a finite covering of the compact set $\mathbf{S}_1 \cap \mathfrak{a}_Z^-/\mathfrak{a}_{Z,E}$, one deduces from this that $f \in \mathcal{C}(Z,\chi)$. This achieves to prove that (ii) implies (iii).

To prove that (iii) implies (i), one proceeds as in the proof that (ii) implies (i) in [31,Theorem 8.5]. ■

7 Transitivity of the constant term

Recall that $\mathcal{A}_{temp}(Z)$ consists of $\mathcal{Z}(\mathfrak{g})$-finite functions. In particular, for each $f \in \mathcal{A}_{temp}(Z)$ there exists a co-finite ideal $\mathcal{J} \subset \mathcal{Z}(\mathfrak{g})$ such that $f \in \mathcal{A}_{temp}(Z : \mathcal{J})$. Hence constant terms f_I are defined for all $f \in \mathcal{A}_{temp}(Z)$.

Proposition 7.1 (Transitivity of the constant term). Let $I \subset J$ be two subsets of S. Then, if $f \in \mathcal{A}_{temp}(Z)$,

$$f_I = (f_J)_I .$$

Proof. By G-equivariance of the maps:

$$\begin{array}{ccc} \mathcal{A}_{temp}(Z) & \to & \mathcal{A}_{temp}(Z_I) \\ f & \mapsto & f_I \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{A}_{temp}(Z_J) & \to & \mathcal{A}_{temp}(Z_I) \\ f & \mapsto & f_I \end{array} ,$$

it is enough to show that, if $f \in \mathcal{A}_{temp}(Z)$, $f_I(z_{0,I}) = (f_J)_I(z_{0,I})$. Recall that $\mathfrak{a}_{Z_J} = \mathfrak{a}_Z$ and

$$\mathfrak{a}_I^{--} = \{X \in \mathfrak{a}_I : \alpha(X) < 0, \alpha \in S \setminus I\}, \quad \mathfrak{a}_{I,J}^{--} = \{X \in \mathfrak{a}_I : \alpha(X) < 0, \alpha \in J \setminus I\} .$$

As $\mathfrak{a}_I = \{X \in \mathfrak{a}_Z : \alpha(X) = 0, \alpha \in I\}$ and $\mathfrak{a}_J = \{X \in \mathfrak{a}_Z : \alpha(X) = 0, \alpha \in J\}$, one has:

$$\mathfrak{a}_J \subset \mathfrak{a}_I, \quad \mathfrak{a}_I^{--} \subset \mathfrak{a}_Z^-, \quad \mathfrak{a}_{I,J}^{--} \subset \mathfrak{a}_Z^- .$$

One remarks that $\mathfrak{a}_I^{--} \subset \mathfrak{a}_{I,J}^{--}$. Let $X \in \mathfrak{a}_J^{--}$ and $Y \in \mathfrak{a}_I^{--}$. Then $X + Y \in \mathfrak{a}_I^{--}$.

Using Theorem 6.1(ii) applied successively to $(Z, I, f, X + Y, 1)$, $(Z, J, f, X, \exp(tY))$ and $(Z_J, I, f_J, Y, \exp(tX))$ instead of (Z, I, f, X, a_Z), one gets that there exist $C > 0$ and $\varepsilon > 0$ such that, for all $t \geq 0$,

$$\alpha_t |f(\exp(t(X + Y))) - f_I(\exp(t(X + Y)))| \leq Ce^{-\varepsilon t} ,$$

$$\alpha_t |f(\exp(tY) \exp(tX)) - f_J(\exp(tY) \exp(tX))| \leq Ce^{-\varepsilon t} ,$$

$$\alpha_t |f_J(\exp(tX) \exp(tY)) - (f_J)_I(\exp(tX) \exp(tY))| \leq Ce^{-\varepsilon t} ,$$

where $\alpha_t = e^{-t\rho_Q(X+Y)}$. Hence, one concludes from the three inequalities above that:

$$\alpha_t|f_I(\exp(t(X+Y))) - (f_J)_I(\exp(t(X+Y)))| \leq 3Ce^{-\varepsilon t}, \quad t \geq 0\,.$$

Hence, $\alpha_t[f_I(\exp(t(X+Y))) - (f_J)_I(\exp(t(X+Y)))]$ tends to zero when t goes to $+\infty$. But, each term of this difference is an exponential polynomial in t with unitary characters. Hence, according to (6.6), the difference of the two occurring exponential polynomials is identically zero. It implies, taking $t = 0$, that $f_I(z_{0,I}) = (f_J)_I(z_{0,I})$. ■

7.1 Application: Tempered embedding theorem

From the constant term approximation in Th. 6.9, the consistency relations (Prop. 6.7) and the transitivity of the constant term (Prop. 7.1) one can quite easily derive an extension of the tempered embedding theorem [31, Th. 9.11] to all real spherical spaces. The details are carried out in [12, Th. 11.12] and we record for later reference:

Theorem 7.2 (Tempered embedding theorem). Let V be an irreducible Harish-Chandra module contained in $C^\infty_{temp}(Z)$. Then there exists $I \subset S$, $w \in \mathcal{W}$ and a unitary character χ of A_I such that there is an $(\mathfrak{g}, K)$-embedding

$$V \hookrightarrow L^2(\widehat{(Z_w)_I}, \chi)\,.$$

8 Uniform estimates

The goal of this section is to obtain a parameter independent version of the main result Theorem 6.9: the bounds become uniform if we restrict ourselves to ideals $\mathcal{I}$ of $\mathbf{D}_0(Z)$ of codimension one. The crucial ingredient is a recent result that infinitesimal characters of tempered representations have integral real parts (see [33] and summarized in Lemma 8.8 below).

Recall the Cartan subalgebra $\mathfrak{j} = \mathfrak{a} \oplus \mathfrak{t} \subset \mathfrak{g}$ with real form $\mathfrak{j}_\mathbb{R} = \mathfrak{a} \oplus i\mathfrak{t} \subset \mathfrak{j}_\mathbb{C}$, associated Weyl group $W_\mathfrak{j}$ and half sum of roots $\rho_\mathfrak{j}$. Note that $\rho_\mathfrak{j}\big|_{\mathfrak{a}_H} = \rho\big|_{\mathfrak{a}_H} = \rho_Q\big|_{\mathfrak{a}_H} = 0$ as Z was requested to be unimodular. In particular $\rho_\mathfrak{j}\big|_\mathfrak{a}$ factors through $\mathfrak{a}_Z$ and coincides with ρ_Q.

If $\Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}$, let χ_Λ be the character of $\mathcal{Z}(\mathfrak{g})$ corresponding to Λ via the Harish-Chandra isomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{j})^{W_\mathfrak{j}}$. More precisely,

$$\chi_\Lambda(u) = (\gamma(u))(\Lambda), \quad u \in \mathcal{Z}(\mathfrak{g})\,.$$

Further, we set $\mathcal{J}_\Lambda := \ker \chi_\Lambda$. We also recall the untwisted Harish-Chandra homomorphism $\gamma_0 : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{j})$ and set $\mathcal{J}_{\Lambda,0} := \gamma_0(\mathcal{J}_\Lambda)$.

According to Chevalley's theorem, $S(\mathfrak{j})$ is a free module of finite rank over $S(\mathfrak{j})^{W_\mathfrak{j}} \simeq \gamma_0(\mathcal{Z}(\mathfrak{g}))$. Hence, we obtain a subspace $U_0 \subset S(\mathfrak{j})$ such that the natural map:

$$\gamma_0(\mathcal{Z}(\mathfrak{g})) \otimes U_0 \to S(\mathfrak{j}), \quad v \otimes u \mapsto vu$$

is an isomorphism. Thus, for any $\Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}$, we obtain with $\gamma_0(\mathcal{Z}(\mathfrak{g})) = \mathcal{J}_{\Lambda,0} + \mathbb{C}1$ that $S(\mathfrak{j})/S(\mathfrak{j})\mathcal{J}_{\Lambda,0} \simeq U_0$ as vector spaces. The natural representation of $S(\mathfrak{j})$ on $S(\mathfrak{j})/S(\mathfrak{j})\mathcal{J}_{\Lambda,0} \simeq U_0$ gives then rise to a $S(\mathfrak{j})$-representation:

$$\sigma_\Lambda : S(\mathfrak{j}) \to \mathrm{End}(U_0)\,.$$

For $\Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}$, let us fix a representative $\lambda \in \mathfrak{j}_\mathbb{C}^*$ such that $\Lambda = W_\mathfrak{j} \cdot \lambda$.

Lemma 8.1. The following assertions hold:

1. The representation (σ_Λ, U_0) is polynomial in Λ, i.e., for all $v \in S(\mathfrak{j})$, the assignment $\Lambda \mapsto \sigma_\Lambda(v)$ is polynomial.
2. One has $\mathrm{Spec}(\sigma_\Lambda) = \rho_\mathfrak{j} + W_\mathfrak{j} \cdot \lambda$.

Proof. We prove both assertions together. Consider the auxiliary $S(\mathfrak{j})$-module $S(\mathfrak{j})/S(\mathfrak{j})\mathcal{J}_\Lambda$ and call the corresponding representation of $S(\mathfrak{j})$ by σ'_Λ. We have $\mathrm{Spec}(\sigma'_\Lambda) = W_\mathfrak{j} \cdot \lambda$. Recall the complement $U_0 \subset S(\mathfrak{j})$ and let $U_1 = U_0(\cdot + \rho_\mathfrak{j}) \subset S(\mathfrak{j})$ obtained from $\rho_\mathfrak{j}$-shift. We model σ'_Λ on U_1 and claim that $v \mapsto \sigma'_\Lambda(v)$ is polynomial in Λ. It suffices to verify the assertion for v of the form $v = \gamma(z)u$ with $z \in \mathcal{Z}(\mathfrak{g})$ and $u \in U_1$. Now

$$v = u(\gamma(z) - \chi_\Lambda(z)) + \chi_\Lambda(z)u$$

with the first sum in the ideal $S(\mathfrak{j})\mathcal{J}_\Lambda$. The claim follows. It remains to relate the representation σ_Λ to σ'_Λ, which is given by $\sigma_\Lambda(v) = \sigma'_\Lambda(v(\cdot + \rho_\mathfrak{j}))$ via the algebra automorphism $S(\mathfrak{j}) \to S(\mathfrak{j})$, $v \mapsto v(\cdot + \rho_\mathfrak{j})$ obtained by the $\rho_\mathfrak{j}$-shift upon identification $S(\mathfrak{j}) \simeq \mathbb{C}[\mathfrak{j}_\mathbb{C}^*]$. ■

Recall that there is a surjective algebra morphism $p : \mathcal{Z}(\mathfrak{g}) \to \mathbf{D}_0(Z)$. Given a codimension one ideal $\mathcal{I}$ in $\mathbf{D}_0(Z)$, its preimage $\mathcal{J} = p^{-1}(\mathcal{I})$ is of codimension one in $\mathcal{Z}(\mathfrak{g})$, hence, of the form $\mathcal{J}_\Lambda$, for some $\Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}$.

Denote by $\mathbb{X} \subset \mathfrak{j}_{\mathbb{C}}^*/W_{\mathfrak{j}}$ the set of Λ's obtained that way. For $\Lambda \in \mathbb{X}$, we set $\mathcal{I}_\Lambda := p(\mathcal{J}_\Lambda)$.

Next we wish to describe the set $\mathbb{X}$ more closely. Since $\mathbf{D}_0(Z)$ is a finitely generated $\mathbb{C}$-algebra without nilpotent elements, its maximal spectrum $\mathrm{specmax}(\mathbf{D}_0(Z))$ is an affine variety and naturally identifies with $\mathbb{X}$. The surjective algebra morphism $p : \mathcal{Z}(\mathfrak{g}) \to \mathbf{D}_0(Z)$ gives rise to the closed embedding:

$$p_* : \mathbb{X} = \mathrm{specmax}(\mathbf{D}_0(Z)) \hookrightarrow \mathfrak{j}_{\mathbb{C}}^*/W_{\mathfrak{j}} = \mathrm{specmax}(\mathcal{Z}(\mathfrak{g}))\,.$$

We recall our choice of $\mathfrak{t}$ and $\mathfrak{t}_H$ before Lemma 5.6.

Lemma 8.2. The affine subvariety $\mathbb{X} \subset \mathfrak{j}_{\mathbb{C}}^*/W_{\mathfrak{j}}$ is given by

$$\mathbb{X} = \{\Lambda \in \mathfrak{j}_{\mathbb{C}}^*/W_{\mathfrak{j}} \mid \exists \mu \in \Lambda = W_{\mathfrak{j}} \cdot \lambda \text{ such that } (\rho_{\mathfrak{j}} + \mu)\big|_{\mathfrak{a}_H + \mathfrak{t}_H} = 0\} \tag{8.1}$$

To prepare the proof of this Lemma, we need to develop a little bit of general theory, which is used later on as well.

We recall that Lemma 5.6 implies that $\mathbf{D}(Z_I)$ is a finitely generated $\mathbb{C}$-algebra without nilpotent elements and thus corresponds to an affine variety $\mathbb{Y}_I = \mathrm{specmax}(\mathbf{D}(Z_I))$. It follows from Lemma 5.2 that the algebra morphism $\mu_I : \mathbf{D}_0(Z) \to \mathbf{D}(Z_I)$ is injective, hence $\mu_{I,*} : \mathbb{Y}_I \to \mathbb{X}$ is a dominant morphism of affine algebraic varieties. Moreover, since $\mathbf{D}(Z_I)$ is a module of finite type over $\mathbf{D}_0(Z)$, it follows in addition that $\mu_{I,*}$ is a finite surjective morphism with uniformly bounded finite fibers (by the going up property in ring theory, see [1,Theorem 5.10] or [32,Proposition 3.2.4]).

Define $\gamma_{00} : \mathcal{Z}(\mathfrak{g}) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H+\mathfrak{t}_H)$, obtained from the composition of γ_0 and the projection $S(\mathfrak{j}) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)$. We recall from (5.12) the injective algebra morphism

$$j_0 : \mathfrak{Z}(Z_\emptyset) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)\,.$$

Now j_0 composed with the natural inclusion $\mathbf{D}(Z_\emptyset) \hookrightarrow \mathfrak{Z}(Z_\emptyset)$ gives rise the injective morphism

$$\iota_\emptyset : \mathbf{D}(Z_\emptyset) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)\,.$$

Next, we recall that $\mathbf{D}(Z_I)$ is naturally a subalgebra of $\mathbf{D}(Z_\emptyset)$ via the monomorphism $\mathbb{D}(Z_I) \hookrightarrow \mathbb{D}(Z_\emptyset)$ of Lemma 5.2 applied to $Z = Z_I$. Composing this injection with $\iota_\emptyset$ we obtain a monomorphism

$$\iota_I : \mathbf{D}(Z_I) \to S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H)\,.$$

With (5.13), we thus arrive at the following commutative diagram of finite module extensions

$$\begin{tikzcd} S(\mathfrak{j}) \arrow[r] & S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H) \arrow[r, equal] & S(\mathfrak{j})/S(\mathfrak{j})(\mathfrak{a}_H + \mathfrak{t}_H) \\ \mathcal{Z}(\mathfrak{g}) \arrow[u, "\gamma_0"] \arrow[r, "p"] & \mathbf{D}_0(Z) \arrow[u, "\iota_0"] \arrow[r, "\mu_I"] & \mathbf{D}(Z_I)\,, \arrow[u, "\iota_I"'] \end{tikzcd} \tag{8.2}$$

with the middle vertical arrow ι_0 uniquely determined by the injectivity of μ_I. In particular, ι_0 is injective. On the level of affine varieties, this corresponds to the commutative diagram

$$\begin{tikzcd} \mathfrak{j}_{\mathbb{C}}^* \arrow[d, "\gamma_{0,*}"'] & (\mathfrak{a}_H + \mathfrak{t}_H)^{\perp} \arrow[l] \arrow[r, equal] \arrow[d, "\iota_{0,*}"'] & (\mathfrak{a}_H + \mathfrak{t}_H)^{\perp} \arrow[d, "\iota_{I,*}"'] \\ \mathfrak{j}_{\mathbb{C}}^*/W_{\mathfrak{j}} & \mathbb{X} \arrow[l, "p_*"'] & \mathbb{Y}_I\,, \arrow[l, "\mu_{I,*}"'] \end{tikzcd} \tag{8.3}$$

where $(\mathfrak{a}_H + \mathfrak{t}_H)^{\perp} \subset \mathfrak{j}_{\mathbb{C}}^*$. Since all vertical arrows in (7.2) are injective and represent finite module extensions, it follows that all vertical arrows in (7.3) are surjective (by application of the going down property as above).

Proof of Lemma 8.2 Immediate from the surjectivity of the vertical maps in the commutative diagram (8.3). ■

Recall the decomposition

$$\mathbf{D}(Z_I) = U_{\Lambda} \oplus \mathbf{D}(Z_I)\mathcal{I}'_{\Lambda}\,,$$

from (5.18), with $U_{\Lambda} := U_{\mathcal{I}_{\Lambda}}$ and $\mathcal{I}'_{\Lambda} := \mu_I(\mathcal{I}_{\Lambda})$.

Recall that $U_{\Lambda} \subset U$ (where U is the finite dimensional subspace of $\mathbf{D}(Z_I)$ independent of Λ satisfying (5.14)) and thus $n := \max_{\Lambda \in \mathbb{X}} \dim U_{\Lambda} \leq \dim U < \infty$. For every $0 \leq j \leq n$, we now set

$$\mathbb{X}^j := \{\Lambda \in \mathbb{X} \mid \dim U_{\Lambda} = j\}$$

and get $\mathbb{X} = \coprod_{j=0}^{n} \mathbb{X}^j$. Now, for every $\Lambda \in \mathbb{X}^j$, the set

$$\mathbb{X}_\Lambda := \{x \in \mathbb{X} \mid U_\Lambda \oplus \mathbf{D}(Z_I)\mathcal{I}'_x = \mathbf{D}(Z_I)\}$$

is a subset of $\mathbb{X}^j$.

Lemma 8.3. The following assertions hold:

1. For any $0 \leq j \leq n$, the set $\bigcup_{k \leq j} \mathbb{X}^k$ is Zariski-open in $\mathbb{X}$. In particular, $\mathbb{X}^j$ is locally closed in $\mathbb{X}$.
2. For $\Lambda \in \mathbb{X}^j$, the set $\mathbb{X}_\Lambda$ is Zariski-open in $\mathbb{X}^j$.

Proof. Note that $\mathbf{D}_0(Z) = \mathcal{O}(\mathbb{X})$ is the coordinate ring of the affine variety $\mathbb{X}$. For any $x \in \mathbb{X}$, we denote by $\mathfrak{m}_x \subset \mathcal{O}(\mathbb{X})$ the corresponding maximal ideal. Since $\mathcal{O}(\mathbb{Y}_I) = \mathbf{D}(Z_I)$ is a finite module of $\mathcal{O}(\mathbb{X})$, we find a finite dimensional subspace $U_x \subset \mathcal{O}(\mathbb{Y}_I)$ such that $\mathcal{O}(\mathbb{Y}_I) = U_x \oplus \mathcal{O}(\mathbb{Y}_I)\mathfrak{m}_x$. The Nakayama lemma implies that there exists an $f \in \mathcal{O}(\mathbb{X})$ with $f(x) \neq 0$ such that $\mathcal{O}(\mathbb{Y}_I)_f = \mathcal{O}(\mathbb{X})_f U_x$. In particular, we have, for all $z \in \mathbb{X}$ with $f(z) \neq 0$, that $\mathcal{O}(\mathbb{Y}_I) = U_x + \mathcal{O}(\mathbb{Y}_I)\mathfrak{m}_z$. This implies that:

$$\mathbb{X} \to \mathbb{N}_0, \quad x \mapsto \dim \mathcal{O}(\mathbb{Y}_I)/\mathcal{O}(\mathbb{Y}_I)\mathfrak{m}_x$$

is upper semi-continuous and, in particular, for any $1 \leq j \leq n$, we have that $\bigcup_{k \leq j} \mathbb{X}^k$ is Zariski-open in $\mathbb{X}$ and (i) follows.

For (ii), we just saw that, for $\Lambda \in \mathbb{X}^j$, we have, for $z \in \mathbb{X}_\Lambda$, that there exists $f \in \mathcal{O}(\mathbb{X})$ such that $f(z) \neq 0$ and $\{y \in \mathbb{X}^j \mid f(y) \neq 0\} \subset \mathbb{X}_\Lambda$. Hence, $\mathbb{X}_\Lambda$ is Zariski-open in $\mathbb{X}^j$. ■

As quasi-affine varieties are quasi-compact for the Zariski topology, it follows that there exists finitely many $\Lambda \in \mathbb{X}$, say $\Lambda_1, \ldots, \Lambda_s$, such that:

$$\mathbb{X} = \bigcup_{j=1}^{s} \mathbb{X}_{\Lambda_j}.$$

For any $1 \leq j \leq s$, we define a fixed finite dimensional vector space $U_j := U_{\Lambda_j}$ as above. This gives us a direct sum decomposition

$$\mathbf{D}(Z_I) = U_j \oplus \mathbf{D}(Z_I)\mathcal{I}'_\Lambda, \quad \Lambda \in \mathbb{X}_{\Lambda_j}, \tag{8.4}$$

and, upon the identification $U_j \simeq \mathbf{D}(Z_I)/\mathcal{I}'_\Lambda$, a representation

$$\rho_\Lambda : \mathbf{D}(Z_I) \to \mathrm{End}(U_j)\,.$$

Lemma 8.4. The following assertions hold:

1. Fix $1 \le j \le s$. For any $v \in \mathbf{D}(Z_I)$, the map

$$\mathbb{X}_{\Lambda_j} \to \mathrm{End}(U_j), \quad \Lambda \mapsto \rho_\Lambda(v)$$

   is regular, i.e., locally the restriction to $\mathbb{X}_{\Lambda_j}$ of a rational function on $\mathbb{X}$. In particular, there exists an open covering $\mathbb{X} = \bigcup_{j=1}^{s} \mathbb{X}_j$ with $\mathbb{X}_j \subset \mathbb{X}_{\Lambda_j}$ such that, for all $v \in \mathbf{D}(Z_I)$, there exists a constant $C_v > 0$ such that

$$\|\rho_\Lambda(v)\| \le C_v(1 + \|\Lambda\|)^N \qquad (\Lambda \in \mathbb{X}_j)\,, \tag{8.5}$$

   for some $N \in \mathbb{N}$ independent of v. Here, $\|\cdot\|$ on the left hand side of (7.5) refers to the operator norm of $\mathrm{End}(U_j)$.
2. With $S(\mathfrak{a}_I) \subset \mathbf{D}(Z_I)$, one has:

$$\mathrm{Spec}_{\mathfrak{a}_I}(\rho_\Lambda) \subset (\rho_Q + W_j \cdot \Lambda)|_{\mathfrak{a}_I}\,.$$

Proof. Recall the terminology we introduced in the proof of Lemma 8.3. Since the assertion is local, we may assume that $\mathbb{X} = \mathbb{X}_{\Lambda_j}$, for some j and $U = U_j$, is such that $\mathcal{O}(\mathbb{Y}_I) = \mathcal{O}(\mathbb{X})U = U \oplus \mathcal{O}(\mathbb{Y}_I)\mathfrak{m}_x$ for all $x \in \mathbb{X}$. This decomposition defines a projection $p_x : \mathcal{O}(\mathbb{Y}_I) \to U$ for any $x \in \mathbb{X}$. Moreover, note that the natural map

$$\mathcal{O}(\mathbb{X}) \otimes U \to \mathcal{O}(\mathbb{Y}_I), \quad g \otimes u \mapsto gu$$

is an isomorphism. Accordingly, every $f \in \mathcal{O}(\mathbb{Y}_I)$ can be expressed uniquely as $f = \sum_i g_i \otimes u_i$ for a fixed basis (u_i) of U. Then

$$p_x(f) = \sum_i g_i(x)u_i$$

is regular in $x \in \mathbb{X}$. This proves the first assertion of (i) and the second assertion in (i) is an immediate consequence thereof.

(ii) Geometrically, it might happen that the fiber of the morphism $\mu_{I,*}: \mathbb{Y}_I \to \mathbb{X}$ over $\Lambda \in \mathbb{X}$ is not reduced, i.e., $\mathbf{D}(Z_I)\mathcal{I}'_\Lambda$ is not a radical ideal in $\mathbf{D}(Z_I)$. However, the set of Λ's, with reduced fibers, is open dense in $\mathbb{X}$. In view of the continuity showed in (i), it suffices to show that $\mathrm{Spec}_{\mathfrak{a}_I}(\rho_\Lambda) \subset (\rho_{\mathfrak{j}} + W \cdot \lambda)\big|_{\mathfrak{a}_I}$ for generic Λ, i.e., Λ reduced.

Next, we recall the diagram (8.3) with all vertical arrows surjective and all fibers being finite. Now, as the fiber $\mu_{I,*}^{-1}(\Lambda)$ was assumed to be reduced, it has $\dim(U_\Lambda)$ elements as the corresponding affine algebra to this finite variety is just the $\mathfrak{a}_I$-module $\mathbf{D}(Z_I)/\mathcal{I}'_\Lambda$. In particular, $\mu_{I,*}^{-1}(\Lambda)$ consists of the $\mathfrak{a}_I$-weights of $U_\Lambda \simeq \mathbf{D}(Z_I)/\mathcal{I}'_\Lambda$.

From (8.3), we obtain the the fiber diagram:

$$\begin{tikzcd}
\rho_{\mathfrak{j}} + W_{\mathfrak{j}} \cdot \lambda \arrow[d] & (\rho_{\mathfrak{j}} + W_{\mathfrak{j}} \cdot \lambda) \cap (\mathfrak{a}_H + \mathfrak{t}_H)^\perp \arrow[l, hook'] \arrow[d] & \iota_{I,*}^{-1}(\mu_{I,*}^{-1}(\Lambda)) \arrow[l, hook'] \arrow[d] \\
W_{\mathfrak{j}} \cdot \lambda = \Lambda \arrow[r, equal] & \Lambda & \mu_{I,*}^{-1}(\Lambda)\,. \arrow[l, two heads]
\end{tikzcd} \tag{8.6}$$

Hence (ii) follows from the $\mathfrak{a}_I$-equivariance of $\iota_{I,*}$. ■

The section $\mathbf{s}$ we use in the sequel is the one where we identify $\mathfrak{a}_Z$ with the subspace $\mathfrak{a}_H^{\perp_{\mathfrak{a}_L}} \subset \mathfrak{a}_L$, the orthogonal being taken with respect to the form κ introduced at the beginning of Section 2.2. Let $\underline{J}(\mathbb{C}) \subset \underline{G}(\mathbb{C})$ be the Cartan subgroup with Lie algebra $\mathfrak{j}_\mathbb{C}$ and $\mathcal{L} := \mathrm{Hom}(\underline{J}(\mathbb{C}), \mathbb{C}^*)$ be its character group. In the sequel, we identify $\mathcal{L}$ with a lattice in $\mathfrak{j}^*$. We call a subspace $V \subset \mathfrak{j}^*$ *rational* provided that $V = \mathbb{R}(V \cap \mathcal{L})$. Likewise, we call a discrete subgroup $\Gamma \subset (\mathfrak{j}^*, +)$ rational if $\Gamma = \Gamma \cap \mathbb{Q}\mathcal{L}$. Using the dual lattice $\mathcal{L}^\vee \subset \mathfrak{j}$, we obtain a notion of rationality for subspaces and discrete subgroups of $\mathfrak{j}$ as well.

Finally, we may and will request that $\kappa\big|_{\mathfrak{j}\times\mathfrak{j}}$ is rational, i.e., with respect to a basis of $\mathfrak{j}$, which lies in $\mathcal{L}^\vee$, its matrix entries are rational.

Lemma 8.5. The following subspaces of $\mathfrak{j}$ are all rational: $\mathfrak{a}_H, \mathfrak{a}_Z$ and $\mathfrak{a}_I$ for $I \subset S$.

Proof. The subspace $\mathfrak{a}_H$ is rational as it corresponds to the Lie algebra of the subtorus $(\underline{A}_L \cap \underline{H})_0 \subset \underline{J}$. Since the form $\kappa\big|_{\mathfrak{j}\times\mathfrak{j}}$ is rational, we obtain that $\mathfrak{a}_Z \subset \mathfrak{a} \subset \mathfrak{j}$ is rational as well. Finally, we obtain from (2.8) that $S \subset \mathbb{Q}\mathcal{L}$ and this gives us the rationality of $\mathfrak{a}_I$ for any $I \subset S$. ■

We recall that $\mathcal{Q}_\Lambda$ denotes the set of $\mathfrak{a}_I$-weights of ρ_Λ and (cf. Lemma 8.4(ii))

$$\mathcal{Q}_\Lambda \subset \{(\rho_Q + w\Lambda)\big|_{\mathfrak{a}_I} \mid w \in W_{\mathfrak{j}}\}\,, \tag{8.7}$$

where we identify $\mathfrak{a}_I$ as a subspace of $\mathfrak{a}$ as above. For $\lambda \in \mathcal{Q}_\Lambda$, we recall the projectors $E_\lambda : U_\Lambda^* \to U_{\Lambda,\lambda}^*$ to the generalized common eigenspace along the supplementary generalized eigenspaces.

In the sequel, we abbreviate and write $\mathcal{A}_{temp}(Z : \Lambda)$ instead of $\mathcal{A}_{temp}(Z : \mathcal{J}_\Lambda)$.

The key to obtain uniform estimates for the constant term approximation is at the core related to polynomial bounds for the truncating spectral projections E_λ.

Proposition 8.6. Let $1 \le j \le s$. There exist constants $C, N > 0$ such that, for all $\Lambda \in \mathbb{X}_j$ with $\mathcal{A}_{temp}(Z : \Lambda) \ne \{0\}$, one has

$$\|E_\lambda\| \le C(1 + \|\Lambda\|)^N, \qquad \lambda \in \mathcal{Q}_\Lambda\,,$$

with $\|E_\lambda\|$ the operator norm on the fixed finite dimensional vector space $\mathrm{End}(U_j)$.

The proof of the Proposition is preceded by two lemmas:

Lemma 8.7. Let $0 < \nu \le 1$, $N \in \mathbb{N}$ and $\mathsf{A} \in \mathrm{Mat}_N(\mathbb{C})$ with $\mathrm{Spec}(\mathsf{A}) = \{\lambda_1, \ldots, \lambda_r\}$ such that $\mathrm{Re}\,\lambda_1 \le \ldots \le \mathrm{Re}\,\lambda_r$. For every $1 \le j \le r$, let $V_j \subset \mathbb{C}^n$ be the generalized eigenspace of A associated to the eigenvalue λ_j. For every $1 \le k \le r$, we let $E_k = \bigoplus_{j=1}^k V_j$ and $\mathsf{P}_k : \mathbb{C}^N \to E_k$ be the projection along $\bigoplus_{j=k+1}^r V_j$. Suppose, for some $1 \le k \le r-1$, that $\mathrm{Re}\,\lambda_{k+1} - \mathrm{Re}\,\lambda_k \ge \nu$. Then there exists a constant $C = C(\nu, N) > 0$ such that

$$\|\mathsf{P}_k\| \le C(1 + \|\mathsf{A}\|)^N\,.$$

Proof. [34,Lemma 6.4]. ■

Lemma 8.8. There exists a $W_\mathfrak{j}$-stable rational lattice Ξ_Z in the vector space $\mathfrak{j}^*$ such that

$$\mathrm{Re}\,\Lambda \in \Xi_Z \tag{8.8}$$

for all $\Lambda \in \mathfrak{j}_\mathbb{C}^*$ with $\mathcal{A}_{temp}(Z : \Lambda) \ne \{0\}$.

Proof. Let $0 \ne f \in \mathcal{A}_{temp}(Z : \Lambda)$ be a K-finite element that generates an irreducible Harish-Chandra module, say V. According to Theorem 7.1 V embeds into a twisted discrete series of some $(Z_w)_I$. Now, we apply [33,Theorem 1.1] and obtain a $W_\mathfrak{j}$-invariant lattice $\Xi_{(Z_w)_I}$, called $\Lambda_{(Z_w)_I}$ in [33], with property (8.8). The lattice is indeed rational by

[33,Theorem 8.3] combined with [33,Lemma 3.4]. The asserted lattice is then obtained by taking the rational lattice generated by the rational lattices $\Xi_{Z_{wI}}$, i.e.

$$\Xi_Z = \langle v \in \Xi_{Z,w_I} : I \subset S, w \in \mathcal{W} \rangle_{\mathbb{Z}-\mathrm{mod}}$$

$\blacksquare$

Proof of Proposition 8.6 According to Lemma 8.5, $\mathfrak{a}_I$ is a rational subspace of $\mathfrak{a} \subset \mathfrak{j}$. Now, we keep in mind the following general fact: if $U \subset \mathfrak{j}$ is a rational subspace and $\Xi \subset \mathfrak{j}^*$ is a rational lattice, then $\Xi\big|_U$ is a rational lattice in U^*. In particular, it follows that $\Xi_{Z,I} := \Xi_Z\big|_{\mathfrak{a}_I}$ is a rational lattice in $\mathfrak{a}_I^*$. Next, observe that Lemma 8.8 combined with (8.7) implies that $\mathrm{Re}\,\mathcal{Q}_\Lambda \subset \rho_Q\big|_{\mathfrak{a}_I} + \Xi_{Z,I}$ for all tempered infinitesimal characters Λ. Denote by $\Xi_{Z,I}^\vee \subset \mathfrak{a}_I$ the dual lattice of $\rho_Q\big|_{\mathfrak{a}_I} + \Xi_{Z,I}$. Since $\mathfrak{a}_I^-$ is a rational cone, we find elements $X_1, \ldots, X_k$ of $\mathfrak{a}_I^- \cap \Xi_{Z,I}^\vee$ such that:

$$\mathfrak{a}_I^- = \sum_{j=1}^k \mathbb{R}_{\geq 0} X_j\,.$$

We identify U_j with $\mathbb{C}^N$ and define matrices $\mathsf{A}_i := \Gamma_\Lambda(X_i) = {}^t\rho_\Lambda(X_i)$. Let $\lambda \in \mathcal{Q}_\Lambda$. Write $E_{\lambda,i}$ for the spectral projection to the generalized eigenspace of A_i with eigenvalue $\lambda(X_i)$. Since the matrices A_i commute with each other and the X_i span $\mathfrak{a}_I$, we obtain that:

$$E_\lambda = E_{\lambda,1} \circ \ldots \circ E_{\lambda,k}\,. \tag{8.9}$$

Hence, we are reduced to prove a polynomial bound for each $E_{\lambda,i}$. As

$$\mathrm{Spec}(\mathsf{A}_i) \subset (\rho_Q + W_{\mathfrak{j}} \cdot \Lambda)(X_i)\,,$$

we get $\mathrm{Re}\,\mathrm{Spec}(\mathsf{A}_i) \subset \mathbb{Z}$. Hence, we can apply Lemma 8.7 to the matrices A_i, with $\nu = 1$, and obtain $\|E_{\lambda,i}\| \leq C(1 + \|\mathsf{A}_i\|)^N$. Now, we recall from (7.5) that

$$\|\Gamma_\Lambda(X)\| \leq C\|X\|(1 + \|\Lambda\|)^N\,,$$

after possible enlargement of C and N. This gives the asserted norm bound for $\|E_{\Lambda,i}\|$ and then for E_λ via (8.9). $\blacksquare$

For $\lambda \in \mathcal{Q}_\Lambda$, we recall the notation

$$E_\lambda(X) = e^{-\lambda(X)} E_\lambda(e^{\Gamma_\Lambda(X)}), \quad X \in \mathfrak{a}_I\,,$$

and recall, from Lemma 5.7(ii), the starting identity:

$$\begin{aligned}\Phi_{f,\lambda}(a_Z \exp(tX_I)) &= e^{t\Gamma_\Lambda(X_I)}\Phi_{f,\lambda}(a_Z) \\ &\quad + \int_0^t E_\lambda e^{(t-s)\Gamma_\Lambda(X_I)}\Psi_{f,X_I}(a_Z \exp(sX_I))\,ds\,, \\ &\qquad a_Z \in A_Z, X_I \in \mathfrak{a}_I, t \in \mathbb{R}\,.\end{aligned}$$

Lemma 8.9. Let $N \in \mathbb{N}$. There exist a continuous semi-norm q on $C^\infty_{temp,N}(Z)$ and $m \in \mathbb{N}$ such that, for all $\Lambda \in \mathfrak{j}^*_\mathbb{C}/W_\mathfrak{j}$ and $f \in \mathcal{A}_{temp,N}(Z : \Lambda)$,

$$\begin{aligned}&\|\Phi_{f,\lambda}(a_Z \exp(tX_I)) - \Phi_{f,\lambda,\infty}(a_Z \exp(tX_I))\| \\ &\leq (a_Z \exp(tX_I))^{\rho_Q} e^{t\delta\beta_I(X_I)}(1 + \|\log a_Z\|)^N (1 + t\|X_I\|)^{\dim U}(1 + \|\Lambda\|)^m q(f), \\ &\qquad \lambda \in \mathcal{Q}_\mathcal{I}, a_Z \in A_Z^-, X_I \in \mathfrak{a}_I^{--}, t \geq 0\,.\end{aligned}$$

Proof. The statement is a uniform version of Lemma 6.1, which rested on Lemma 5.8, Lemma 5.9 and Proposition 5.11. Now Proposition 8.6 makes the bound in Lemma 5.8 for the norm of the spectral projections E_λ uniform at the cost of an additional polynomial factor, a power of $(1 + \|\Lambda\|)$. This takes care of the uniform estimates for the E_λ in Proposition 5.11. It remains to obtain uniform estimates for Φ_f and $\Psi_{f,X}$ in Lemma 5.8. This Lemma was obtained for a fixed ideal $\mathcal{I}$ and fixed complement $U_\mathcal{I}$. Now, by Lemma 8.4 we can in fact get by with finitely many choices of complements $U_1, \ldots, U_s$ at the cost of another polynomial factor of a power of $(1 + \|\Lambda\|)$. As a result the estimate in Lemma 6.1 becomes uniform at the cost of a polynomial factor of the type $(1 + \|\Lambda\|)^m$, which is recored at the right hand side of the asserted estimate. ■

Having said all that, it is now clear that all bounds from Sections 5 and 6 become uniform at the cost of an extra polynomial factor in $\|\Lambda\|$. Polynomial behavior in $\|\Lambda\|$ can be subsumed in raising the Sobolev order of the corresponding semi-norms. In more detail, if p is a continuous semi-norm on an SF-module V^∞ with infinitesimal character Λ, then we claim that there exists $C > 0, k \in \mathbb{N}$ independent of p, V and Λ such that

$$(1 + \|\Lambda\|)p(v) \le Cp_k(v) \qquad (v \in V)\,, \tag{8.10}$$

where p_k denotes the k-th Sobolev norm of p with respect to a fixed basis of $\mathfrak{g}$. For that we first note that

$$|\chi_\Lambda(z)|p(v) = p(zv) \le C_z p_{\deg z}(v), \qquad v \in V^\infty\,, \tag{8.11}$$

for all $z \in \mathcal{Z}(\mathfrak{g})$ and a constant $C_z > 0$. Now for any $X \in \mathfrak{j}_\mathbb{C}$ we define a $W_\mathfrak{j}$-invariant polynomial function on $\mathfrak{j}_\mathbb{C}^*$ by $f_X(\Lambda) := \prod_{w\in W_\mathfrak{j}} \Lambda(w \cdot X)$. Note that for any $\Lambda \neq 0$ we find an $X \in \mathfrak{j}_\mathbb{C}$ such that $f_X(\Lambda) \neq 0$, i.e. choose $X \in \mathfrak{j}_\mathbb{C} \setminus \bigcup_{w\in W_\mathfrak{j}} \ker \Lambda \circ w$. By the homogeneity of the f_X and the compactness of the unit sphere in $\mathfrak{j}_\mathbb{C}^*$ we thus find finitely many $X_1, \ldots, X_m$ such that

$$\max_{1\le j\le m} |f_{X_j}(\Lambda)| \ge c\|\Lambda\|^{|W_\mathfrak{j}|} \qquad (\Lambda \in \mathfrak{j}_\mathbb{C}^*)\,. \tag{8.12}$$

Let now $z_1, \ldots, z_m \in \mathcal{Z}(\mathfrak{g}_\mathbb{C})$ be such that $\chi_\Lambda(z_j) = f_{X_j}(\Lambda)$ for all $\Lambda \in \mathfrak{j}_\mathbb{C}^*$. Thus, combining (8.11) and (8.12) we obtain the claim (8.10) for $k = |W_\mathfrak{j}|$.

The preceding reasoning now implies the following parameter independent version of Theorem 6.9:

Theorem 8.10 (Uniform constant term approximation). Let $N \in \mathbb{N}$, $I \subset S$ and $\mathcal{C}_I$ be a compact subset of $\mathfrak{a}_I^{--}$. Let $w_I \in \mathcal{W}_I$ and $w = \mathbf{m}(w_I) \in \mathcal{W}$. Then there exist $\varepsilon > 0$ and a continuous semi-norm p on $C^\infty_{temp,N}(Z)$ such that, for all $f \in \mathcal{A}_{temp,N}(Z : \Lambda)$, $\Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}$:

$$\begin{aligned}&(a_Z \exp(tX))^{-\rho_Q}|f(ga_Z \exp(tX)w \cdot z_0) - f_I(ga_Z \exp(tX)w_I \cdot z_{0,I})|\\ &\le e^{-\varepsilon t}(1 + \|\log a_Z\|)^N p(f), \qquad a_Z \in A_Z^-, X \in \mathcal{C}_I, g \in \Omega, t \ge 0\,.\end{aligned}$$

Moreover, let $N_1 := \max_\Lambda \dim(\mathbf{D}(Z_I)/\mathbf{D}(Z_I)\mu_I(\mathcal{I}_\Lambda)) \in \mathbb{N}$ and q be a continuous semi-norm on $C^\infty_{temp,N+N_1}(Z_I)$. Then there exists a continuous semi-norm p on $C^\infty_{temp,N}(Z)$ such that:

$$q(f_I) \le p(f), \qquad f \in \mathcal{A}_{temp,N}(Z : \Lambda), \Lambda \in \mathfrak{j}_\mathbb{C}^*/W_\mathfrak{j}\,.$$

A Rapid convergence

Definition A.1. Let $a \geq 0$ and (x_s) be a family of elements of a normed vector space with $s \in [a, +\infty[$. One says that (x_s) converges rapidly to l if
there exist $\varepsilon > 0, C > 0, s_0 \in [a, +\infty[$ such that, for any $s \geq s_0$, $\|x_s - l\| \leq Ce^{-\varepsilon s}$.

To shorten, we will write $x_s \xrightarrow[s\to\infty]{rapid} l$.

Lemma A.2. Let $a \geq 0$, E, F be two Euclidean spaces and $l \in E$. Let ϕ be an F-valued map of class C^1 on a neighborhood U of l and such that the differential $d\phi(l)$ of ϕ at l is injective. If $(x_s)_{s\in[a,+\infty[}$ is a family of elements of E such that $\phi(x_s) \xrightarrow[s\to\infty]{rapid} \phi(l)$ and (x_s) converges to l when s tends to $+\infty$, then

$$x_s \xrightarrow[s\to\infty]{rapid} l\,.$$

Proof. Choose a left inverse $A \in \mathrm{Hom}(F, E)$ to $d\phi(l)$ and replace ϕ by $A \circ \phi$. In this way we reduce to the case where $E = F$ with $d\phi(l)$ an isomorphism. By the inverse function theorem we may, after shrinking U, assume further that $\phi : U \to E$ is diffeomorphic onto its open image $\phi(U) \subset E$. Applying the Taylor expansion of ϕ^{-1} at $\phi(l)$, one has for s large enough such that $x_s \in V$:

$$\begin{aligned} \|x_s - l\| &= \|\phi^{-1}(\phi(x_s)) - \phi^{-1}(\phi(l))\| \\ &\leq \|d\phi^{-1}(\phi(l))\|\, \|\phi(x_s) - \phi(l)\| + o(\|\phi(x_s) - \phi(l)\|)\,. \end{aligned}$$

Our claim follows from the rapid convergence of $(\phi(x_s))$. ■

Definition A.3. Let $a \geq 0$, X be a d-dimensional smooth manifold and $(x_s)_{s\in[a,+\infty[}$ be a family of elements of X. One says that (x_s) converges rapidly in X if there exist $l \in X$ and a chart (U, ϕ) around l such that:

$(\phi(x_s))$ converges rapidly to $\phi(l)$.

Remark A.4. This notion is independent of the choice of the chart (U, ϕ). Indeed, let $(\tilde{U}, \tilde{\phi})$ be another chart around l. Then, from Lemma A.2, $((\phi \circ \tilde{\phi}^{-1})^{-1}(\phi(x_s)))$ converges rapidly to $\tilde{\phi}(l)$, which means that $(\tilde{\phi}(x_s))$ converges rapidly to $\tilde{\phi}(l)$. Also if $\Psi : X \to Y$ is a differentiable map between C^∞ manifolds and (x_s) converges rapidly to x in X, then $\Psi((x_s))$ converges rapidly to $\Psi(x)$ in Y.

B Real points of elementary group actions

We assume that $\underline{G}$ is a reductive group defined over $\mathbb{R}$ and let $\underline{H}$ be an $\mathbb{R}$-algebraic subgroup of $\underline{G}$. We form the homogeneous space $\underline{Z} = \underline{G}/\underline{H}$ and our concern is to what extent $\underline{Z}(\mathbb{R})$ coincides with $\underline{G}(\mathbb{R})/\underline{H}(\mathbb{R})$.

We say that $\underline{G}$ is anisotropic provided $\underline{G}(\mathbb{R})$ is compact and recall from [25,Proposition 13.1] the following fact:

Lemma B.1. If $\underline{G}$ is anisotropic, then $\underline{Z}(\mathbb{R}) = \underline{G}(\mathbb{R})/\underline{H}(\mathbb{R})$.

In the sequel, we assume that $\underline{G}$ is a connected elementary group (defined over $\mathbb{R}$), that is:

- $\underline{G} = \underline{MA}$ for normal $\mathbb{R}$-subgroups $\underline{A}$ and $\underline{M}$,
- $\underline{M}$ is anisotropic,
- $\underline{A}$ is a split torus, i.e., $\underline{A}(\mathbb{R}) \simeq (\mathbb{R}^\times)^n$.

Consider now $\underline{Z} = \underline{G}/\underline{H}$, with $\underline{G}$ elementary. We set $\underline{M_H} := \underline{M} \cap \underline{H}$ and, likewise, $\underline{A_H} := \underline{A} \cap \underline{H}$. Furthermore, we set $\underline{A_Z} := \underline{A}/\underline{A_H}$ and $\underline{M_Z} := \underline{M}/\underline{M_H}$, which we view as subvarieties of $\underline{Z}$. From Lemma B.1, we already know that $\underline{M_Z}(\mathbb{R}) = \underline{M}(\mathbb{R})/\underline{M_H}(\mathbb{R})$. Consider now the fiber bundle

$$\underline{A_Z} \to \underline{Z} \to \underline{G}/\underline{HA}$$

and take real points

$$\underline{A_Z}(\mathbb{R}) \to \underline{Z}(\mathbb{R}) \to (\underline{G}/\underline{HA})(\mathbb{R})\,. \tag{B.1}$$

We claim that the natural map

$$\underline{M_Z}(\mathbb{R}) \times \underline{A_Z}(\mathbb{R}) \to \underline{Z}(\mathbb{R}) \tag{B.2}$$

is surjective. In fact, observe that $\underline{G}/\underline{HA} \simeq \underline{M}/(\underline{M} \cap (\underline{HA}))$ is homogeneous for the anisotropic group $\underline{M}$. Hence, $(\underline{G}/\underline{HA})(\mathbb{R}) \simeq \underline{M}(\mathbb{R})/(\underline{M} \cap (\underline{HA}))(\mathbb{R})$ and our claim follows from (B1).

We remain with the determination of the fiber of the map (B2). Since $\underline{M}$ and $\underline{A}$ commute, we obtain with

$$\underline{\widehat{M}_H} := \{m \in \underline{M} \mid m\underline{H} \in \underline{A_Z} \subset \underline{Z}\}$$

a closed $\mathbb{R}$-subgroup of $\underline{M}$, which acts on $\underline{A_Z}$ by morphisms (translations). The kernel of this action is $\underline{M_H}$ and this identifies $\underline{M_H}$ as a normal subgroup of $\underline{\widehat{M}_H}$. In particular, we obtain an embedding $\widehat{\underline{M}}_{\underline{H}}/\underline{M}_{\underline{H}} \to \underline{A}_{\underline{Z}}$ and, taking real points, we obtain, as $\underline{M}$ is anisotropic and $\widehat{\underline{M}}_{\underline{H}}$ is closed in $\underline{M}$, a closed embedding

$$F_{\underline{M}(\mathbb{R})} := \widehat{\underline{M}}_{\underline{H}}(\mathbb{R})/\underline{M}_{\underline{H}}(\mathbb{R}) \to \underline{A}_{\underline{Z}}(\mathbb{R})\,.$$

The image of $F_{\underline{M}(\mathbb{R})}$ is compact, hence, a 2-group of $\underline{A}_{\underline{Z}}(\mathbb{R}) \simeq (\mathbb{R}^\times)^k$. In summary, we have shown:

Proposition B.2. Let $\underline{Z} = \underline{G}/\underline{H}$ be a homogeneous space for an elementary group $\underline{G} = \underline{MA}$ with respect to an $\mathbb{R}$-algebraic subgroup $\underline{H}$. Then $F_{\underline{M}(\mathbb{R})}$ is a finite 2-group and the map

$$[\underline{M}(\mathbb{R})/\underline{M}_{\underline{H}}(\mathbb{R})] \times^{F_{\underline{M}(\mathbb{R})}} \underline{A}_{\underline{Z}}(\mathbb{R}) \to \underline{Z}(\mathbb{R}), \quad [m\underline{M}_{\underline{H}}(\mathbb{R}), a_Z] \mapsto ma_Z$$

is an isomorphism of real manifolds.

Corollary B.3. Under the assumptions of Proposition B.2, the $\underline{G}(\mathbb{R})$-orbits in $\underline{Z}(\mathbb{R})$ are in bijection with $\underline{A}_{\underline{Z}}(\mathbb{R})_2/F_{\underline{M}(\mathbb{R})}$, where $\underline{A}_{\underline{Z}}(\mathbb{R})_2$ is the group of 2-torsion points in $\underline{A}_{\underline{Z}}(\mathbb{R})$. The isomorphism is given explicitly by:

$$\underline{A}_{\underline{Z}}(\mathbb{R})_2/F_{\underline{M}(\mathbb{R})} \to \underline{G}(\mathbb{R})\backslash\underline{Z}(\mathbb{R}), \quad F_{\underline{M}(\mathbb{R})}a_Z \mapsto \underline{G}(\mathbb{R})a_Z\,.$$

C Invariant differential operators on Z and Z_I (by Raphaël Beuzart-Plessis)

In the beginning we let $Z = G/H$ be a general homogeneous space attached to a Lie group G and a closed subgroup $H \subset G$. A bit later we specialize to real spherical spaces as in the main body of the text. Our concern is with the algebra of G-invariant differential operators $\mathbb{D}(Z)$ and we start with a recall of the standard description of $\mathbb{D}(Z)$ in terms of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}_\mathbb{C}$. As usual, we denote the right regular representation of G on $C^\infty(G)$ by R and, by slight abuse of notation, the induced action of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ by the same letter; in symbols:

$$R : \mathcal{U}(\mathfrak{g}) \to \mathrm{End}(C^\infty(G))\,.$$

Now, for an element $u \in \mathcal{U}(\mathfrak{g})$, the operator $R(u)$ descends to a differential operator on Z if and only if $u \in \mathcal{U}_H(\mathfrak{g})$, where

$$\mathcal{U}_H(\mathfrak{g}) := \{u \in \mathcal{U}(\mathfrak{g}) \mid \mathrm{Ad}(h)u - u \in \mathcal{U}(\mathfrak{g})\mathfrak{h}, \quad h \in H\}\,.$$

Notice that $\mathcal{U}_H(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})$ is a subalgebra of $\mathcal{U}(\mathfrak{g})$, which features $\mathcal{U}(\mathfrak{g})\mathfrak{h} \subset \mathcal{U}_H(\mathfrak{g})$ as a two-sided ideal. The following Lemma goes back to Helgason in case there exists an $\mathrm{Ad}(H)$-stable vector complement to $\mathfrak{h}$ in $\mathfrak{g}$. The general case is an easy adaption and probably known to a larger part in the community. Since we could not find a reference we include a proof.

Lemma C.1. The right regular action induces a natural isomorphism

$$\mathbb{D}(Z) \simeq \mathcal{U}_H(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{h}\,, \tag{C.1}$$

Proof. (Compare [20,Proof of Lemma 16]) Let $\pi : G \to Z$ be the natural projection. For every function $f \in C^\infty(Z)$, we set $\pi^* f = f \circ \pi \in C^\infty(G)$. The map $f \mapsto \pi^* f$ induces an isomorphism $C^\infty(Z) \simeq C^\infty(G)^H$ with the space of H-right-invariant functions in $C^\infty(G)$. Let $u \in \mathcal{U}_H(\mathfrak{g})$. Then, $R(u)$ preserves $C^\infty(G)^H$ and therefore induces an endomorphism of $C^\infty(Z)$ obviously given by a G-invariant differential operator. Thus, we have an algebra homomorphism

$$u \in \mathcal{U}_H(\mathfrak{g}) \mapsto D_u \in \mathbb{D}(Z) \tag{C.2}$$

characterized by the property that

$$R(u)\pi^* f = \pi^*(D_u f) \tag{C.3}$$

for every $u \in \mathcal{U}_H(\mathfrak{g}), f \in C^\infty(Z)$. It remains to show that this morphism is onto with kernel $\mathcal{U}(\mathfrak{g})\mathfrak{h}$.

First we show that

$$\{u \in \mathcal{U}(\mathfrak{g}) \mid R(u)\pi^* C^\infty(Z) = 0\} = \mathcal{U}(\mathfrak{g})\mathfrak{h}. \tag{C.4}$$

Note that this fact immediately implies that the kernel of (C2) is $\mathcal{U}(\mathfrak{g})\mathfrak{h}$.

Choose a complementary subspace $\mathfrak{m}$ of $\mathfrak{h}$ in $\mathfrak{g}$ and let

$$\mathrm{Symm} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$$

be the symmetrization map. By Poincaré-Birkhoff-Witt, we have

$$\mathcal{U}(\mathfrak{g}) = \mathrm{Symm}(S(\mathfrak{m})) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{h}.$$

Hence, we just need to show that if $u \in \mathrm{Symm}(S(\mathfrak{m}))$ is such that $R(u)\pi^* C^\infty(Z) = 0$ then $u = 0$. Let u be such an element. It can be written as $u = \mathrm{Symm}(v)$ for some $v \in S(\mathfrak{m})$. If U is a sufficiently small open neighborhood of 0 in $\mathfrak{m}$, the map

$$\psi : X \in U \mapsto \pi(\exp(X)) \in Z$$

is an open embedding. Therefore,

$$\psi^* C^\infty(Z) = C^\infty(U). \tag{C.5}$$

On the other hand, by a well-known characterization of the symmetrization map (see [20,eq. (3.9)]), we have

$$(R(u)\pi^*f)(1) = (\partial(v)\psi^*f)(0)$$

for every $f \in C^\infty(Z)$ where $\partial(v)$ is the differential operator with constant coefficients on $\mathfrak{m}$ associated to v. By (C5), this last equality implies $v = 0$ hence $u = 0$ and this ends the proof of (C4).

It only remains to prove that (C2) is surjective. Let $D \in \mathbb{D}(Z)$. As ψ is an open embedding, there exists $v \in S(\mathfrak{m})$ such that

$$(Df)(z_0) = (\partial(v)\psi^*f)(0)$$

for every $f \in C^\infty(Z)$ where $z_0 = \pi(1)$ is the natural base-point of Z. Set $u = \mathrm{Symm}(v)$. As before, the above identity can be rewritten as

$$(Df)(z_0) = (R(u)\pi^*f)(1).$$

Since D is G-invariant, it follows that

$$(Df)(gz_0) = (DL(g^{-1})f)(z_0) = (R(u)\pi^*L(g^{-1})f)(1) = (L(g^{-1})R(u)\pi^*f)(1) = (R(u)\pi^*f)(g)$$

for all $f \in C^\infty(Z)$ and $g \in G$. Otherwise said, we have

$$\pi^*(Df) = R(u)\pi^*(f), \;\; f \in C^\infty(Z). \tag{C.6}$$

Since $R(h)\pi^*(f) = \pi^*(f)$ for every $h \in H$ and $f \in C^\infty(Z)$, we deduce that $R(\mathrm{Ad}(h)u - u)\pi^*C^\infty(Z) = 0$, hence $\mathrm{Ad}(h)u - u \in \mathcal{U}(\mathfrak{g})\mathfrak{h}$ by (C4). This shows that $u \in \mathcal{U}_H(\mathfrak{g})$ and comparing (C3) with (C6) we have $D = D_u$. Therefore, the map (C2) is surjective. ∎

For $u \in \mathcal{U}_H(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\mathfrak{h}$, we denote by $R_H(u) \in \mathbb{D}(Z)$ the correponding invariant differential operator. Suppose furthermore that there is a subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$ (not necessarily direct). Then Poincaré-Birkhoff-Witt (PBW) implies that $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{b}) + \mathcal{U}(\mathfrak{g})\mathfrak{h}$ and setting $\mathcal{U}_H(\mathfrak{b}) = \mathcal{U}(\mathfrak{b}) \cap \mathcal{U}_H(\mathfrak{g})$, we obtain from (C1) an isomorphism

$$\mathbb{D}(Z) \simeq \mathcal{U}_H(\mathfrak{b})/\mathcal{U}(\mathfrak{b})(\mathfrak{h} \cap \mathfrak{b})\,. \tag{C.7}$$

Remark C.2. (a) Recall that we expressed by H_0 the identity component of H. It is then clear that $\mathcal{U}_H(\mathfrak{g}) \subset \mathcal{U}_{H_0}(\mathfrak{g})$. Hence, we obtain from $\mathrm{Lie}\, H = \mathrm{Lie}\, H_0$ and (C1) that

$$\mathbb{D}(Z) \subset \mathbb{D}(G/H_0)$$

naturally. Moreover, we record that

$$\begin{aligned}\mathcal{U}_{H_0}(\mathfrak{g}) &= \{u \in \mathcal{U}(\mathfrak{g}) \mid [X,u] \in \mathcal{U}(\mathfrak{g})\mathfrak{h}, \quad X \in \mathfrak{h}\} \\ &= \{u \in \mathcal{U}(\mathfrak{g}) \mid Xu \in \mathcal{U}(\mathfrak{g})\mathfrak{h}, \quad X \in \mathfrak{h}\}\,.\end{aligned}$$

(b) Assume that $G = \underline{G}(\mathbb{R})$ is the group of $\mathbb{R}$-points of a linear algebraic group $\underline{G}$ over $\mathbb{R}$. Let H_{alg} be the Zariski closure of H in G and assume that H_{alg} and H have the same Lie algebra (this happens, e.g., if H has finite index in the group of $\mathbb{R}$-points of an algebraic subgroup of $\underline{G}$). Then, by (C7),

$$\mathbb{D}(Z) = \mathbb{D}(G/H_{alg})\,.$$

We now return to the spherical setup and request from now that $Z = G/H$ is real spherical and unimodular where as in the main body of the text, $G = \underline{G}(\mathbb{R})$ is the group of $\mathbb{R}$-points of a connected real reductive group and H is open in the $\mathbb{R}$-points of an algebraic subgroup of $\underline{G}$. The topic of this section is then to study the relationship of $\mathbb{D}(Z)$ to $\mathbb{D}(Z_I)$ for $I \subset S$. Recall that the authors of this paper have defined H_I to be connected. We abbreviate notation and write R_I for R_{H_I} and $\mathcal{U}_I(\mathfrak{g}) = \mathcal{U}_{H_I}(\mathfrak{g})$ etc. With

$$\mathfrak{b} := \mathfrak{a} + \mathfrak{m} + \mathfrak{u}\,,$$

we obtain a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}_I$ for all $I \subset S$. Further, we have $\mathfrak{b} \cap \mathfrak{h}_I = \mathfrak{a}_H + \mathfrak{m}_H =: \mathfrak{b}_H$. In particular, we obtain an algebra isomorphism

$$p_I : \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H \to \mathbb{D}(Z_I), \quad u \mapsto R_I(u)\,. \tag{C.8}$$

Via this algebra isomorphism, we identify from now on $\mathbb{D}(Z_I)$ with $\mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. Remark that, as A_I normalizes H_I, we obtain a natural inclusion $S(\mathfrak{a}_I) \hookrightarrow \mathbb{D}(Z_I)$ induced from the right action of A_I on Z_I. Note that $Z_S = G/H_0$ and $\mathfrak{a}_S = \mathfrak{a}_{Z,E}$.

Lemma C.3. The symmetric algebra $S(\mathfrak{a}_S)$ embeds in the center of $\mathbb{D}(Z_S)$.

Proof. By slight abuse of notation, let us denote by $\underline{H}$ the algebraic closure of H_0 in $\underline{G}$ and let $H = \underline{H}(\mathbb{R})$. In view of Remark C.2(b), we may replace H_0 by H in the following.

Let $\underline{Z}(\mathbb{C}) = \underline{G}(\mathbb{C})/\underline{H}(\mathbb{C})$. Since Z is unimodular, $\underline{Z}(\mathbb{C})$ is a quasi-affine algebraic variety (see Lemma 5.4) and there is a natural embedding

$$\mathbb{D}(Z) \hookrightarrow \mathrm{End}_G(\mathbb{C}[\underline{Z}(\mathbb{C})]) \simeq \bigoplus_V \mathrm{End}(V^H)\,,$$

where the direct sum runs over all isomorphism classes of algebraic finite dimensional irreducible G-modules. Moreover, the image of $S(\mathfrak{a}_S)$ in $\mathrm{End}(V^H)$ by this morphism corresponds to the natural action of $\mathfrak{a}_S$ on V^H. Therefore, we only need to show that this action is scalar for every finite dimensional irreducible G-module V. Set $V(U) = \mathfrak{u}V$. Then $V(U)$ is a proper Q-submodule of V and the quotient $V/V(U)$ is an irreducible L-module on which the split center $\mathfrak{a}_L$ acts by a certain weight $\mu \in \mathfrak{a}_L^*$. Identify $\mathfrak{a}_S$ with a subspace of $\mathfrak{a}_L$ through the choice of a splitting of $\mathfrak{a}_Z$ in $\mathfrak{a}_L$. Then the claim would follow if we can show that the only weight of $\mathfrak{a}_S$ in V^H is the restriction of μ. We have

$$V^H \cap V(U) = 0. \tag{C.9}$$

Indeed, if $v \in V^H \cap V(U)$ then $\underline{Q}(\mathbb{C})\underline{H}(\mathbb{C}).v \subset V(U)$ and, as $\underline{Q}(\mathbb{C})\underline{H}(\mathbb{C})$ is Zariski dense in $\underline{G}(\mathbb{C})$, this implies that the $\underline{G}(\mathbb{C})$-invariant subspace generated by v is included in $V(U)$ hence $v = 0$ since V is irreducible and $V(U) \neq V$. By (C9), the restriction of the projection $V \to V/V(U)$ yields an injective $\mathfrak{a}_S$-equivariant morphism $V^H \hookrightarrow V/V(U)$. The result follows. ■

In the sequel, we view $\mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ as a subspace of $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, which is naturally a module for A/A_H, hence for A_Z. In particular, we can speak of the $\mathfrak{a}_Z$-weights of an element in $\mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. Recall that $\mathfrak{a}_S = \mathfrak{a}_{Z,E} \subset \mathfrak{a}_I$ for all $I \subset S$.

Let $I \subset S$ and $(\mathfrak{a}_I^*)^+$ be the cone of elements $\lambda \in \mathfrak{a}_I^*$ such that

$$\lambda(X) \leqslant 0, \quad X \in \mathfrak{a}_I^- .$$

Let $u_S \in \mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ and $u_S = \sum_{\lambda \in \mathfrak{a}_I^*} u_{S,\lambda}$ be its decomposition (in $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$) into $\mathfrak{a}_I$-eigenvectors. Let $\mathbf{W}_I(u_S)$ be the set of $\lambda \in \mathfrak{a}_I^*$ such that $u_{S,\lambda} \neq 0$. Then there exists a unique minimal subset $\mathbf{W}_I(u_S)_{max}$ of $\mathbf{W}_I(u_S)$ such that

$$\mathrm{conv}(\mathbf{W}_I(u_S)_{max} + (\mathfrak{a}_I^*)^+) = \mathrm{conv}(\mathbf{W}_I(u_S) + (\mathfrak{a}_I^*)^+)\,,$$

where $\mathrm{conv}(D)$ denotes the convex hull of a subset $D \subset \mathfrak{a}_I^*$ (indeed, $\mathbf{W}_I(u_S)_{max}$ is just the set of extremal points of $\mathrm{conv}(\mathbf{W}_I(u_S) + (\mathfrak{a}_I^*)^+)$; this follows from a version of the Krein–Milman theorem for convex subsets invariant by a cone, see, e.g., [13]).

Lemma C.4. Let $\lambda_{max} \in \mathbf{W}_I(u_S)_{max}$. Then $u_{S,\lambda_{max}} \in \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$.

Proof. Choose, for every $\lambda \in \mathfrak{a}_I^*$, a lift $\widetilde{u}_{S,\lambda} \in \mathcal{U}_S(\mathfrak{b})$ of $u_{S,\lambda}$, which is again an $\mathfrak{a}_I$-eigenvector of weight λ and with $\widetilde{u}_{S,\lambda} = 0$ if $u_{S,\lambda} = 0$. Set

$$\widetilde{u}_S = \sum_{\lambda \in \mathfrak{a}_I^*} \widetilde{u}_{S,\lambda}$$

(a lift of u_S). Then we want to show that $\widetilde{u}_{S,\lambda_{max}} \in \mathcal{U}_I(\mathfrak{b})$. By the choice of λ_{max}, there exists $X \in \mathfrak{a}_I^{--}$ such that $\lambda(X) < \lambda_{max}(X)$ for every $\lambda \in \mathfrak{a}_I^*$ with $\widetilde{u}_{S,\lambda} \neq 0$ and $\lambda \neq \lambda_{max}$. Therefore, we have

$$\lim_{t\to\infty} e^{-t\lambda_{max}(X)} e^{t\,\mathrm{ad}\,X} \widetilde{u}_S = \widetilde{u}_{S,\lambda_{max}}.$$

Since $\lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\mathfrak{h} = \mathfrak{h}_I$ in the Grassmannian $\mathrm{Gr}(\mathfrak{g})$, we easily check that for every $n \geqslant 0$ the limit $\lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\mathcal{U}_S(\mathfrak{g})_{\leqslant n}$ in the Grassmannian $\mathrm{Gr}(\mathcal{U}(\mathfrak{g})_{\leqslant n})$ (which always exists) is a subspace of $\mathcal{U}_I(\mathfrak{g})_{\leqslant n}$. Since $\widetilde{u}_S \in \mathcal{U}_S(\mathfrak{g})$, this shows that $\widetilde{u}_{S,\lambda_{max}} \in \mathcal{U}_I(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{b}) = \mathcal{U}_I(\mathfrak{b})$. ■

Notice that, for every $I \subset S$, we have a morphism $\mathcal{Z}(\mathfrak{g}) \to \mathbb{D}(Z_I)$ induced by the "right" action of $\mathcal{Z}(\mathfrak{g})$ on smooth functions on Z_I. We can now state the main theorem of this appendix.

Theorem C.5. For every $u_S \in \mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, the limit

$$u_I = \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X} u_S \tag{C.10}$$

exists in $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ for every $X \in \mathfrak{a}_I^{--}$ and is independent of X. The map $u \mapsto u_I$ induces an injective morphism of algebras

$$\mu_I : \mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H = \mathbb{D}(Z_S) \quad\longrightarrow\quad \mathbb{D}(Z_I) = \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H.$$

Moreover, the following assertions hold:

1. a. the $\mathfrak{a}_Z$-weights of u_S are non-positive on $\mathfrak{a}_Z^-$,
   b. the $\mathfrak{a}_Z$-weights of $u_I - u_S$ are negative on $\mathfrak{a}_I^{--}$.
2. The morphism μ_I fits into commutative squares

$$\begin{tikzcd} \mathcal{Z}(\mathfrak{g}) \arrow[r, equal] \arrow[d] & \mathcal{Z}(\mathfrak{g}) \arrow[d] \\ \mathbb{D}(Z_S) \arrow[r] & \mathbb{D}(Z_I) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} S(\mathfrak{a}_S) \arrow[r] \arrow[d] & S(\mathfrak{a}_I) \arrow[d] \\ \mathbb{D}(Z_S) \arrow[r] & \mathbb{D}(Z_I) \end{tikzcd} ,$$

   where the vertical arrows in the first and second diagrams are the natural ones.

Proof. By Lemma C.3 (applied to Z_I instead of Z) and Lemma C.4, we see that, for any nonzero $u_S \in \mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, we have $\mathbf{W}_I(u_S)_{max} = \{0\}$ (in particular, $u_{S,0} \neq 0$). This implies that the limit in (C10) exists, is independent of X and is nonzero if $u_S \neq 0$. This readily implies that μ_I is a monomorphism of algebras. Moving on to (i), we deduce (a) and (b) from the fact that the limit (C10) exists.

The second square of assertion (ii) is commutative since the image of $S(\mathfrak{a}_S)$ in $\mathcal{U}(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ is obviously in the 0-weight space of $\mathfrak{a}_I$. It only remains to show that the first square of (ii) is commutative. Let $z \in \mathcal{Z}(\mathfrak{g})$. Let $\widetilde{z}_S \in \mathcal{U}(\mathfrak{b})$ and $\widetilde{z}^S \in \mathcal{U}(\mathfrak{g})\mathfrak{h}$ be such that $z = \widetilde{z}_S + \widetilde{z}^S$. Then $\widetilde{z}_S \in \mathcal{U}_S(\mathfrak{b})$ and through our identification $\mathbb{D}(Z_S) \simeq \mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, z gets mapped to the image z_S of $\widetilde{z}_S$ in $\mathcal{U}_S(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$. By (i), up to translating $\widetilde{z}_S$ by an element of $\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, we may assume that the limit

$$\widetilde{z}_I = \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\widetilde{z}_S$$

exists in $\mathcal{U}(\mathfrak{b})$ for every $X \in \mathfrak{a}_I^{--}$ and that it is independent of X. Moreover, $\widetilde{z}_I \in \mathcal{U}_I(\mathfrak{b})$ and its image z_I in $\mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$ coincides with the image of z_S by μ_I. As z is fixed by any inner automorphism, the limit

$$\widetilde{z}^I = \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\widetilde{z}^S$$

also exists in $\mathcal{U}(\mathfrak{g})$ for every $X \in \mathfrak{a}_I^{--}$, is independent of X and $z = \widetilde{z}_I + \widetilde{z}^I$. Since $\widetilde{z}^S \in \mathcal{U}(\mathfrak{g})\mathfrak{h}$ and $\lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\mathfrak{h} = \mathfrak{h}_I$ in the Grassmannian $\mathrm{Gr}(\mathfrak{g})$, we have $\widetilde{z}^I \in \mathcal{U}(\mathfrak{g})\mathfrak{h}_I$. Therefore, by definition of the identification $\mathbb{D}(Z_I) \simeq \mathcal{U}_I(\mathfrak{b})/\mathcal{U}(\mathfrak{b})\mathfrak{b}_H$, z_I is also the image of z in $\mathbb{D}(Z_I)$. The commutativity of the first square follows. ■

Acknowledgement

We thank Raphaël Beuzart-Plessis and Friedrich Knop for their generous help on certain technicalities of the paper. We appreciate comments of Yiannis Sakellaridis regarding the exposition of an earlier version of this article. Finally we thank a professional team of three referees for their careful and complete screening of our work.

References

[1] Atiyah, M. F. and I. G. Macdonald. *Introduction to Commutative algebra*. Perseus Books Publishing, 1969.

[2] van den Ban, E. P. and H. Schlichtkrull. "The Plancherel decomposition for a reductive symmetric space." *I-II. Invent. Math* 161, no. 3 (2005): 453–566 and 567–628.

[3] Bernstein, J. N. "On the support of Plancherel measure." *J. Geom. Phys.* 5, no. 4 (1988): 663–710.

[4] Bernstein, J. N. and B. Krötz. "Smooth Fréchet globalizations of Harish-Chandra modules." *Israel J. Math.* 199, no. 1 (2014): 45–111.

[5] Bernstein, J. N. and A. Reznikov. "Estimates of automorphic functions." *Moscow Math. J.* 4, no. 1 (2004): 19–37.

[6] Brion, M. and D. Luna. "Espaces homogènes sphériques." *Invent. Math.* 84, no. 3 (1986): 617–32.

[7] Carmona, J. "Terme constant des fonctions tempérées Sur un espace symétrique réductif." *J. Reine Angew. Math.*, no. 491 (1997): 17–63.

[8] Casselman, W. "Canonical extensions of Harish-Chandra modules to representations of *G*." *Canad. J. Math.* 41, no. 3 (1989): 385–438.

[9] Cox, D., J. Little, and H. Schenck. "*Toric varieties*." *Graduate Studies in Mathematics*, Vol. 124. Providence, RI: American Mathematical Society, 2011.

[10] Danielson, T., B. Krötz, and H. Schlichtkrull. "Decomposition theorems for triple spaces." *Geom. Dedicata* 174 (2015): 145–54.

[11] Delorme, P. "Formule de Plancherel pour les espaces symétriques réductifs." *Ann. of Math.* 147 (1998): 417–52.

[12] Delorme, P., F. Knop, B. Krötz, and H. Schlichtkrull. "Plancherel theory for real spherical spaces: construction of the Bernstein morphisms." *J. Amer. Math. Soc.*, to appear, arXiv1807.07541.

[13] Goubault-Larrecq, J. A Cone-Theoretic Krein-Milman Theorem. *Research Report LSV-08-18, Laboratoire Spécification et Vérification*, ENS Cachan, France, June 2008. 8 pages.

[14] Harish-Chandra "The characters of semisimple lie groups." *Trans. Amer. Math. Soc.* 83 (1956): 98–163.

[15] Harish-Chandra "Discrete series for semisimple lie groups. I. Construction of invariant eigendistributions." *Acta Math.* 113 (1965): 241–318.

[16] Harish-Chandra "Discrete series for semisimple lie groups. II. Explicit determination of the characters." *Acta Math.* 116 (1966): 1–111.

[17] Harish-Chandra "Harmonic analysis on real reductive groups. I. the theory of the constant term." *J. Functional Analysis* 19 (1975): 104–204.

[18] Harish-Chandra "Harmonic analysis on real reductive groups. II. Wavepackets in the Schwartz space." *Invent. Math.* 36 (1976): 1–55.

[19] Harish-Chandra "Harmonic analysis on real reductive groups. III. The Maass-Selberg relations and the Plancherel formula." *Ann. of Math.* 104, no. 1 (1976): 117–201.

[20] Helgason, S. "Differential operators on homogeneous spaces." *Acta Math.* 102 (1959): 239–99.

[21] Humphreys, J. E. *Linear Algebraic Groups*, Vol. 21. Springer, Graduate Texts in Mathematics.

[22] Knapp, A. W. and D. A. Vogan. "Cohomological induction and unitary representations." *Princeton University Press.*

[23] Knop, F. "The asymptotic behavior of invariant collective motion." *Invent. Math.* 16, no. 1–3 (1994): 309–28.

[24] Knop, F. "A Harish-Chandra homomorphism for reductive group actions." *Ann. of Math.* 140 (1994): 253–88.

[25] Knop, F. and B. Krötz. "Reductive group actions." *arXiv* 1604.01005v2 (2016).

[26] Knop, F., B. Krötz, T. Pecher, and H. Schlichtkrull. "Classification of reductive real spherical pairs I. the simple case." *Transform. Groups* 24, no. 1 (2019): 67–114.

[27] Knop, F., B. Krötz, T. Pecher, and H. Schlichtkrull. "Classification of reductive real spherical pairs II. The semisimple case." *Transform. Groups* 24, no. 2 (2019): 467–510.

[28] Knop, F., B. Krötz, E. Sayag, and H. Schlichtkrull. "Simple compactifications and polar decomposition of homogeneous real spherical spaces." *Selecta Math. (N.S.)* 21, no. 3 (2015): 1071–97.

[29] Knop, F., B. Krötz, E. Sayag, and H. Schlichtkrull. "Volume growth, temperedness and integrability of matrix coefficients on a real spherical space." *J. Funct. Anal.* 271, no. 1 (2016): 12–36.

[30] Knop, F., B. Krötz, and H. Schlichtkrull. "The local structure theorem for real spherical varieties." *Compositio Math.* 151, no. 11 (2015): 2145–59.

[31] Knop, F., B. Krötz, and H. Schlichtkrull. "The tempered spectrum of a real spherical space." *Acta Math.* 218, no. 2 (2017): 319–83.

[32] Kraft, H. "Basics from algebraic geometry." *Online lecture notes.*

[33] Krötz, B., J. Kuit, E. Opdam, and H. Schlichtkrull. "The infinitesimal characters of discrete series for real spherical spaces." *Geom. Funct. Anal.* 30, no. 3 (2020): 804–857.

[34] Krötz, B., E. Sayag, and H. Schlichtkrull. "Geometric counting on Wavefront real spherical spaces." *Acta Mathematica Sinica (English Series)* 34, no. 3 (2018): 488–531.

[35] Krötz, B. and H. Schlichtkrull. "Multiplicity bounds and the subrepresentation theorem for real spherical spaces." *Trans. Amer. Math. Soc.* 368, no. 4 (2016): 2749–62.

[36] Krötz, B. and H. Schlichtkrull. "Harmonic analysis for real spherical spaces." *Acta Mathematica Sinica (English Series)* 34, no. 3 (2018): 341–70.

[37] Oshima, T. and T. Matsuki. "A description of discrete series for semisimple symmetric spaces." *Adv. Stud. Pure Math.* 4 (1984): 331–90.

[38] Repka, J. "*Tensor Products of Unitary Representations of* $\mathrm{SL}_2(\mathrm{R})$." *Amer. J. Math.* 100, no. 4 (1978): 747–74.

[39] Sakellaridis, Y. and A. Venkatesh. "Periods and harmonic analysis on spherical varieties." *Astérisque* 396 (2017).

[40] Waldspurger, J.-L. "La formule de Plancherel pour les groupes p-adiques (d'aprés Harish-Chandra)." *J. Inst. Math. Jussieu* 2, no. 3 (2003): 235–333.

[41] Wallach, N. R. *Real reductive groups. I, volume 132 of Pure and Applied Mathematics*. Boston, MA: Academic Press, Inc., 1988.

[42] Wallach, N. R. *Real reductive groups. II, volume 132 -II of Pure and Applied Mathematics*. Boston, MA: Academic Press Inc., 1992.