

PLANCHEREL THEORY FOR REAL SPHERICAL SPACES: CONSTRUCTION OF THE BERNSTEIN MORPHISMS

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1. INTRODUCTION

Our concern is with a homogeneous real spherical space $Z = G/H$. We assume that Z is algebraic, i.e. there exists a connected reductive group \underline{G} , defined over \mathbb{R} , and an algebraic subgroup $\underline{H} \subset \underline{G}$, defined over \mathbb{R} as well, such that $G = \underline{G}(\mathbb{R})$ and $H = \underline{H}(\mathbb{R})$. Then Z is a G -orbit of the variety $\underline{Z}(\mathbb{R})$ where $\underline{Z} = \underline{G}/\underline{H}$. We denote by $z_0 = eH \in Z \subset \underline{Z}(\mathbb{R})$ the standard base point and recall that Z is called real spherical if there is a minimal parabolic subgroup $P \subset G$ such that $P \cdot z_0$ is open in Z .

The goal of this paper is to develop the basic Plancherel theory for $L^2(Z)$, i.e. to establish the foundational Bernstein-decomposition of $L^2(Z)$ into different series of representations. Although the main body of the text is written in terms of Z , we focus in this introduction on $\underline{Z}(\mathbb{R})$ and the Bernstein decomposition for $L^2(\underline{Z}(\mathbb{R}))$, for which our results are easier to state. On a technical level we obtain the information for $\underline{Z}(\mathbb{R})$ by collecting the data of all G -orbits in $\underline{Z}(\mathbb{R})$.

Real spherical varieties $\underline{Z}(\mathbb{R})$ have a well understood G -equivariant compactification theory, which is constructed out of the combinatorial data of \underline{Z} originating from the local structure theorem. We recall from [28] that attached to \underline{Z} there is a torus $\underline{A}_Z = \underline{A}/\underline{A} \cap \underline{H}$, homogeneous for a maximal split torus \underline{A} of \underline{G} contained in \underline{P} . Let A_Z be the identity component of $\underline{A}_Z(\mathbb{R})$, and \mathfrak{a}_Z its Lie algebra. Inside \mathfrak{a}_Z one finds a co-simplicial cone \mathfrak{a}_Z^- , called the compression cone, which is a fundamental domain for a finite reflection group W_Z [26]. In particular there is a set $S \subset \mathfrak{a}_Z^*$, of the so-called spherical roots, such that the faces of \mathfrak{a}_Z^- are given by $\mathfrak{a}_I^- := \mathfrak{a}_Z \cap \mathfrak{a}_I$ with $I \subset S$ and $\mathfrak{a}_I := I^\perp \subset \mathfrak{a}_Z$. For the simplicity of exposition we assume in this introduction that S is a basis of the character group $\Xi_Z \simeq \mathbb{Z}^n$ of the torus \underline{A}_Z , the so-called wonderful case.

Now there exists a (wonderful) smooth G -equivariant compactification $\widehat{\underline{Z}}(\mathbb{R})$ of $\underline{Z}(\mathbb{R})$ featuring a stratification in G -manifolds,

$$\widehat{\underline{Z}}(\mathbb{R}) = \coprod_{I \subset S} \widehat{\underline{Z}}_I(\mathbb{R}),$$

parametrized by subsets $I \subset S$ of spherical roots [26] and with $\underline{Z}(\mathbb{R}) = \widehat{\underline{Z}}_S(\mathbb{R})$. The strata $\widehat{\underline{Z}}_I(\mathbb{R})$ for $I \subset S$ arise as follows. For every element X in the relative interior

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\mathfrak{a}_I^- of the face \mathfrak{a}_I^- of \mathfrak{a}_Z^- , the radial limit

$$\widehat{z}_{0,I} := \lim_{t \rightarrow \infty} \exp(tX) \cdot z_0 \in \widehat{Z}(\mathbb{R})$$

exists and is independent of X . Then \widehat{H}_I , the G -stabilizer of $\widehat{z}_{0,I}$, is real algebraic, i.e. $\widehat{H}_I = \widehat{H}_I(\mathbb{R})$, and $\widehat{Z}_I(\mathbb{R}) := [\underline{G} \cdot \widehat{z}_{0,I}](\mathbb{R})$ is the set of real points in the boundary orbit $\underline{G} \cdot \widehat{z}_{0,I}$. The group \widehat{H}_I acts on the normal space to the stratum $\widehat{Z}_I(\mathbb{R})$ at $\widehat{z}_{0,I}$. The kernel of this isotropy action defines an algebraic normal subgroup $\underline{H}_I \triangleleft \widehat{H}_I$ with torus quotient $\underline{A}_I = \widehat{H}_I/\underline{H}_I$. The real spherical space $\underline{Z}_I(\mathbb{R}) := (\underline{G}/\underline{H}_I)(\mathbb{R})$ is in fact canonically attached to $\underline{Z}(\mathbb{R})$, i.e. it does not depend on the particular compactification. Geometrically $\underline{Z}_I(\mathbb{R})$ is a deformation of $\underline{Z}(\mathbb{R})$ which approximates $\underline{Z}(\mathbb{R})$ asymptotically near the vertex $\widehat{z}_{0,I}$. We denote by A_I the identity component of $\underline{A}_I(\mathbb{R})$ and note that its Lie algebra is \mathfrak{a}_I defined above.

We assume now that Z and hence also $\underline{Z}(\mathbb{R})$ is unimodular, i.e. it carries a G -invariant positive Radon measure. As $\underline{Z}_I(\mathbb{R})$ is a deformation of $\underline{Z}(\mathbb{R})$ for each $I \subset S$, it follows that $\underline{Z}_I(\mathbb{R})$ carries a natural G -invariant measure as well. On $\underline{Z}_I(\mathbb{R})$ the group $G \times A_I$ acts from left times right. The left G -action defines a unitary representation L of G on $L^2(\underline{Z}_I(\mathbb{R}))$ given by $(L(g)f)(z) = f(g^{-1} \cdot z)$ for $g \in G$, $z \in \underline{Z}_I(\mathbb{R})$ and $f \in L^2(\underline{Z}_I(\mathbb{R}))$. The right action of A_I on $\underline{Z}_I(\mathbb{R})$ defines a normalized unitary representation $\mathcal{R}(a_I)f(z) = a_I^{-\rho} f(z \cdot a_I)$ for $a_I \in A_I$ and f, z as before. The decomposition of $L^2(\underline{Z}_I(\mathbb{R}))$ with respect to \mathcal{R} yields the disintegration in unitary G -modules

$$L^2(\underline{Z}_I(\mathbb{R})) = \int_{\widehat{A}_I} L^2(\underline{Z}_I(\mathbb{R}), \chi) d\chi$$

with \widehat{A}_I the unitary character group of the non-compact torus A_I . The space $L^2(\underline{Z}_I(\mathbb{R}), \chi)$ is the space of square integrable densities with respect to χ and we denote by $L^2(\underline{Z}_I(\mathbb{R}), \chi)_d$ the discrete spectrum of this unitary G -module. We define the twisted discrete spectrum of $L^2(\underline{Z}_I(\mathbb{R}))$ by

$$L^2(\underline{Z}_I(\mathbb{R}))_{td} := \int_{\widehat{A}_I} L^2(\underline{Z}_I(\mathbb{R}), \chi)_d d\chi.$$

The main result of this work (see Theorem 11.11 where \mathbf{B} of (1.1) is denoted by $B_{\mathbb{R}, \text{res}}$) is the construction of a G -equivariant surjective map

$$(1.1) \quad \mathbf{B} : \bigoplus_{I \subset S} L^2(\underline{Z}_I(\mathbb{R}))_{td} \rightarrow L^2(\underline{Z}(\mathbb{R}))$$

such that source and image have equivalent Plancherel measures, i.e. belong to the same measure class. Further each $\mathbf{B}_I := \mathbf{B}|_{L^2(\underline{Z}_I(\mathbb{R}))_{td}}$ is a sum of partial isometries. The latter property translates into the Maass-Selberg relations, see Theorem 9.6, and will be explained in more detail below. The existence of such a map originates from ideas of J. Bernstein, and accordingly we call \mathbf{B} the Bernstein morphism. Let us remark that in the main text we derive a more general (but more complicated to state) result, namely a Bernstein decomposition for $L^2(Z)$ (see Theorem 11.1 and Theorem 11.9) from which we derive (1.1) by collecting the data for the various G -orbits in $\underline{Z}(\mathbb{R})$.

For absolutely spherical spaces of wavefront type over a p -adic field k a Bernstein map for $L^2(\underline{Z}(k))$ with the same properties as above was constructed by Sakellaridis

and Venkatesh in [42] under the assumption of certain properties of the discrete series, see [42, Conjecture 9.4.6]. A novel point of view in [42], which we have adopted, is the observation that the decomposition of $L^2(\underline{Z}(k))$ into the various series of representations is reflected in the boundary geometry of a smooth compactification $\widehat{\underline{Z}}(k)$ of $\underline{Z}(k)$. Another new insight of [42] is that no explicit knowledge of the discrete series is needed to derive the Bernstein decomposition: the bottom line is the existence of a spectral gap for the discrete series. Since a spectral gap theorem is established in full generality for real spherical spaces in [32], we do not have to make any assumptions on the discrete spectrum as in [42].

With the implementation of the Bernstein decomposition the Plancherel theorem for $L^2(\underline{Z}(\mathbb{R}))$ essentially reduces to the understanding of the twisted discrete spectrum for each $\underline{Z}_I(\mathbb{R})$, and the determination of $\ker \mathbf{B}$. Since the Bernstein map is isospectral and surjective, it follows that the measure class of the Plancherel measure of $L^2(\underline{Z}(\mathbb{R}))$ is given by countably many copies of the Haar measures on the tori A_I .

Let us consider the example $Z = \underline{Z}(\mathbb{R}) = G \times G / \text{diag } G \simeq G$ of a real semisimple algebraic Lie group. Here the spherical roots S are identified with the simple roots with respect to \mathfrak{a} , the Lie algebra of A of a maximal split torus of G . Recall that subsets $I \subset S$ parametrize the parabolic subgroups $P_I = L_I U_I$ of G . Then we have $H_I = \text{diag}(L_I)(U_I \times \overline{U}_I)$ with $\overline{P}_I = L_I \overline{U}_I$ the parabolic opposed to P_I and in particular

$$\underline{Z}_I(\mathbb{R}) = [G/U_I \times G/\overline{U}_I] / \text{diag}(L_I).$$

Write $L_I = M_I A_I$ as usual. Now, via induction by stages, we readily obtain

$$(1.2) \quad L^2(\underline{Z}_I(\mathbb{R}))_{\text{td}} \underset{G \times G}{\simeq} \sum_{\sigma \in \widehat{M}_{I, \text{disc}}} \int_{i\mathfrak{a}_I^*} \pi_{\sigma, \lambda} \otimes \pi_{\sigma, \lambda}^* d\lambda,$$

where $\pi_{\sigma, \lambda} = \text{Ind}_{P_I}^G(\lambda \otimes \sigma)$ is the unitarily induced representation of G with respect to the unitary character of A_I defined by λ , and σ is a discrete series representation of M_I . Via basic intertwining theory we then group the occurring representations in (1.2) into equivalence classes and obtain Harish-Chandra’s Plancherel formula up to the classification of the discrete spectrum of the inducing datum (see Section 14). Likewise holds for the Plancherel theorem for symmetric spaces as obtained by Delorme [9] and van den Ban-Schlichtkrull [3] and we refer to Section 15 for the complete account.

As in the work of Harish-Chandra on the Plancherel theorem for a real reductive group, a constant term approximation [17] lies at the heart of the proof. Let us explain that. A Harish-Chandra module V endowed with a linear functional η , such that η extends to a continuous H -invariant functional on the unique smooth moderate growth completion V^∞ , will be called a spherical pair and denoted (V, η) . The continuous dual of V^∞ is denoted $V^{-\infty}$, and from [29] originates a natural linear map

$$(1.3) \quad (V^{-\infty})^H \rightarrow (V^{-\infty})^{H_I}, \quad \eta \mapsto \eta^I.$$

Attached to η are the generalized matrix coefficients $m_{v, \eta}(gH) = \eta(g^{-1}v)$ which define smooth functions on Z for all $v \in V^\infty$. Likewise we obtain smooth functions m_{v, η^I} on $Z_I := G/H_I \subset \underline{Z}_I(\mathbb{R})$. An appropriate notion of temperedness for functions on a real spherical spaces was defined in [27], and accordingly η is called

tempered if all associated matrix coefficients are tempered functions. The map (1.3) then gives rise to a linear map of tempered functionals

$$(V^{-\infty})_{\text{temp}}^H \rightarrow (V^{-\infty})_{\text{temp}}^{H_I}.$$

The constant term approximation [10] measures the differences

$$|m_{v,\eta}(g \exp(tX)H) - m_{v,\eta^I}(g \exp(tX)H_I)|$$

for $g \in \Omega$, a compact subset of G , and $t \rightarrow \infty$ for $X \in \mathfrak{a}_I^-$. We refer to Theorem 7.1 for the detailed statement.

In case of the group Harish-Chandra obtained in [17] such an approximation for a fixed representation. Using his strong results on the discrete series [16] it was made uniform for all tempered representations in [18]. For spherical spaces the uniformity of the constant term approximation is obtained in [10] via the spectral gap theorem of [32] for the twisted discrete spectrum.

Let us mention that our constant term approximation is also uniform in the category of smooth vectors so that there is no need for expansion of functions in terms of K -types. On a geometric level this allows us to view Z_I and Z in terms of the orbit geometry of the minimal parabolic subgroup P . In more detail, we show that there is a natural injective map of open P -orbits $(P \backslash Z_I)_{\text{open}} \rightarrow (P \backslash Z)_{\text{open}}$. This in turn allows us to identify Z_I inside Z , up to measure zero via the open P -orbits. We refer to Section 8 for the analytic implementation of this P -equivariant point of view. Let us point out that the auxiliary “exponential maps” of [42], which allowed an identification of $\underline{Z}(k)$ and $\widehat{Z}_I(k)$ near the vertex $\widehat{z}_{0,I}$, are no longer needed in our context of P -equivariant matching of Z_I with Z up to measure zero.

For almost all irreducible Harish-Chandra modules in the spectrum of $L^2(Z_I)$ the multiplicity space $(V^{-\infty})_{\text{temp}}^{H_I}$ is a finite dimensional semisimple module for \mathfrak{a}_I and accordingly every $\eta^I \in (V^{-\infty})_{\text{temp}}^{H_I}$ decomposes into eigenvectors

$$\eta^I = \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} \eta^{I,\lambda}.$$

Our Maass-Selberg relations are then expressed in the form that $\eta \mapsto \eta^{I,\lambda}$ is a partial isometry, see Theorem 9.6. Notice that the $\eta^{I,\lambda}$ reflect the asymptotics of the matrix coefficients $m_{v,\eta}$ through the constant term approximation. Finally we define the Bernstein morphisms spectrally via the technique of tempered embedding developed in [29, Sect. 9].

As a corollary of the Bernstein decomposition we obtain a partial geometric characterization of the existence of the discrete spectrum:

$$(1.4) \quad \text{int } \mathfrak{h}_{\text{ell}}^\perp \neq \emptyset \quad \Rightarrow \quad L^2(Z)_d \neq \emptyset,$$

see Theorem 12.1. This formulation reflects the known geometric characterization for groups and symmetric spaces, going back to Harish-Chandra [16] and Flensted-Jensen [12]. Actually we expect that the converse implication in (1.4) holds as well, and we provide a geometric analogue of the expected equivalence via moment map geometry in Theorem 13.1.

2. NOTIONS AND GENERALITIES

Throughout this paper we use upper case Latin letters $A, B, C \dots$ to denote Lie groups and write $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ for their corresponding Lie algebras. If G is a Lie group, then we denote by G_0 its identity component.

If M is a set and \sim is an equivalence relation on M , then we denote by $[m]$ the equivalence class of $m \in M$. Often the equivalence class is obtained by orbits of a group G acting on M . More specifically if X, Y are sets and G is a group which acts on X from the right and acts on Y from the left, then we obtain a left G -action on $X \times Y$ by $g \cdot (x, y) := (x \cdot g^{-1}, g \cdot y)$ whose set of equivalence classes we denote by $X \times_G Y$. We often abbreviate and simply write $[x, y]$ instead of $[(x, y)]$ to denote the equivalence class of (x, y) .

Given a group G and subgroup $H \subset G$ we use for $g \in G$ the notation $H_g := gHg^{-1}$, i.e. H_g is the G -stabilizer of the point $gH \in G/H$.

For a Lie algebra \mathfrak{g} we write $\mathcal{U}(\mathfrak{g})$ for the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. Further we denote by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$.

If \underline{Z} is an algebraic variety defined over \mathbb{R} and $k \supset \mathbb{R}$ is a field, then we denote by $\underline{Z}(k)$ the set of k -points. Since we only consider fields $k = \mathbb{R}, \mathbb{C}$ in this paper we abbreviate in the sequel and simply set $\underline{Z} := \underline{Z}(\mathbb{C})$.

Let now \underline{G} be a connected reductive algebraic group defined over \mathbb{R} and let $G := \underline{G}(\mathbb{R})$. As a general rule we use the following notation: if \underline{R} is an algebraic subgroup of \underline{G} and defined over \mathbb{R} , then we set $R := \underline{R}(\mathbb{R})$ and note that R is closed Lie subgroup of G . We regard $G \subset \underline{G}$ and then $R = G \cap \underline{R}$. We let $\underline{H} < \underline{G}$ be an algebraic subgroup defined over \mathbb{R} , and define $H < G$ according to this rule. For intersections with \underline{H} we adopt the notation $\underline{R}_{\underline{H}} := \underline{R} \cap \underline{H}$ and likewise $R_H := R \cap H = \underline{R}_{\underline{H}}(\mathbb{R})$.

Set $\underline{Z} := \underline{G}/\underline{H}$ and observe that \underline{Z} is a smooth \underline{G} -variety defined over \mathbb{R} . Set $Z := G/H$ and observe that Z is a G -orbit of $\underline{Z}(\mathbb{R})$. In general $\underline{Z}(\mathbb{R})$ is a finite union of G -orbits, but typically not equal to Z . For example if $\underline{G} = \text{SL}(n, \mathbb{C})$ and $\underline{H} = \text{SO}(n, \mathbb{C})$ then $\underline{Z}(\mathbb{R}) \simeq \bigcup_{2k \leq n} \text{SL}(n, \mathbb{R})/\text{SO}(n - 2k, 2k)$ identifies with the real symmetric matrices with unit determinant, whereas Z comprises the set of positive definite symmetric matrices therein. In particular, in this case $Z = G/H \subsetneq \underline{Z}(\mathbb{R})$. This shows, when taking real points of the principal bundle

$$(2.1) \quad \mathbf{1} \rightarrow \underline{H} \rightarrow \underline{G} \rightarrow \underline{Z}$$

we have to act with care, as the functor of taking real points in (2.1) is only left exact

$$(2.2) \quad \mathbf{1} \rightarrow H \rightarrow G \rightarrow \underline{Z}(\mathbb{R})$$

and extends to a long exact sequence of pointed sets [43, I.5.4, Prop. 36] in Galois cohomology

$$(2.3) \quad \mathbf{1} \rightarrow H \rightarrow G \rightarrow \underline{Z}(\mathbb{R}) \rightarrow H^1(\text{Gal}(\mathbb{C}|\mathbb{R}), \underline{H}) \rightarrow H^1(\text{Gal}(\mathbb{C}|\mathbb{R}), \underline{G}).$$

In this context we recall from [26, Prop. 13.1] that:

Lemma 2.1. *If \underline{G} is anisotropic over \mathbb{R} , i.e. $\underline{G}(\mathbb{R})$ is compact, then (2.2) is right exact.*

We denote by $z_0 = \underline{H}$ the standard base point of \underline{Z} and observe the G -equivariant embedding

$$Z \rightarrow \underline{Z} = \underline{G}/\underline{H}, \quad gH \mapsto g\underline{H} = g \cdot z_0.$$

If \underline{R} is a unipotent group, then note that R is connected for the Euclidean topology. This is because unipotent groups \underline{R} are isomorphic (as varieties) to their Lie algebras $\mathfrak{t}_{\mathbb{C}}$ via the algebraic exponential map.

2.1. Real spherical spaces and the local structure theorem. Let $\underline{P} < \underline{G}$ be a parabolic subgroup of \underline{G} which is minimal with respect to being defined over \mathbb{R} . We denote by \underline{N} the unipotent radical of \underline{P} .

We assume that Z is *real spherical*, that is, the action of P on Z admits an open orbit. After replacing P by a conjugate we will assume that $P \cdot z_0$ is open in Z . The local structure theorem (see [28, Th. 2.3] and [26, Cor. 4.11]) asserts the existence of a parabolic subgroup $\underline{Q} \supset \underline{P}$ with Levi-decomposition $\underline{Q} = \underline{L} \ltimes \underline{U}$ defined over \mathbb{R} such that one has

$$(2.4) \quad \underline{P} \cdot z_0 = \underline{Q} \cdot z_0$$

$$(2.5) \quad \underline{Q}_{\underline{H}} = \underline{L}_{\underline{H}}$$

$$(2.6) \quad \underline{L}_{\mathfrak{n}} \subset \underline{L}_{\underline{H}}$$

where $\underline{L}_{\mathfrak{n}}$ is the unique connected normal \mathbb{R} -subgroup of \underline{L} such that the Lie algebra $\mathfrak{l}_{\mathfrak{n}}$ is the sum of all non-compact, non-abelian simple ideals of \mathfrak{l} .

Remark 2.2. In addition to (2.4)–(2.6) we request from our choice of \underline{L} that it is obtained via the constructive proof of the local structure theorem. In case that $\underline{Z} = \underline{G}/\underline{H}$ is quasi-affine, this means that there exists $\xi \in \mathfrak{h}^{\perp} \subset \mathfrak{g}$ such that

$$\underline{L} = Z_{\underline{G}}(\xi) = \{g \in \underline{G} \mid \text{Ad}^*(g)\xi = \xi\}.$$

In case \underline{Z} is not quasi-affine one uses a quasi-affine cover (cone construction) to reduce to the quasi-affine case: extend \underline{G} to $\underline{G}_1 = \underline{G} \times \mathbb{C}^{\times}$ and let $\psi : \underline{H} \rightarrow \mathbb{C}^{\times}$ be a character defined over \mathbb{R} which is obtained from a Chevalley embedding of \underline{Z} into projective space which is defined over \mathbb{R} . With $\underline{H}_1 = \{(h, \psi(h)) \mid h \in \underline{H}\}$ we obtain a real spherical subgroup $\underline{H}_1 \subset \underline{G}_1$ such that $\underline{Z}_1 = \underline{G}_1/\underline{H}_1$ is quasi-affine. The local structure theorem for \underline{Z}_1 then descends to a local structure theorem for \underline{Z} .

With this choice of \underline{L} it is then guaranteed that the slice $\underline{L}/\underline{L}_{\underline{H}}$ can be extended to suitable compactifications of \underline{Z} which will be used later in this text.

In particular, we obtain from (2.4)–(2.5) via the obvious multiplication map

$$(2.7) \quad \underline{P} \cdot z_0 \simeq \underline{U} \times \underline{L}/\underline{L}_{\underline{H}}$$

an isomorphism of algebraic varieties defined over \mathbb{R} . If we take real points in (2.7) we get

$$(2.8) \quad [\underline{P} \cdot z_0](\mathbb{R}) \simeq U \times (\underline{L}/\underline{L}_{\underline{H}})(\mathbb{R}).$$

In the next step we wish to describe $(\underline{L}/\underline{L}_{\underline{H}})(\mathbb{R})$ in more detail. For that let $\underline{A} \subset \underline{L} \cap \underline{P}$ be a maximal split torus and set $\underline{A}_{\underline{Z}} := \underline{A}/\underline{A}_{\underline{H}}$. We also view the torus $\underline{A}_{\underline{Z}}$ as a subvariety of \underline{Z} . Further we define A_Z to be the identity component of $\underline{A}_{\underline{Z}}(\mathbb{R})$.

The number $r := \text{rank}_{\mathbb{R}} Z := \dim A_Z$ is an invariant of Z and referred to as the *real rank of Z* .

Let K be a maximal compact subgroup of G . Note that K is algebraic, i.e. $K = \underline{K}(\mathbb{R})$. Further we denote by $\mathfrak{z}(\mathfrak{g})$ the center of \mathfrak{g} , and we fix with $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ a non-degenerate $\text{Ad}(G)$ -invariant bilinear form which yields an orthogonal decomposition

of the center $\mathfrak{z}(\mathfrak{g}) = (\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{a}) \oplus (\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k})$. In case \mathfrak{g} is semi-simple, the Cartan-Killing form can be used for κ . It is a standing further requirement for K that $\mathfrak{k} \perp \mathfrak{a}$. Then $\underline{M} := Z_K(\underline{A})$, the centralizer of \underline{A} in \underline{K} , does not depend on the particular choice of K with $\mathfrak{k} \perp \mathfrak{a}$.

Notice that $Z_G(\underline{A})$ is a Levi-subgroup of \underline{P} and as such connected. Moreover we have $Z_G(\underline{A}) = \underline{MA}$. Notice that (2.6) implies that \underline{MA} acts transitively on $\underline{L}/\underline{L}_H$.

In the next two paragraphs we recall some elementary facts from [10, Sect. 1 and App. B]. Define

$$\widehat{M}_H = \{m \in \underline{M} \mid m \cdot z_0 \in \underline{A}_Z\}$$

and note that \widehat{M}_H is the isotropy group for the action of \underline{M} on $\underline{L}/\underline{L}_H \underline{A}$. In particular, \widehat{M}_H is an algebraic subgroup of \underline{G} defined over \mathbb{R} . Moreover, \widehat{M}_H contains \underline{M}_H as a normal subgroup such that $F_M := \widehat{M}_H/\underline{M}_H \simeq \widehat{M}_H/M_H$ is a finite 2-group. Here $\widehat{M}_H = \widehat{M}_H(\mathbb{R}) \subset M$ by our notational conventions.

Now $\underline{L}/\underline{L}_H$ is homogeneous for \underline{MA} and thus

$$(2.9) \quad \underline{L}/\underline{L}_H \simeq \underline{M} \times_{\widehat{M}_H} \underline{A}_Z = \underline{M}/\underline{M}_H \times_{F_M} \underline{A}_Z.$$

In particular, by [10, Prop. B.2]

$$(2.10) \quad (\underline{L}/\underline{L}_H)(\mathbb{R}) \simeq M \times_{\widehat{M}_H} \underline{A}_Z(\mathbb{R}) = M/M_H \times_{F_M} \underline{A}_Z(\mathbb{R})$$

where \simeq refers to an isomorphism of real algebraic varieties.

From (2.7) and (2.10) we obtain the following form of the local structure theorem, which we will use later on:

$$(2.11) \quad [\underline{P} \cdot z_0](\mathbb{R}) \simeq U \times [M/M_H \times_{F_M} \underline{A}_Z(\mathbb{R})].$$

Recall that $\underline{A}_Z(\mathbb{R}) \simeq (\mathbb{R}^\times)^r$, with $r = \text{rank}_{\mathbb{R}}(Z)$, is a split torus viewed as a subvariety of $\underline{Z}(\mathbb{R})$. Set

$$A_{Z,\mathbb{R}} := \underline{A}_Z(\mathbb{R}) \cap Z.$$

Then it is clear that $A_Z \subset A_{Z,\mathbb{R}} \subset \underline{A}_Z(\mathbb{R})$. In general however, $A_{Z,\mathbb{R}}$ is not a group, but carries only the structure of an A_Z -set (see Example 4.10 for $Z = \text{SL}(3, \mathbb{R})/\text{SO}(2, 1)$).

Let $F_{\mathbb{R}} = \{-1, 1\}^r < \underline{A}_Z(\mathbb{R}) = (\mathbb{R}^\times)^r$ be the 2-torsion subgroup of $\underline{A}_Z(\mathbb{R})$. Since A_Z is defined to be the identity component of $\underline{A}_Z(\mathbb{R})$ we obtain the following isomorphism of groups

$$(2.12) \quad \underline{A}_Z(\mathbb{R}) = A_Z F_{\mathbb{R}} \simeq A_Z \times F_{\mathbb{R}}.$$

Let $F \subset F_{\mathbb{R}}$ be the subset such that $A_{Z,\mathbb{R}} = A_Z F$, i.e. $F = F_{\mathbb{R}} \cap A_{Z,\mathbb{R}}$. Set $T_Z := \exp_{\underline{A}}(i\mathfrak{a}_H^\perp) < \underline{A}$ and note that $F_{\mathbb{R}} \subset T_Z \cdot z_0$ as $T_Z \cdot z_0$ contains all torsion elements of \underline{A}_Z .

Since F_M maps faithfully into $F_{\mathbb{R}}$ we view it in the sequel as a subgroup of $F_{\mathbb{R}}$. Note that $F_M \subset F$ and that F_M acts on F .

With this terminology we obtain from (2.10) that

$$(2.13) \quad Z \cap (\underline{L}/\underline{L}_H)(\mathbb{R}) \simeq M/M_H \times_{F_M} A_{Z,\mathbb{R}},$$

and accordingly from (2.11)

$$(2.14) \quad Z \cap [\underline{P} \cdot z_0](\mathbb{R}) \simeq U \times [M/M_H \times_{F_M} A_{Z,\mathbb{R}}].$$

The set of open P -orbits in Z , resp. $\underline{Z}(\mathbb{R})$, is an important geometric invariant and plays a dominant role in the harmonic analysis on Z , resp. $\underline{Z}(\mathbb{R})$. For a symmetric space it is known from [38] that the open P -orbits are parametrized by a quotient of a Weyl group with a subgroup. Although no such parametrization is known in general we denote

$$W_{\mathbb{R}} := (P \backslash \underline{Z}(\mathbb{R}))_{\text{open}} \quad \text{and} \quad W := (P \backslash Z)_{\text{open}},$$

motivated by the special case.

From (2.11) and (2.14) we deduce:

Lemma 2.3. *The maps*

$$F_M \backslash F_{\mathbb{R}} \rightarrow W_{\mathbb{R}}, \quad \mathfrak{t} = F_M t \mapsto Pt$$

and

$$F_M \backslash F \rightarrow W, \quad \mathfrak{t} = F_M t \mapsto Pt$$

are bijections.

It is often convenient to select representatives of W in G . For any $\mathfrak{t} \in F_M \backslash F$ we pick a representative $t \in F$ such that $\mathfrak{t} = F_M t$. Then $Pt \in W$ and $t \in Z = G \cdot z_0$ implies that there is a lift $w = w(\mathfrak{t}) \in G$ of t to G such that $t = w \cdot z_0$. If $\mathcal{W} = \{w(\mathfrak{t}) \mid \mathfrak{t} \in F_M \backslash F\}$, then the assignment

$$\mathcal{W} \rightarrow W, \quad w \mapsto Pw \cdot z_0$$

is a bijection.

Let $w = w(\mathfrak{t}) \in \mathcal{W}$ and let $\tilde{t} \in T_Z$ be a lift of t , i.e. $\tilde{t} \cdot z_0 = t$. Then

$$(2.15) \quad w = \tilde{t}h$$

for some $h \in \underline{H}$.

2.2. Spherical roots and the compression cone. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system for the pair $(\mathfrak{g}, \mathfrak{a})$ and let

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^{\alpha}$$

be the attached root space decomposition. Write $(\mathfrak{l} \cap \mathfrak{h})^{\perp} \subset \mathfrak{l}$ for the orthogonal complement of $\mathfrak{l} \cap \mathfrak{h}$ in \mathfrak{l} with respect to κ . From $\mathfrak{g} = \mathfrak{q} + \mathfrak{h} = \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp} \oplus \mathfrak{h}$ and $\mathfrak{g} = \mathfrak{q} \oplus \bar{\mathfrak{u}}$ we infer the existence of a linear map $T : \bar{\mathfrak{u}} \rightarrow \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp}$ such that $\mathfrak{h} = (\mathfrak{l} \cap \mathfrak{h}) \oplus \mathcal{G}(T)$ with $\mathcal{G}(T) \subset \bar{\mathfrak{u}} \oplus \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp}$ the graph of T .

Set $\Sigma_{\mathfrak{u}} := \Sigma(\mathfrak{u}, \mathfrak{a}) \subset \Sigma$. For $\alpha \in \Sigma_{\mathfrak{u}}$ and $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ let $T(X_{-\alpha}) = \sum_{\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}} X_{\alpha, \beta}$ with $X_{\alpha, \beta} \in \mathfrak{g}^{\beta}$ for $\beta \in \Sigma_{\mathfrak{u}}$ and $X_{\alpha, 0} \in (\mathfrak{l} \cap \mathfrak{h})^{\perp}$. Let $\mathcal{M} \subset \mathfrak{a}^* \setminus \{0\}$ be the additive semi-group generated by

$$\{\alpha + \beta \mid \alpha \in \Sigma_{\mathfrak{u}}, \exists X_{-\alpha} : X_{\alpha, \beta} \neq 0\}.$$

Note that all elements of \mathcal{M} vanish on \mathfrak{a}_H so that we can view \mathcal{M} as a subset of \mathfrak{a}_Z^* . A bit more precisely the elements of \mathcal{M} , seen as characters of \underline{A} , are trivial when restricted to \underline{A}_H and therefore factor to characters of \underline{A}_Z . Thus if we denote by $\Xi_Z := \text{Hom}(\underline{A}_Z, \mathbb{C}^{\times}) \simeq \mathbb{Z}^r$ the character group, seen as a lattice in \mathfrak{a}_Z^* , we have $\mathcal{M} \subset \Xi_Z$.

Define

$$\mathfrak{a}_{Z, E} := \{X \in \mathfrak{a}_Z \mid (\forall \alpha \in \mathcal{M}) \alpha(X) = 0\}$$

and note that \mathcal{M} belongs to $\mathfrak{a}_{Z,E}^\perp \subset \mathfrak{a}_Z^*$. Next, according to [26, Cor. 9.7], the convex cone $\mathbb{R}_{\geq 0}\mathcal{M}$ is simplicial in $\mathfrak{a}_{Z,E}^\perp$. Generators of this cone, suitably normalized, will be called spherical roots and denoted S .

The standard normalization of S is that a generator σ of $\mathbb{R}_{\geq 0}\mathcal{M}$ belongs to S provided it is integral and indivisible, i.e. $\sigma \in \Xi_Z$ and $\frac{1}{n}\sigma \notin \Xi_Z$ for all $n \geq 2$.

Next we define the *compression cone* by

$$\mathfrak{a}_Z^- := \{X \in \mathfrak{a}_Z \mid (\forall \alpha \in S) \alpha(X) \leq 0\}.$$

Remark 2.4. The set of spherical roots S and the associated co-simplicial compression cone \mathfrak{a}_Z^- make up an algebro-geometric invariant of the real spherical space Z , see [26]. This is important for this article, as the Bernstein morphisms defined later have an inherent parametrization by subsets $I \subset S$, i.e. faces of \mathfrak{a}_Z^- .

Let us also mention that there is an alternative elementary approach to the compression cone as a fundamental domain of a finite Coxeter group, see [36].

Let us define by $\mathfrak{a}_{Z,E} = \mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-)$ the edge of \mathfrak{a}_Z^- and record

$$\#S = \dim \mathfrak{a}_Z / \mathfrak{a}_{Z,E}.$$

Following [29, Sect. 3] we define for $I \subset S$ the boundary degeneration of \mathfrak{h}_I of \mathfrak{h} by

$$(2.16) \quad \mathfrak{h}_I := \mathfrak{l} \cap \mathfrak{h} \oplus \mathcal{G}(T_I)$$

where

$$(2.17) \quad T_I(X_{-\alpha}) := \sum_{\alpha+\beta \in \mathbb{N}_0[I]} X_{\alpha,\beta}.$$

Observe that $\mathfrak{h}_S = \mathfrak{h}$ and $\mathfrak{h}_\emptyset = \bar{\mathfrak{u}} \oplus \mathfrak{l} \cap \mathfrak{h}$.

We also set

$$(2.18) \quad \mathfrak{a}_I := \{X \in \mathfrak{a}_Z \mid (\forall \alpha \in I) \alpha(X) = 0\},$$

$$(2.19) \quad \mathfrak{a}_I^- := \{X \in \mathfrak{a}_I \mid (\forall \alpha \in S \setminus I) \alpha(X) \leq 0\},$$

$$(2.20) \quad \mathfrak{a}_I^{- -} := \{X \in \mathfrak{a}_I \mid (\forall \alpha \in S \setminus I) \alpha(X) < 0\}.$$

We recall from [29, Sect. 3] that for all $X \in \mathfrak{a}_I^{- -}$

$$(2.21) \quad \mathfrak{h}_I = \lim_{t \rightarrow \infty} e^{t \operatorname{ad} X} \mathfrak{h}.$$

Notice that $\mathfrak{a}_Z = \mathfrak{a} / \mathfrak{a}_H$ is a quotient and not canonically a subalgebra of \mathfrak{a} . In general it is convenient and notation saving to identify \mathfrak{a}_Z as a subalgebra of \mathfrak{a} by means of the identification $\mathfrak{a}_Z \simeq \mathfrak{a}_H^{\perp \mathfrak{a}}$. Then \mathfrak{a}_I normalizes \mathfrak{h}_I and we obtain with

$$\widehat{\mathfrak{h}}_I := \mathfrak{a}_I + \mathfrak{h}_I$$

a Lie subalgebra of \mathfrak{g} . It follows from (2.21) that L_H normalizes each \mathfrak{h}_I . Further we define $A_Z = \exp(\mathfrak{a}_Z) \subset A$ as a connected subgroup of A and set $A_Z^- := \exp(\mathfrak{a}_Z^-)$.

3. EQUIVARIANT SMOOTH COMPACTIFICATIONS OF $\underline{Z}(\mathbb{R})$

In this section we explain and recall the principles of G -equivariant compactification theory of $\underline{Z}(\mathbb{R})$ as developed in [26, Sect. 7].

The main idea is to use a partial toric completion of the torus \underline{A}_Z via a fan \mathcal{F} supported in all of \mathfrak{a}_Z^- (in [26] these fans are called *complete*). Let us call this partial completion $\underline{A}_Z(\mathcal{F})$.

Given a complete fan supported in $\mathfrak{a}_{\mathbb{Z}}^-$, we inflate (2.7) and form the \underline{P} -variety

$$\underline{Z}_0(\mathcal{F}) := \underline{U} \times (\underline{L}/\underline{L}_H \times_{\underline{A}_{\mathbb{Z}}} \underline{A}_{\mathbb{Z}}(\mathcal{F})).$$

Now it is the content of [26, Th. 7.1] that there exists a \underline{G} -variety $\underline{Z}(\mathcal{F})$ of the form $\underline{Z}(\mathcal{F}) = \underline{G} \cdot \underline{Z}_0(\mathcal{F})$ containing $\underline{Z}_0(\mathcal{F})$ as an open subset. Note that $\underline{Z}(\mathcal{F})(\mathbb{R})$ is compact by [26, Cor. 7.12]. The compactifications $\underline{Z}(\mathcal{F})$ of \underline{Z} just constructed are usually called *toroidal* as they origin from partial compactifications of the torus $\underline{A}_{\mathbb{Z}}$.

For a cone $\mathcal{C} \in \mathcal{F}$ in the fan \mathcal{F} we denote by $\text{int } \mathcal{C}$ its relative interior, i.e. the interior with respect to $\mathfrak{a}_{\mathcal{C}} := \text{span}_{\mathbb{R}} \mathcal{C} \subset \mathfrak{a}_{\mathbb{Z}}$. Now to every cone $\mathcal{C} \in \mathcal{F}$ corresponds a radial limit $\widehat{z}_{\mathcal{C}} \in \underline{A}_{\mathbb{Z}}(\mathcal{F}) \subset \underline{Z}(\mathcal{F})$ defined as follows. The limit

$$\widehat{z}_{\mathcal{C}} := \lim_{s \rightarrow \infty} \exp(sX) \cdot z_0$$

exists for every $X \in \text{int } \mathcal{C}$ and is independent of X . Moreover, the \underline{G} -orbits in $\underline{Z}(\mathcal{F})$ are parametrized by the cones $\mathcal{C} \in \mathcal{F}$ by way of $\mathcal{C} \mapsto \widehat{\underline{Z}}_{\mathcal{C}} := \underline{G} \cdot \widehat{z}_{\mathcal{C}}$, see [26, Cor. 7.5].

Define $\underline{A}_{\mathcal{C}} \subset \underline{A}_{\mathbb{Z}}$ as the torus which fixes $\widehat{z}_{\mathcal{C}}$ and note that its Lie algebra is given by the complexification of $\mathfrak{a}_{\mathcal{C}}$ defined above. Hence if $I = I(\mathcal{C})$ is the set of spherical roots vanishing on \mathcal{C} , then $\mathfrak{a}_{\mathcal{C}} \subset \mathfrak{a}_I$. Further if we denote by $\widehat{H}_{\mathcal{C}}$ the \underline{G} -stabilizer of $\widehat{z}_{\mathcal{C}}$, then we have the following relation for Lie algebras:

$$(3.1) \quad \widehat{\mathfrak{h}}_{\mathcal{C}} = \mathfrak{h}_I + \mathfrak{a}_{\mathcal{C}}$$

with \mathfrak{h}_I defined as in (2.21). In case $\underline{Z}(\mathcal{F})$ is smooth we provide a simple argument for (3.1) below.

For our purpose we need that $\underline{Z}(\mathcal{F})$ is a smooth manifold. By the construction of $\underline{Z}(\mathcal{F})$ this is the case if and only if $\underline{A}_{\mathbb{Z}}(\mathcal{F})$ is smooth. Let us now provide a standard construction of a complete fan which yields a smooth partial completion $\underline{A}_{\mathbb{Z}}(\mathcal{F})$. For that we denote by $\Xi_{\mathbb{Z}} = \text{Hom}(\underline{A}_{\mathbb{Z}}, \mathbb{C}^*) \simeq \mathbb{Z}^r$ the character group of $\underline{A}_{\mathbb{Z}}$. Likewise we let $\Xi_{\mathbb{Z}}^{\vee} = \text{Hom}(\mathbb{C}^*, \underline{A}_{\mathbb{Z}})$ be the co-character group and note the natural identification $\mathfrak{a}_{\mathbb{Z}} \simeq \Xi_{\mathbb{Z}}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R}$.

Best results are obtained when S is a \mathbb{Z} -basis for the character lattice $\Xi_{\mathbb{Z}}$. In this case the standard fan \mathcal{F}_{st} obtained by the faces of $\mathfrak{a}_{\mathbb{Z}}^-$ is smooth and $\underline{Z}(\mathcal{F}_{\text{st}})$ is the wonderful compactification of \underline{Z} (see [26, Definition 11.4]).

Remark 3.1. In general S is not a basis of $\Xi_{\mathbb{Z}}$. This can have several natural reasons, for example if $\mathfrak{a}_{Z,E} \neq 0$ as $\#S := \dim \mathfrak{a}_{\mathbb{Z}} / \mathfrak{a}_{Z,E} < r = \text{rank}_{\mathbb{R}}(Z) = \dim \mathfrak{a}_{\mathbb{Z}}$. One might overcome this by passing from \underline{H} to $\underline{H} = \underline{H} \cdot \underline{A}_{Z,E}$. But even if $\#S = r$ it might happen that there is torsion, i.e. $\Xi_{\mathbb{Z}} / \mathbb{Z}[S] \neq 0$ which destroys smoothness of $\underline{A}_{\mathbb{Z}}(\mathcal{F})$ for \mathcal{F} the fan generated by $\mathfrak{a}_{\mathbb{Z}}^-$.

One can overcome both issues indicated in Remark 3.1 simultaneously by subdividing $\mathfrak{a}_{\mathbb{Z}}^-$ into finitely many simple simplicial cones C_1, \dots, C_N such that

- $\mathfrak{a}_{\mathbb{Z}}^- = \bigcup_{j=1}^N C_j$,
- $C_i \cap C_j$ is a common face of both C_i and C_j for all $1 \leq i, j \leq N$,
- For each $1 \leq j \leq N$ there exists a basis $(\psi_{ji})_{1 \leq i \leq r}$ of $\Xi_{\mathbb{Z}}$ such that $C_j = \{X \in \mathfrak{a}_{\mathbb{Z}} \mid (\forall 1 \leq i \leq r) \psi_{ji}(X) \leq 0\}$.

The existence of such a decomposition is a standard fact of toric geometry, see [21, Ch. 3]. Let us denote by \mathcal{F}_i the fan generated by C_i , i.e. the set of all faces of C_i . Then define the fan $\mathcal{F} := \bigcup_{i=1}^N \mathcal{F}_i$. Notice that $\underline{A}_{\mathbb{Z}}(\mathcal{F})$ is smooth and is obtained

from gluing together the various open pieces $\underline{A}_Z(\mathcal{F}_i) \simeq \mathbb{C}^r$. From $\mathfrak{a}_Z^- = \bigcup_{j=1}^N C_j$ we obtain $\mathfrak{a}_I^- = \bigcup_{j=1} C_j \cap \mathfrak{a}_I^-$. Now for every $I \subset S$ we let $J_I \subset \{1, \dots, N\}$ be the set of indices j for which $C_j \cap \mathfrak{a}_I^- \neq \emptyset$. Then $\mathfrak{a}_I^- = \bigcup_{j \in J_I} (C_j \cap \mathfrak{a}_I^-)$. Note that in general J_I is not a singleton as for example $J_\emptyset = \{1, \dots, N\}$.

We fix now a simplicial subdivision as above and the corresponding complete fan \mathcal{F} . To abbreviate notation we set $\widehat{\underline{Z}}_0 := \underline{Z}_0(\mathcal{F})$ and $\widehat{\underline{Z}} := \underline{Z}(\mathcal{F})$. We denote by \widehat{Z} the closure of Z in $\widehat{\underline{Z}}(\mathbb{R})$ which is then a manifold with corners [26, Sect. 14].

For every $I \subset S$ we fix now an $j_I \in J_I$ and let $\mathfrak{c}_I^- = C_{j_I} \cap \mathfrak{a}_I^- \subset \mathfrak{a}_I^-$. We denote by \mathfrak{c}_I^- the relative interior of \mathfrak{c}_I^- . We recall $z_0 = \underline{H} \in \underline{Z}$ the standard base point. Then for $X \in \mathfrak{c}_I^-$ the limit

$$(3.2) \quad \widehat{z}_{0,I} := \lim_{s \rightarrow \infty} \exp(sX) \cdot z_0 \in \underline{A}_Z(\mathcal{F}_{j_I})(\mathbb{R}) \subset \widehat{\underline{Z}}_0(\mathbb{R})$$

exists and is independent of the choice of $X \in \mathfrak{c}_I^-$ (but depends on j_I).

Remark 3.2. Our choice of $j_I \in J_I$ yielding \mathfrak{c}_I^- can also be seen in the following context. Set

$$(3.3) \quad \mathcal{F}_I := \{\mathcal{C} \in \mathcal{F} \mid \mathfrak{a}_{\mathcal{C}} = \mathfrak{a}_I\}.$$

Then our choice of $j_I \in J_I$ picks an element $\mathfrak{c}_I^- \in \mathcal{F}_I$ together with an $1 \leq j_I \leq N$ such that $\mathfrak{c}_I^- \subset C_{j_I}$.

Let us denote by \widehat{H}_I the stabilizer of $\widehat{z}_{0,I}$ in \underline{G} . Note that \widehat{H}_I is defined over \mathbb{R} . We claim that \widehat{H}_I has Lie algebra

$$(3.4) \quad \widehat{\mathfrak{h}}_I = \mathfrak{h}_I + \mathfrak{a}_I$$

with \mathfrak{h}_I defined in (2.16). In order to see that we note that the G -stabilizer of $z_t := \exp(tX) \cdot z_0$ is $H_t := \exp(tX)H \exp(-tX)$. Moreover the fact that $z_t \rightarrow \widehat{z}_{0,I}$ in the smooth manifold $\widehat{\underline{Z}}(\mathbb{R})$ implies that the stabilizer Lie algebra of the limit $\widehat{z}_{0,I}$ contains the limit \mathfrak{h}_I of (2.21). Now the claim follows from [26, Th. 7.3].

We define $\widehat{\underline{Z}}_I = \underline{G} \cdot \widehat{z}_{0,I} \simeq \underline{G}/\widehat{H}_I$. Proposition 3.3 shows that this definition is independent of the choice of $\mathfrak{c}_I^- \in \mathcal{F}_I$.

Proposition 3.3. *We have $\widehat{H}_{\mathcal{C}} = \widehat{H}_I$ for all $\mathcal{C} \in \mathcal{F}_I$. Moreover, \widehat{H}_I does not depend on the choice of the smooth complete fan \mathcal{F} defining the smooth toroidal compactification $\widehat{\underline{Z}} = \underline{Z}(\mathcal{F})$ of \underline{Z} . In other words, for every $I \subset S$ the \underline{G} -variety $\widehat{\underline{Z}}_I = \underline{G}/\widehat{H}_I$ is up to \underline{G} -isomorphism canonically attached to the \underline{G} -variety $\underline{Z} = \underline{G}/\underline{H}$.*

Proof. We prove the first assertion by induction on $n = \#S$. We start with $n = 0$, the case of horospherical varieties, see [26, Sect. 8]. In this situation \underline{A} normalizes \underline{H} and moreover $\underline{H} = (\underline{H} \cap \underline{L})\underline{U}^{\text{opp}}$ with $\underline{U}^{\text{opp}}$ the opposite of \underline{U} . In particular, $\underline{A}_Z = \underline{A}/\underline{A}_{\underline{H}}$ acts naturally on the right of $\underline{Z} = \underline{G}/\underline{H}$. By the construction of the toroidal compactification as the unique minimal \underline{G} -extension of $\underline{Z}_0(\mathcal{F}) = [\underline{Q}/\underline{Q}_{\underline{H}}] \times_{\underline{A}_Z} \underline{A}_Z(\mathcal{F})$ we obtain that

$$\underline{Z}(\mathcal{F}) = \underline{G}/\underline{H} \times_{\underline{A}_Z} \underline{A}_Z(\mathcal{F})$$

and hence $\widehat{H}_{\mathcal{C}} = \underline{H}\underline{A}$ for all $\mathcal{C} \in \mathcal{F}_S$.

Let now $n > 0$ and $I \subset S$. We first treat the case for $I = S$. Then $\mathfrak{a}_S = \mathfrak{a}_{Z,E}$ and we note for all $\mathcal{C} \in \mathcal{F}_S$ the natural isomorphism

$$\underline{A}_Z \cdot \widehat{z}_{\mathcal{C}} \simeq \underline{A}_Z/\underline{A}_{Z,E}.$$

Hence we obtain that

$$\underline{Q} \cdot \widehat{z}_C = [\underline{Q}/\underline{Q}_H] \times_{\underline{A}_Z} [\underline{A}_Z \cdot \widehat{z}_C] \simeq \underline{Q}/(\underline{Q} \cap \underline{H})\underline{A}_{Z,E}.$$

This means for the \underline{G} -extension $\widehat{\underline{Z}}_C$ of $\underline{Z}_0(\mathcal{C}) = \underline{Q} \cdot \widehat{z}_C$

$$\widehat{\underline{Z}}_C \simeq \underline{G}/\underline{H}\underline{A}_{Z,E},$$

i.e. $\widehat{H}_C = \underline{H}\underline{A}_{Z,E}$.

Suppose now that $I \subsetneq S$ and let $\mathcal{C}, \mathcal{C}' \in \mathcal{F}_I$. We connect now \mathcal{C} and \mathcal{C}' in \mathfrak{a}_I^- face to face, i.e. we find $\mathcal{C}_1, \dots, \mathcal{C}_m \in \mathcal{F}_I$ such that $I(\mathcal{C} \cap \mathcal{C}_1) = I$, $I(\mathcal{C}_i \cap \mathcal{C}_{i+1}) = I$ for $1 \leq i \leq m-1$ and $I(\mathcal{C}' \cap \mathcal{C}_m) = I$. Hence we may assume that $I(\mathcal{C} \cap \mathcal{C}') = I$. Set $\mathcal{C}_0 := \mathcal{C} \cap \mathcal{C}'$. Set

$$\mathcal{F}(\mathcal{C}_0) := \{\mathcal{C} \in \mathcal{F} \mid \mathfrak{a}_{\mathcal{C}_0} \subset \mathfrak{a}_{\mathcal{C}}\}$$

and note that $\mathcal{C}, \mathcal{C}' \in \mathcal{F}(\mathcal{C}_0)$. Set $\underline{Z}_0 := \underline{G} \cdot \widehat{z}_{\mathcal{C}_0} \simeq \underline{G}/\underline{H}_{\mathcal{C}_0}$ and note that $\mathfrak{a}_{Z_0} = \mathfrak{a}_Z/\mathfrak{a}_{\mathcal{C}_0}$. Moreover, $\mathcal{F}_0 := \mathcal{F}(\mathcal{C}_0)/\mathfrak{a}_{\mathcal{C}_0}$ is a complete smooth fan for \underline{Z}_0 featuring $\underline{Z}_0(\mathcal{F}_0) \subset \underline{Z}(\mathcal{F})$ as the Zariski closure of \underline{Z}_0 in $\underline{Z}(\mathcal{F})$. Now $S_0 = S(\underline{Z}_0) = I \subsetneq S$ and we obtain by induction that $\widehat{H}_C = \widehat{H}_{C'}$.

Finally we note that if \mathcal{F}_1 and \mathcal{F}_2 are smooth fans, then there exists a smooth fan \mathcal{F}_3 containing both \mathcal{F}_1 and \mathcal{F}_2 , i.e. $\underline{Z}(\mathcal{F}_1), \underline{Z}(\mathcal{F}_2) \subset \underline{Z}(\mathcal{F}_3)$. This completes the proof of the proposition. \square

For the purpose of this paper our interest is not so much with $\widehat{\underline{Z}}_I$ but with the real G -orbit $\widehat{Z}_I = G \cdot \widehat{z}_{0,I} \simeq G/\widehat{H}_I$. Note that $\widehat{Z}_I \subset \widehat{Z}$.

For $I \subset S$ we denote by \underline{A}_I the subtorus of \underline{A}_Z corresponding to $\mathfrak{a}_I \subset \mathfrak{a}_Z$. For our fixed $j = j_I \in J_I$ with regard to \mathfrak{c}_I^- we now set $\psi_i^I := \psi_{j_i}$ for $1 \leq i \leq r$. Let $k := r - |I|$. We may order the basis $(\psi_i^I)_{1 \leq i \leq r}$ then in such a way that $\mathbb{Q}[I] = \mathbb{Q}[\psi_{k+1}^I, \dots, \psi_r^I]$ and then

$$\mathfrak{c}_I^- = \{X \in \mathfrak{a}_I^- \mid (\forall i \leq k) \psi_i^I(X) < 0\}.$$

With the basis $(\psi_i^I)_i$ we identify \underline{A}_Z with $(\mathbb{C}^\times)^r$ via

$$(3.5) \quad \underline{A}_Z \rightarrow (\mathbb{C}^\times)^r, \quad a \mapsto (a^{\psi_i^I})_{1 \leq i \leq r}.$$

In these coordinates A_I corresponds to the subgroup $(\mathbb{C}^\times)^{r-k} \simeq \mathbf{1} \times (\mathbb{C}^\times)^{r-k} \subset (\mathbb{C}^\times)^r$.

Let us denote by $(\mathbf{e}_i^I)_{1 \leq i \leq r} \subset \mathfrak{a}_Z$ the basis dual to $(\psi_i^I)_{1 \leq i \leq r}$. We define the $\underline{A}_Z(\mathbb{R})$ -modules $V_I := \bigoplus_{i \leq k} \mathbb{R}\mathbf{e}_i^I \simeq \mathfrak{a}_I$ and $V_I^\perp := \bigoplus_{i > k} \mathbb{R}\mathbf{e}_i^I$, which are both diagonal with respect to the fixed basis $(\psi_i^I)_{1 \leq i \leq r}$ of Ξ_Z . Via the coordinates of (3.5) we view \underline{A}_Z as open subset of $V_C = \mathbb{C}^r = \underline{A}_Z(\mathcal{F}_{j_I})$ and obtain in particular that

$$(3.6) \quad \underline{Z}_0(\mathcal{F}_{j_I}) = \underline{U} \times [\underline{M}/\underline{M}_H \times_{F_M} V_C]$$

where we view $F_M = \widehat{M}_H/M_H$ as a subgroup of $\{-1, 1\}^r$ acting on V_C by sign changes in the coordinates. Set $V = \mathbb{R}^r$.

Lemma 3.4. *The real points of $\underline{Z}_0(\mathcal{F}_{j_I}) = \underline{U} \times [\underline{M}/\underline{M}_H \times_{F_M} V_C]$ are given by*

$$(3.7) \quad \underline{Z}_0(\mathcal{F}_{j_I})(\mathbb{R}) = U \times [M/M_H \times_{F_M} V].$$

Proof. Let $x = (u, [m\underline{M}_H, v]) \in \underline{Z}_0(\mathcal{F}_{j_I})$ where $u \in \underline{U}$, $m \in \underline{M}$ and $v \in V_{\mathbb{C}}$. Then x is real if and only if $\bar{x} = x$, that is

$$(\bar{u}, [\bar{m}\underline{M}_H, \bar{v}]) = (u, [m\underline{M}_H, v])$$

and in particular $u = \bar{u}$. Moreover, as F_M has representatives in \widehat{M}_H , we obtain that $\bar{m}\underline{M}_H \in m\widehat{M}_H\underline{M}_H$. Now it follows from Lemma 2.1 that the polar map

$$M \times_{M_H} \mathfrak{m}_H^\perp \rightarrow \underline{M}/\underline{M}_H, [g, X] \mapsto g \exp(iX)\underline{M}_H$$

is a diffeomorphism. Hence if $y = m\underline{M}_H = [g, X]$ is such that $\bar{y} \in m\widehat{M}_H\underline{M}_H$ we obtain $\bar{y} = [g, -X] = [g\widehat{m}^{-1}, \text{Ad}(\widehat{m})X]$ for some $\widehat{m} \in \widehat{M}_H$. But this gives $\widehat{m} \in M_H$ and thus $\bar{y} = y$, i.e. $X = 0$. Therefore $y = m\underline{M}_H = [g, 0]$ and we may choose $m = g \in M$. This yields in turn that $\bar{v} = v$ which concludes the proof of the lemma. \square

Let $\mathbf{e}_I := \sum_{j=1}^k \mathbf{e}_j^I \in V_I$. Set $F_{M,I} := F_M \cap \underline{A}_I$ and note that $F_{M,I}$ is the F_M -stabilizer of $\mathbf{e}_I \in V_I$. Further put $F_M^I := F_M/F_{M,I}$. Denote by $V_I^{\perp, \times} \subset V_I$ the subset with all coordinates non-zero and observe that

$$V_{I, \mathbb{C}}^{\perp, \times} = \underline{A}_Z \cdot \mathbf{e}_I \simeq \underline{A}_Z/\underline{A}_I.$$

Then we obtain from (3.6) and (3.7) the isomorphisms

$$\begin{aligned} (3.8) \quad P \cdot \widehat{z}_{0,I} &\simeq \underline{U} \times \left[\left[\underline{M}/\underline{M}_H F_{M,I} \right] \times_{F_M^I} V_{I, \mathbb{C}}^{\perp, \times} \right] \\ &\simeq \underline{U} \times \left[\left[\underline{M}/\underline{M}_H F_{M,I} \right] \times_{F_M^I} \left[\underline{A}_Z/\underline{A}_I \right] \right] \end{aligned}$$

and

$$\begin{aligned} (3.9) \quad [P \cdot \widehat{z}_{0,I}](\mathbb{R}) &\simeq U \times \left[[M/M_H F_{M,I}] \times_{F_M^I} V_I^{\perp, \times} \right] \\ &\simeq U \times \left[[M/M_H F_{M,I}] \times_{F_M^I} [\underline{A}_Z(\mathbb{R})/\underline{A}_I(\mathbb{R})] \right] \end{aligned}$$

which are given in coordinates as

$$(u, [mM_H F_{M,I}, v]) = umv \cdot \widehat{z}_{0,I}.$$

3.1. Relatively open P -orbits in \widehat{Z} . The structure of the finite set of G -orbits in \widehat{Z} is in general complicated and the G -orbits through the boundary points $\widehat{z}_{0,I} \subset \widehat{Z}$ do typically not give all G -orbits in \widehat{Z} (see Example 4.10 below).

In general, let us call a P -orbit $P \cdot \widehat{z} \subset \widehat{Z}$ *relatively open* provided $P \cdot \widehat{z}$ is open in the G -orbit $G \cdot \widehat{z}$. The goal of this subsection is to describe the set of all relatively open P -orbits in \widehat{Z} , denoted by $(P \backslash \widehat{Z})_{\text{rel-op}}$ in the sequel.

Recall from the end of Section 2.1 the set $\mathcal{W} \subset G$ which parametrizes $(P \backslash Z)_{\text{open}}$. In addition we remind that elements $w \in \mathcal{W}$ have a representation as $w = \tilde{t}h$ with $\tilde{t} \in T_Z = \exp(i\mathfrak{a}_H^\perp) \subset \underline{A}$ and $h \in \underline{H}$ such that $t := \tilde{t} \cdot z_0 \in F$ where $F = F_{\mathbb{R}} \cap Z$ and $F_{\mathbb{R}}$ the finite group of 2-torsion points of $\underline{A}_Z(\mathbb{R}) \subset \underline{Z}(\mathbb{R})$ (see Subsection 2.1 for the notation).

For $w \in \mathcal{W}$ we now define the shifted base points:

$$z_w := w \cdot z_0 = \tilde{t} \cdot z_0 = t \in F \subset Z.$$

Likewise for $I \subset S$ and $X \in \mathfrak{c}_I^-$ we define in analogy to (3.2)

$$\widehat{z}_{w,I} := \lim_{s \rightarrow \infty} \exp(sX) \cdot z_w = \tilde{t} \cdot \widehat{z}_{0,I}$$

and note that the second equality (immediate from the definitions) implies that $\widehat{z}_{w,I}$ is independent of the choice of $X \in \mathfrak{c}_I^-$. As $\tilde{t} \cdot \widehat{z}_{0,I}$ is independent of the chosen lift \tilde{t} of t we can define for $t \in F_{\mathbb{R}}$

$$t \cdot \widehat{z}_{0,I} := \tilde{t} \cdot \widehat{z}_{0,I}.$$

Since the limit defining $\widehat{z}_{w,I}$ exists and $z_w \in Z$ we infer that $\widehat{z}_{w,I} \in \widehat{Z}$. Moreover, as $\widehat{z}_{w,I} \in F_{\mathbb{R}} \cdot \widehat{z}_{0,I}$ with the notation defined above, we infer from the local structure theorem as recorded in (3.9) that $P \cdot \widehat{z}_{w,I}$ is open in $G \cdot \widehat{z}_{w,I}$. With that we obtain in fact all relatively open P -orbits in the wonderful situation:

Lemma 3.5. *Suppose that \widehat{Z} is wonderful. Then the set of relatively open P -orbits in \widehat{Z} is given by*

$$(P \backslash \widehat{Z})_{\text{rel-op}} = \{P \cdot \widehat{z}_{w,I} \mid w \in \mathcal{W}, I \subset S\}.$$

Proof. The inclusion \supset was already seen above. In the wonderful situation the \underline{G} -orbits in \widehat{Z} are precisely the $\underline{G} \cdot \widehat{z}_{0,I} \simeq \underline{G}/\widehat{H}_I$ for $I \subset S$ and accordingly every relatively open P -orbit in $\widehat{Z}(\mathbb{R})$ lies in some $[P \cdot \widehat{z}_{0,I}](\mathbb{R})$. Hence any relatively open P -orbit in \widehat{Z} is of the form $Pt_1 \cdot \widehat{z}_{0,I}$ for some $t_1 \in F_{\mathbb{R}}$ by (3.9) and (2.12). Since \widehat{Z} is G -invariant, and in particular P -invariant, it follows that $t_1 \cdot \widehat{z}_{0,I} \in \widehat{Z}$. Further the local structure theorem (3.9) implies that $t_1 \cdot \widehat{z}_{0,I} \in \partial Z$ is approached by a curve in Z of the form $\exp(sX)t_2 \in Z$ for some $t_2 \in F$ and $X \in \mathfrak{a}_I^- = \mathfrak{c}_I^-$, for $s \rightarrow \infty$. In other words $t_1 \cdot \widehat{z}_{0,I} = \lim_{s \rightarrow \infty} \exp(sX)t_2 = t_2 \cdot \widehat{z}_{0,I}$. With Lemma 2.3 this concludes the proof. \square

Remark 3.6.

(a) In the wonderful case we have a stratification $\widehat{Z}(\mathbb{R}) = \coprod_{I \subset S} \widehat{Z}_I(\mathbb{R})$ of $\widehat{Z}(\mathbb{R})$ in real spherical G -manifolds with $P \cdot \widehat{z}_{w,I} \subset \widehat{Z}_I(\mathbb{R})$ for each $w \in \mathcal{W}$. In particular if $I \neq J \subset S$ we have $P \cdot \widehat{z}_{w,I} \neq P \cdot \widehat{z}_{w',J}$ for all $w, w' \in \mathcal{W}$. However, for fixed I it can and will happen that $P \cdot \widehat{z}_{w,I} = P \cdot \widehat{z}_{w',I}$ for some $w \neq w'$. The extremal case is $I = \emptyset$ where $\widehat{z}_{\emptyset} = \widehat{z}_{w,\emptyset}$ does not depend on $w \in \mathcal{W}$ at all.

(b) In case \widehat{Z} is not wonderful, the assertion in Lemma 3.5 needs to be modified as follows. For every cone $\mathcal{C} \in \mathcal{F}$ and $w \in \mathcal{W}$ let us define

$$(3.10) \quad \widehat{z}_{w,\mathcal{C}} := \lim_{s \rightarrow \infty} \exp(sX) \cdot z_w$$

which does not depend on $X \in \text{int } \mathcal{C}$. Recall that the \underline{G} -orbits in the toroidal compactification \widehat{Z} are parametrized by $\mathcal{C} \in \mathcal{F}$ and explicitly given by $\underline{G} \cdot \widehat{z}_{\mathcal{C}}$. Then for each $\mathcal{C} \in \mathcal{F}$ the relatively open P -orbits in ∂Z contained in $\widehat{Z}_{\mathcal{C}} = \underline{G} \cdot \widehat{z}_{\mathcal{C}}$ are given by the $P \cdot \widehat{z}_{w,\mathcal{C}}$ with $w \in \mathcal{W}$.

(c) As every open G -orbit in $\widehat{Z}_{\mathcal{C}}$ is open and contains an open P -orbit, we deduce from (b) that

$$\widehat{Z}_{\mathcal{C}} = \bigcup_{w \in \mathcal{W}} G \cdot \widehat{z}_{w,\mathcal{C}}.$$

4. NORMAL BUNDLES TO BOUNDARY ORBITS IN A SMOOTH COMPACTIFICATION

Let X be a manifold and $Y \subset X$ be a submanifold. We denote by TX and TY the associated tangent bundles of X and Y . The normal bundle of Y is then defined to be

$$N_Y := TX|_Y/TY.$$

Note that $N_Y \rightarrow Y$ is a vector bundle with fibers $(N_Y)_y = T_y X / T_y Y$.

We are mainly interested in the case where X is a smooth G -manifold for a Lie group G , and $Y := G \cdot y$ is a locally closed orbit. In this case we have a natural action of the stabilizer G_y on $(N_Y)_y$ and

$$(4.1) \quad N_Y = G \times_{G_y} (N_Y)_y$$

reveals the G -structure of N_Y .

4.1. Normal bundles to boundary orbits. After this interlude on normal bundles we return to our basic setting with G a real reductive algebraic group, and let $X := \widehat{Z}(\mathbb{R})$ be a smooth G -equivariant compactification of Z as constructed in Section 3.

Fix $I \subset S$ and let $Y := \widehat{Z}_I \subset X$ be a boundary orbit with base point $y := \widehat{z}_{0,I}$. Recall the basis $(\psi_i^I)_{1 \leq i \leq r}$ of Ξ_Z , its dual basis $(e_i^I)_i$ and $V_I := \bigoplus_{i \leq k} \mathbb{R} e_i^I$. By means of the basis it is often convenient to identify V_I with \mathbb{R}^k where $k = r - |I|$. Define $V_I^\times := \bigoplus_{i \leq k} \mathbb{R}^\times e_i^I$ and $V_I^0 \subset V_I^\times$ by

$$V_I^0 := \bigoplus_{i \leq k} \mathbb{R}^+ e_i^I \simeq (\mathbb{R}^+)^k.$$

Set $V := V_I \oplus V_I^\perp$ and recall $e_I = \sum_{j=1}^k e_j^I \in V_I^0$.

Let $\mathcal{U}_M \subset M/M_H$ be an open neighborhood of the base point $M_H \in M/M_H$ such that $\mathcal{U}_M \cap \mathcal{U}_M \cdot x = \emptyset$ for $x \in F_M, x \neq 1$.

Recall that $V_I^{\perp, \times} = \underline{A}_Z(\mathbb{R}) / \underline{A}_I(\mathbb{R})$. According to (3.9) the mapping

$$\Psi_1 : U \times \mathcal{U}_M \times V_I^{\perp, \times} \rightarrow [\underline{P} \cdot \widehat{z}_{0,I}](\mathbb{R}) = U \times [[M/M_H F_{M,I}] \times_{F_M^I} V_I^{\perp, \times}]$$

given by

$$(u, mM_H, v) \mapsto (u, [mM_H F_{M,I}, v])$$

is a diffeomorphism onto an open subset of $[\underline{P} \cdot \widehat{z}_{0,I}](\mathbb{R})$ and hence also of $\widehat{Z}_I(\mathbb{R})$.

Set

$$\mathcal{V} := \Psi_1^{-1}(Y).$$

Thus we obtain two diffeomorphisms onto their images

$$\Psi_0 : \mathcal{V} \rightarrow Y = \widehat{Z}_I, \quad (u, mM_H, a \underline{A}_I(\mathbb{R})) \mapsto uma \cdot y$$

and

$$\Psi : \mathcal{V} \times V_I \rightarrow U \times [M/M_H \times_{F_M} V] \subset X = \widehat{Z}(\mathbb{R}),$$

the latter one being given by

$$(u, mM_H, a \underline{A}_I(\mathbb{R}), v_I) \rightarrow (u, [m, a \cdot e_I + v_I]).$$

Set $F_I := F \cap \underline{A}_I(\mathbb{R})$ and note that F_I identifies with a subset of $\{-1, 1\}^k$ upon identification of $\underline{A}_I(\mathbb{R}) \simeq (\mathbb{R}^\times)^k$. From the definition of Ψ we then get

$$(4.2) \quad \Psi^{-1}(Z) = \mathcal{V} \times F_I \cdot V_I^0.$$

It is worth to note that

$$(4.3) \quad \Psi(y, e_I) = z_0.$$

With Ψ being diffeomorphic we record the following property of transversality

$$(4.4) \quad d\Psi(x, 0)(0 \times V_I) \oplus T_x Y = T_x X \quad (x \in \mathcal{V}).$$

In the sequel we use (4.4) to identify the spaces $V_I \simeq (N_Y)_y$ for $y = \widehat{z}_{0,I}$. On $V_I = (N_Y)_y$ there is a natural linear action of $G_y = \widehat{H}_I$, the isotropy representation, which we call

$$\rho : \widehat{H}_I \rightarrow \text{GL}(V_I).$$

The representation ρ is algebraic, i.e. it originates from the complex isotropy representation

$$\underline{\rho} : \widehat{\underline{H}}_I \rightarrow \text{GL}(V_{I,\mathbb{C}}).$$

We write $\underline{H}_I = \ker \underline{\rho}$ and note that $H_I = \underline{H}_I(\mathbb{R})$ is given by $H_I = \ker \rho$. Observe that $\underline{H}_I \triangleleft \widehat{\underline{H}}_I$ and $H_I \triangleleft \widehat{H}_I$ are closed normal subgroups.

Theorem 4.1. *The following assertions hold:*

- (1) *The Lie algebra of H_I is given by \mathfrak{h}_I , as defined in (2.16).*
- (2) *$\widehat{\underline{H}}_I/\underline{H}_I \simeq \underline{A}_I$.*

The proof of Theorem 4.1 will be prepared by several intermediate steps. The key is Lemma 4.2 and the techniques contained in its proof.

Lemma 4.2. *The Lie algebra of H_I contains \mathfrak{h}_I , as defined by (2.16).*

Proof. Let $Y \in \mathfrak{h}_I$, then $h_I := \exp(Y) \in \widehat{H}_I$ as explained in Proposition 3.3. We claim that $\rho(h_I) = \mathbf{1}$.

For all $X \in \mathfrak{c}_I^- \cap \Xi_Z^\vee$ we consider the curve

$$\gamma_X : [0, 1] \rightarrow \mathbf{X} = \widehat{Z}(\mathbb{R}), \quad s \mapsto \exp(-(\log s)X) \cdot z_0,$$

which connects $\widehat{z}_{0,I}$ to z_0 . Note, that in coordinates of (3.5) we have $\underline{A}_Z(\mathcal{F}_{j_I})(\mathbb{R}) \simeq \mathbb{R}^k$ (with $j_I \in J_I$ the selected element for \mathfrak{c}_I^-), and

$$\gamma_X(s) = (s^{m_1}, \dots, s^{m_k}) \in V_I$$

for some $m_i \in \mathbb{N}$. Notice that all tuples of $m_i \in \mathbb{N}$ occur for some X . Hence γ_X is differentiable with $\gamma_X(0) = y = \widehat{z}_{0,I}$ and $\gamma'_X(0) = (\delta_1, \dots, \delta_k)$ with $\delta_i = 1$ if $m_i = 1$ and $\delta_i = 0$ otherwise.

Since $\rho(h_I)(\gamma'_X(0)) = \frac{d}{ds} \Big|_{s=0} h_I \gamma_X(s)$, the lemma will follow provided we can show that $\frac{d}{ds} \Big|_{s=0} h_I \gamma_X(s) = \gamma'_X(0)$ for all X as above. Now for $h_I \in L \cap H$ this is clear and thus we may assume that Y is of the form (see (2.17))

$$Y = \sum_{\alpha \in \Sigma(\mathfrak{a}, \mathfrak{u})} (X_{-\alpha} + \sum_{\alpha + \beta \in \mathbb{N}_0[I]} X_{\alpha, \beta}).$$

Set now for $s > 0$

$$Y_s := \sum_{\alpha \in \Sigma(\mathfrak{a}, \mathfrak{u})} (X_{-\alpha} + \sum_{\beta} e^{-(\log s)(\alpha + \beta)(X)} X_{\alpha, \beta}) \in \text{Ad}(\gamma_X(s))\mathfrak{h}.$$

Note that $Y_s \rightarrow Y$ for $s \rightarrow 0$. Likewise we set $h_{I,s} := \exp(Y_s)$ and note $h_{I,s} \rightarrow h_I$. Now we use that $\mathcal{M} \subset \Xi_Z$ in order to conclude that $h_{I,s}$ is right differentiable at $s = 0$. The Leibniz-rule yields

$$\frac{d}{ds} \Big|_{s=0} h_{I,s} \gamma_X(s) = \frac{d}{ds} \Big|_{s=0} h_I \gamma_X(s) + \underbrace{\frac{d}{ds} \Big|_{s=0} h_{I,s} y}_{\in T_y \mathbf{Y}}$$

and thus we get

$$\frac{d}{ds} \Big|_{s=0} h_I \gamma_X(s) = \text{P} \left(\frac{d}{ds} \Big|_{s=0} h_{I,s} \gamma_X(s) \right)$$

with P the projection $T_y X \rightarrow V_I$ along $T_y Y$. Now observe that

$$\begin{aligned} h_{I,s} \gamma_X(s) &= h_{I,s} \exp(-(\log s)X) \cdot z_0 \\ &= \exp(-(\log s)X) \underbrace{\exp((\log s)X) h_{I,s} \exp(-(\log s)X)}_{\in H} \cdot z_0 \\ &= \gamma_X(s) \end{aligned}$$

and the lemma follows. □

4.1.1. *Normal curves and the proof of Theorem 4.1.* Recall the base points $\widehat{z}_{0,I} \in \widehat{Z}_I \subset \widehat{Z}$. Now $\widehat{z}_{0,I} \in \partial Z$ is a boundary point of Z provided that $I \subsetneq S$ or $\mathfrak{a}_{Z,E} \neq 0$ – in case $\mathfrak{a}_{Z,E} = \{0\}$ we have $\widehat{Z}_S = Z$ and $\widehat{z}_S = z_0$ is not a boundary point.

The proof of Lemma 4.2 contains an important concept, namely smooth curves in Z which approach the point $\widehat{z}_{0,I} \in \widehat{Z}_I$ in normal direction. Let $X_I \in \mathfrak{a}_I^-$ correspond to $-\mathbf{e}_I \in V_I$ after the natural identification of \mathfrak{a}_I with V_I . Then we saw that the curve

$$\gamma_I : [0, 1] \rightarrow \widehat{Z}(\mathbb{R}), \quad s \mapsto \exp(-(\log s)X_I) \cdot z_0$$

is smooth with the following properties

- $\gamma_I((0, 1]) \subset Z$,
- $\gamma_I(0) = \widehat{z}_{0,I}$,
- $\gamma'_I(0) = \mathbf{e}_I \in V_I \subset T_{\widehat{z}_{0,I}} \widehat{Z}(\mathbb{R})$.

More generally, let $v \in V_{I,\mathbb{C}}^\times$. Then there exists a unique $a(v) \in \underline{A}_I$ such that $v = a(v) \cdot \mathbf{e}_I$. If we consider now $\underline{A}_I \subset \underline{Z}$, then we obtain a smooth curve

$$\gamma_v : [0, 1] \rightarrow \widehat{Z}, \quad s \mapsto \exp(-(\log s)X_I) \cdot a(v)$$

such that

- $\gamma_v((0, 1]) \subset \underline{Z}$,
- $\gamma_v(0) = \widehat{z}_{0,I}$,
- $\gamma'_v(0) = v \in V_{I,\mathbb{C}} \subset T_{\widehat{z}_{0,I}} \widehat{Z}$.

For $g \in \underline{G}$ we now shift γ_v by g , i.e. we set

$$\gamma_{g,v}(s) := g \cdot \gamma_v(s) \in \widehat{Z} \quad s \in [0, \epsilon].$$

Notice that $\gamma_{g,v}(0) = g \cdot \widehat{z}_{0,I}$ and $\gamma'_{g,v}(0) = dL_g(\widehat{z}_{0,I})v \in T_{g \cdot \widehat{z}_{0,I}} \widehat{Z}$, where dL_g denotes the differential of the displacement $L_g(z) = g \cdot z$. If $v = \mathbf{e}_I$ we simply set $\gamma_{g,I} := \gamma_{g,v}$. Specifically we are interested when $g \in \widehat{H}_I$ so that $\gamma_{g,v}(0) = \widehat{z}_{0,I}$.

Next we recall the decomposition of the complex tangent spaces

$$T_{\widehat{z}_{0,I}} \widehat{Z} = T_{\widehat{z}_{0,I}} \widehat{Z}_I \oplus V_{I,\mathbb{C}}$$

which identifies $V_{I,\mathbb{C}}$ with the complex normal space to the boundary orbit $\widehat{Z}_I = \underline{G} \cdot \widehat{z}_{0,I}$ at the point $\widehat{z}_{0,I}$. We denote by

$$P : T_{\widehat{z}_{0,I}} \widehat{Z} \rightarrow V_{I,\mathbb{C}}, \quad u \mapsto u_n$$

the projection of a tangent vector $u \in T_{\widehat{z}_{0,I}} \widehat{Z}$ to its normal part u_n . With this notation we then obtain from the definition of $\underline{H}_I = \ker \rho$ that

$$(4.5) \quad \underline{H}_I = \{g \in \widehat{H}_I \mid (\forall v \in V_I^\times) [\gamma'_{g,v}(0)]_n = v\}.$$

Proof of Theorem 4.1. First recall that $\widehat{\mathfrak{h}}_I = \mathfrak{h}_I + \mathfrak{a}_I$ from (3.4). As further $\underline{A}_H \subset \ker \underline{\rho}$ we see that $\underline{\rho}$ induces a representation of \underline{A}_I on $V_{I,\mathbb{C}}$ which is given by the faithful standard representation $\underline{\rho}(a)(v) = a \cdot v$. In fact, if we denote by $\tilde{a} \in \underline{A}$ any lift of $a \in \underline{A}_Z$ for the projection $\pi : \underline{A} \rightarrow \underline{A}_Z$, then for $a \in \underline{A}_I$ we have $\underline{\rho}(a)(v) = \tilde{a} \cdot \gamma'_v(0) = a \cdot v$. Notice that $\underline{\rho}(\underline{A}_I) \simeq \text{diag}(k, \mathbb{C}^\times)$ within our identification $V_{I,\mathbb{C}} \simeq \mathbb{C}^k$. It follows in particular that $\mathfrak{a}_I \cap \text{Lie}(H_I) = \{0\}$ and thus $\mathfrak{h}_I = \text{Lie}(H_I)$ by Lemma 4.2. This shows (1).

Moving on to (2) we first observe that $\underline{P}\widehat{H} = \underline{P}H$ for any spherical subgroup \underline{H} . In fact, since \widehat{H} normalizes \underline{H} it follows that $\underline{P}\widehat{H}$ is a union of open right \underline{H} -orbits. Since \underline{G} is connected the identity $\underline{P}\widehat{H} = \underline{P}H$ follows. Equivalently,

$$(4.6) \quad \widehat{H} = (\underline{P} \cap \widehat{H})\underline{H}.$$

We apply this to the spherical subgroup \underline{H}_I . Now if $p \in \underline{P} \cap \widehat{H}_I$ then

$$(4.7) \quad p \cdot \widehat{z}_{0,I} = \widehat{z}_{0,I}.$$

Let $\widetilde{\underline{A}}_I := \pi^{-1}(\underline{A}_I)$. Then (4.7) and the local structure theorem in the form of (3.8) implies $p \in \underline{M}_H \widetilde{\underline{A}}_I \subset \underline{H}_I \widetilde{\underline{A}}_I$, and hence $\widehat{H}_I = \underline{H}_I \widetilde{\underline{A}}_I$ by (4.6). \square

In particular it follows from Theorem 4.1 that $\underline{\rho}(\widehat{H}_I) \simeq \text{diag}(k, \mathbb{C}^\times)$ and thus for $g \in \widehat{H}_I$ that $\underline{\rho}(g) = \mathbf{1}$ if and only if $\underline{\rho}(g)(v) = v$ for some $v \in V_I^\times$. Thus we obtain the following strengthening of (4.5) to

$$(4.8) \quad \begin{aligned} \underline{H}_I &= \{g \in \widehat{H}_I \mid [\gamma'_{g,I}(0)]_n = \mathbf{e}_I\} \\ &= \{g \in \widehat{H}_I \mid [\gamma'_{g,v}(0)]_n = v\} \quad (v \in V_I^\times). \end{aligned}$$

4.2. The part of the normal bundle which points to Z . We denote by A_I the identity component of $\underline{A}_I(\mathbb{R})$.

According to Theorem 4.1 there is the exact sequence

$$(4.9) \quad \mathbf{1} \rightarrow \underline{H}_I \rightarrow \widehat{H}_I \rightarrow \underline{A}_I \rightarrow \mathbf{1}.$$

In (4.9) we take real points, which is only left exact, and obtain

$$(4.10) \quad \mathbf{1} \rightarrow H_I \rightarrow \widehat{H}_I \rightarrow \underline{A}_I(\mathbb{R}).$$

The image of the last arrow in (4.10) is an open subgroup since taking real points is exact on the level of Lie algebras. We denote this open subgroup by $A(I)$ and record the exact sequence

$$(4.11) \quad \mathbf{1} \rightarrow H_I \rightarrow \widehat{H}_I \rightarrow A(I) \rightarrow \mathbf{1}.$$

In particular,

$$(4.12) \quad A(I) = A_I F(I),$$

where $F(I) < \{-1, 1\}^k \subset \underline{A}_I(\mathbb{R})$ is a subgroup of the 2-torsion group $\{-1, 1\}^k$ of $\underline{A}_I(\mathbb{R}) \simeq (\mathbb{R}^\times)^k$.

Remark 4.3.

(a) The non-compact torus $A(I) \simeq \widehat{H}_I/H_I$ acts naturally on $Z_I = G/H_I$ from the right and thus commutes with the left G -action on Z_I .

(b) Since $\underline{A}_I \subset \underline{A}_Z$ we obtain that $A(I)$ is naturally a subgroup of $\underline{A}_Z(\mathbb{R})$. In particular we stress that it is not possible in general to realize $A(I)$ as a subgroup of $A = \underline{A}(\mathbb{R}) \subset G$.

We return to the normal bundle of the boundary orbit $Y = \widehat{Z}_I$:

$$N_Y = G \times_{G_y} V_I = G \times_{\widehat{H}_I} V_I.$$

From (4.11) we obtain that

$$(4.13) \quad \rho(\widehat{H}_I)\mathbf{e}_I = A(I) \cdot \mathbf{e}_I = F(I) \cdot V_I^0.$$

Recall the set $F_I = F \cap \underline{A}_I(\mathbb{R}) \subset \{-1, 1\}^k$ with $F(I) \subset F_I$. We define an $A(I)$ -stable open cone in V_I by

$$(4.14) \quad V_{Z,I} = F_I \cdot V_I^0 = F_I \cdot (\mathbb{R}^+)^k,$$

and we define the cone-bundle

$$(4.15) \quad N_Y^Z := G \times_{G_y} V_{Z,I}$$

as part of the normal bundle N_Y which *points to* Z . To explain the term “points to Z ” we recall the curves γ_v and note that $\gamma_v((0, \epsilon)) \subset Z$ if and only if $a(v) \in A(I)$, that is $a(v) \cdot \mathbf{e}_I = v \in F_I V_I^0$.

Observe that the coset space $F_I := F(I) \backslash F_I$ identifies with the N_Y^Z/G . For every $\mathfrak{t} = F(I)t \in F_I$ with $t \in F_I$ we now denote by

$$N_Y^{Z,\mathfrak{t}} = G \times_{G_y} [F(I)t \cdot V_I^0]$$

and note that

$$N_Y^Z = \coprod_{\mathfrak{t} \in F_I} N_Y^{Z,\mathfrak{t}}$$

is the disjoint decomposition into G -orbits.

Presently we do not have a good understanding of $F(I)$ and the coset space $F_I = F(I) \backslash F_I$, except when G is complex, where $F_I = F(I)$ for all $I \subset S$. Here are two further instructive examples:

Example 4.4 (cf. [26, Ex. 14.6]).

(a) Let $G = \text{SL}(2, \mathbb{R})$ and $H = \text{SO}(1, 1)$. We identify $Z = G/H$ with the one sheeted hyperboloid

$$Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = -1\}.$$

We note that $Z = \underline{Z}(\mathbb{R})$ and we embed Z into the projective space $\mathbb{P}(\mathbb{R}^4)$. The closure of Z in projective space is given by

$$\widehat{Z} = \{(x_1, x_2, x_3, x_4) \in \mathbb{P}(\mathbb{R}^4) \mid x_1^2 + x_4^2 = x_2^2 + x_3^2\} \simeq \mathbb{S}^1 \times \mathbb{S}^1$$

and coincides with the wonderful compactification $\widehat{\underline{Z}}(\mathbb{R})$. In the identification $\widehat{Z} = \mathbb{S}^1 \times \mathbb{S}^1$ from above, the unique closed G -orbit is given by $Y = \{1\} \times \mathbb{S}^1$ and

$$\widehat{Z} = Z \cup Y.$$

In particular both directions of the normal bundle N_Y point to Z . In our notation above this means that $F_I = \{-1, 1\}$, $F(I) = \{1\}$ and

$$N_Y = N_Y^Z = N_Y^{Z,+1} \amalg N_Y^{Z,-1}.$$

(b) The situation becomes different when we consider $G = \text{SL}(2, \mathbb{R})$ with $H = \text{SO}(2)$. We identify $Z = G/H$ with the upper component of the two sheeted hyperboloid $\underline{Z}(\mathbb{R})$, in formulae:

$$Z = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 1, x_1 > 0\},$$

and

$$\mathcal{Z}(\mathbb{R}) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 1\}.$$

We emphasize that $\mathcal{Z}(\mathbb{R})$ has two connected components, one of them being Z . As before we view $\mathcal{Z}(\mathbb{R})$ in the projective space $\mathbb{P}(\mathbb{R}^4)$ and obtain the wonderful compactification $\widehat{\mathcal{Z}}(\mathbb{R})$ as the closure

$$\widehat{\mathcal{Z}}(\mathbb{R}) = \{[x_1, x_2, x_3, x_4] \in \mathbb{P}(\mathbb{R}^4) \mid x_1^2 = x_2^2 + x_3^2 + x_4^2\} \simeq \mathbb{S}^2.$$

The unique closed orbit $Y = \mathbb{S}^1$ is identified with the great circle $\mathbb{S}^1 \subset \mathbb{S}^2$ which divides $\widehat{\mathcal{Z}}(\mathbb{R})$ into the two open G -orbits. In particular, only one direction of the normal bundle N_Y points to Z . We obtain that $F_I = F(I) = \{1\}$ with

$$N_Y \supsetneq N_Y^Z.$$

By this we end Example 4.4.

Define

$$Z_I := G/H_I$$

and write $z_{0,I} = H_I$ for its standard base point.

Let $\mathfrak{t} = F_I$ and fix with $t \in F_I$ a representative so that $\mathfrak{t} = F(I)t$. We then claim that

$$G/H_I \rightarrow N_Y^{Z,\mathfrak{t}}, \quad gH_I \mapsto [g, t \cdot \mathbf{e}_I]$$

defines a G -equivariant diffeomorphism for each $\mathfrak{t} \in F_I$. In fact, with $A(I) \simeq F(I)V_I^0$ via $a \mapsto a \cdot \mathbf{e}_I$, this follows from:

$$(4.16) \quad N_Y^{Z,\mathfrak{t}} \simeq G/H_I \times_{A(I)} (A(I) \cdot t \cdot \mathbf{e}_I) \simeq G/H_I \times_{A_I} V_I^0 \simeq G/H_I.$$

4.3. Speed of convergence. Next we wish to describe a quantitative version of the fact that \underline{H}_I asymptotically preserves normal limits, i.e. of (4.8). For that recall the curves $\gamma_{g,v}$.

Lemma 4.5. *Let $g \in \widehat{H}_I$ and $v \in V_I^\times$. Then there exists a smooth curve $[0, \epsilon) \rightarrow \underline{P}$, $s \mapsto p_s$ such that*

$$\gamma_{g,v}(s) = p_s \cdot \gamma_v(s) \quad (s \in [0, \epsilon))$$

and:

- (1) $p_0 \in \underline{A}_I$.
- (2) If $g \in \underline{H}_I$ then $p_0 = \mathbf{1}$.
- (3) If $g \in \widehat{H}_I$ we can assume that $p_s \in P$.

Proof. Note that $g \cdot \widehat{z}_{0,I} = \widehat{z}_{0,I}$ by assumption, and hence $\gamma_{g,v}(s) \rightarrow \widehat{z}_{0,I}$ for $s \rightarrow 0^+$ in a smooth fashion.

The local structure theorem gives us coordinates near $\widehat{z}_{0,I}$, see (3.8). In particular, it implies that we can find a smooth curve $s \mapsto \tilde{p}_s \in \underline{P}$ such that $\tilde{p}_s \cdot \gamma_v(s) = \gamma_{g,v}(s)$. Note that $\tilde{p}_0 \cdot \widehat{z}_{0,I} = \widehat{z}_{0,I}$ and hence $\tilde{p}_0 \in \underline{A}_I(\underline{P} \cap \underline{H})$ by (3.8). With that we obtain an element $p_H \in \underline{P} \cap \underline{H}$ such that $p_s := \tilde{p}_s p_H$ satisfies (1). Here we used the fact that $\gamma_v = \gamma_{p,v}$ for all $p \in \underline{P} \cap \underline{H}$.

We move on to (2). For that we recall the decomposition of the tangent space

$$T_{\widehat{z}_{0,I}} \widehat{\mathcal{Z}} = T_{\widehat{z}_{0,I}} \widehat{\mathcal{Z}}_I \oplus V_{I,\mathbb{C}}$$

and the normal part $u_n \in V_{I,\mathbb{C}}$ of a tangent vector $u \in T_{\widehat{z}_{0,I}} \widehat{\mathcal{Z}}$. Now if $g \in \underline{H}_I$, then by the definition of \underline{H}_I as the kernel of the isotropy representation, we obtain that

$[\gamma'_{g,v}(0)]_n = v$. On the other hand, using the identity $\gamma_{g,v}(s) = p_s \cdot \gamma_v(s)$ we obtain $[\gamma'_{g,v}(0)]_n = p_0 \cdot v$. As $p_0 \in \underline{A}_I$, this implies $p_0 = \mathbf{1}$.

The last assertion (3) is proved using the real version of the argument for (1). \square

Let $d_{\underline{G}}$ be a left invariant Riemannian metric on \underline{G} . Then the quantitative version of (4.8) reads as follows:

Corollary 4.6. *Let $X_I \in \mathfrak{c}_I^- \subset \mathfrak{a}_I^-$ correspond to $-\mathbf{e}_I \in V_I$ in the identification $V_I \simeq \mathfrak{a}_I$. Set $a_t := \exp(tX_I)$ for $t \geq 0$. Let $h_I \in \underline{H}_I$. Then there exist constants $C, \epsilon, t_0 > 0$ and for each $t \geq t_0$ an element $x_t \in \underline{P}$ such that $d_{\underline{G}}(x_t, \mathbf{1}) \leq Ce^{-\epsilon t}$ and*

$$(4.17) \quad h_I a_t \cdot z_0 = x_t a_t \cdot z_0.$$

If further, $h_I \in H_I$, then we can choose $x_t \in P$.

Proof. Apply Lemma 4.5 to $g = h_I$ and $v = e_I$. Set $x_t := p_{e^{-t}}$ and use that $p_0 = \mathbf{1}$ and $s \mapsto p_s$ is differentiable at $s = 0^+$. \square

4.4. The intersection of \underline{H}_I with \underline{L} . For later reference we record the following fact, which is more or less immediate from (3.8). Since it is crucial for the paper we include a detailed argument.

Lemma 4.7. *For all $I \subset S$ one has*

$$(4.18) \quad \underline{L} \cap \underline{H} = \underline{L} \cap \underline{H}_I.$$

Proof. First note that $\underline{L} = \underline{M}\underline{A}\underline{L}_n$ and from $\underline{L}_n \subset \underline{H} \cap \underline{H}_I$ we obtain that $\underline{H} \cap \underline{L} = \underline{L}_n[(\underline{M}\underline{A}) \cap \underline{H}]$ and likewise $\underline{H}_I \cap \underline{L} = \underline{L}_n[(\underline{M}\underline{A}) \cap \underline{H}_I]$. Hence it suffices to show that $\underline{H} \cap (\underline{M}\underline{A}) = \underline{H}_I \cap (\underline{M}\underline{A})$.

We first show that $\underline{H} \cap (\underline{M}\underline{A}) \subset \underline{H}_I \cap (\underline{M}\underline{A})$. For that we recall the isotropy representation ρ which we view here as a representation of $\widehat{\underline{H}}_I$ so that $\underline{H}_I = \ker \rho$. Recall the curves γ_X from the proof of Lemma 4.2. Now for $g \in (\underline{M}\underline{A}) \cap H$ we have $g\gamma_X(s) = \gamma_X(s)$ and thus $g\gamma'_X(0) = \gamma'_X(0)$. Hence $g \in \ker \rho = \underline{H}_I$ and “ \subset ” is established.

For the converse inclusion we first note that both $\underline{H} \cap (\underline{M}\underline{A})$ and $\underline{H}_I \cap (\underline{M}\underline{A})$ are elementary algebraic groups (see [26] or [10, Appendix B] for the notion “elementary”). Together with $\mathfrak{l} \cap \mathfrak{h} = \mathfrak{l} \cap \mathfrak{h}_I$ (which we obtain from (2.16)) we infer that $\underline{H} \cap (\underline{M}\underline{A})$ and $\underline{H}_I \cap (\underline{M}\underline{A})$ have the same Lie algebra, namely $[\mathfrak{m}_H + \mathfrak{a}_H]_{\mathbb{C}}$. Further as $\underline{M}\underline{A}$ is an elementary group we obtain $\underline{H}_I \cap (\underline{M}\underline{A}) = (\underline{M} \cap \underline{H}_I)(\underline{A}_{\underline{H}_I})_0$, see [10, Appendix B].

From $\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a} \cap \mathfrak{h}_I$ again obtained from (2.16) we derive $(\underline{A}_{\underline{H}_I})_0 = (\underline{A}_H)_0$. Hence we only need to show that $\underline{M} \cap \underline{H}_I \subset \underline{M} \cap \underline{H}$. Let now $m \in \underline{M} \cap \underline{H}_I$. In particular $m \in \widehat{\underline{H}}_I$ fixes $\widehat{z}_{0,I}$ and thus we obtain from (3.9) that $m \in \underline{M}_{\underline{H}} F_{M,I}$. Hence we may assume that $m \in F_{M,I}$. From $\rho(m) = \mathbf{1}$ we then obtain that $m \in F_{M,I} \subset \{-1, 1\}^k$ needs to have all coordinates to be 1, i.e. $m = \mathbf{1}$ and the proof is complete. \square

4.5. The structure of $\underline{Z}_I(\mathbb{R})$. From Theorem 4.1 and Proposition 3.3 we obtain:

Lemma 4.8. *For any $I \subset S$, the \underline{G} -isomorphism class of the variety \underline{Z}_I is canonically attached to \underline{Z} , i.e. independent of the particular smooth toroidal compactification $\widehat{\underline{Z}} = \underline{Z}(\mathcal{F})$ of \underline{Z} .*

In particular, it follows that up to G -isomorphism $\underline{Z}_I(\mathbb{R})$ is canonically attached to $\underline{Z}(\mathbb{R})$. However, for Z_I the situation is different. We recall the shifted base

points $z_w = w \cdot z_0$ and $\widehat{z}_{w,I}$ from Subsection 3.1. For $I \subset S$ we then define the set of G -orbits

$$C_I := \{G \cdot \widehat{z}_{w,I} \mid w \in \mathcal{W}\},$$

and note that different orbits in C_I may not be isomorphic, see Example 4.10. In particular, the isomorphism class of $Z_I = G/H_I$ is not canonically attached to Z . In this sense only the collection of G -spaces $\{G/(H_w)_I \mid w \in \mathcal{W}\}$ (where $(\widehat{H_w})_I$ is the stabilizer of $\widehat{z}_{w,I}$) is canonically attached to the G -space $Z = G/H$.

Remark 4.9. In case \widehat{Z} is wonderful the set C_I equals the set of G -orbits $\partial Z \cap \widehat{Z}_I(\mathbb{R})$. This follows from Lemma 3.5. The general case is a bit more complicated, see Remark 3.6(b). Recall the boundary points $\widehat{z}_{w,C}$ from (3.10). Then

$$\begin{aligned} D_I &:= \{G \cdot \widehat{z}_{w,C} \mid G \cdot \widehat{z}_{w,C} \subset \widehat{Z}_I(\mathbb{R}), w \in \mathcal{W}, C \in \mathcal{F}\} \\ &= \{G \cdot \widehat{z}_{w,C} \mid w \in \mathcal{W}, C \in \mathcal{F}_I\} \end{aligned}$$

yields all G -orbits in $\partial Z \cap \widehat{Z}_I(\mathbb{R})$.

For $c \in C_I$ we set

$$\mathcal{W}_c := \{w \in \mathcal{W} \mid G \cdot \widehat{z}_{w,I} = c\}$$

and obtain the partition

$$(4.19) \quad \mathcal{W} = \coprod_{c \in C_I} \mathcal{W}_c.$$

Given $c \in C_I$ we choose a representative $w(c) \in \mathcal{W}_c$. In case $c = G \cdot \widehat{z}_{0,I} = \widehat{Z}_I$ we make the request that $w(c) = 1$. We then define

$$H_{I,c} := (H_{w(c)})_I.$$

We will see in Lemma 5.16 that the G -conjugacy class of $H_{I,c}$ is independent of the representative $w(c)$ used for its definition.

Example 4.10. Consider $\underline{Z} = \text{SL}(3, \mathbb{C})/\text{SO}(3, \mathbb{C})$ which is defined over \mathbb{R} . We will use the identification

$$\underline{Z} = \text{Sym}(3 \times 3, \mathbb{C})_{\det=1}$$

with Sym denoting the symmetric matrices. Hence

$$\underline{Z}(\mathbb{R}) = G/K \amalg G/H$$

consists of two G -orbits with $K = \text{SO}(3, \mathbb{R})$ and $H = \text{SO}(1, 2)$, both real forms of $\underline{H} = \text{SO}(3, \mathbb{C})$. Our interest is here with $Z = G/H$. If we identify \underline{A}_Z with the diagonal matrices in \underline{Z} , then $F_{\mathbb{R}}$, the 2-torsion group of $\underline{A}_Z(\mathbb{R})$, is given by

$$F_{\mathbb{R}} = \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\} = \{t_0, t_1, t_2, t_3\}$$

which in this case parametrize the open P -orbits in $\underline{Z}(\mathbb{R})$ – we have $F_M = \{1\}$ in this example. Notice that $t_0 \in G/K$ whereas $t_1, t_2, t_3 \in Z = G/H$. In particular $F = \{t_1, t_2, t_3\}$ is not a group. Let us denote by $w_1, w_2, w_3 \in G$ lifts of t_i to G so that $\mathcal{W} = \{w_1, w_2, w_3\}$.

In this case the spherical roots comprise a system of type A_2 . With $I = \{\alpha_2\}$ we can take $a_t = \text{diag}(t^{-2}, t, t)$ for our ray.

Our example \underline{Z} has a wonderful compactification which is given by the closure of the image of its standard embedding into projective space

$$\underline{Z} = \text{Sym}(3 \times 3, \mathbb{C})_{\det=1} \rightarrow \mathbb{P}(\text{Sym}(3 \times 3, \mathbb{C}) \times \text{Sym}(3 \times 3, \mathbb{C}))$$

$$X \mapsto \mathbb{C} \cdot (X, X^{-1}).$$

Note $H_{w_1} = H$ and an elementary calculation in the above model for \widehat{Z} yields $H_I = (H_{w_1})_I = S(O(1)O(2))U_I$ and $(H_{w_2})_I = (H_{w_3})_I = S(O(1)O(1,1))U_I$ where

$$U_I = \begin{pmatrix} 1 & & \\ * & 1 & \\ * & & 1 \end{pmatrix} \subset G.$$

In particular, we see $H_{I,1} := H_I$ is not conjugate to $H_{I,2} := (H_{w_2})_I$.

We further note that $\widehat{Z}_I(\mathbb{R}) = \partial Z \cap \widehat{Z}_I(\mathbb{R})$ consists of two G -orbits $\widehat{Z}_{I,1} = G/\widehat{H}_{I,1}$ and $\widehat{Z}_{I,2} = G/\widehat{H}_{I,2}$, and accordingly $\mathbf{C}_I \simeq \{1, 2\}$ has two elements. Note that $\widehat{Z}_I(\mathbb{R})$ is G -isomorphic to the projective space of the rank two real symmetric matrices. Within this identification $\widehat{Z}_{I,1} \subset \widehat{Z}(\mathbb{R})$ consists of the rank two symmetric matrices (viewed projectively) with equal signature (i.e. $0++$ or $0--$), and $\widehat{Z}_{I,2} \subset \widehat{Z}(\mathbb{R})$ of the rank two symmetric matrices with signature $0+-$. Finally note that $\mathcal{W}_1 = \{w_1\}$ and $\mathcal{W}_2 = \{w_2, w_3\}$.

5. OPEN P -ORBITS ON Z_I AND Z

Recall the set $\mathcal{W} \subset G$ of representatives for $W = (P \backslash Z)_{\text{open}}$. Let $W_I = (P \backslash Z_I)_{\text{open}}$, the set of open P -orbits in Z_I . The objective of this section is to obtain a good set \mathcal{W}_I of representatives for W_I which results in a natural injective map $\mathbf{m} : \mathcal{W}_I \rightarrow \mathcal{W}$ (or $W_I \rightarrow W$ if one wishes), and thus matches each open P -orbit in Z_I with a particular open P -orbit in Z . This map is important for various constructions of the paper.

In general the map \mathbf{m} is not surjective and this originates from the fact that Z_I is only one G -orbit in $\underline{Z}_I(\mathbb{R})$ which points to Z . We will show in this section that the G -orbits in $\underline{Z}_I(\mathbb{R})$ which point to Z are given by

$$(5.1) \quad \widetilde{Z}_I := \coprod_{\mathbf{c} \in \mathbf{C}_I} \coprod_{\mathbf{t} \in \mathbf{F}_{I,\mathbf{c}}} Z_{I,\mathbf{c},\mathbf{t}}.$$

Here $Z_{I,\mathbf{c},\mathbf{t}} \simeq Z_{I,\mathbf{c}} = G/H_{I,\mathbf{c}}$ for all $\mathbf{t} \in \mathbf{F}_{I,\mathbf{c}}$ with $\mathbf{F}_{I,\mathbf{c}}$ the set corresponding to $\mathbf{F}_I = F_I/F(I)$ when Z_I is replaced by $Z_{I,\mathbf{c}}$. For every pair \mathbf{c}, \mathbf{t} this then leads to an injective matching map $\mathbf{m}_{\mathbf{c},\mathbf{t}} : \mathcal{W}_{I,\mathbf{c}} \rightarrow \mathcal{W}$ with $\mathcal{W}_{I,\mathbf{c}}$ a parameter set for $W_{I,\mathbf{c}} = (P \backslash Z_{I,\mathbf{c}})_{\text{open}}$. The case of $\mathbf{c} = \mathbf{t} = \mathbf{1}$ corresponds to the original map $\mathbf{m} = \mathbf{m}_{\mathbf{1},\mathbf{1}}$. The decomposition (5.1) then leads to a partition

$$(5.2) \quad \mathcal{W} = \coprod_{\mathbf{c} \in \mathbf{C}_I} \coprod_{\mathbf{t} \in \mathbf{F}_{I,\mathbf{c}}} \mathbf{m}_{\mathbf{c},\mathbf{t}}(\mathcal{W}_{I,\mathbf{c}})$$

refining (4.19).

This section has several parts. It starts with the construction of the injective map $\mathbf{m} : \mathcal{W}_I \rightarrow \mathcal{W}$. For a better understanding of the matching map \mathbf{m} we then illustrate the case where Z is a symmetric space and relate \mathbf{m} to Matsuki's description [38] of the open $P \times H$ -double cosets in G in terms of Weyl groups. After that we derive the general partition of \mathcal{W} in terms of the $\mathbf{m}_{\mathbf{c},\mathbf{t}}$. This last part is a bit more technical and can be skipped at a first reading.

Throughout this section $I \subset S$ is fixed.

5.1. Relating W_I to W . We recall from Lemma 2.3 the natural bijection of $W_{\mathbb{R}} = (P \backslash \underline{Z}(\mathbb{R}))_{\text{open}}$ with $F_M \backslash F_{\mathbb{R}}$ where $F_{\mathbb{R}} = \underline{A}_{\underline{Z}}(\mathbb{R})_2$ denotes the 2-torsion subgroup of $\underline{A}_{\underline{Z}}(\mathbb{R})$. On the other hand we recall from Lemma 4.7 that $\underline{L} \cap \underline{H} = \underline{L} \cap \underline{H}_I$. Intersecting this identity with \underline{A} we obtain that $\underline{A} \cap \underline{H} = \underline{A} \cap \underline{H}_I$ and hence an identity of homogeneous spaces

$$\underline{A}_{\underline{Z}} = \underline{A} / \underline{A} \cap \underline{H} = \underline{A} / \underline{A} \cap \underline{H}_I = \underline{A}_{\underline{Z}_I}.$$

In particular, $\underline{A}_{\underline{Z}}(\mathbb{R})$ and $\underline{A}_{\underline{Z}_I}(\mathbb{R})$ have the same 2-torsion groups, namely $F_{\mathbb{R}}$. In addition $\underline{L} \cap \underline{H} = \underline{L} \cap \underline{H}_I$ implies that the two open P -orbits $P \cdot z_0 \subset \underline{Z}$ and $P \cdot z_{0,I} \subset \underline{Z}_I$ carry canonically isomorphic local structure theorems, see (2.7), (2.9) and (2.10). Hence the group F_M is identical in both cases and we obtain a natural bijection (the identity map)

$$\mathbf{m}_{\mathbb{R}} : W_{I,\mathbb{R}} = (P \backslash \underline{Z}_I(\mathbb{R}))_{\text{open}} \rightarrow (P \backslash \underline{Z}(\mathbb{R}))_{\text{open}} = W_{\mathbb{R}}.$$

Remark 5.1. On the one hand we have an identity of homogeneous spaces $\underline{A}_{\underline{Z}} = \underline{A}_{\underline{Z}_I}$, but on the other hand we also view $\underline{A}_{\underline{Z}}$ as a subvariety of \underline{Z} and $\underline{A}_{\underline{Z}_I}$ as a subvariety of \underline{Z}_I . In the latter picture the identity of homogeneous spaces yields a natural identification of subvarieties of \underline{Z} and \underline{Z}_I .

Proposition 5.2. *One has $\mathbf{m}_{\mathbb{R}}(W_I) \subset W$.*

In order to prove this proposition we first recall another natural map $\mathbf{m} : W_I \rightarrow W$ which first arose in [29, Sect. 3]. We fix with $\mathcal{W}_I \subset G$ a set of representatives of W_I with elements $w_I \in \mathcal{W}_I$ of the form $w_I = \tilde{t}_I h_I$ where $h_I \in \underline{H}_I$ and $\tilde{t}_I \in T_Z$. Upon the identification of varieties $\underline{A}_{\underline{Z}}(\mathbb{R}) \simeq \underline{A}_{\underline{Z}_I}(\mathbb{R})$ we view $t_I := \tilde{t}_I \cdot z_{0,I} = w_I \cdot z_{0,I}$ as an element of $F_{\mathbb{R}}$.

Let now $Pw_I \cdot z_{0,I} \in W_I$ be an open P -orbit in Z_I with $w_I \in \mathcal{W}_I$. Next let $X \in \mathfrak{a}_I^{-}$ and set

$$a_s := \exp(sX) \in A_I^{-} \subset A \quad (s > 0).$$

It follows from [29, Lemma 3.9] that there exist $s_0 = s_0(X) > 0$ and a unique $w = \tilde{t}h \in \mathcal{W}$ such that

$$(5.3) \quad Pw_I a_s \cdot z_0 = Pw \cdot z_0 \quad (s \geq s_0).$$

Lemma 5.3. *Given $w_I = \tilde{t}_I h_I \in \mathcal{W}_I$ as above, the element $w \in \mathcal{W}$ such that (5.3) holds does not depend on the choice of $X \in \mathfrak{a}_I^{-}$.*

Proof. In order to record the possible dependence on X we write $a_s(X) = \exp(sX)$ and $w(X)$ for the corresponding w . Now we recall the argument of [29, Lemma 3.9]: For fixed X we have $\lim_{s \rightarrow \infty} e^{s \text{ad } X} \mathfrak{h} = \mathfrak{h}_I$ by (2.21). Thus there exists an $s_0(X)$ such that $\mathfrak{p} + \text{Ad}(w_I) e^{s \text{ad } X} \mathfrak{h} = \mathfrak{g}$ for all $s \geq s_0(X)$. In particular, we obtain that $Pw_I a_s(X) \cdot z_0$ is open for all $s \geq s_0(X)$. Since the limit (2.21) is locally uniform in $X \in \mathfrak{a}_I^{-}$, it follows that $w(X)$ is locally constant. The lemma follows. \square

With this lemma we obtain in particular a natural map

$$(5.4) \quad \mathbf{m} : \mathcal{W}_I \rightarrow \mathcal{W}, \quad w_I \mapsto w = \mathbf{m}(w_I).$$

With the identifications $W_I \simeq \mathcal{W}_I$ and $W \simeq \mathcal{W}$ we view \mathbf{m} also as a map $W_I \rightarrow W$, which by slight abuse of notation is denoted as well by \mathbf{m} .

Since the choice of $X \in \mathfrak{a}_I^{-}$ was irrelevant for the definition of \mathbf{m} we may henceforth assume that $X = X_I \in \mathfrak{c}_I^{-} \subset \mathfrak{a}_I^{-}$ was such that it corresponds to $-\mathbf{e}_I \in V_I$ under the identification $V_I \simeq \mathfrak{a}_I$.

Proposition 5.2 will now follow from:

Lemma 5.4. $\mathfrak{m}_{\mathbb{R}}|_{\mathcal{W}_I} = \mathfrak{m}$.

Proof. Let $w_I = \tilde{t}_I h_I \in \mathcal{W}_I$ and $w = \mathfrak{m}(w_I) = \tilde{t}h$. From Lemma 4.5 for $g = h_I \in \underline{H}_I$ we obtain a C^1 -curve $[0, 1] \rightarrow \underline{P}$, $u \mapsto p_u$ with $p_u \rightarrow \mathbf{1}$ for $u \rightarrow 0^+$ and

$$h_I a_s \cdot z_0 = p_{e^{-s}} a_s \cdot z_0 \quad (s \geq s_0).$$

Hence with $p'_s := \tilde{t}_I p_{e^{-s}} \tilde{t}_I^{-1}$ we obtain that

$$w_I a_s \cdot z_0 = p'_s \cdot t_I \in Z.$$

Since $t_I \in Z$ and $p'_s \rightarrow \mathbf{1}$ for $s \rightarrow \infty$ we may assume that $p'_s \in P$ is real as well (use the local structure theorem of the form (2.11)). On the other hand the matching property (5.3) yields

$$w_I a_s \cdot z_0 = p''_s \cdot t$$

for some $p''_s \in P$. Thus we get

$$P \cdot t_I = P \cdot t.$$

This implies the lemma, and with that Proposition 5.2. □

In the sequel we adjust $\mathcal{W} \subset G$ (by possibly multiplying the previous $w = \tilde{t}h$ by an element of F_M) in such a way that for each $w_I = \tilde{t}_I h_I \in \mathcal{W}_I$ one has

$$(5.5) \quad t = t_I \quad \text{when } w = \tilde{t}h = \mathfrak{m}(w_I).$$

We note that this adjustment of \mathcal{W} depends on our fixed choice of \mathcal{W}_I and hence on I .

Remark 5.5. Notice that we typically have $\mathfrak{m}(\mathcal{W}_I) \subsetneq \mathcal{W}$ as the example of $Z = \text{SL}(2, \mathbb{R})/\text{SO}(1, 1)$ with $I = \emptyset$ already shows (cf. Example 4.4(a)). Here we have $H_\emptyset = M\overline{N}$ and $\widehat{H}_\emptyset = MAN\overline{N}$ and thus $\mathcal{W}_\emptyset = \{\mathbf{1}\}$ while $\mathcal{W} = \{1, w\}$ has two elements.

Proposition 5.6 (Consistency relations for stabilizers). *Let $w_I \in \mathcal{W}_I$ and $w = \mathfrak{m}(w_I) \in \mathcal{W}$. Then*

$$(5.6) \quad (H_w)_I = (H_I)_{w_I}.$$

Proof. Recall from (4.8) that H_I is the subgroup of G which asymptotically preserves the curves γ_v in normal direction, i.e. is the group of elements $g \in G$ with $g \cdot [\gamma'_v(0)]_n = [\gamma'_v(0)]_n = v$. Hence $(H_I)_{w_I} \subset G$ is the group of elements $g \in G$ with $g \cdot [\gamma'_{w_I, v}(0)]_n = [\gamma'_{w_I, v}(0)]_n = t_I \cdot v$. On the other hand we can characterize $(H_w)_I$ as follows: define the curve

$$\sigma_{w, v}(s) := \tilde{a}(v) \exp(-(\log s)X_I) \cdot z_w = \tilde{a}(v) \exp(-(\log s)X_I) \cdot t$$

where $\tilde{a}(v) \in \underline{A}$ is any lift of $a(v) \in \underline{A}_I$ with respect to the projection $\pi : \underline{A} \rightarrow \underline{A}_Z$. Then $(H_w)_I$ is the group of elements $g \in G$ with $g \cdot [\sigma'_{w, v}(0)]_n = [\sigma'_{w, v}(0)]_n = t \cdot v$. As $t = t_I$, the desired equality of groups follows. □

5.2. Symmetric spaces. The nature of the map \mathbf{m} becomes quite clear in the special case where Z is a symmetric space. In this special situation we can make the matching map explicit in terms of certain Weyl groups.

For this subsection $Z = G/H$ is symmetric, that is, there exists an involution $\tau : \underline{G} \rightarrow \underline{G}$, defined over \mathbb{R} , such that \underline{H} is an open subgroup of the τ -fixed point group \underline{G}^τ . We choose our maximal anisotropic group $\underline{K} \subset \underline{G}$ in such a way that the Cartan involution θ , which defines \underline{K} , commutes with τ . By slight abuse of notation we use τ and θ for the induced derived involutions on \mathfrak{g} as well.

5.2.1. The adapted parabolic. With $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}^\perp$, resp. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$, we obtain the decomposition of \mathfrak{g} in eigenspaces of τ , resp. θ , with eigenvalues $+1$ and -1 . We let $\mathfrak{a}_Z \subset \mathfrak{h}^\perp \cap \mathfrak{k}^\perp$ be a maximal abelian subspace and extend \mathfrak{a}_Z to a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{k}^\perp$. Now, according to Rossmann, the root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ restricts to a root system

$$\Sigma_Z = \Sigma|_{\mathfrak{a}_Z} \setminus \{0\}$$

on \mathfrak{a}_Z . The Weyl group of Σ_Z is denoted by $W = W_Z$.

Let $\Sigma_Z^+ \subset \Sigma_Z$ be a positive system, and let $\Sigma^+ \subset \Sigma$ be a positive system such that $\Sigma^+|_{\mathfrak{a}_Z} \setminus \{0\} = \Sigma_Z^+$. Then $PH \subset G$ is open for the minimal parabolic subgroup $P = MAN$, for which \mathfrak{n} is the sum of the positive root spaces. The adapted parabolic $Q = LU \supset P$ is then characterized by $L = Z_G(\mathfrak{a}_Z)$. It is the unique minimal $\theta\tau$ -stable parabolic subgroup of G containing P .

5.2.2. The deformations H_I . The spherical roots $S \subset \mathfrak{a}_Z^*$ are given by the simple roots in Σ_Z with respect to Σ_Z^+ . Hence for any $I \subset S$ we obtain parabolic subgroups $P_I \supset Q$ with $L_I = Z_G(\mathfrak{a}_I)$. As before we realize $\mathfrak{a}_I \subset \mathfrak{a}$ so that $A_I = \exp(\mathfrak{a}_I)$ becomes a subgroup of A . Then $L_I = M_I A_I \simeq M_I \times A_I$ for a unique τ -stable subgroup $M_I \subset L_I$. Now the deformations H_I are given by

$$H_I = (M_I \cap H) \overline{U}_I$$

with $M_I \cap H \subset M_I$ a symmetric subgroup, i.e. $M_I / M_I \cap H \subset G/H$ is a symmetric subspace. Note that the H_w , $w \in \mathcal{W}$, can be treated on the same footing, i.e. $(H_w)_I = (M_I \cap H_w) \overline{U}_I$ and $M_I \cap H_w \subset M_I$ a symmetric subgroup. As seen in Example 4.10 the subgroups $M_I \cap H$ and $M_I \cap H_w$ are not necessarily conjugate in M_I .

5.2.3. Open double cosets. For later reference in Section 15 (where we derive the Plancherel formula for symmetric spaces) we consider here both $(P \setminus Z)_{\text{open}}$ and $(P_I \setminus Z)_{\text{open}}$ together.

Recall that for symmetric spaces the set $W = (P \setminus Z)_{\text{open}}$ allows a description in terms of Weyl groups. For that we identify $W = W_Z \simeq [N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_Z)]/M$ and define a subgroup of W by $W_H = [N_{K \cap H}(\mathfrak{a}) \cap N_{K \cap H}(\mathfrak{a}_Z)]/M$. Then Matsuki [38] has shown that

$$(5.7) \quad W/W_H \rightarrow (P \setminus Z)_{\text{open}}, \quad wW_H \mapsto PwH$$

is a bijection. In particular, $W \simeq W/W_H$.

When applied to the symmetric space $M_I/(M_I \cap H)$, Matsuki's result becomes

$$(5.8) \quad W(I)/(W(I) \cap W_H) \simeq ((P \cap M_I) \setminus M_I / (M_I \cap H))_{\text{open}}.$$

Now

$$(5.9) \quad (P \setminus Z)_{\text{open}} \rightarrow (P_I \setminus Z)_{\text{open}}, \quad Pz \mapsto P_I z$$

is surjective. It follows from (5.8) that the composition of (5.7) with (5.9) factorizes to a bijection (see also [39])

$$W(I)\backslash W/W_H \rightarrow (P_I\backslash Z)_{\text{open}}$$

where $W(I) < W = W_Z$ is the subgroup generated by the reflections s_α for $\alpha \in I$.

In particular we obtain an action of $W(I)$ on $\mathcal{W} \simeq W/W_H$ and record:

Lemma 5.7. *For $I \subset S$ the following assertions hold:*

- (1) $(P_I\backslash Z)_{\text{open}} \simeq W(I)\backslash \mathcal{W}$.
- (2) $(P\backslash Z_I)_{\text{open}} \simeq \mathcal{W}_I \simeq W(I)/(W(I) \cap W_H)$.

Proof. We have just shown the first assertion. For the second, recall first that $H_I = (M_I \cap H)\overline{U}_I$. Hence the Bruhat decomposition yields that

$$(P\backslash Z_I)_{\text{open}} \simeq ((P \cap M_I)\backslash M_I/(M_I \cap H))_{\text{open}},$$

so that (2) follows from (5.8). □

Lemma 5.8. *Upon identifying $W(I)/W(I) \cap W_H$ with \mathcal{W}_I and W/W_H with \mathcal{W} , the map $\mathbf{m} : \mathcal{W}_I \rightarrow \mathcal{W}$ corresponds to the natural inclusion map $W(I)/(W(I) \cap W_H) \hookrightarrow W/W_H$.*

Proof. We recall the construction of the map \mathbf{m} via considering the limits of the double cosets $Pw_I a_s H$. So let $w_I \in W(I)$ and observe that $W(I)$ keeps \mathfrak{a}_I pointwise fixed. Thus we have $Pw_I a_s H = Pw_I H$ and the lemma follows. □

Also of later relevance are the open $H \times \overline{P}_I$ -double cosets in G which we treat here as well. Since the anti-involution

$$G \rightarrow G, \quad g \mapsto g^{-\theta} := \theta(g^{-1})$$

leaves H invariant and maps P_I to its opposite \overline{P}_I , we obtain a bijection of double cosets

$$P_I\backslash G/H \rightarrow H\backslash G/\overline{P}_I, \quad P_I g H \mapsto H g^{-\theta} \overline{P}_I.$$

With Lemma 5.7 we thus obtain a bijection

$$(5.10) \quad W(I)\backslash \mathcal{W} \rightarrow (H\backslash G/\overline{P}_I)_{\text{open}}, \quad W(I)w \mapsto Hw^{-\theta}\overline{P}_I.$$

5.3. Relating W_I to \widehat{W}_I . We now return to the setup of a general real spherical space. In this subsection we provide some complementary material on the relation of W_I to $\widehat{W}_I := (P\backslash \widehat{Z}_I)_{\text{open}}$. This will lead to a better geometric understanding of what to come next.

Recall that A_I is the connected component of $A(I) = A_I F(I) \simeq A_I \times F(I)$. Notice that $A(I)$ acts naturally on the right of $Z_I = G/H_I$ and thus induces an action of $A(I)$ on $W_I = (P\backslash Z_I)_{\text{open}}$. Lemma 5.9 is then a consequence of the fact that the connected group A_I acts trivially on the finite set W_I .

Lemma 5.9. *The natural map*

$$W_I = (P\backslash Z_I)_{\text{open}} \rightarrow \widehat{W}_I = (P\backslash \widehat{Z}_I)_{\text{open}}, \quad PwH_I \mapsto Pw\widehat{H}_I$$

is surjective and induces an isomorphism $W_I/F(I) \simeq \widehat{W}_I$.

Proof. Let $Pw\widehat{H}_I \subset G$ be open for some $w \in G$. We first show that $PwH_I A_I = PwH_I$. According to (2.15) applied to the real spherical space \widehat{Z}_I we may write $w = \tilde{t}\tilde{h}$ with $\tilde{t} \in T_Z$ and $\tilde{h} \in \widehat{H}_I$. Since $\widehat{H}_I = \exp_{\underline{G}}(\mathfrak{a}_{I,\mathbb{C}})\underline{H}_I$ by (4.9), we have $\tilde{h} = \tilde{t}_I h$ with $h \in \underline{H}_I$ and $\tilde{t}_I \in \exp_{\underline{G}}(\mathfrak{a}_{I,\mathbb{C}})$. Let $a \in A_I \subset A$. Then

$$(aw)^{-1}wa = (a\tilde{t}\tilde{t}_I h)^{-1}\tilde{t}\tilde{t}_I h a = h^{-1}a^{-1}h a.$$

Now we observe that $aw \in G$ and $a^{-1}h a \in \underline{H}_I$. Hence $(aw)^{-1}wa \in H_I$, and thus $PwH_I a = PwaH_I = PwH_I$ as claimed.

Since $\widehat{H}_I = H_I A_I F(I)$ we obtain that $Pw\widehat{H}_I = \bigcup_{t \in F(I)} PwH_I t$. In particular PwH_I is open. Hence the map $W_I \rightarrow \widehat{W}_I$ is onto. The last assertion also follows. \square

Similar to $W \simeq F_M \backslash F$ (see Lemma 2.3) we obtain with

$$F_I^\perp := F \underline{A}_I(\mathbb{R}) / \underline{A}_I(\mathbb{R}) \subset \underline{A}_Z(\mathbb{R}) / \underline{A}_I(\mathbb{R})$$

that

$$\widehat{W}_I \simeq F_M \backslash F_I^\perp$$

as a consequence of (3.9). We further recall that we view $F_I \subset \{-1, 1\}^r \cap V_I \subset V$ and accordingly the group $F_M \cap F_I = F_M \cap F(I)$ acts on F_I . Thus we obtain an exact sequence of pointed sets

$$(F_M \cap F(I)) \backslash F_I \hookrightarrow F_M \backslash F \twoheadrightarrow F_M \backslash F_I^\perp$$

or, equivalently,

$$(F_M \cap F(I)) \backslash F_I \hookrightarrow W \twoheadrightarrow \widehat{W}_I.$$

From the injectivity of \mathfrak{m} and Lemma 5.9 we thus obtain the commutative diagram:

$$(5.11) \quad \begin{array}{ccccc} (F_M \cap F(I)) \backslash F_I & \hookrightarrow & W & \twoheadrightarrow & \widehat{W}_I \\ \uparrow & & \uparrow \mathfrak{m} & & \parallel \\ (F_M \cap F(I)) \backslash F(I) & \hookrightarrow & \widehat{W}_I & \twoheadrightarrow & \widehat{W}_I \end{array}$$

Remark 5.10. Let us emphasize that the upper horizontal sequence in (5.11) is exact in the category of pointed sets, but not in the category of sets, i.e. we do not have $W \simeq \widehat{W}_I \times (F_M \cap F(I)) \backslash F_I$ as sets, see Example 5.18.

This phenomenon disappears if we consider $W_{\mathbb{R}}$ and $\widehat{W}_{I,\mathbb{R}} = (P \backslash \widehat{Z}_I(\mathbb{R}))_{\text{open}}$ instead of W and \widehat{W}_I . In more detail, recall the basis $(\psi_i^I)_{1 \leq i \leq r}$ by means of which we get a decomposition (see (3.5))

$$(5.12) \quad \underline{A}_Z(\mathbb{R}) = \underbrace{\underline{A}_I(\mathbb{R})}_{\simeq (\mathbb{R}^\times)^k} \times \underbrace{\underline{A}_I^\perp(\mathbb{R})}_{\simeq (\mathbb{R}^\times)^{r-k}} \simeq (\mathbb{R}^\times)^r$$

analogous to the decomposition $V = V_I \oplus V_I^\perp$. In particular, $F_{\mathbb{R}}$, the 2-torsion subgroup of $\underline{A}_Z(\mathbb{R})$, decomposes as $F_{\mathbb{R}} = F_{I,\mathbb{R}} \times F_{I,\mathbb{R}}^\perp$ in self explaining notation. Hence any $t \in F_{\mathbb{R}}$ decomposes as $t = t^\parallel t^\perp$ with $t^\parallel \in F_{I,\mathbb{R}}$ and $t^\perp \in F_{I,\mathbb{R}}^\perp$.

In this situation we obtain that the map

$$F_{\mathbb{R}} \rightarrow F_{I,\mathbb{R}}^\perp, \quad t \mapsto t^\perp$$

induces an epimorphism

$$W_{\mathbb{R}} \simeq F_M \backslash F_{\mathbb{R}} \rightarrow \widehat{W}_{I, \mathbb{R}} \simeq F_M \backslash F_{I, \mathbb{R}}^{\perp}, \quad F_M t \mapsto F_M \cdot t^{\perp},$$

leading to a decomposition

$$W_{\mathbb{R}} \simeq \widehat{W}_{I, \mathbb{R}} \times (F_{I, \mathbb{R}} \cap F_M) \backslash F_{I, \mathbb{R}}.$$

5.4. The fine partition of W with respect to I . Our next goal is to explore the issue of non-surjectivity of \mathbf{m} .

We recall from Subsection 4.5 the set C_I , the partition

$$\mathcal{W} = \coprod_{\mathbf{c} \in C_I} \mathcal{W}_{\mathbf{c}}$$

and the groups $H_{I, \mathbf{c}}$ for $\mathbf{c} \in C_I$. Thus the understanding of \mathcal{W} with respect to I comes down to understanding the various $\mathcal{W}_{\mathbf{c}}$. Once we have fixed \mathbf{c} we will see below that we obtain a natural geometric splitting of $\mathcal{W}_{\mathbf{c}} \simeq \widehat{W}_{\mathbf{c}} \times (F_M \cap F_I) \backslash F_I$ contrary to what happens for \mathcal{W} (see Remark 5.10 and (5.13)).

For expository reasons we start with $\mathbf{c} = \mathbf{1} \in C_I$, by which we mean $\mathbf{c} = \widehat{Z}_I = G \cdot \widehat{z}_{0, I}$. Thereupon we consider the other cases by replacing H_I with $H_{I, \mathbf{c}}$ and adding a further index \mathbf{c} to the notation.

Remark 5.11. Even in case $C_I = \{\mathbf{1}\}$ and $\mathcal{W} = \mathcal{W}_{\mathbf{1}}$ it can happen that $\mathbf{m}(\mathcal{W}_I) \subsetneq \mathcal{W}$. As we will see below this is related to the set $F_I = F(I) \backslash F_I$ originating from the normal bundle geometry in Subsection 4.2.

5.4.1. The case $\mathbf{c} = \mathbf{1}$. We assume that $w \in \mathcal{W}_{\mathbf{1}}$, i.e. $\widehat{z}_{w, I} \in \widehat{Z}_I$. Let $W_{\mathbf{1}} = \{P \cdot w \mid w \in \mathcal{W}_{\mathbf{1}}\}$. Let $F_{\mathbf{1}} := \{t \in F \mid Pt \in W_{\mathbf{1}}\} \subset F_{\mathbb{R}}$. Then we can describe $F_{\mathbf{1}}$ and thus $\mathcal{W}_{\mathbf{1}} \simeq F_M \backslash F_{\mathbf{1}}$ geometrically as follows.

Recall from (5.12) that any $t \in F_{\mathbb{R}}$ decomposes as $t = t^{\parallel} t^{\perp}$ with $t^{\parallel} \in F_{I, \mathbb{R}}$ and $t^{\perp} \in F_{I, \mathbb{R}}^{\perp}$. Let $w \in \mathcal{W}_{\mathbf{1}}$, write it as $w = \tilde{t}h$, and decompose $\tilde{t} = \tilde{t}^{\parallel} \tilde{t}^{\perp}$ such that $\tilde{t}^{\parallel} \cdot z_0 = t^{\parallel}$ and $\tilde{t}^{\perp} \cdot z_0 = t^{\perp}$. Consider the curve $s \mapsto a_s \cdot z_w = a_s \cdot t$ where $a_s = \exp(sX)$ with $X \in \mathfrak{c}_I^{-}$. Then, as $t^{\parallel} \in \underline{A}_I(\mathbb{R})$ fixes $\widehat{z}_{0, I}$, we obtain in the limit for $s \rightarrow \infty$ that $\tilde{t} \cdot \widehat{z}_{0, I} = t^{\perp} \cdot \widehat{z}_{0, I}$, and as $w \in \mathcal{W}_{\mathbf{1}}$ this limit belongs to an open P -orbit of \widehat{Z}_I .

Furthermore the coordinate $t^{\parallel} \in F_{I, \mathbb{R}}$ tells us in which direction we approach the limit $t^{\perp} \cdot \widehat{z}_{0, I}$, i.e. in which component of the cone $V_{Z, I} = F_I V_I^0$ we approach the limit. With $F_{I, \mathbf{1}}^{\perp} := F_{\mathbf{1}} \underline{A}_I(\mathbb{R}) / \underline{A}_I(\mathbb{R}) \simeq F_{\mathbf{1}} F_{I, \mathbb{R}} / F_{I, \mathbb{R}}$ we obtain the following.

Lemma 5.12. *By restriction the map $t \mapsto (t^{\parallel}, t^{\perp})$ yields a bijection $F_{\mathbf{1}} \simeq F_I \times F_{I, \mathbf{1}}^{\perp}$.*

Proof. First we claim that $F_I = F_{\mathbf{1}} \cap \underline{A}_I(\mathbb{R})$. The inclusion \supset is clear since by definition $F_I = F \cap \underline{A}_I(\mathbb{R})$. Conversely, each $t \in F_I$ corresponds to a $w = \tilde{t}h \in \mathcal{W}$ with $t \in \underline{A}_I(\mathbb{R})$. Then $\widehat{z}_{w, I} = \widehat{z}_{0, I}$, and hence $t \in F_{\mathbf{1}}$ as claimed.

In particular it follows that $(t^{\parallel}, t^{\perp}) \in F_I \times F_{I, \mathbf{1}}^{\perp}$ for all $t \in F_{\mathbf{1}}$. Since $t \mapsto (t^{\parallel}, t^{\perp})$ is injective by its definition in (5.12), it remains to see that $t = t^{\parallel} t^{\perp} \in F_{\mathbf{1}}$ for all pairs $(t^{\parallel}, t^{\perp}) \in F_I \times F_{I, \mathbf{1}}^{\perp}$. Since $t^{\parallel} \in \underline{A}_I(\mathbb{R})$ we know that $t \cdot \widehat{z}_{0, I} = t^{\perp} \cdot \widehat{z}_{0, I} \in \widehat{Z}_I$ which is the limit of the curve $\gamma(s) = a_s \cdot t$ for $s \rightarrow \infty$. The coordinate $t^{\parallel} \in F_I$ shows that γ approaches the limit $t \cdot \widehat{z}_{0, I}$ in a direction pointing to Z (see also the end of the proof of Lemma 3.5 for a more formal argument). Hence $t \in F$ and $Pt \in W_{\mathbf{1}}$. □

Lemma 5.12 implies the splitting

$$(5.13) \quad \mathcal{W}_1 \simeq \widehat{\mathcal{W}}_I \times (F_M \cap F(I)) \backslash F_I$$

and we can rephrase Lemma 5.9 as:

Lemma 5.13. *We have $\mathbf{m}(\mathcal{W}_I) \subset \mathcal{W}_1$ and under the identification (5.13) we have*

$$(5.14) \quad \mathbf{m}(\mathcal{W}_I) \simeq \widehat{\mathcal{W}}_I \times (F_M \cap F(I)) \backslash F(I)$$

From (5.13) and (5.14) we obtain that

$$(5.15) \quad \mathcal{W}_1 \simeq \mathbf{m}(\mathcal{W}_I) \times F_I$$

with $F_I = F(I) \backslash F_I$.

Remark 5.14. It is instructive for the following to recall from Subsection 4.2 the part $N_Y^Z = \coprod_{\mathfrak{t} \in F_I} N_Y^{Z, \mathfrak{t}}$ of the normal bundle N_Y which points to Z . Here

$$Z_{I, \mathfrak{t}} := N_Y^{Z, \mathfrak{t}} \simeq G/H_I$$

by the isomorphism (4.16), and $F_I = F(I) \backslash F_I$ parametrizes the components of N_Y^Z .

Note that $F_{I, \mathbb{R}}$ is a \mathbb{Z}_2 -vector space and thus we can find a splitting $F_{I, \mathbb{R}} = F(I) \oplus F_{I, \mathbb{R}}^0$ of vector spaces. In particular, we obtain $F_I = F(I) \oplus F_I^0$ for a subset $F_I^0 \subset F_{I, \mathbb{R}}^0$. In particular the map

$$F_I^0 \rightarrow F_I, \quad t \mapsto \mathfrak{t} := tF(I)$$

is a bijection.

Now, using the isomorphism (5.15) and the identification $F_I \simeq F_I^0$ we obtain injective maps

$$\mathbf{m}_t : \mathcal{W}_I \rightarrow \mathcal{W}_1 \simeq \mathbf{m}(\mathcal{W}_I) \times F_I^0, \quad w_I \mapsto (\mathbf{m}(w_I), t)$$

which yields the partition

$$(5.16) \quad \mathcal{W}_1 = \coprod_{\mathfrak{t} \in F_I} \mathbf{m}_t(\mathcal{W}_I).$$

Let us explain the map \mathbf{m}_t more geometrically using the normal bundle, see Remark 5.14. The subset $\mathbf{m}_t(\mathcal{W}_I) \subset \mathcal{W}_1$ corresponds to those $w = \tilde{t}_w h \in \mathcal{W}_1$ for which the curve $s \mapsto a_s \cdot z_w = a_s t_w \cdot z_0$ approaches the boundary point $\widehat{z}_{w, I} = t_w \cdot \widehat{z}_{0, I} = t_w^\perp \cdot \widehat{z}_{0, I}$ in direction of $tF(I)V_I^0 \subset V_{Z, I}$. Let us emphasize that our initial map \mathbf{m} corresponds then to the case where $\mathfrak{t} = F(I)$ is the identity coset.

Recall that $\mathfrak{t} \in F_I$ corresponds to a unique $t \in F_I^0$. Further we let $\tilde{t} \in T_Z$ be a lift of t , i.e. $\tilde{t} \cdot z_0 = t$. We assume that $t = \mathbf{1}$ in case $\mathfrak{t} = F(I)$.

Remark 5.15. Let $w_I = \tilde{t}_I h_I$ and $w_1 = \mathbf{m}(w_I) = \tilde{t}_I h \in \mathcal{W}$. Then note that

$$\mathbf{m}_t(w_I) = \tilde{t}_M \tilde{t} \tilde{t}_I \tilde{h}$$

for some $\tilde{t}_M \in F_M$, depending on the choice of representatives for $w := \mathbf{m}_t(w_I) \in \mathcal{W}$, and $\tilde{h} \in \underline{H}$. Thus by changing $w = \mathbf{m}_t(w_I) \in \mathcal{W}$ to $\tilde{t}_M w \tilde{h} \in G$ for some $h' \in \underline{H}$ we may assume that the compatibility conditions

$$\mathbf{m}_t(w_I) = \tilde{t}_I h''$$

hold for some $h'' \in \underline{H}$. In particular, we have

$$\mathbf{m}_t(w_I) \cdot z_0 = \tilde{t} \mathbf{m}(w_I) \cdot z_0 = t t_1$$

for all $\mathfrak{t} \in F_I, w_I \in \mathcal{W}_I$.

Notice that this correction of choice of \mathcal{W} (by harmless left displacements of elements of F_M) with respect to \mathcal{W}_I depends on I . In general it seems to be not possible to make a consistent choice of \mathcal{W} which would be valid for all I simultaneously.

Recall the notation $H_g = gHg^{-1}$ for a subgroup H in a group G and $g \in G$. Then note that $\underline{H}_{\tilde{t}}$ is defined over \mathbb{R} and $H_{\tilde{t}} := (\underline{H}_{\tilde{t}})(\mathbb{R})$ is conjugate to H as $t \in Z$. Likewise we define $z_{t,I} := \tilde{t} \cdot z_{0,I} \in \underline{Z}_I(\mathbb{R})$ and note that G -stabilizer of $z_{t,I}$ is H_I as we have $(H_{\tilde{t}})_I = H_I$ as a consequence of the fact that \tilde{t} fixes the vertex $\widehat{z}_{0,I}$.

We then obtain the following extension of the consistency relations from Proposition 5.6:

Lemma 5.16. *Let $w_I \in \mathcal{W}_I, \mathfrak{t} \in F_I$ and $w = \mathfrak{m}_{\mathfrak{t}}(w_I) \in \mathcal{W}_1$. Then*

$$(5.17) \quad (H_w)_I = (H_I)_{w_I}.$$

In particular, $(H_w)_I$ only depends on w_I and is independent of \mathfrak{t} .

Proof. For $w_I = \tilde{t}_I h_I$ we have $w_1 := \mathfrak{m}(w_I) = \tilde{t}_I h'$ for some $h' \in \underline{H}$. Hence $\mathfrak{m}_{\mathfrak{t}}(w_I) = \tilde{t} h$ for some $h \in \underline{H}$. We further have

$$(H_w)_I = ((H_{w_1})_{\tilde{t}})_I = (H_{w_1})_I$$

and now Proposition 5.6 applies. □

5.4.2. *The general decomposition of \mathcal{W} .* In general we obtain a partition

$$(5.18) \quad \mathcal{W} = \prod_{\mathfrak{c} \in \mathbf{C}_I} \prod_{\mathfrak{t} \in F_{I,\mathfrak{c}}} \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathcal{W}_{I,\mathfrak{c}})$$

where $\mathcal{W}_{I,\mathfrak{c}}$ are the open P -orbits for $Z_{I,\mathfrak{c}} := G/H_{I,\mathfrak{c}}$ parametrized as in the previous section with H_I replaced by $H_{I,\mathfrak{c}}$. The set $F_{I,\mathfrak{c}}$ is then F_I , but for H_I replaced by $H_{I,\mathfrak{c}}$. We define $\mathfrak{m}_{\mathfrak{c},\mathfrak{t}}$ similarly. Regarding our choices $w(\mathfrak{c}) \in \mathcal{W}$ which defined $H_{I,\mathfrak{c}}$ we normalize $\mathfrak{m}_{\mathfrak{c},\mathbf{1}}$ such that $\mathfrak{m}_{\mathfrak{c},\mathbf{1}}(\mathbf{1}) = w(\mathfrak{c})$.

Remark 5.17. If we let $F_{\mathfrak{c}} \subset F$ correspond to $\mathcal{W}_{\mathfrak{c}} \subset \mathcal{W}$ we define as before $F_{I,\mathfrak{c}} := F_{\mathfrak{c}} \cap \underline{A}_I(\mathbb{R})$ and $F_{I,\mathfrak{c}}^{\perp} := F_{\mathfrak{c}} F_{I,\mathbb{R}} / F_{I,\mathbb{R}}$. As in Lemma 5.12 we then obtain

- $F_{I,\mathfrak{c}} = F_I$.
- $F_{\mathfrak{c}} \simeq F_{I,\mathfrak{c}} \times F_{I,\mathfrak{c}}^{\perp}$ under $t \mapsto (t^{\parallel}, t^{\perp})$.

The first item tells us that $F_{I,\mathfrak{c}}$ is independent of \mathfrak{c} . However $F(I)_{\mathfrak{c}}$ does depend on \mathfrak{c} as Example 5.18 shows. In particular the dependence of $F_{I,\mathfrak{c}} = F(I)_{\mathfrak{c}} \setminus F_I$ on \mathfrak{c} is caused by the \mathfrak{c} -dependence of $F(I)_{\mathfrak{c}}$ only.

Further we denote by $z_{0,I,\mathfrak{c}} = H_{I,\mathfrak{c}}$ the standard base point of $Z_{I,\mathfrak{c}}$, and state the general version of (5.17): let $\mathfrak{c} \in \mathbf{C}_I$ and $\mathfrak{t} \in F_{I,\mathfrak{c}}$ such that $w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}}) \in \mathcal{W}_{\mathfrak{c}}$ for some $w_{I,\mathfrak{c}} \in \mathcal{W}_{I,\mathfrak{c}}$. Then $(H_w)_I$ does not depend on \mathfrak{t} and

$$(5.19) \quad (H_w)_I = (H_{I,\mathfrak{c}})_{w_{I,\mathfrak{c}}} \quad (w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}})).$$

If we define $w(\mathfrak{c}, \mathfrak{t}) := \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathbf{1}) \in \mathcal{W}$ and set $Z_{I,\mathfrak{c},\mathfrak{t}} = (H_{w(\mathfrak{c},\mathfrak{t})})_I$, then $Z_{I,\mathfrak{c},\mathfrak{t}} = Z_{I,\mathfrak{c}}$ and the decomposition (5.1) follows.

Example 5.18. We continue Example 4.10 of $Z = \text{SL}(3, \mathbb{R}) / \text{SO}(1, 2)$ with $\mathcal{W} = \{w_1, w_2, w_3\}$ and $H_{w_1} = H$. We chose $I = \{\alpha_2\}$ and obtained $\mathbf{C}_I = \{1, 2\}$ with $\mathcal{W}_1 = \{w_1\}$ and $\mathcal{W}_2 = \{w_2, w_3\}$. Further we had $H_{I,1} = H_I = S(\text{O}(1) \text{O}(2))U_I$ and $H_{I,2} = (H_{w_2})_I = (H_{w_3})_I = S(\text{O}(1) \text{O}(1, 1))U_I$.

Next we claim that both $\mathcal{W}_{I,1} = \{\mathbf{1}\}$ and $\mathcal{W}_{I,2} = \{\mathbf{1}\}$ are one-elemented. In fact this follows from the fact that the open P -orbits in $G/H_{I,j}$ are induced: if we denote by $G_I \simeq \mathrm{GL}(2, \mathbb{R})$ the Levi for the parabolic defined by I , then the open P -orbits on $G/H_{I,j}$ correspond to the open $P \cap G_I$ orbits in $\mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2)$ respectively $\mathrm{GL}(2, \mathbb{R})/\mathrm{O}(1, 1)$. Both cases feature only one open orbit for $P \cap G_I$ and establish our claim.

Finally we determine $F_{I,1}$ and $F_{I,2}$. Since $F = \{t_1, t_2, t_3\}$ with $t_i t_j = t_k$ for all i, j, k pairwise different, we readily deduce that $F_{\mathbb{R}, I} = F_{I,1} = F_{I,2} \simeq \mathbb{Z}_2$ is a group. Recall that we described $\widehat{H}_{I,1}$ and $\widehat{H}_{I,2}$ already in Example 4.10. From that we deduce that $\widehat{H}_{I,1}/H_{I,1} \simeq A_I$ is connected and thus $F(I)_1 = \{\mathbf{1}\}$. In particular, $F_{I,1} \simeq \mathbb{Z}_2$.

On the other hand we have

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \in \widehat{H}_{I,2}$$

as it preserves the diagonal quadratic form $(0, 1, -1)$ projectively (i.e. up to sign). Since $u \notin H_{I,2}$ and commutes with $A_I = \{\mathrm{diag}(t^{-2}, t, t) : t > 0\}$ we thus have $F(I)_2 \simeq \mathbb{Z}_2$. In particular, $F_{I,2} = \{\mathbf{1}\}$.

Remark 5.19. The above example shows that the group $A(I) = A_I \times F(I)$ is sensitive to the orbit type in C_I . More explicitly, we do not have $A(I) \simeq \widehat{H}_{I,c}/H_{I,c}$ for all $c \in C_I$.

6. ABSTRACT PLANCHEREL THEOREM AND TEMPERED REPRESENTATIONS

This section has several parts. We begin with a brief recall on Banach representations and their smooth vectors, followed by a recap of smooth completions of Harish-Chandra modules. Then we turn our attention to the abstract Plancherel theorem for real spherical spaces. In fact there is no much difference to the case of a general unimodular homogeneous space and “real spherical” only enters via finite multiplicities. Finally we recall the basic tempered theory for homogeneous spaces, initiated by Bernstein [4] in a general setup, and then made concrete for real spherical spaces in [27].

6.1. Generalities on Banach representations and their smooth vectors.

We begin with a few facts on Banach representations of a Lie group G . By a Banach (or a Fréchet) representation of a Lie group G we understand a continuous linear action

$$G \times E \rightarrow E, \quad (g, v) \mapsto \pi(g)v$$

on a Banach (or Fréchet) space E . As customary we use the symbolic pair (π, E) to denote the representation. Sometimes we abbreviate and use $g \cdot v$ instead of $\pi(g)v$.

Let now (π, E) be a Banach representation. Further we fix with p a norm which induces the topology on E . In case E is a Hilbert space and p originates from the defining scalar product, then we say p is the Hermitian norm on E . As the space E does not necessarily allow an action of the Lie algebra we pass to the subspace $E^\infty \subset E$ of smooth vectors. Here $v \in E$ is called smooth provided the E -valued orbit map $f_v : G \rightarrow E, \quad g \mapsto \pi(g)v$ is smooth. In this sense we obtain a G -invariant subspace $E^\infty \subset E$ which is dense in E . The space E^∞ carries a Fréchet topology for which the G -action is smooth. For further reference we briefly recall

a few standard possibilities on how to define the Fréchet topology. To begin with let $\mathcal{B} := \{X_1, \dots, X_n\}$ be an ordered basis of \mathfrak{g} . For a multi-index $\alpha \in \mathbb{N}_0^n$ we set $\mathbf{X}^\alpha := X_1^{\alpha_1} \cdot \dots \cdot X_n^{\alpha_n} \in \mathcal{U}(\mathfrak{g})$. For each $k \in \mathbb{N}_0$ we now define a norm on E^∞ by

$$p_{\mathcal{B},k}(v) := \left(\sum_{\substack{\alpha \in \mathbb{N}_0^n \\ |\alpha| \leq k}} p(\mathbf{X}^\alpha \cdot v)^2 \right)^{\frac{1}{2}} \quad (v \in E^\infty).$$

Notice that $p_{\mathcal{B},k}$ is Hermitian in case p is Hermitian. If \mathcal{C} is any other choice of ordered basis we note that there exist constants $C_k = C_k(\mathcal{B}, \mathcal{C}) > 0$, depending on \mathcal{B} and \mathcal{C} but not on the space E and its norm, such that $\frac{1}{C_k} p_{\mathcal{B},k} \leq p_{\mathcal{C},k} \leq C_k p_{\mathcal{B},k}$ for all $k \in \mathbb{N}_0$. In particular the locally convex topology on E^∞ induced from the family $(p_{\mathcal{B},k})_{k \in \mathbb{N}_0}$ does not depend on the particular choice of \mathcal{B} . In the sequel we fix a basis \mathcal{B} , set $p_k := p_{\mathcal{B},k}$, and refer to p_k as a k -th Sobolev norm of p . We denote by E_k the completion of E^∞ with respect to the norm p_k . Note that G leaves E_k invariant and defines a Banach representation (π_k, E_k) of G . It follows that the Fréchet representation (π^∞, E^∞) is of moderate growth (see [5, Lemma 2.10]).

A second possibility to define the Fréchet structure is by Laplace Sobolev norms. Let

$$(6.1) \quad \Delta := -(X_1^2 + \dots + X_n^2) \in \mathcal{U}(\mathfrak{g})$$

be a Laplace element attached to the basis \mathcal{B} , and set

$$(6.2) \quad \Delta_R = \Delta + R^2 \cdot \mathbf{1}$$

for $R \in \mathbb{R}$. We recall the following from [13, Cor. 3.3, Rem. 3.4].

Lemma 6.1. *Let (π, E) be a Banach representation of a unimodular Lie group G . Then there exists a constant $R_E \geq 0$ such that for all $R > R_E$ the operator*

$$d\pi(\Delta_R) : E^\infty \rightarrow E^\infty$$

is an isomorphism of Fréchet spaces. Moreover, one can take $R_E = 0$ in case (π, E) is unitary.

From now on we assume that G is a unimodular Lie group. For a Banach representation (π, E) and fixed $R > R_E$, we define Laplace Sobolev norms of even order for any $k \in \mathbb{Z}$ by

$$(6.3) \quad \Delta p_{2k}(v) := p(\Delta_R^k v) \quad (v \in E^\infty).$$

Strictly speaking Δp_{2k} depends on $R > R_E$ but we suppress this in the notation. In case (π, E) is unitary we use $R = 1$ and thus $\Delta p_{2k}(v) = p(\Delta_1^k v)$.

For $k \geq 0$, it is clear that $\Delta p_{2k} \leq c_k \cdot p_{2k}$ for a constant $c_k > 0$ which is independent of p and E .

Further, for $k \geq 0$ [13, Prop. 4.12] yields constants $C_k > 0$ only depending on \mathcal{B} and not on E or p such that

$$(6.4) \quad p_{2k}(v) \leq C_k \cdot \Delta p_{2k+n^*}(v) \quad (v \in E^\infty)$$

where

$$(6.5) \quad n^* = \min\{k \in 2\mathbb{N} \mid 1 + \dim G \leq k\}$$

For the rest of this section we request that G is real reductive and $G < \text{GL}(m, \mathbb{R})$ for some m . In this situation we take the basis $\mathcal{B} = \{X_1, \dots, X_n\}$ such that the

Laplace element Δ as defined in (6.1) satisfies

$$\Delta = -\mathcal{C}_G + 2\mathcal{C}_K$$

with \mathcal{C}_G and \mathcal{C}_K appropriate Casimir elements (unique if \mathfrak{g} and \mathfrak{k} are semisimple).

Lemma 6.2. *Assume (π, E) is irreducible and unitary and let p be any continuous K -invariant Hermitian norm on E^∞ . Let $R > 0$. Then for each $k \in \mathbb{N}$ there exists a constant $C = C(k, R) > 0$, independent of p and π , such that*

$$p(\Delta_R^k v) \leq Cp(\Delta_1^k v) \quad (v \in E^\infty).$$

Proof. It suffices to prove this for $k = 1$. Notice that any $v \in E^\infty$ admits a convergent expansion $v = \sum_{\tau \in \widehat{K}} v_\tau$ in K -types which is orthogonal with respect to any K -invariant Hermitian norm on E^∞ . Since Δ_R is K -invariant, the norm $p(\Delta_R \cdot)$ is K -invariant and Hermitian. Hence it suffices to show that $p(\Delta_R v) \leq Cp(\Delta_1 v)$ for v belonging to a K -type $E[\tau]$. Then both \mathcal{C}_G and \mathcal{C}_K act by scalars on $E[\tau]$. Hence $\Delta_R v = (c_\tau + R^2)v$ for some scalar c_τ , which has to be ≥ 0 as the representation π was unitary: use $\langle \Delta v, v \rangle \geq 0$ for all $v \in E^\infty$ and $\langle \cdot, \cdot \rangle$ a unitary inner product on E . Then

$$p(\Delta_R v) = (c_\tau + R^2)p(v) \leq C(c_\tau + 1)p(v) = Cp(\Delta_1 v)$$

for $C = \max\{1, R^2\}$ and the lemma follows. □

6.2. Smooth completions of Harish-Chandra modules and spherical pairs.

We move on to Harish-Chandra modules and their canonical smooth completions. A useful reference for the following summary might be [5].

If V is a complex vector space and p is a norm on V , then we denote by V_p the Banach completion of the normed space (V, p) .

Let V be a Harish-Chandra module (with regard to a fixed choice of a maximal compact group K of G). A norm p on V is called G -continuous provided the infinitesimal action of \mathfrak{g} on V exponentiates to a Banach representation of G on V_p . Note that every Harish-Chandra module admits a G -continuous norm, as a consequence of the Casselman embedding theorem.

The Casselman-Wallach globalization theorem asserts that the space of smooth vectors V_p^∞ is independent of the particular G -continuous norm p , i.e. if q is another G -continuous norm, then the identity map $V \rightarrow V$ extends to a G -equivariant isomorphism of Fréchet spaces $V_p^\infty \rightarrow V_q^\infty$. Stated differently, up to G -isomorphism of Fréchet spaces, there is a unique Fréchet completion V^∞ of V such that the G -action on V^∞ is smooth and of moderate growth.

We extend \mathfrak{a} to an abelian subalgebra $\mathfrak{j} = \mathfrak{a} + i\mathfrak{t} \subset \mathfrak{g}_\mathbb{C}$ with $\mathfrak{t} \subset \mathfrak{m}$ a maximal torus. Note that $\mathfrak{j}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ for which the roots are real valued on \mathfrak{j} , i.e. $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \subset \mathfrak{j}^*$. We denote by $W_j = W(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C})$ the corresponding Weyl group and let $\rho_j \in \mathfrak{j}^*$ be a half-sum with $\rho_j|_{\mathfrak{a}} = \rho$, where ρ is the half sum defined by \mathfrak{n} .

Assume now that V is an irreducible Harish-Chandra module and denote by $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. By the Schur-Dixmier lemma the elements of $\mathcal{Z}(\mathfrak{g})$ act by scalars on V and we thus obtain an algebra morphism $\chi_V : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathbb{C}$, the infinitesimal character of V . Via the Harish-Chandra isomorphism we identify $\mathcal{Z}(\mathfrak{g}) \simeq S(\mathfrak{j}_\mathbb{C})^{W_j}$, and consequently we may identify χ_V with an element of $\mathfrak{j}_\mathbb{C}^*/W_j$.

Let V be an irreducible Harish-Chandra module and V^∞ its canonical smooth completion. Further let $V^{-\infty} := (V^\infty)'$ be the continuous dual of V^∞ and let

$\eta \in (V^{-\infty})^H$ be an H -fixed element. We refer to (V, η) as a *spherical pair* provided $\eta \neq 0$.

Let now (V, η) be a spherical pair and $v \in V^\infty$. We form the generalized matrix coefficient

$$m_{v,\eta}(g \cdot z_0) := \eta(g^{-1} \cdot v) \quad (g \in G)$$

which is a smooth function on Z .

6.3. Abstract Plancherel theory. We denote by \widehat{G} the unitary dual of G and pick for every equivalence class $[\pi]$ a representative (π, \mathcal{H}_π) , i.e. \mathcal{H}_π is a Hilbert space and $\pi : G \rightarrow U(\mathcal{H}_\pi)$ is an irreducible unitary representation in the equivalence class of $[\pi]$. We denote by $(\overline{\pi}, \mathcal{H}_{\overline{\pi}})$ the dual representation. We recall the G -equivariant antilinear equivalence

$$\mathcal{H}_\pi \rightarrow \mathcal{H}_{\overline{\pi}}, \quad v \mapsto \overline{v} := \langle \cdot, v \rangle_{\mathcal{H}_\pi}$$

which induces the G -equivariant antilinear isomorphism:

$$\mathcal{H}_\pi^{-\infty} \rightarrow \mathcal{H}_{\overline{\pi}}^{-\infty}, \quad \eta \mapsto \overline{\eta}; \quad \overline{\eta}(\overline{v}) := \overline{\eta(v)}$$

and a linear embedding $\mathcal{H}_\pi^\infty \hookrightarrow \mathcal{H}_{\overline{\pi}}^{-\infty}$.

In this context we recall the mollifying map

$$C_c^\infty(G) \otimes \mathcal{H}_{\overline{\pi}}^{-\infty} \rightarrow \mathcal{H}_\pi^\infty \subset \mathcal{H}_{\overline{\pi}}^{-\infty}, \quad f \otimes \overline{\eta} \mapsto \overline{\pi}(f)\overline{\eta} := \int_G f(g)\overline{\eta}(\overline{\pi}(g)^{-1}\cdot) dg.$$

The mollifying map restricted to H -invariants induces a map

$$C_c^\infty(G/H) \otimes (\mathcal{H}_{\overline{\pi}}^{-\infty})^H \rightarrow \mathcal{H}_\pi^\infty,$$

$$F \otimes \overline{\eta} \mapsto \overline{\pi}(F)\overline{\eta} := \int_{G/H} f(gH)\overline{\eta}(\overline{\pi}(g)^{-1}\cdot) d(gH).$$

The abstract Plancherel Theorem for the unimodular real spherical space $Z = G/H$ asserts the following (see [41], [11], or [35, Section 8]): There exists a Radon measure μ on \widehat{G} and for every $[\pi] \in \widehat{G}$ a Hilbert space $\mathcal{M}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$, depending measurably on $[\pi]$ (note that $(\mathcal{H}_\pi^{-\infty})^H$ is finite dimensional [30], [34]), such that with the induced Hilbert space structure on $\text{Hom}(\mathcal{M}_{\overline{\pi}}, \mathcal{H}_\pi) \simeq \mathcal{M}_\pi \otimes \mathcal{H}_\pi$ the Fourier transform

$$\mathcal{F} : C_c^\infty(Z) \rightarrow \int_{\widehat{G}}^\oplus \text{Hom}(\mathcal{M}_{\overline{\pi}}, \mathcal{H}_\pi) d\mu(\pi)$$

$$F \mapsto \mathcal{F}(F) = (\mathcal{F}(F)_\pi)_{\pi \in \widehat{G}}; \quad \mathcal{F}(F)_\pi(\overline{\eta}) := \overline{\pi}(F)\overline{\eta} \in \mathcal{H}_\pi^\infty$$

extends to a unitary G -isomorphism from $L^2(Z)$ onto $\int_{\widehat{G}}^\oplus \text{Hom}(\mathcal{M}_{\overline{\pi}}, \mathcal{H}_\pi) d\mu(\pi)$.

Moreover the measure class of μ is uniquely determined by Z and we call μ a *Plancherel measure* for Z . Unique are also the *multiplicity subspaces* $\mathcal{M}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$ for almost all π together with their inner products up to positive scalar.

Note that by definition

$$(6.6) \quad \langle F, m_{v,\eta} \rangle_{L^2(Z)} = \langle \mathcal{F}(F)_\pi(\overline{\eta}), v \rangle \quad (F \in C_c^\infty(Z)),$$

for all $\eta \in \mathcal{M}_\pi, v \in \mathcal{H}_\pi^\infty$, and furthermore the *Parseval formula*

$$(6.7) \quad \|F\|_{L^2(Z)}^2 = \int_{\widehat{G}} \mathbf{H}_\pi(F) d\mu(\pi) \quad (F \in C_c^\infty(Z)),$$

where H_π denotes the Hermitian form on $C_c^\infty(Z)$ defined by

$$(6.8) \quad H_\pi(F) = \sum_{j=1}^{m_\pi} \|\overline{\pi}(F)\overline{\eta}_j\|_{\mathcal{H}_\pi}^2$$

for $\overline{\eta}_1, \dots, \overline{\eta}_{m_\pi}$ an orthonormal basis of \mathcal{M}_π . Observe that $H_\pi(F)$ is the Hilbert-Schmidt norm squared of the operator $\mathcal{F}(F)_\pi : \mathcal{M}_\pi \rightarrow \mathcal{H}_\pi$ and hence does not depend on the choice of the particular orthonormal basis.

Remark 6.3 (Normalization of Plancherel measure). As mentioned, only the measure class of $[\mu]$ of μ is unique. With a choice of Plancherel measure $\mu \in [\mu]$ we pin down uniquely the G -invariant Hermitian forms H_π on $\mathcal{H}_\pi \otimes \mathcal{M}_\pi$ for almost all π . In particular, together with a choice of an inner product on \mathcal{H}_π (unique up to scalar by Schur’s Lemma) we pin down the scalar product on \mathcal{M}_π uniquely.

Typically the \mathcal{H}_π are induced representations with a preferred inner product, but in practice there are several meaningful choices for the inner product on the multiplicity space (see Section 14 and Section 15.) A different choice of inner product on \mathcal{M}_π then leads to a rescaling of μ in its measure class.

Remark 6.4 (Fourier inversion). Let $f \in C_c^\infty(Z)$ be of the form $f = (F^* * F)^H$ where $F \in C_c^\infty(G)$, $F^*(g) = \overline{F(g^{-1})}$ and the upper index H denoting the right H -average of $F^* * F$. Then $f(z_0) = \|F^H\|_{L^2(Z)}^2$. Hence we deduce from the Parseval formula (6.7) for all $f \in C_c^\infty(Z)$ the inversion formula

$$(6.9) \quad f(z_0) = \int_{\widehat{G}} \sum_{i=1}^{m_\pi} \Theta_\pi^i(f) \, d\mu(\pi)$$

where Θ_π^i is the *spherical character*, i.e. the left H -invariant distribution

$$\Theta_\pi^i(f) = \eta_i(\overline{\pi}(f)\overline{\eta}_i) \quad (f \in C_c^\infty(Z)).$$

6.4. Tempered norms. We recall the standard tempered norms on Z . Using the weight functions \mathbf{w} and \mathbf{v} from [27, Sections 3 and 4], the following norms on $C_c^\infty(Z)$ are attached to a parameter $N \in \mathbb{R}$:

$$q_N(f) := \sup_{z \in Z} |f(z)| \mathbf{v}(z)^{\frac{1}{2}} (1 + \mathbf{w}(z))^N,$$

$$p_N(f) := \left(\int_Z |f(z)|^2 (1 + \mathbf{w}(z))^N \, dz \right)^{\frac{1}{2}}.$$

Note that the norm p_N is G -continuous, K -invariant, and Hermitian. We recall that the two families of Sobolev norms $q_{N;k}$ and $p_{N;k}$ for $(N, k) \in \mathbb{R} \times \mathbb{N}_0$ define the same topology on $C_c^\infty(Z)$, and specifically for $k > \frac{\dim G}{2}$ we recall the inequality

$$(6.10) \quad q_N(f) \leq C p_{N;k}(f) \quad (f \in C_c^\infty(Z))$$

for a constant C only depending on k and N (see [35, Lemma 9.5] and its proof).

We denote by $L^2_{N;k}(Z)$ the completion of $C_c^\infty(Z)$ with respect to $p_{N;k}$. We wish to define $L^2_{N;k}(Z)$ and $p_{N;k}$ as well for $k \in -\mathbb{N}$, and we do that by duality. Given the invariant measure on Z , the dual $L^2_N(Z)'$ is canonically isometric isomorphic to $L^2_{-N}(Z)$ via the equivariant bilinear pairing

$$L^2_N(Z) \times L^2_{-N}(Z) \rightarrow \mathbb{C}, \quad (f, g) \mapsto \int_Z f(z)g(z) \, dz.$$

This leads to the definition

$$(6.11) \quad L^2_{N;-k}(Z) := L^2_{-N;k}(Z)' \quad (k \in \mathbb{N})$$

with

$$(6.12) \quad p_{N;-k}(f) := \sup_{\substack{\phi \in L^2_{-N;k}(Z) \\ p_{-N;k}(\phi) \leq 1}} \left| \int_Z f(z)\phi(z) dz \right|.$$

6.5. Negative Sobolev norms. The definition of the negative Sobolev norms $p_{N;-k}$ for the norm p_N fits into a general pattern which we recall in this Subsection. Given a Banach representation (π, E) and a G -continuous norm p on E we define the dual norm p' of p on the continuous dual E' as usual:

$$p'(\lambda) = \sup_{p(v) \leq 1} |\lambda(v)| \quad (\lambda \in E').$$

In the sequel we assume that p is a Hermitian norm. This guarantees in particular that the dual action of G on E' is continuous, i.e. (π', E') is a representation. Further we retrieve p from p' via $p = (p)'$. For any $k \in \mathbb{N}_0$ we write $p'_k := (p')_k$ for the k -th Sobolev norm of the dual norm p' and define the negative Sobolev norm p_{-k} of p by

$$(6.13) \quad p_{-k}(v) := (p'_k)'(v) \quad (v \in E).$$

Recall that we define Laplace Sobolev norms Δp_{2k} for all integers $k \in \mathbb{Z}$.

Lemma 6.5. *Let (π, E) be a Hilbert representation of G and p a corresponding Hermitian norm. Then for all $k \in \mathbb{N}_0$ there exists a constant $C_k > 0$ such that*

$$\Delta p_{-2k-n^*}(v) \leq C_k p_{-2k}(v) \quad (v \in E^\infty).$$

Proof. In view of the definition of the negative Sobolev norm p_{-2k} in (6.13) this follows from (6.4) applied to the dual norm p' and the observation that

$$(\Delta p'_{2k})' = \Delta p_{-2k}$$

for all $k \in \mathbb{N}_0$. □

Lemma 6.6. *Let (V, η) be a spherical pair where $V = V_\pi$ is the Harish-Chandra module of a unitary irreducible representation π , and let $N \in \mathbb{R}$ be such that $p_N(m_{v,\eta}) < \infty$ for all $v \in V^\infty$. Then for each $2k > n^*$ there exists a constant $C > 0$, depending on k but not on (V, η) and N , such that*

$$(6.14) \quad p_N(m_{v,\eta}) \leq C p_{N;-2k+n^*}(m_{\Delta_1^k v,\eta}) \quad (v \in V^\infty).$$

Proof. In general we have for all $f \in E^\infty = L^2_N(Z)^\infty$ and fixed $R > R_E$

$$p_N(f) = p_N(\Delta_R^{-k} \Delta_R^k f) = \Delta p_{N;-2k}(\Delta_R^k f).$$

Upon applying Lemma 6.5 we obtain that

$$p_N(f) \leq C p_{N;-2k+n^*}(\Delta_R^k f).$$

Specifically for $f = m_{v,\eta}$ we arrive at

$$p_N(m_{v,\eta}) \leq C p_{N;-2k+n^*}(m_{\Delta_R^k v,\eta}) \quad (v \in V^\infty).$$

Now $q(v) := p_{N;-2k+n^*}(m_{v,\eta})$ defines a K -invariant continuous Hermitian norm on V^∞ and thus we may replace R by 1 according to Lemma 6.2. □

6.6. Tempered pairs. We now define

$$(6.15) \quad N_Z := 2 \operatorname{rank}_{\mathbb{R}} Z + 1 \quad k_Z := \frac{1}{2} \dim \mathfrak{g}.$$

Then for all $N \geq N_Z$ and $k > k_Z$ it follows from [35, Prop. 9.6] combined with [4, Th. 1.5] that for μ -almost all $[\pi] \in \widehat{G}$, the π -Fourier transform

$$\mathcal{F}_\pi : C_c^\infty(Z) \rightarrow \operatorname{Hom}(\mathcal{M}_\pi, \mathcal{H}_\pi)$$

extends continuously to $L^2_{N;k}(Z)$ and that the corresponding inclusion

$$(6.16) \quad L^2_{N;k}(Z) \rightarrow \int_{\widehat{G}}^\oplus \operatorname{Hom}(\mathcal{M}_\pi, \mathcal{H}_\pi) \, d\mu(\pi)$$

is Hilbert-Schmidt (in the sequel HS for short).

We wish to make this fact a bit more concrete in the context of the Hermitian forms \mathbf{H}_π . For that purpose we fix N and k as above and denote by $\|\mathbf{H}_\pi\|_{\text{HS},N;k}$ the HS-norm of the operator $F \otimes \bar{\eta} \mapsto \bar{\pi}(F)\bar{\eta}$ from $L^2_{N;k}(Z) \otimes \mathcal{M}_\pi$ to \mathcal{H}_π , that is

$$\|\mathbf{H}_\pi\|_{\text{HS},N;k}^2 := \sum_{n \in \mathbb{N}} \mathbf{H}_\pi(F_n)$$

for any orthonormal basis $(F_n)_{n \in \mathbb{N}}$ of $L^2_{N;k}(Z)$. The fact that (6.16) is HS then translates into the *a priori bound*

$$(6.17) \quad \int_{\widehat{G}} \|\mathbf{H}_\pi\|_{\text{HS},N;k}^2 \, d\mu(\pi) < \infty.$$

By (6.6) we further infer

$$(6.18) \quad \sum_{j=1}^{m_\pi} p_{-N;-k}(m_{v,\eta_j})^2 = \sum_{j=1}^{m_\pi} \sup_{\substack{F \in C_c^\infty(Z) \\ p_{N;k}(F) \leq 1}} |\langle \mathcal{F}(F)_\pi(\bar{\eta}_j), v \rangle|^2 \leq \|\mathbf{H}_\pi\|_{\text{HS},N;k}^2 \|v\|_{\mathcal{H}_\pi}^2$$

for μ -almost all $[\pi] \in \widehat{G}$, all $v \in \mathcal{H}_\pi^\infty$, and $\eta_1, \dots, \eta_{m_\pi}$ an orthonormal basis of \mathcal{M}_π .

Hence it follows from (6.17) that

$$(6.19) \quad \int_{\widehat{G}} \sup_{\substack{\eta \in \mathcal{M}_\pi \\ \|\eta\| \leq 1}} \sup_{\substack{v \in \mathcal{H}_\pi^\infty \\ \|v\| \leq 1}} p_{-N;-k}(m_{v,\eta})^2 \, d\mu(\pi) < \infty.$$

Consequently $p_{-N;-k}(m_{v,\eta}) < \infty$ for $N \geq N_Z$ and $k > k_Z$, for all $v \in \mathcal{H}_\pi^\infty$, $\eta \in \mathcal{M}_\pi$, and μ -almost all $[\pi]$.

In particular with any k with $2k - n^* > k_Z$ for $N \geq N_Z$ we obtain from Lemma 6.5 that

$$\begin{aligned} p_{-N}(m_{v,\eta}) &= p_{-N}(\Delta_R^{-k} \Delta_R^k m_{v,\eta}) = \Delta p_{-N;2k}(\Delta_R^k m_{v,\eta}) \\ &\leq C p_{-N;-2k+n^*}(m_{\Delta_R^k v,\eta}) < \infty \end{aligned}$$

for all $v \in \mathcal{H}_\pi^\infty$ and μ -almost all $[\pi]$.

Definition 6.7 (cf. [27, Def. 5.3] and [10, Sect. 3.3]). Let (V, η) be a spherical pair. We say that η is *tempered* or (V, η) is a *tempered pair* provided that

$$p_{-N}(m_{v,\eta}) < \infty \quad (v \in V^\infty)$$

for some $N \in \mathbb{R}$.

The tempered functionals make up a subspace of $(V^{-\infty})^H$ which we denote by $(V^{-\infty})^H_{\text{temp}}$. We conclude that $\mathcal{M}_\pi \subset (V^{-\infty})^H_{\text{temp}}$ for almost all π .

Remark 6.8.

(a) (About the inclusion $(V^{-\infty})_{\text{temp}}^H \subset (V^{-\infty})^H$). For a tempered pair (V, η) the inclusion $\{0\} \neq (V^{-\infty})_{\text{temp}}^H \subset (V^{-\infty})^H$ can be strict. This already appears for the rank one symmetric spaces $Z = \text{SO}_0(1, n)/\text{SO}_0(1, n - 1)$ when $n \geq 4$, in which case there exists an irreducible Harish-Chandra module which has multiplicity one in $L^p(Z)$ for $p \leq n - 1$ and multiplicity two for $p > n - 1$. For details of this example we refer to [33].

(b) (Tempered Frobenius reciprocity). If we denote by $C_{\text{temp}}^\infty(Z) = \bigcup_{N \in \mathbb{R}} L_N^2(Z)^\infty$ the G -module of smooth functions of moderate growth on Z , then we recall from [10, 3.10] the following variant of Frobenius reciprocity for Harish-Chandra modules V :

$$\text{Hom}(V^\infty, C_{\text{temp}}^\infty(Z)) \simeq (V^{-\infty})_{\text{temp}}^H$$

with Hom referring to continuous morphisms of G -modules.

(c) (About the inclusion $\mathcal{M}_\pi \subset (V^{-\infty})^H$). For symmetric spaces one has equality

$$(6.20) \quad \mathcal{M}_\pi = (V^{-\infty})_{\text{temp}}^H \quad \text{for almost all } \pi.$$

This was established by forming wave packets, which was a central technical step in the proof of the Plancherel formula for symmetric spaces. Since we follow another approach towards the Plancherel formula in this article, the equality (6.20) together with an explicit description of $(V^{-\infty})^H$ is not an issue in the underlying treatment. However, we do expect that in general $\mathcal{M}_\pi = (V^{-\infty})_{\text{temp}}^H$ for almost all π .

7. CONSTANT TERM APPROXIMATIONS

In this section we review the constant term approximation of [10] which is a central technical tool for this paper. In fact, by using our geometric results from Section 4.5 on the stabilizer H_I , and our combinatorial results on the open P -orbits of Section 5, we are able to refine slightly the results from [10].

Recall from (5.18) that the set of open P -orbits \mathcal{W} of Z admits a combinatorial decomposition $\mathcal{W} = \prod_{\mathbf{c} \in \mathcal{C}_I} \prod_{\mathbf{t} \in \mathcal{F}_{I, \mathbf{c}}} \mathbf{m}_{\mathbf{c}, \mathbf{t}}(\mathcal{W}_{I, \mathbf{c}})$. For the sake of readability we first consider the part $\mathbf{m}(\mathcal{W}_I) \subset \mathcal{W}$ corresponding to $\mathbf{c} = \mathbf{t} = \mathbf{1}$ and treat the notationally heavier case later.

7.1. Notation. Let V be an irreducible Harish-Chandra module with smooth completion V^∞ and dual $V^{-\infty}$.

We recall that $(V^{-\infty})^H$ is a finite dimensional space for any real spherical subgroup $H \subset G$. Also we recall that A_I normalizes H_I . Hence for any $I \subset S$ we obtain an action of A_I on $(V^{-\infty})^{H_I}$ by $a_I \cdot \xi = \xi(a_I^{-1} \cdot)$ for $\xi \in (V^{-\infty})^{H_I}$. Accordingly we can decompose ξ into generalized eigenvectors:

$$\xi = \sum_{\lambda \in \mathfrak{a}_{I, \mathbf{c}}^*} \xi^\lambda,$$

where ξ^λ has generalized eigenvalue λ . We set

$$(7.1) \quad \mathcal{E}_\xi := \{\lambda \in \mathfrak{a}_{I, \mathbf{c}}^* \mid \xi^\lambda \neq 0\}.$$

For $\eta \in (V^{-\infty})^H$ and $w \in \mathcal{W}$ we set $\eta_w := w \cdot \xi$ and note that η_w is H_w -fixed.

7.2. Base points from $\mathfrak{m}(\mathcal{W}_I)$. We recall from (5.4) the injective map $\mathfrak{m} : \mathcal{W}_I \rightarrow \mathcal{W}$. Let now $w_I \in \mathcal{W}_I$ and $w = \mathfrak{m}(w_I)$. Then, given $\xi \in (V^{-\infty})^{H_I}$ we note that $\xi_{w_I} = w_I \cdot \xi$ is fixed under $(H_I)_{w_I} = (H_w)_I$, see (5.6). Moreover A_I normalizes $(H_I)_{w_I}$ and we obtain from (a slight adaption of) [29, Lemma 6.2] that $(\xi^\lambda)_{w_I}$ is a generalized eigenvector for the \mathfrak{a}_I -action to the same spectral value λ .

We recall that $\rho|_{\mathfrak{a}_H} = 0$ by the request that Z is unimodular, see [27, Lemma 4.2]. This allows us to consider ρ as a functional on $\mathfrak{a}_Z = \mathfrak{a}/\mathfrak{a}_H$ as well. In the sequel if not stated otherwise we take $N = N_Z$ (see (6.15)).

Theorem 7.1 (Constant term approximation). *Let $Z = G/H$ be a unimodular real spherical space and $I \subset S$. Then for all irreducible Harish-Chandra modules V there exists a unique linear map*

$$(V^{-\infty})_{\text{temp}}^H \rightarrow (V^{-\infty})_{\text{temp}}^{H_I}, \quad \eta \mapsto \eta^I$$

with the following property. For all compact sets $\Omega \subset G$ and $\mathcal{C}_I \subset \mathfrak{a}_I^{-}$ there exist $k \in \mathbb{N}$, $\epsilon > 0$, and $C > 0$, such that

$$(7.2) \quad |m_{v,\eta}(ga_I w \cdot z_0) - m_{v,\eta^I}(ga_I w_I \cdot z_{0,I})| \leq C a_I^{(1+\epsilon)\rho} p_{-N;k}(m_{v,\eta})$$

for all $\eta \in (V^{-\infty})_{\text{temp}}^H$, $v \in V^\infty$, $g \in \Omega$, $a_I \in A_I^{-}$ with $\log a_I \in \mathbb{R}_{\geq 0} \mathcal{C}_I$, and $w = \mathfrak{m}(w_I) \in \mathfrak{m}(\mathcal{W}_I) \subset \mathcal{W}$. The constants k , ϵ , and C can be chosen independently of V .

Moreover, with $\chi_V \in \mathfrak{j}_{\mathbb{C}}^*/\mathcal{W}_{\mathfrak{j}}$ the infinitesimal character of V one has

$$(7.3) \quad \mathcal{E}_{\eta^I} \subset (\rho|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*) \cap (\rho - \mathcal{W}_{\mathfrak{j}} \cdot \chi_V)|_{\mathfrak{a}_I},$$

where \mathcal{E}_{η^I} is defined by (7.1). Finally there is the consistency relation

$$(7.4) \quad (\eta_w)^I = (\eta^I)_{w_I} \quad (w = \mathfrak{m}(w_I) \in W).$$

The constant term assignment

$$(V^{-\infty})_{\text{temp}}^H \rightarrow (V^{-\infty})_{\text{temp}}^{H_I}, \quad \eta \mapsto \eta^I$$

is typically neither injective nor surjective. Let us illustrate that in two examples before giving the proof of the theorem.

Example 7.2.

(a) Let $H = K$ be a maximal compact subgroup of G and $I = \emptyset$. Then $H_\emptyset = M\bar{N}$. Now let V be a K -spherical tempered Harish-Chandra module. Then $\dim V^K = 1$. However, for generic V we have $\dim(V^{-\infty})^{M\bar{N}} = |\mathcal{W}_{\mathfrak{a}}|$ with $\mathcal{W}_{\mathfrak{a}}$ the Weyl group of the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$. This shows that the constant term assignment is typically not surjective.

(b) Tempered pairs (V, η) of the twisted discrete series can be characterized by the vanishing of the constant term assignments for $I \neq S$, see [10, Th. 5.12]. In particular, if (V, η) belongs to the discrete series of Z , then we have $\eta^I = 0$ for all $I \neq S$. Hence the constant term assignment is typically not injective.

Proof. The existence of an $\eta^I \in (V^{-\infty})_{\text{temp}}^{H_I}$ satisfying (7.2), (7.3) and (7.4) is proved in [10], with the exception that invariance of η^I is only shown for the identity component of H_I . In more precision, (7.2) for H_I replaced by $(H_I)_0$ is [10, Th. 7.10] with the caveat that in [10] the norms to bound the right hand side of (7.2) are Sobolev norms of q_{-N} and not of p_{-N} . However, the passage between q_{-N} and p_{-N} is justified by the comparison of Sobolev norms in (6.10) which is valid for any

$N \in \mathbb{R}$. The inclusion of exponents (7.3) is part of the general theory in [10] and the consistency relation in (7.4) is [10, Prop. 5.7].

We turn to the uniqueness of the map $\eta \rightarrow \eta^I$. We recall that $(V^{-\infty})^{H_I}$ is a finite dimensional A_I -module and thus (7.3) implies that for any fixed $g \in G$ and $v \in V^\infty$ the map

$$A_I \ni a_I \mapsto m_{v,\eta^I}(ga_I \cdot z_{0,I}) = m_{v,a_I \cdot \eta^I}(g \cdot z_{0,I})$$

is an exponential polynomial with normalized unitary exponents and hence unique as constant term approximation of $m_{v,\eta}(ga \cdot z_0)$, see Remark 7.3. In particular, η^I is then uniquely determined by the approximation property (7.2).

Finally we will show that η^I is in fact H_I -invariant for all $\eta \in (V^{-\infty})_{\text{temp}}^H$. We do this for the case of $w_I = w = \mathbf{1}$, the more general case being an easy adaption. We recall Lemma 4.6 and the notation used therein.

Let $X_I \in \mathfrak{c}_I^{-}$ corresponding to $-\mathbf{e}_I$ under the identification $\mathfrak{a}_I \simeq V_I$. Set $a_t := \exp(tX_I)$ for $t \geq 0$. First notice that both $m_{v,\eta^I}(gh_I a_t \cdot z_{0,I})$ and $m_{v,\eta^I}(g x_t a_t \cdot z_{0,I})$ approximate

$$m_{v,\eta}(gh_I a_t \cdot z_0) = m_{v,\eta}(g x_t a_t \cdot z_0)$$

via (7.2), and thus we get

$$(7.5) \quad a_t^{-\rho} |m_{v,\eta^I}(gh_I a_t \cdot z_{0,I}) - m_{v,\eta^I}(g x_t a_t \cdot z_{0,I})| \leq C e^{-\epsilon t}$$

for some $C, \epsilon > 0$. On the other hand, the coefficients of the exponential polynomial

$$a_I \mapsto a_I^{-\rho} m_{v,\eta^I}(g x_t a_I \cdot z_{0,I}) = a_I^{-\rho} m_{(g x_t)^{-1} v, a_I \cdot \eta^I}(z_{0,I})$$

with unitary exponents depend smoothly on $g x_t$. Hence it follows, after possibly shrinking ϵ , from (4.17) that

$$(7.6) \quad |a^{-\rho} m_{v,\eta^I}(g x_t a \cdot z_{0,I}) - a^{-\rho} m_{v,\eta^I}(g a \cdot z_{0,I})| \leq C e^{-\epsilon t}$$

for all $a \in A_I$. Now the H_I -invariance of η^I follows from combining (7.5) and (7.6) together with the before mentioned uniqueness. \square

Remark 7.3 (Uniqueness of the constant term). Let $f(a)$ be a function on A_I and

$$F(a) = a^\rho \sum_{\lambda \in \mathcal{E}} q_\lambda(\log a) a^\lambda \quad (a \in A_I)$$

an exponential polynomial with unitary exponents, i.e. $\mathcal{E} \subset i\mathfrak{a}_I^*$ is finite and q_λ are polynomial functions on \mathfrak{a}_I . In case there exists an $\epsilon > 0$ such that

$$(7.7) \quad |f(a) - F(a)| \leq C a^{(1+\epsilon)\rho} \quad (a \in A_I^-),$$

then F is the unique exponential polynomial with normalized unitary exponents having the approximation property (7.7). This is a consequence of the following basic lemma, which we record without proof.

Lemma 7.4. *Let $\Lambda \subset \mathbb{R}$ be a finite set and for each $\lambda \in \Lambda$ let $q_\lambda \in \mathbb{C}[t]$ be a polynomial. If there exist constants $\epsilon, C > 0$ such that*

$$\left| \sum_{\lambda \in \Lambda} q_\lambda(t) e^{i\lambda t} \right| < C e^{-\epsilon t} \quad (t \geq 0)$$

then $q_\lambda = 0$ for all $\lambda \in \Lambda$.

7.3. General base points. So far we have treated the constant term approximation through the base points $z_w = w \cdot z_0$ for $w \in \mathfrak{m}(\mathcal{W}_I)$. The general case is obtained by adapting the notation to the partition $\mathcal{W} = \coprod_{\mathfrak{c} \in \mathbf{C}_I} \coprod_{\mathfrak{t} \in \mathbf{F}_{I,\mathfrak{c}}} \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathcal{W}_{I,\mathfrak{c}})$ from (5.18).

For $\mathfrak{c} \in \mathbf{C}_I$ and $\mathfrak{t} \in \mathbf{F}_{I,\mathfrak{c}}$ we define $w(\mathfrak{c}, \mathfrak{t}) := \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathbf{1}) \in \mathcal{W}$ and set $z_{\mathfrak{c},\mathfrak{t}} = w(\mathfrak{c}, \mathfrak{t}) \cdot z_0$. Further we set $w(\mathfrak{c}) = \mathfrak{m}_{\mathfrak{c},\mathbf{1}}(\mathbf{1}) \in \mathcal{W}$ and $z_{\mathfrak{c}} = w(\mathfrak{c}) \cdot z_0$. Let $H_{\mathfrak{c},\mathfrak{t}}$ and $H_{\mathfrak{c}}$ denote the G -stabilizers of $z_{\mathfrak{c},\mathfrak{t}}$ and $z_{\mathfrak{c}}$ respectively.

Define for $\eta \in (V^{-\infty})^H$ accordingly $\eta_{\mathfrak{c},\mathfrak{t}} := w(\mathfrak{c}, \mathfrak{t}) \cdot \eta$. Notice that $\eta_{\mathfrak{c},\mathfrak{t}}^I$ is invariant under $(H_{\mathfrak{c},\mathfrak{t}})_I$. From (5.19) we infer further that $(H_{\mathfrak{c},\mathfrak{t}})_I = H_{I,\mathfrak{c}}$ does not depend on \mathfrak{t} .

As before we obtain that A_I normalizes $(H_{I,\mathfrak{c},\mathfrak{t}})_{w_I} = (H_{I,\mathfrak{c}})_{w_I}$, so that A_I acts naturally on $(H_{I,\mathfrak{c}})_{w_I}$ -invariant distribution vectors ξ and yields generalized eigenspace decompositions $\xi = \sum_{\lambda \in \mathfrak{a}_I^*} \xi^\lambda$. Within the introduced terminology the general case of the constant term approximation then reads as follows:

Theorem 7.5 (Constant term approximation - general version). *Let $Z = G/H$ be a unimodular real spherical space and $I \subset S$. Fix $\mathfrak{c} \in \mathbf{C}_I$ and $\mathfrak{t} \in \mathbf{F}_{I,\mathfrak{c}}$. Then for all irreducible Harish-Chandra modules V there exists a unique linear map*

$$(V^{-\infty})_{\text{temp}}^H \rightarrow (V^{-\infty})_{\text{temp}}^{H_{I,\mathfrak{c}}}, \quad \eta \mapsto \eta_{\mathfrak{c},\mathfrak{t}}^I$$

with the following property: There exist constants $\epsilon > 0$, $k \in \mathbb{N}$, such that for all compact subsets $\mathcal{C}_I \subset \mathfrak{a}_I^-$ and $\Omega \subset G$ there exists a constant $C > 0$, such that

$$(7.8) \quad |m_{v,\eta}(g a_I w \cdot z_0) - m_{v,\eta_{\mathfrak{c},\mathfrak{t}}^I}(g a_I w_{I,\mathfrak{c}} \cdot z_{0,I,\mathfrak{c}})| \leq C a_I^{(1+\epsilon)\rho} p_{-N;k}(m_{v,\eta})$$

for all $\eta \in (V^{-\infty})_{\text{temp}}^H$, $v \in V^\infty$, $g \in \Omega$, $a_I \in A_I^-$ with $\log a_I \in \mathbb{R}_{\geq 0} \mathcal{C}_I$, and $w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}}) \in \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathcal{W}_{I,\mathfrak{c}}) \subset \mathcal{W}$. The constants ϵ , k , and C can all be chosen independently of V .

Moreover, with $\chi_V \in \mathfrak{j}_{\mathbb{C}}^*/\mathfrak{W}_j$ the infinitesimal character of V one has

$$(7.9) \quad \mathcal{E}_{\eta_{\mathfrak{c},\mathfrak{t}}^I} \subset (\rho|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*) \cap (\rho - \mathfrak{W}_j \cdot \chi_V)|_{\mathfrak{a}_I}.$$

Finally there is the consistency relation

$$(7.10) \quad (\eta_w)^I = (\eta_{\mathfrak{c},\mathfrak{t}}^I)_{w_{I,\mathfrak{c}}} \quad (w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}}) \in W).$$

Proof. By replacing $H_{I,\mathfrak{c}}$ with H_I we may assume that $\mathfrak{c} = \mathbf{1}$. Let then $w \in \mathfrak{m}_{\mathfrak{t}}(W_I)$. The passage to $\mathfrak{t} = \mathbf{1}$ is obtained via the material in Subsection 5.4.1 and via the further base point shift $z_0 \rightarrow z_{\mathfrak{t}}$. By this we obtain a reduction to Theorem 7.1. \square

8. THE MAIN REMAINDER ESTIMATE

In this section we derive an important uniform estimate which is the key technical tool for the results in the next section. The estimate is based on the constant term approximation of Section 7.

8.1. Adjustment of Haar measures. We assume that $Z = G/H$ carries a G -invariant measure. Then, according to [29, Lemma 3.12], the same holds for $Z_I := G/H_I$. Since $L \cap H = L \cap H_I$ by Lemma 4.7, we see that the P -orbits through z_0 and $z_{0,I}$ are isomorphic as homogeneous spaces for Q , i.e.

$$(8.1) \quad P \cdot z_0 = Q \cdot z_0 \simeq Q/L \cap H \simeq Q \cdot z_{0,I} = P \cdot z_{0,I}.$$

We fix the normalizations of the G -invariant measures on Z and Z_I such that on these open pieces they coincide with a common Haar measure on $Q/L \cap H$, and we denote these measures on Z and Z_I by dz and dz_I , respectively.

8.2. Right action by $A(I)$. As $A(I)$ normalizes H_I we obtain a right action of $A(I)$ on functions f on Z_I given by

$$(R(a_I)f)(g \cdot z_{0,I}) := f(ga_I \cdot z_{0,I}) \quad (g \in G, a_I \in A(I)).$$

Lemma 8.1. *Let $f \in L^1(Z_I)$ and $a_I \in A(I)$. Then*

$$(8.2) \quad \int_{Z_I} (R(a_I)f)(z_I) dz_I = |a_I^{2\rho}| \int_{Z_I} f(z_I) dz_I$$

In particular, the normalized action $f \mapsto |a_I^{-\rho}|R(a_I)f$ of $A(I)$ is unitary on $L^2(Z_I)$.

Proof. First note that $|a^\rho| = 1$ for all $a \in T_Z = \exp(i\mathfrak{a}_H^\perp) \subset \underline{A}$. Since elements of $F(I)$ have finite order it is sufficient to consider $a_I \in A_I \subset A(I)$. The first assertion then follows from [29, Lemma 8.4], and the second assertion is a consequence of the first. \square

Fix an element $X \in \mathfrak{a}_I^-$ and set $a_t := \exp(tX)$ for $t \in \mathbb{R}$. Let $f \in L^2(Z_I)$ and define

$$(8.3) \quad f_t(z) := a_t^\rho (R(a_t^{-1})f)(z), \quad (z \in Z_I).$$

Notice that the assignment $f \mapsto f_t$ is G -equivariant and unitary by Lemma 8.1. In particular

$$(8.4) \quad \|f_t\|_{L^2(Z_I)} = \|f\|_{L^2(Z_I)} \quad (t \in \mathbb{R})$$

and, in case f is smooth,

$$(8.5) \quad L_u f_t = (L_u f)_t \quad (u \in \mathcal{U}(\mathfrak{g})).$$

8.3. Matching of functions. We recall from Section 5 the injective map $\mathbf{m} : \mathcal{W}_I \rightarrow \mathcal{W}$ which matches the open Q -orbit $Qw_I \cdot z_{0,I} = Pw_I \cdot z_{0,I}$ in Z_I with the open Q -orbit $Qw \cdot z_0 = Pw \cdot z_0$ in Z where $w = \mathbf{m}(w_I)$. As in (8.1) we have

$$(8.6) \quad Qw \cdot z_0 \simeq Q/L \cap H \simeq Qw_I \cdot z_{0,I}.$$

Given a smooth function f on Z_I with compact support in $Qw_I \cdot z_{0,I} \subset Z_I$ we define via (8.6) a ‘matching’ smooth function $F = \Phi(f)$ on Z with compact support in $Q\mathbf{m}(\mathcal{W}_I) \cdot z_0 \subset Z$ by

$$(8.7) \quad F(q\mathbf{m}(w_I) \cdot z_0) := f(qw_I \cdot z_{0,I}) \quad (q \in Q).$$

Observe that the space spanned by the smooth functions on Z_I with compact support contained in the union of the open Q -orbits $Qw_I \cdot z_{0,I}$ is dense in $L^2(Z_I)$.

Since the invariant measures on Z and Z_I coincide on the open Q -orbits we get

$$\|\Phi(f)\|_{L^2(Z)} = \|f\|_{L^2(Z_I)}.$$

Together with (8.4) this implies for the function f_t defined in (8.3)

$$(8.8) \quad \|\Phi(f_t)\|_{L^2(Z)} = \|f\|_{L^2(Z_I)}$$

for all $t \in \mathbb{R}$.

The main result of this section now reads as follows. Let $N = N_Z$ from (6.15).

Theorem 8.2 (Main remainder estimate). *There exists $\epsilon > 0$ with the following property. Let $\Omega \subset Q$ be a compact set. Then for every $s \in \mathbb{R}$ there exist $C > 0$ and $m \in \mathbb{N}$ such that for all $f \in C_c^\infty(Z_I)$ with $\text{supp } f \subset \Omega \mathcal{W}_I \cdot z_{0,I}$, all tempered pairs (V, η) , and all $v \in V^\infty$ the following equality holds*

$$\langle \Phi(f_t), m_{v,\eta} \rangle_{L^2(Z)} = \langle f_t, m_{v,\eta^I} \rangle_{L^2(Z_I)} + R(t) \quad (t \geq 0),$$

with the remainder bounded by

$$|R(t)| \leq C e^{-t\epsilon} p_{-N; -s}(m_{v,\eta}) p_{N;m}(\Phi(f)).$$

Before giving the proof we observe Corollary 8.3. Recall from (6.8) the Hermitian forms H_π on $C_c^\infty(Z)$. We fix an orthonormal basis $\eta_1, \dots, \eta_{m_\pi}$ of \mathcal{M}_π and define a preliminary Hermitian form $H_\pi^{I,\text{pre}}$ on $C_c^\infty(Z_I)$ by

$$(8.9) \quad H_\pi^{I,\text{pre}}(f) = \sum_{j=1}^{m_\pi} \|\bar{\pi}(f)\bar{\eta}_j^I\|_{\mathcal{H}_\pi}^2 \quad (f \in C_c^\infty(Z_I)).$$

Notice that $H_\pi^{I,\text{pre}}$ is independent from the particular choice of the orthonormal basis $\eta_1, \dots, \eta_{m_\pi}$, being the Hilbert-Schmidt norm squared of the linear map

$$\mathcal{M}_\pi \rightarrow \mathcal{H}_\pi, \quad \bar{\eta} \mapsto \bar{\pi}(f)\eta^I.$$

We derive from Theorem 8.2 and the global a priori bound (6.17) that:

Corollary 8.3. *Let $\epsilon > 0$ be as in Theorem 8.2 and let $f \in C_c^\infty(Z_I)$ with support in $Q\mathcal{W}_I \cdot z_{0,I}$. Then there exists a constant $C > 0$ such that*

$$\|f\|_{L^2(Z_I)}^2 = \int_{\widehat{G}} H_\pi^{I,\text{pre}}(f_t) \, d\mu(\pi) + R(t)$$

with $|R(t)| \leq C e^{-\epsilon t}$ for all $t \geq 0$.

Proof. We first observe that by (8.8) and (6.7)–(6.8)

$$(8.10) \quad \|f\|_{L^2(Z_I)}^2 = \int_{\widehat{G}} H_\pi(\Phi(f_t)) \, d\mu(\pi) = \int_{\widehat{G}} \sum_{j=1}^{m_\pi} \|\bar{\pi}(\Phi(f_t))\bar{\eta}_j\|_{\mathcal{H}_\pi}^2 \, d\mu(\pi).$$

Hence we need to estimate the integral over $\pi \in \widehat{G}$ of

$$\sum_{j=1}^{m_\pi} \left(\|\bar{\pi}(\Phi(f_t))\bar{\eta}_j\|_{\mathcal{H}_\pi}^2 - \|\bar{\pi}(f)\bar{\eta}_j^I\|_{\mathcal{H}_\pi}^2 \right).$$

Using the identity $a^2 - b^2 = 2a(a - b) - (a - b)^2$ together with Cauchy-Schwarz and (8.10), we see that it suffices to show

$$(8.11) \quad \left[\int_{\widehat{G}} \sum_{j=1}^{m_\pi} \left(\|\bar{\pi}(\Phi(f_t))\bar{\eta}_j\|_{\mathcal{H}_\pi} - \|\bar{\pi}(f)\bar{\eta}_j^I\|_{\mathcal{H}_\pi} \right)^2 \, d\mu(\pi) \right]^{1/2} \leq C e^{-\epsilon t}.$$

From the dense inclusion $\mathcal{H}_\pi^\infty \subset \mathcal{H}_\pi$ and (6.6) we obtain that

$$\|\bar{\pi}(\Phi(f_t))\bar{\eta}_j\|_{\mathcal{H}_\pi} = \sup_{\substack{v \in \mathcal{H}_\pi^\infty \\ \|v\|=1}} \langle \bar{\pi}(\Phi(f_t))\bar{\eta}_j, v \rangle_{\mathcal{H}_\pi} = \sup_{\substack{v \in \mathcal{H}_\pi^\infty \\ \|v\|=1}} \langle \Phi(f_t), m_{v,\eta_j} \rangle_{L^2(Z)}$$

and similarly

$$\|\bar{\pi}(f)\bar{\eta}_j^I\|_{\mathcal{H}_\pi} = \sup_{\substack{v \in \mathcal{H}_\pi^\infty \\ \|v\|=1}} \langle f, m_{v,\eta_j^I} \rangle_{L^2(Z_I)}.$$

Let $s > k_Z$ (see (6.15)). Now application of Theorem 8.2 implies for all $t > 0$

$$\left| \|\bar{\pi}(\Phi(f_t))\bar{\eta}_j\|_{\mathcal{H}_\pi} - \|\bar{\pi}(f)\bar{\eta}_j^I\|_{\mathcal{H}_\pi} \right| \leq C e^{-te} \sup_{\substack{v \in \mathcal{H}_\pi^\infty \\ \|v\|=1}} p_{-N; -s}(m_{v, \eta_j}),$$

where $C > 0$ depends on f , but not on t or π . Hence (8.11) follows from (6.18) and (6.17). □

8.4. Comparing Haar measures. In the proof of Theorem 8.2 we will assume for simplicity that $\text{supp } f \subset \Omega \cdot z_{0,I}$. The general case is obtained using the following observation. Recall that the Haar measures of Z and Z_I are both adjusted to agree with a fixed Haar measure of Q/Q_H on the Q -orbits through z_0 and $z_{0,I}$.

Recall from the local structure theorem that

$$(8.12) \quad Qw \cdot z_0 \simeq Q/Q_H \simeq U \times L/L_H$$

and by (8.6) likewise $Qw_I \cdot z_{0,I} \simeq Q/Q_H$. We claim that the Haar measures of Z and Z_I coincide on every open Q -orbit with the fixed normalized measure on Q/Q_H . Let us verify this for Z , the proof for Z_I being analogous. We first implement the Haar measure on Q/Q_H via a density $|\omega_Z|$ obtained from a top degree differential form $\omega_Z \in \bigwedge^{\text{top}}(\mathfrak{q}/\mathfrak{q} \cap \mathfrak{h})^*$. As usual we decompose $w = \tilde{t}h$ with $\tilde{t} \in T_Z$ and $h \in \underline{H}$, see (2.15). Then $\text{Ad}(\tilde{t})$ preserves $(\mathfrak{q}/\mathfrak{q} \cap \mathfrak{h})_{\mathbb{C}}$ and thus acts on $\bigwedge^{\text{top}}(\mathfrak{q}/\mathfrak{q} \cap \mathfrak{h})_{\mathbb{C}}^*$ by a unit scalar. Since the scalar has to be real, the claim follows.

8.5. Matching derivatives. Before we can give the proof of Theorem 8.2 we need Lemma 8.4.

Lemma 8.4. *Let $\Omega \subset Q$ be a compact subset. Then the following assertions hold:*

- (1) *Let $u \in \mathcal{U}(\mathfrak{g})$. There exist $u_1, \dots, u_k \in \mathcal{U}(\mathfrak{q})$ with $\deg u_j \leq \deg u$ and a constant $C = C(\Omega, u)$ such that*

$$(8.13) \quad \left| [\Phi(L_u(f_t)) - L_u(\Phi(f_t))](z) \right| \leq C \max_{\sigma \in S \setminus I} a_t^\sigma \sum_{j=1}^k |L_{u_j}(\Phi(f_t))(z)|$$

for all $f \in C_c^\infty(Z_I)$ with support in $\Omega W_I \cdot z_{0,I}$, and all $z \in Z, t \geq 0$.

- (2) *Let p_0 denote the L^2 -norm on $L^2(Z)$. Then for every $k \in \mathbb{N}_0$ there exists a constant $C = C(\Omega, k) > 0$ such that*

$$(8.14) \quad p_{0;k}(\Phi(f_t)) \leq C p_{0;k}(\Phi(f))$$

for all $f \in C_c^\infty(Z_I)$ with support in $\Omega W_I \cdot z_{0,I}$ and $t \geq 0$.

Proof. Since the map

$$\Phi : C_c^\infty(QW_I \cdot z_{0,I}) \rightarrow C_c^\infty(QW \cdot z_0), \quad f \mapsto \Phi(f)$$

is Q -equivariant we have

$$(8.15) \quad \Phi(L_Y f) = L_Y \Phi(f)$$

for all $Y \in \mathfrak{q}$.

For simplicity we consider the case $\text{supp } f \subset \Omega \cdot z_{0,I}$. We first calculate $L_X(\Phi(f_t))$ ($qa_t \cdot z_0$) and $\Phi(L_X(f_t))(qa_t \cdot z_0)$ for $X \in \mathfrak{g}$.

For that we recall that $\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{q}$ is a direct sum. More generally for all $q \in Q$ the sum $\mathfrak{g} = \text{Ad}(q)\bar{\mathfrak{u}} + \mathfrak{q}$ is direct. Accordingly we can decompose any $X \in \mathfrak{g}$ as

$$X = \sum_{\alpha, k} c_{\alpha, k}(q) \text{Ad}(q) X_{-\alpha}^k + \sum_j d_j(q) X_j$$

where $(X_{-\alpha}^k)_k$ is a basis of $\mathfrak{g}^{-\alpha}$, $\alpha \in \Sigma_u$, and $(X_j)_j$ is a basis of \mathfrak{q} . The coefficients $c_{\alpha,k}(q), d_j(q) \in \mathbb{R}$ depend smoothly on q .

Recall that $X_{-\alpha}^k + \sum_{\beta} X_{\alpha,\beta}^k \in \mathfrak{h}$ by (2.17) with $I = S$. Thus we get for every smooth function F on Z and every $q \in Q, a \in A_Z$ that

$$L_X F(qa \cdot z_0) = \sum_j d_j(q) L_{X_j} F(qa \cdot z_0) - \sum_{\alpha,\beta,k} c_{\alpha,k}(q) a^{\alpha+\beta} L_{\text{Ad}(q)X_{\alpha,\beta}^k} F(qa \cdot z_0).$$

By expanding each $\text{Ad}(q)X_{\alpha,\beta}^k$ in terms of the X_j we can rephrase this identity as

$$(8.16) \quad L_X F(qa \cdot z_0) = \sum_j \left[d_j(q) - \sum_{\alpha,\beta} c_{j,\alpha,\beta}(q) a^{\alpha+\beta} \right] L_{X_j} F(qa \cdot z_0)$$

with coefficients $c_{j,\alpha,\beta}$ depending smoothly on q .

On the other hand by (2.17) we also have $X_{-\alpha}^k + \sum_{\alpha+\beta \in \langle I \rangle} X_{\alpha,\beta}^k \in \mathfrak{h}_I$ which then similarly yields for every smooth function f on Z_I

$$L_X f(qa \cdot z_{0,I}) = \sum_j \left[d_j(q) - \sum_{\substack{\alpha,\beta \\ \alpha+\beta \in \langle I \rangle}} c_{j,\alpha,\beta}(q) a^{\alpha+\beta} \right] L_{X_j} f(qa \cdot z_{0,I})$$

with exactly the same coefficients as before, but for fewer α and β . We apply Φ to this equation with f replaced by f_t . With (8.15) this gives

$$(8.17) \quad \Phi(L_X f_t)(qa \cdot z_0) = \sum_j \left[d_j(q) - \sum_{\substack{\alpha,\beta \\ \alpha+\beta \in \langle I \rangle}} c_{j,\alpha,\beta}(q) a^{\alpha+\beta} \right] L_{X_j}(\Phi(f_t))(qa \cdot z_0).$$

From this equation we subtract (8.16) with $F = \Phi(f_t)$. With $a = a_t$ we obtain

$$(8.18) \quad [\Phi(L_X(f_t)) - L_X(\Phi(f_t))](qa_t \cdot z_0) = \sum_j c_j(q, t) [L_{X_j}(\Phi(f_t))(qa_t \cdot z_0)]$$

with coefficients $c_j(q, t)$, each being a linear combination $\sum_{\mu} c_{\mu}(q) a_t^{\mu}$ of functions a_t^{μ} with $\mu \in \langle S \rangle \setminus \langle I \rangle$, and with coefficients $c_{\mu} \in C^{\infty}(Q)$ supported in Ω . In particular (8.13) follows for $\text{deg } u = 1$.

We now prove by induction on $\text{deg } u$ that

$$(8.19) \quad [L_u(\Phi(f_t)) - \Phi(L_u(f_t))](qa_t \cdot z_0) = \sum_j c_j(q, t) [L_{u_j}(\Phi(f_t))(qa_t \cdot z_0)]$$

for some $u_j \in \mathcal{U}(\mathfrak{g})$ with $\text{deg } u_j \leq \text{deg } u$ and coefficients $c_j(q, t)$ of the same type as required in (8.18). Note that the set of coefficients of this type is stable under differentiation by elements from \mathfrak{q} .

Let $u = Xv$ with $X \in \mathfrak{g}$ and $\text{deg } v < \text{deg } u$. We write

$$L_u(\Phi(f_t)) - \Phi(L_u(f_t)) = L_X [L_v(\Phi(f_t)) - \Phi(L_v(f_t))] + [L_X \Phi(L_v f_t) - \Phi(L_X(L_v f_t))].$$

For the first term we apply (8.16) to L_X in order to replace the differentiation with $X \in \mathfrak{g}$ by differentiation with the $X_j \in \mathfrak{q}$. We then apply the induction hypothesis (8.19) to $[L_v(\Phi(f_t)) - \Phi(L_v(f_t))]$. After the differentiations by X_j we then obtain for the first term an expression of the required form. For the second term we apply (8.18) with f_t replaced by $(L_v f_t)_t = L_v f_t$. This gives

$$\sum_j c_j(q, t) [L_{X_j}(\Phi(L_v f_t))(qa_t \cdot z_0)].$$

Once more we apply the induction hypothesis to v , which allows us to replace this expression by

$$\sum_j c_j(q, t)[L_{X_j} L_v(\Phi(f_t))(qa_t \cdot z_0)]$$

at the cost of additional terms. Since all these terms have the required form this completes the proof of (8.19).

In order to complete the proof of (8.13) we need to replace the $u_j \in \mathcal{U}(\mathfrak{g})$ in (8.19) by elements from $\mathcal{U}(\mathfrak{q})$. By induction on the degree, similar to the one before, we obtain from (8.16) for every $u \in \mathcal{U}(\mathfrak{g})$ a set of elements $u_1, \dots, u_n \in \mathcal{U}(\mathfrak{q})$ with $\deg u_j \leq \deg u$ such that

$$(8.20) \quad L_u \Phi(f_t)(qa_t \cdot z_0) = \sum_j e_j(q, t) L_{u_j} \Phi(f_t)(qa_t \cdot z_0),$$

with coefficients $e_j(q, t)$, each being a linear combination $\sum_\mu c_\mu(q) a_t^\mu$ of functions a_t^μ with $\mu \in \langle S \rangle$, and with coefficients $c_\mu \in C^\infty(Q)$ supported in Ω . This finally implies (8.13) and with that the proof of (1) has been completed.

For (2) we note that (8.20) and (8.15) imply:

$$p_0(L_u \Phi(f_t)) \leq C_u \sum_j p_0(\Phi(L_{u_j}(f_t))).$$

If we denote by q_0 the L^2 -norm on $L^2(Z_I)$ we obtain from (8.5) and (8.8)

$$p_0(\Phi(L_{u_j} f_t)) = q_0(L_{u_j} f) = p_0(\Phi(L_{u_j} f)) = p_0(L_{u_j}(\Phi(f))).$$

Combining this with the preceding inequality, (2) follows. □

8.6. Proof of Theorem 8.2.

Proof. In view of the consistency relations $w_I \cdot \eta^I = (\eta_{\mathfrak{m}(w_I)})^I$ for all $w \in \mathcal{W}_I$ (see (7.4)), the assertion readily reduces to the case where $\text{supp } f \subset Q \cdot z_{0,I}$. Let us assume that in the sequel.

Recall that $a_t = \exp(tX)$ with $X \in \mathfrak{a}_I^-$ fixed. For simplicity we assume again that $\text{supp } f \subset \Omega \cdot z_{0,I}$, and then $\text{supp } \Phi(f_t) \subset \Omega a_t \cdot z_0$.

Recall the Laplace element $\Delta_1 \in \mathcal{U}(\mathfrak{g})$ from (6.2). In what follows we will apply the Sobolev inequality of Lemma 6.6 to V , and for this we observe (see Theorem 11.2) that V is unitarizable since (V, η) is tempered.

In the sequel we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{L^2(Z)}$ and $\langle \cdot, \cdot \rangle_I$ for $\langle \cdot, \cdot \rangle_{L^2(Z_I)}$ to save notation.

Let $n \in \mathbb{N}$, to be specified at the end of the proof. It will depend on s , but apart from that only on the space Z . We start with the identity $v = \Delta_1^n \Delta_1^{-n} v$ which yields

$$(8.21) \quad \langle \Phi(f_t), m_{v,\eta} \rangle = \langle L_{\Delta_1^n} \Phi(f_t), m_{\Delta_1^{-n} v, \eta} \rangle.$$

Next we have to address the subtle point that $\Phi(L_{\Delta_1^n} f_t)$ does not necessarily equal $L_{\Delta_1^n} \Phi(f_t)$. However from Lemma 8.4(1) we obtain constants $\epsilon > 0$, $C > 0$, and elements $u_j \in \mathcal{U}(\mathfrak{g})$ of degree $\leq 2n$ such that for all f supported by $\Omega \cdot z_{0,I}$

$$(8.22) \quad |L_{\Delta_1^n} \Phi(f_t)(z) - \Phi(L_{\Delta_1^n} f_t)(z)| \leq C e^{-t\epsilon} \sum_j |L_{u_j}(\Phi(f_t))(z)| \quad (z \in Z, t \geq 0).$$

We rewrite (8.21) as

$$(8.23) \quad \langle \Phi(f_t), m_{v,\eta} \rangle = \langle \Phi(L_{\Delta_1^n} f_t), m_{\Delta_1^{-n}v,\eta} \rangle + R_1(t)$$

with $R_1(t) = \langle L_{\Delta_1^n} \Phi(f_t) - \Phi(L_{\Delta_1^n} f_t), m_{\Delta_1^{-n}v,\eta} \rangle$. We claim, after shrinking ϵ to $\frac{\epsilon}{2}$, that for $2n > n^*$ where n^* is the even integer given by (6.5)

$$(R1) \quad |R_1(t)| \leq C e^{-t\epsilon} p_{N;2n}(\Phi(f)) p_{-N;-2n+n^*}(m_{v,\eta})$$

with a constant $C > 0$ that depends on Ω and n , but not on f . From (8.22) and Cauchy-Schwarz we obtain

$$(8.24) \quad |R_1(t)| \leq C e^{-t\epsilon} p_{N;2n}(\Phi(f_t)) p_{-N}(m_{\Delta_1^{-n}v,\eta}).$$

We obtain from [27, Prop. 3.4 (2)] that $|\mathbf{w}(z)| \leq C(1+t)$ for all $z \in \text{supp } \Phi(f_t)$ for a constant C only depending on Ω . Hence it follows with Lemma 8.4(2) that

$$(8.25) \quad \begin{aligned} p_{N;2n}(\Phi(f_t)) &\leq C(1+t)^{\frac{N}{2}} p_{0;2n}(\Phi(f_t)) \\ &\leq C(1+t)^{\frac{N}{2}} p_{0;2n}(\Phi(f)) \leq C(1+t)^{\frac{N}{2}} p_{N;2n}(\Phi(f)) \end{aligned}$$

with positive constants C (possibly not equal to each other). Note that these constants C depend on n .

Furthermore it follows from (6.14) that for $2n > n^*$

$$(8.26) \quad p_{-N}(m_{\Delta_1^{-n}v,\eta}) \leq C p_{-N;-2n+n^*}(m_{v,\eta}).$$

If we insert (8.25) and (8.26) into (8.24) we obtain the claim (R1) by noting that $(1+t)^{\frac{N}{2}} e^{-\frac{\epsilon}{2}t}$ is bounded for all $t \geq 0$.

We move on with the identity (8.23) and wish to analyze $\langle \Phi(L_{\Delta_1^n} f_t), m_{\Delta_1^{-n}v,\eta} \rangle$ further. By the definitions of Φ and f_t

$$(8.27) \quad \begin{aligned} \langle \Phi(L_{\Delta_1^n} f_t), m_{\Delta_1^{-n}v,\eta} \rangle &= \int_{Q/Q_H} (L_{\Delta_1^n} f)(qa_t \cdot z_{0,I}) a_t^\rho \overline{m_{\Delta_1^{-n}v,\eta}(q \cdot z_0)} d(qQ_H) \\ &= \int_{Q/Q_H} (L_{\Delta_1^n} f)(q \cdot z_{0,I}) a_t^{-\rho} \overline{m_{\Delta_1^{-n}v,\eta}(qa_t \cdot z_0)} d(qQ_H). \end{aligned}$$

Likewise

$$(8.28) \quad \langle L_{\Delta_1^n} f_t, m_{\Delta_1^{-n}v,\eta^I} \rangle_I = \int_{Q/Q_H} (L_{\Delta_1^n} f)(q \cdot z_{0,I}) a_t^{-\rho} \overline{m_{\Delta_1^{-n}v,\eta^I}(qa_t \cdot z_{0,I})} d(qQ_H).$$

Next we wish to replace $m_{\Delta_1^{-n}v,\eta^I}$ by the constant term approximation $m_{\Delta_1^{-n}v,\eta^I}$ via Theorem 7.1. We then obtain constants $\epsilon > 0, k \in \mathbb{N}$, depending only on Z , and a constant $C > 0$ depending also on Ω and n , such that with $l := k + n^*$ one has for all $q \in \Omega$ and all $v \in V^\infty$

$$(8.29) \quad \begin{aligned} |m_{\Delta_1^{-n}v,\eta}(qa_t \cdot z_0) - m_{\Delta_1^{-n}v,\eta^I}(qa_t \cdot z_{0,I})| &\leq C a_t^{(1+\epsilon)\rho} p_{-N;k}(m_{\Delta_1^{-n}v,\eta}) \\ &\leq C a_t^{(1+\epsilon)\rho} p_{-N;l-2n}(m_{v,\eta}). \end{aligned}$$

In the passage to the second line of (8.29) we used (8.26).

Now note that (8.5) implies

$$\langle (L_{\Delta_1^n} f)_t, m_{\Delta_1^{-n}v,\eta^I} \rangle_I = \langle f_t, m_{v,\eta^I} \rangle_I,$$

and thus if we insert the bound (8.29) into the difference between (8.27) and (8.28), we obtain the identity

$$(8.30) \quad \langle \Phi(L_{\Delta_1^n} f_t), m_{\Delta_1^{-n} v, \eta} \rangle = \langle f_t, m_{v, \eta^t} \rangle_I + R_2(t)$$

with

$$(8.31) \quad |R_2(t)| \leq C e^{-t\epsilon} p_{-N; l-2n}(m_{v, \eta}) \|L_{\Delta_1^n} f\|_{L^2(Z_I)} \sqrt{\text{vol}_{Z_I}(\Omega \cdot z_{0, I})}.$$

Now, as in (8.20) we convert derivatives,

$$L_{\Delta_1^n} f(q \cdot z_{0, I}) = \sum c_j(q) L_{u_j} f(q \cdot z_{0, I})$$

with $u_j \in \mathcal{U}(\mathfrak{q})$ of $\deg u_j \leq 2n$ and smooth coefficients c_j . Hence

$$\|L_{\Delta_1^n} f\|_{L^2(Z_I)} \leq C p_{0; 2n}(\Phi(f)) \leq C p_{N; 2n}(\Phi(f))$$

with constants C depending only on Ω . Hence we obtain

$$(R2) \quad |R_2(t)| \leq C e^{-t\epsilon} p_{-N; l-2n}(m_{v, \eta}) p_{N; 2n}(\Phi(f)).$$

Now the theorem follows from the two remainder estimates (R1) and (R2), by choosing the number n such that $m = 2n \geq s + k + n^*$. \square

8.7. Matching with respect to \tilde{Z}_I . We conclude this section with a slight extension of the preceding results, when we consider instead of Z_I the union of all G -orbits in $\underline{Z}_I(\mathbb{R})$ which point to Z , i.e. the space $\tilde{Z}_I = \coprod_{c \in c_I} \coprod_{t \in F_{I, c}} Z_{I, c, t}$ from (5.1) which gives rise to the full partition $\mathcal{W} = \coprod_{c \in c_I} \coprod_{t \in F_{I, c}} \mathbf{m}_{c, t}(\mathcal{W}_{I, c})$ from (5.2).

Observe that $f \in C_c^\infty(\tilde{Z}_I)$ corresponds to a family $f = (f_{c, t})_{c, t}$ with $f_{c, t} \in C_c^\infty(Z_{I, c, t})$ and $Z_{I, c, t} = Z_{I, c}$ as homogeneous spaces. Suppose now that $\text{supp } f_{c, t} \subset Qw_{I, c} \cdot z_{0, I, c} \subset Z_{I, c} = Z_{I, c, t}$ for all c, t . With (5.2) the function f can then be matched with a function $F = \Phi(f) \in C_c^\infty(Z)$ by requesting

$$F(q\mathbf{m}_{c, t}(w_{I, c}) \cdot z_0) = f_{c, t}(qw_{I, c} \cdot z_{0, I, c}) \quad (q \in Q).$$

Then Corollary 8.3 extends to all $f \in C_c^\infty(\tilde{Z}_I)$ with $\text{supp } f_{c, t} \subset Qw_{I, c} \cdot z_{0, I, c}$, and yields constants $C, \epsilon > 0$ such that

$$(8.32) \quad \|f\|_{L^2(\tilde{Z}_I)}^2 = \int_{\tilde{G}} \sum_{c, t} H_{\pi, c, t}^{I, \text{pre}}((f_{c, t})_t) d\mu(\pi) + R(t)$$

with $|R(t)| \leq C e^{-\epsilon t}$ for all $t \geq 0$. Here $H_{\pi, c, t}^{I, \text{pre}}$ refers to $H_{\pi}^{I, \text{pre}}$ for Z_I replaced by $Z_{I, c, t}$; explicitly

$$(8.33) \quad H_{\pi, c, t}^{I, \text{pre}}(f_{c, t}) = \sum_{j=1}^{m_{\pi}} \|\bar{\pi}(f_{c, t})((\bar{\eta}_j)_{c, t}^I)\|_{\mathcal{H}_{\pi}}^2 \quad (f_{c, t} \in C_c^\infty(Z_{I, c, t})).$$

9. INDUCED PLANCHEREL MEASURES

In this section we show that the Plancherel measure of $L^2(Z_I)$ is induced from the Plancherel measure of $L^2(Z)$ in a natural manner, see Theorem 9.5. A consequence thereof is a certain variant of the Maass-Selberg relations as recorded in Theorem 9.6. Statements and approach are largely motivated by the reasoning in Sakellaridis-Venkatesh [42, Sect. 11.1–11.4], which originates from ideas of Joseph Bernstein. The main technical ingredient is our remainder estimate of Corollary 8.3.

Given a point $[\pi] \in \widehat{G}$ we denote by $\mathcal{U}_{[\pi]}$ the neighborhood filter of $[\pi]$ in \widehat{G} . Let $I \subset S$ and recall from (8.9) the definition of the Hermitian form $H_\pi^{I,\text{pre}}$. Attached to the Plancherel measure μ we define its I -support by

$$(9.1) \quad \text{supp}^I(\mu) := \{[\pi] \in \widehat{G} \mid (\forall U \in \mathcal{U}_{[\pi]}) \mu(\{\sigma \in U : H_\sigma^{I,\text{pre}} \neq 0\}) > 0\}.$$

We denote by μ^I the restriction of μ to $\text{supp}^I(\mu)$. In the sequel we let (π, \mathcal{H}_π) be such that $[\pi] \in \text{supp}^I(\mu)$. Define

$$(9.2) \quad \mathcal{M}_\pi^I := \text{span}\{a \cdot \eta^I : \eta \in \mathcal{M}_\pi, a \in A_I\} \subset (\mathcal{H}_\pi^{-\infty})_{\text{temp}}^{H_I}$$

where the latter inclusion is part of Theorem 7.1.

The elements $\xi \in \mathcal{M}_\pi^I$ decompose into generalized eigenvectors for the A_I -action,

$$(9.3) \quad \xi = \sum_{\lambda \in \mathcal{E}_\xi} \xi^\lambda,$$

and we recall from (7.3) that the generalized eigenvalues λ satisfy

$$(9.4) \quad \mathcal{E}_\xi \subset (\rho - \mathbf{W}_j \cdot \chi_\pi)|_{\mathfrak{a}_I} \cap (\rho|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*).$$

It will be seen later that the A_I -action is semisimple for almost all $\pi \in \text{supp}^I(\mu)$.

Recall that the conjugation $\mathcal{H}_\pi^{-\infty} \rightarrow \mathcal{H}_\pi^{-\infty}, \eta \mapsto \bar{\eta}$ is a G -equivariant isomorphism of topological vector spaces. The conjugation map induces an antilinear A_I -equivariant isomorphism $\mathcal{M}_\pi^I \simeq \mathcal{M}_\pi^I$. In particular, \mathcal{M}_π^I is semisimple if and only if \mathcal{M}_π^I is semisimple.

9.1. Averaging. What follows is motivated by the techniques of [42, Sect. 10]. Let $X \in \mathfrak{a}_I^-$ and set $a_t = \exp(tX)$ as usual. Throughout this section we let (π, \mathcal{H}_π) be a representation occurring in $\text{supp}^I(\mu)$. We recall the notion f_t from (8.3).

Lemma 9.1 (Averaging lemma). *Let $X \in \mathfrak{a}_I^-$. Then the following assertions hold:*

- (1) *Suppose that \mathcal{M}_π^I is X -semisimple. Then we have for all $f \in C_c^\infty(Z_I)$ and $\xi \in \mathcal{M}_\pi^I$ that*

$$(9.5) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} \|\pi(f_t)\xi\|^2 &= \sum_{\lambda \in \mathcal{E}_\xi} \|\pi(f)\xi^\lambda\|^2 + \\ &+ 2 \operatorname{Re} \sum_{\substack{\lambda \neq \lambda' \in \mathcal{E}_\xi \\ (\lambda - \lambda')(X) \in 2\pi i\mathbb{Z}}} \langle \pi(f)\xi^\lambda, \pi(f)\xi^{\lambda'} \rangle. \end{aligned}$$

In particular, if $(\lambda - \lambda')(X) \notin 2\pi i\mathbb{Z}$ for all $\lambda, \lambda' \in \mathcal{E}_\xi$ with $\lambda \neq \lambda'$, we have

$$(9.6) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} \|\pi(f_t)\xi\|^2 = \sum_{\lambda \in \mathcal{E}_\xi} \|\pi(f)\xi^\lambda\|^2.$$

- (2) *Suppose that \mathcal{M}_π^I is not X -semisimple. Then for every $\xi \in \mathcal{M}_\pi^I$ which does not belong to the sum of eigenspaces for X we have*

$$(9.7) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} \|\pi(f_t)\xi\|^2 = \infty$$

for every $f \in C_c^\infty(Z_I)$ for which $\pi(f)|_{\mathcal{M}_\pi^I}$ is injective.

Proof. (1) Assume \mathcal{M}_π^I is diagonalizable for X and let $\xi \in \mathcal{M}_\pi^I$. It then follows from (9.3) and (9.4) that

$$\pi(a_t)\xi = \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} a_t^\lambda \xi^\lambda.$$

In particular we obtain from Lemma 8.1 for all $f \in C_c^\infty(Z_I)$ and $t \geq 0$ that

$$\pi(f_t)\xi = \sum_{\lambda \in \rho + i\mathfrak{a}^*} a_t^{\lambda - \rho} \pi(f)\xi^\lambda$$

and thus

$$\begin{aligned} \|\pi(f_t)\xi\|^2 &= \left\| \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} a_t^{\lambda - \rho} \pi(f)\xi^\lambda \right\|^2 \\ &= \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} \|\pi(f)\xi^\lambda\|^2 + 2 \operatorname{Re} \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_\xi \\ \lambda \neq \lambda'}} a_t^{\lambda - \lambda'} \langle \pi(f)\xi^\lambda, \pi(f)\xi^{\lambda'} \rangle. \end{aligned}$$

Now for any $\gamma \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ we have $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} e^{it\gamma} = 0$ and (1) follows.

For (2) we remark that with the mentioned assumption on ξ we have for some $m \in \mathbb{N}$ and each $\lambda \in \rho + i\mathfrak{a}^*$ that

$$(9.8) \quad \pi(f_t)\xi^\lambda = a_t^{\lambda - \rho} \pi(f) \sum_{j=0}^m \frac{t^j}{j!} \xi^{\lambda, j}$$

where $\xi^{\lambda, 0} = \xi^\lambda, \xi^{\lambda, 1}, \dots, \xi^{\lambda, m} \in \mathcal{M}_\pi^I$. Moreover we can assume $\xi^{\lambda, m} \neq 0$ for some λ . Now (9.7) becomes a simple matter on polynomial asymptotics: set

$$\xi_t^{\text{top}} := \sum_{\lambda} a_t^{\lambda - \rho} \xi^{\lambda, m} \quad (t \geq 0)$$

and note that $|a_t^{\lambda - \rho}| = 1$ implies that the vectors $\xi_t^{\text{top}}, t \geq 0$, stay away from 0 in the finite dimensional space \mathcal{M}_π^I . Thus we obtain from (9.8) and the injectivity of $\pi(f)|_{\mathcal{M}_\pi^I}$ that

$$\|\pi(f_t)\xi\| \sim t^m \|\pi(f)\xi_t^{\text{top}}\|$$

from which (9.7) follows. □

Suppose that $X \in \mathfrak{a}_I^-$ is such that $(\lambda - \lambda')(X) \notin 2\pi i\mathbb{Z}$ for all $\lambda, \lambda' \in \mathcal{E}_\xi, \xi \in \mathcal{M}_\pi^I$, with $\lambda \neq \lambda'$. Then we obtain from (8.9) and Lemma 9.1 that

$$(9.9) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} \mathbf{H}_\pi^{I, \text{pre}}(f_t) = \begin{cases} \sum_{j=1}^{m_\pi} \sum_{\lambda \in \mathcal{E}_{\eta_j^I}} \|\overline{\pi}(f)\overline{\eta}_j^{I, \lambda}\|^2 & \text{if } \mathcal{M}_\pi^I \text{ is } X\text{-semisimple} \\ \infty & \text{if otherwise and} \\ \overline{\pi}(f)|_{\mathcal{M}_\pi^I} \text{ is injective} & \end{cases}$$

where $\eta_1, \dots, \eta_{m_\pi}$ is an orthonormal basis for \mathcal{M}_π .

This motivates the following definition of \mathbf{H}_π^I . In case \mathcal{M}_π^I is a semisimple A_I -module we set

$$(9.10) \quad \mathbf{H}_\pi^I(f) := \sum_{j=1}^{m_\pi} \sum_{\lambda \in \mathcal{E}_{\eta_j^I}} \|\overline{\pi}(f)\overline{\eta}_j^{I, \lambda}\|^2 \quad (f \in C_c^\infty(Z_I)),$$

and otherwise $H_\pi^I := 0$. Observe that the Hermitian form H_π^I is left G -invariant, and normalized-right A_I -invariant. Set

$$\text{supp}_{\text{fin}}^I(\mu) := \{[\pi] \in \text{supp}^I(\mu) \mid \mathcal{M}_\pi^I \text{ is } \mathfrak{a}_I\text{-semisimple}\}.$$

9.1.1. *Mollifying on multiplicity spaces.* Throughout this subsection we let V be an irreducible Harish-Chandra module and V^∞ its unique SF -completion. Let $\mathcal{S}(G)$ be the Schwartz algebra of rapidly decreasing functions on G (see [5]) and recall the following variant of the Casselman-Wallach theorem: if $0 \neq v \in V$, then

$$(9.11) \quad \mathcal{S}(G) * v = V^\infty$$

by [5, Th. 8.1], where for $f \in \mathcal{S}(G)$ and $v \in V^\infty$ we use the standard notation

$$f * v = \int_G f(g)g \cdot v \, dg$$

with the right hand side being a convergent integral in the Fréchet space V^∞ . Assertion (9.11) can be strengthened further as follows. Let \tilde{V} be the Harish-Chandra module dual to V . Then we first record the mollifying property $\mathcal{S}(G) * \tilde{V}^{-\infty} \subset V^\infty$ which in view of (9.11) strengthens to

$$(9.12) \quad \mathcal{S}(G) * \eta = V^\infty \quad (0 \neq \eta \in \tilde{V}^{-\infty})$$

In fact, choose first a left K -finite function $f \in C_c^\infty(G)$ such that $0 \neq f * \eta \in V$ and then apply (9.11) with $\mathcal{S}(G) * C_c^\infty(G) \subset \mathcal{S}(G)$. Let now $H \subset G$ be any closed unimodular subgroup of G . Then we define $\mathcal{S}(G/H)$ as the space of right H -averages of functions $F \in \mathcal{S}(G)$, i.e. $f \in \mathcal{S}(G/H)$ if and only if there exists an $F \in \mathcal{S}(G)$ such that

$$f(gH) = F^H(g) := \int_H F(gh) \, dh \quad (g \in G).$$

With that we can define for $\eta \in (\tilde{V}^{-\infty})^H$ and $f = F^H \in \mathcal{S}(G/H)$:

$$f * \eta := F * \eta$$

as the right hand side of this equation is independent of the particular lift F of f . Then we have the following generalization of (9.12).

Lemma 9.2. *Let $H \subset G$ be a closed unimodular subgroup and let $E \subset (\tilde{V}^{-\infty})^H$ be a finite dimensional subspace. Then the map*

$$\Phi_E : \mathcal{S}(G/H) \rightarrow \text{Hom}(E, V^\infty), \quad f \mapsto (\eta \mapsto f * \eta)$$

is continuous and surjective. Moreover E is uniquely determined by $\ker \Phi_E$.

Proof. First of all it is clear that Φ_E is continuous. Next we observe that the statement reduces to $H = \{1\}$ which we will assume from now on.

Notice that Φ_E is an $\mathcal{S}(G)$ -module morphism with $\mathcal{S}(G)$ acting on $\text{Hom}(E, V^\infty)$ on the target V^∞ , i.e. for $f \in \mathcal{S}(G)$ and $T \in \text{Hom}(E, V^\infty)$ we set $(f * T)(\eta) := f * (T(\eta))$.

Suppose that Φ_E were not surjective. Then $\text{im } \Phi_E \subset \text{Hom}(E, V^\infty)$ would be a proper $\mathcal{S}(G)$ -invariant subspace. Upon the identification $\text{Hom}(E, V^\infty) = E^* \otimes V^\infty$ we then derive from the fact that V^∞ is an algebraically simple module for $\mathcal{S}(G)$ (a consequence of (9.11)) that $\text{im } \Phi_E = F^\perp \otimes V^\infty$ for a subspace $0 \neq F \subset E$. This then means

$$\text{im } \Phi_E = \{T \in \text{Hom}(E, V^\infty) \mid T|_F = 0\}$$

which contradicts the fact that $\mathcal{S}(G) * F = V^\infty \neq \{0\}$ as $F \neq 0$.

Finally from $\mathcal{S}(G)/\ker \Phi_E \simeq E^* \otimes V^\infty$ we obtain the asserted uniqueness. Indeed, suppose you have $\ker \Phi_{E_1} = \ker \Phi_{E_2}$. Then $\ker \Phi_{E_i} = \ker \Phi_{E_1+E_2}$ for $i = 1, 2$ and thus $\dim(E_1 + E_2)^* = \dim E_i^*$ for $i = 1, 2$, i.e. $E_1 = E_2$. \square

We apply Lemma 9.2 to the Hermitian forms H_π^I of (9.10) as follows. Let $E = \mathcal{M}_\pi^I$.

Corollary 9.3. *Let $[\pi] \in \text{supp}_{\text{fin}}^I(\mu)$. There exists a unique Hermitian form H on*

$$\text{Hom}(\mathcal{M}_\pi^I, \mathcal{H}_\pi^\infty) \simeq \mathcal{H}_\pi^\infty \otimes \mathcal{M}_\pi^I$$

for which $H(\Phi_E(f)) = H_\pi^I(f)$ for all $f \in \mathcal{S}(G/H)$. This form is G -invariant and positive definite.

Proof. Clearly $f \in \ker \Phi_E \Rightarrow H_\pi^I(f) = 0$. Moreover, since $[\pi] \in \text{supp}_{\text{fin}}^I(\mu)$ we have

$$E = \mathcal{M}_\pi^I = \text{span}\{\eta_j^{I,\lambda} \mid 1 \leq j \leq m_\pi, \lambda \in \rho|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*\}$$

from which we deduce the converse implication. \square

We use the symbol H_π^I also for the form H introduced in the corollary. Now a variant of Schur’s Lemma implies that H_π^I viewed as a form on $\mathcal{H}_\pi^\infty \otimes \mathcal{M}_\pi^I$ is given by

$$(9.13) \quad H_\pi^I(v \otimes \xi) = \langle v, v \rangle_{\mathcal{H}_\pi} \langle \xi, \xi \rangle_{\mathcal{M}_\pi^I}$$

for a unique Hilbert inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}_\pi^I}$ on \mathcal{M}_π^I .

We conclude this intermediate subsection with a simple observation of later use.

Lemma 9.4. *Keep the assumptions of Lemma 9.2 and let $(f_n)_{n \in \mathbb{N}}$ be a Dirac-sequence in $C_c^\infty(G/H)$. Then there exists an $N = N(E)$ such that the map*

$$\Phi_E(f_n) : E \rightarrow V^\infty, \quad \eta \mapsto f_n * \eta$$

is injective for all $n \geq N$.

Proof. This is a special case of a more general fact. Let X be a locally convex topological vector space and $E \subset X$ a finite dimensional subspace. Let $T_n : E \rightarrow X$ be a family of linear continuous maps with $\lim_{n \rightarrow \infty} T_n(x) = x$ for all $x \in E$. We claim that there exists $N \in \mathbb{N}$ such that T_n is injective for all $n \geq N$. To see that we choose a closed complement to E and obtain a continuous projection $p_E : X \rightarrow E$. With $S_n := p_E \circ T_n$ we then obtain a sequence $S_n \in \text{End}(E)$ such that $S_n \rightarrow \mathbf{1}$. This proves the claim. The lemma follows with $X = \tilde{V}^{-\infty}$ and $T_n(x) = f_n * x$. \square

9.2. Induced Plancherel measure. Theorem 9.5 was largely motivated by [42, Th. 11.3].

Theorem 9.5 (Induced Plancherel measure). *For all $f \in C_c^\infty(Z_I)$ one has*

$$(9.14) \quad \|f\|_{L^2(Z_I)}^2 = \int_{\text{supp}^I(\mu)} H_\pi^I(f) \, d\mu(\pi).$$

In particular, the Plancherel measure μ_I of $L^2(Z_I)$ is equivalent to μ restricted to $\text{supp}^I(\mu)$, and \mathcal{M}_π^I as defined in (9.2) and equipped with the Hermitian form obtained from (9.13) provides a multiplicity space for μ_I -almost all π . In other words

$$(9.15) \quad L^2(Z_I) \underset{G \times A_I}{\simeq} \int_{\text{supp}^I(\mu)} \mathcal{H}_\pi \otimes \mathcal{M}_\pi^I \, d\mu(\pi),$$

with the just described inner product on \mathcal{M}_π^I , is a Plancherel decomposition for Z_I . Finally, the complement of $\text{supp}_{\text{fin}}^I(\mu)$ in $\text{supp}^I(\mu)$ is a null set.

Proof. It is sufficient to prove this identity for test functions f with support in $PW_I \cdot z_{0,I}$ because $PW_I \cdot z_{0,I}$ exhausts Z_I up to measure zero. Let such a test function f be given.

Fix $X \in \mathfrak{a}_I^-$. It follows from the exponential decay of $R(t)$ in Corollary 8.3 that

$$(9.16) \quad \frac{1}{n} \sum_{t=n+1}^{2n} \int_{\widehat{G}} H_\pi^{I,\text{pre}}(f_t) \, d\mu(\pi) \rightarrow \|f\|_{L^2(Z_I)}^2$$

as $n \rightarrow \infty$. Define

$$H_\pi^{I,X-\text{inv}}(f) := \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{t=n+1}^{2n} H_\pi^{I,\text{pre}}(f_t) \in [0, \infty].$$

Then (9.16) and Fatou’s lemma imply

$$(9.17) \quad \int_{\widehat{G}} H_\pi^{I,X-\text{inv}}(f) \, d\mu(\pi) \leq \|f\|_{L^2(Z_I)}^2 < \infty.$$

Next set

$$\widehat{G}_X := \{[\pi] \in \widehat{G} \mid \mathcal{M}_\pi^I \neq \{0\} \text{ and } \mathcal{M}_\pi^I \text{ is } X\text{-semisimple}\}.$$

By choosing a Dirac sequence f_1, f_2, \dots of $C_c^\infty(Z_I)$ which is supported in $P \cdot z_{0,I}$ we obtain from Lemma 9.4 for each $[\pi] \in \widehat{G}$ that $\overline{\pi}(f_j)|_{\mathcal{M}_\pi^I}$ is injective for some j . Hence by countable additivity it follows from (9.17) together with (9.9) and the definition of $\text{supp}^I(\mu)$ in (9.1) that $\mu(\text{supp}^I(\mu) \setminus \widehat{G}_X) = 0$. Further for $[\pi] \in \widehat{G}_X$ we have $H_\pi^{I,X-\text{inv}}(f) < \infty$ and from (9.5) we infer

$$(9.18) \quad \begin{aligned} H_\pi^{I,X-\text{inv}}(f) &= \sum_{j=1}^{m_\pi} \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} \|\overline{\pi}(f)\overline{\eta}_j^{I,\lambda}\|^2 + \\ &+ \sum_{j=1}^{m_\pi} 2 \operatorname{Re} \sum_{\substack{\lambda \neq \lambda' \in \mathcal{E}_\xi \\ (\lambda - \lambda')(X) \in 2\pi i\mathbb{Z}}} \langle \overline{\pi}(f)\overline{\eta}_j^{I,\lambda}, \overline{\pi}(f)\overline{\eta}_j^{I,\lambda'} \rangle. \end{aligned}$$

Next we define

$$\widehat{G}_{X,\text{reg}} := \{[\pi] \in \widehat{G}_X \mid (\forall \lambda \neq \lambda' \in (\rho - W_j \chi_\pi)|_{\mathfrak{a}_I}) : (\lambda - \lambda')(X) \notin 2\pi i\mathbb{Z}\}$$

and deduce from (9.17), (9.18), and (9.10) that

$$(9.19) \quad \|f\|_{L^2(Z_I)}^2 \geq \int_{\widehat{G}_{X,\text{reg}}} H_\pi^I(f) \, d\mu(\pi) + \int_{\widehat{G}_X \setminus \widehat{G}_{X,\text{reg}}} H_\pi^{I,X-\text{inv}}(f) \, d\mu(\pi).$$

Now we start iterating (9.19) with finitely many $X \in \mathfrak{a}_I^-$. In more precision, let $X_1 := X$ and set $X_2 := \sqrt{2}X_1$. Now the iteration of (9.19) starts with $a_t := \exp(tX_2)$ while observing $\|f\|_{L^2(Z_I)}^2 = \|f_t\|_{L^2(Z_I)}^2$ and taking weighted averages as before. Another application of Fatou’s Lemma then yields

$$\begin{aligned} \|f\|_{L^2(Z_I)}^2 &\geq \int_{\bigcup_{j=1}^2 \widehat{G}_{X_j,\text{reg}}} H_\pi^I(f) \, d\mu(\pi) + \\ &+ \int_{(\bigcap_{j=1}^2 \widehat{G}_{X_j}) \setminus (\bigcup_{j=1}^2 \widehat{G}_{X_j,\text{reg}})} H_\pi^{I,\{X_1, X_2\}-\text{inv}}(f) \, d\mu(\pi) \end{aligned}$$

with

$$\begin{aligned}
 \mathbf{H}_\pi^I, \{X_1, X_2\}^{-\text{inv}}(f) &= \sum_{j=1}^{m_\pi} \sum_{\lambda \in \rho + i\mathfrak{a}_I^*} \|\bar{\pi}(f)\bar{\eta}_j^{I, \lambda}\|^2 + \\
 (9.20) \qquad \qquad \qquad &+ \sum_{j=1}^{m_\pi} 2 \operatorname{Re} \sum_{\substack{\lambda, \lambda' \in \mathcal{E}_{\bar{\eta}_j^I} \\ \lambda \neq \lambda', (\lambda - \lambda')(X_1) = 0}} \langle \bar{\pi}(f)\bar{\eta}_j^{I, \lambda}, \bar{\pi}(f)\bar{\eta}_j^{I, \lambda'} \rangle
 \end{aligned}$$

as a result of making (9.18) also invariant under X_2 . Here we used that $(\lambda - \lambda')(X_i) \in 2\pi\mathbb{Z}$ for $i = 1, 2$ means $(\lambda - \lambda')(X_i) = 0$.

Next take $X_3 \in \mathfrak{a}_I^-$ linearly independent to X_1 and then $X_4 := \sqrt{2}X_3$. This we continue until $X_1, X_3, \dots, X_{2m-1}$ is a basis of \mathfrak{a}_I contained in \mathfrak{a}_I^- .

Notice that iterating (9.20) yields that

$$\mathbf{H}_\pi^I, \{X_1, \dots, X_{2m}\}^{-\text{inv}}(f) = \mathbf{H}_\pi^I(f)$$

and we finally arrive at

$$(9.21) \qquad \qquad \qquad \|f\|_{L^2(Z_I)}^2 \geq \int_{\widehat{G}} \mathbf{H}_\pi^I(f) \, d\mu(\pi)$$

together with the fact $\mu(\operatorname{supp}^I(\mu) \setminus \operatorname{supp}_{\text{fin}}^I(\mu)) = 0$ as $\operatorname{supp}_{\text{fin}}^I = \bigcap_j \widehat{G}_{X_j}$.

To conclude the proof we observe for $X = X_1$ and any $\pi \in \widehat{G}$ that $\|\bar{\pi}(f_t)\bar{\eta}^I\| \leq \sum_{\lambda \in \mathcal{E}_{\bar{\eta}^I}} \|\bar{\pi}(f)\bar{\eta}^{I, \lambda}\|$ and thus

$$\|\bar{\pi}(f_t)\bar{\eta}^I\|^2 \leq |\mathbf{W}_j| \sum_{\lambda \in \mathcal{E}_{\bar{\eta}^I}} \|\bar{\pi}(f)\bar{\eta}^{I, \lambda}\|^2$$

as $|\mathcal{E}_{\bar{\eta}^I}| \leq |\mathbf{W}_j|$. Summing over t and the $\bar{\eta}_j^I$ this implies via (9.18) for all $[\pi] \in \operatorname{supp}_{\text{fin}}^I \mu$ that

$$\frac{1}{n} \sum_{t=n+1}^{2n} \mathbf{H}_\pi^{I, \text{pre}}(f_t) \leq |\mathbf{W}_j| \mathbf{H}_\pi^I(f)$$

for all $n > 0$. Thus by (9.17) and dominated convergence we can interchange limit and integral in (9.16) and obtain actual equality in (9.17):

$$(9.22) \qquad \qquad \qquad \int_{\widehat{G}} \mathbf{H}_\pi^{I, X^{-\text{inv}}}(f) \, d\mu(\pi) = \|f\|_{L^2(Z_I)}^2.$$

The just described iteration applied to (9.22) then yields

$$\int_{\widehat{G}} \mathbf{H}_\pi^I(f) \, d\mu(\pi) = \|f\|_{L^2(Z_I)}^2$$

and finishes the proof of the theorem.

The final statements follow from uniqueness of the Plancherel measure together with (9.13). □

9.3. Extension to \widetilde{Z}_I . In view of Section 8.7 we can extend Theorem 9.5 to all $f \in C_c^\infty(\widetilde{Z}_I)$:

$$(9.23) \qquad \qquad \qquad \|f\|_{L^2(\widetilde{Z}_I)}^2 = \sum_{\mathfrak{c}, \mathfrak{t}} \int_{\operatorname{supp}^{I, \mathfrak{c}, \mathfrak{t}}(\mu)} \mathbf{H}_{\pi, \mathfrak{c}, \mathfrak{t}}^I(f_{\mathfrak{c}, \mathfrak{t}}) \, d\mu(\pi)$$

where we put an extra index \mathbf{c}, \mathbf{t} when we consider objects, initially defined for Z_I , now for $Z_{I,\mathbf{c},\mathbf{t}}$. Let us further denote by $\mathcal{M}_{\pi,\mathbf{c},\mathbf{t}}^I \subset (\mathcal{H}_{\pi}^{-\infty})^{H_{I,\mathbf{c}}}$ the Hilbert space \mathcal{M}_{π}^I (with the inner product obtained from (9.13)), but for Z_I replaced by $Z_{I,\mathbf{c},\mathbf{t}} = Z_{I,\mathbf{c}}$. We then form the direct sum of Hilbert spaces

$$\widetilde{\mathcal{M}}_{\pi}^I = \bigoplus_{\mathbf{c},\mathbf{t}} \mathcal{M}_{\pi,\mathbf{c},\mathbf{t}}^I,$$

and equip this space with the diagonal action of A_I , i.e. for $\xi = (\xi_{\mathbf{c},\mathbf{t}})_{\mathbf{c},\mathbf{t}} \in \widetilde{\mathcal{M}}_{\pi}^I$ we have $a \cdot \xi = (a \cdot \xi_{\mathbf{c},\mathbf{t}})_{\mathbf{c},\mathbf{t}}$. Then we obtain the following extension of (9.15) to

$$(9.24) \quad L^2(\widetilde{Z}_I) \underset{G \times A_I}{\simeq} \int_{\widetilde{G}} \mathcal{H}_{\pi} \otimes \widetilde{\mathcal{M}}_{\pi}^I \, d\mu(\pi)$$

9.4. The Maass-Selberg relations. The multiplicity space $\widetilde{\mathcal{M}}_{\pi}^I$ is A_I -semisimple for μ -almost all $[\pi]$ and thus admits a direct sum decomposition $\widetilde{\mathcal{M}}_{\pi}^I = \bigoplus_{\lambda \in \rho + i\mathfrak{a}_I^*} \widetilde{\mathcal{M}}_{\pi}^{I,\lambda}$ with

$$\widetilde{\mathcal{M}}_{\pi}^{I,\lambda} = \{ \xi \in \widetilde{\mathcal{M}}_{\pi}^I \mid (\forall a \in A_I) \, a \cdot \xi = a^{\lambda} \xi \}.$$

Since the normalized right action of A_I on $L^2(Z_I)$ is unitary it follows that the Hermitian structure on $\widetilde{\mathcal{M}}_{\pi}^I$ is such that this decomposition of $\widetilde{\mathcal{M}}_{\pi}^I$ is orthogonal for μ -almost all $[\pi]$.

Theorem 9.6 (Maass-Selberg relations). *Let $\lambda \in \rho|_{\mathfrak{a}_I} + i\mathfrak{a}_I^*$. Then for almost all $[\pi] \in \bigcup_{\mathbf{c},\mathbf{t}} \text{supp}_{\text{fin}}^{I,\mathbf{c},\mathbf{t}}(\mu)$ the map*

$$I^{\lambda} : \mathcal{M}_{\pi} \rightarrow \widetilde{\mathcal{M}}_{\pi}^{I,\lambda}, \quad \eta \mapsto (\eta_{\mathbf{c},\mathbf{t}}^{I,\lambda})_{\mathbf{c},\mathbf{t}}$$

is a surjective partial isometry, i.e. its Hermitian adjoint is a unitary isometry.

Proof. Let us denote by $\langle \cdot, \cdot \rangle$ the scalar product on $\widetilde{\mathcal{M}}_{\pi}^I$. By definition it is given by (9.13) (summed over all \mathbf{c}, \mathbf{t}) for almost all $[\pi]$. Now summation of (9.10) over all \mathbf{c}, \mathbf{t} implies for all $x \in \widetilde{\mathcal{M}}_{\pi}^I$ that

$$(9.25) \quad \|x\|_{\mathcal{M}_{\pi}^I}^2 = \sum_{\mathbf{c},\mathbf{t}} \sum_{j=1}^{m_{\pi}} \sum_{\lambda \in \mathcal{E}_{\eta_j^I}} | \langle x, (\eta_j)_{\mathbf{c},\mathbf{t}}^{I,\lambda} \rangle |^2.$$

In particular, for $x \in \widetilde{\mathcal{M}}_{\pi}^{I,\lambda}$ this is condition (9.26) so that Lemma 9.8 applies. \square

Remark 9.7. Of particular interest is the case of a multiplicity one space, i.e. where we have $\dim \mathcal{M}_{\pi} \leq 1$ for almost all $\pi \in \text{supp } \mu$. This is for instance satisfied in the group case $Z = G \times G / \text{diag } G \simeq G$, for complex symmetric spaces, and in the Riemannian situation $Z = G/K$.

For a symmetric space the condition that $\dim \mathcal{M}_{\pi} \leq 1$ for almost all π implies $\mathcal{W} = \{ \mathbf{1} \}$. To see that we first observe that there are $|\mathcal{W}|$ -many open H -orbits $\mathcal{O} \subset G/Q$, each isomorphic to H/L_H as a unimodular H -space. Integration over these open H -orbits yields at least $|\mathcal{W}|$ -many tempered functionals for representations π with generic parameters in the most-continuous spectrum of Z , say $\eta_{\pi,w}$ for $w \in \mathcal{W}$. Now there is a subtle point that a priori we only have $\mathcal{M}_{\pi} \subset (V_{\pi}^{-\infty})_{\text{temp}}^H$. But forming wave packets finally yields that these $\eta_{\pi,w}$ indeed contribute a.e. to the L^2 -spectrum. For this one needs an estimate of $\eta_{\pi,w}$ which is locally uniform

with respect to π . For the case of a symmetric space Z such an estimate is given in [2, Thm. 9.1]. The statement follows.

The statement above implies that $\mathcal{M}_\pi^I = \widetilde{\mathcal{M}}_\pi^I$. Our Maass-Selberg relations in Theorem 9.6 then assert for $\eta \in \mathcal{M}_\pi$ with $\|\eta\| = 1$ that $(\eta^{I,\lambda})_\lambda$ is an orthonormal basis of $\mathcal{M}_\pi^I = \widetilde{\mathcal{M}}_\pi^I$ (where we only count those λ for which $\mathcal{M}_\pi^{I,\lambda} \neq \{0\}$). In particular, for the group case this leads to the Maass-Selberg relations of Harish-Chandra [19, p. 146].

We finish this section with an elementary lemma about finite dimensional Hilbert spaces. It was used for Theorem 9.6.

Lemma 9.8. *Let $J : \mathcal{M} \rightarrow \mathcal{N}$ a linear map between two finite dimensional Hilbert spaces. Assume that for some orthonormal basis η_1, \dots, η_n for \mathcal{M} one has*

$$(9.26) \quad \langle x, x \rangle = \sum_{j=1}^n |\langle x, J\eta_j \rangle|^2, \quad (x \in \mathcal{N}).$$

Then the adjoint of J is an isometry.

Proof. It follows from (9.26) that $\|x\|^2 = \sum_{j=1}^n |\langle J^*x, \eta_j \rangle|^2 = \|J^*x\|^2$. □

10. SPECTRAL RADON TRANSFORMS AND TWISTED DISCRETE SPECTRUM

The constant term assignments

$$\mathcal{M}_\pi \ni \eta \mapsto \eta^I \in \mathcal{M}_\pi^I$$

give rise to spectral Radon transform $R_I : L^2(Z) \rightarrow L^2(Z_I)$ which is the topic of this section. With the help of this transform we can characterize the twisted discrete series $L^2(Z)_{\text{td}}$ of $L^2(Z)$ spectrally. The section starts with a brief recall on the twisted discrete series, see also [32] and [28, Sect. 9].

10.1. Twisted discrete series. Let us denote by $L^2(Z)_{\text{d}}$ the discrete spectrum of $L^2(Z)$, i.e. the direct sum of all irreducible subspaces. Now in case $\mathfrak{a}_{Z,E} \neq \{0\}$, it is easy to see that $L^2(Z)_{\text{d}} = \emptyset$, see [32, Lemma 3.3]. In particular, for $I \subsetneq S$ we have $L^2(G/H_I)_{\text{d}} = \emptyset$ as $\mathfrak{a}_{Z_I,E} = \mathfrak{a}_I \neq \{0\}$.

Recall that the subspace $\mathfrak{a}_{Z,E} = \mathfrak{a}_S \subset \mathfrak{a}_Z$ normalizes \mathfrak{h} and gives rise to the subalgebra $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathfrak{a}_{Z,E}$. Hence $A_{Z,E} := A_S \subset A$ normalizes H and acts unitarily on $L^2(G/H)$ via the normalized right regular action

$$(\mathcal{R}(a)f)(gH) = a^{-\rho} f(gaH) \quad (g \in G, a \in A_{Z,E}, f \in L^2(Z)).$$

Disintegration of $L^2(G/H)$ with respect to the right action of $A_{Z,E}$ then yields the unitary equivalence of G -modules

$$(10.1) \quad L^2(Z) = \int_{\widehat{A}_{Z,E}} L^2(G/\widehat{H}, \chi) \, d\chi,$$

where $\widehat{A}_{Z,E}$ denotes the unitary dual of the abelian Lie group $A_{Z,E}$, and for each unitary character $\chi : A_{Z,E} \rightarrow \mathbb{S}^1$ the G -module $L^2(G/\widehat{H}, \chi)$ is a certain Hilbert space of densities explained in [29, Sect. 8] or [32, Sect. 3.2]. A spherical pair (V, η) which embeds into some $L^2(G/\widehat{H}, \chi)$ will be referred to as a representation

of the *twisted discrete series* of Z . Further we denote by $L^2(G/\widehat{H}, \chi)_d$ the discrete spectrum and define the *twisted discrete series* by

$$(10.2) \quad L^2(Z)_{td} = \int_{\widehat{A}_{Z,E}} L^2(G/\widehat{H}, \chi)_d \, d\chi$$

made more rigorous in Subsection 10.3.

10.2. Spectral Radon transforms. For $w \in \mathcal{W}$ we set $Z_w = G/H_w$. Note that

$$L^2(Z) \rightarrow L^2(Z_w), \quad f \mapsto (gH_w \rightarrow f(gwH))$$

is a unitary equivalence of G -representations. Hence the abstract Plancherel formula for $L^2(Z)$ induces one for $L^2(Z_w)$ with the same Plancherel measure and isometries

$$\mathcal{M}_\pi \rightarrow \mathcal{M}_{\pi,w}, \quad \eta \mapsto \eta_w.$$

For every $I \subset S$ and $w \in \mathcal{W}$ we set $Z_{I,w} := G/(H_w)_I$ and keep in mind that for fixed I , the various $(H_w)_I$ need not be G -conjugate (cf. Example 4.10).

Now given $\eta \in \mathcal{M}_\pi$ and $w \in \mathcal{W}$ we note that $\eta_w^I = (w \cdot \eta)^I$ is fixed by $(H_w)_I$ and we use notation $\mathcal{M}_{\pi,w}^I$ for \mathcal{M}_π^I with respect to $(H_w)_I$. In the sequel we assume that $[\pi] \in \text{supp } \mu \subset \widehat{G}$ is *generic*, that is $\mathcal{M}_{\pi,w}^I$ is \mathfrak{a}_I -semisimple for all $I \subset S$ and $w \in \mathcal{W}$. By Theorem 9.5 with H replaced by H_w we obtain that the complement of the generic elements is a null set with respect to μ . We endow $\mathcal{M}_{\pi,w}^I$ with the Hilbert space structure induced from \mathcal{M}_π via Theorem 9.5.

Our concern is with the *spectral Radon transforms* induced from the constant term maps:

$$r_{\pi,I,w} : \mathcal{M}_\pi \rightarrow \mathcal{M}_{\pi,w}^I, \quad r_{\pi,I,w}(\eta) = \eta_w^I,$$

and for $J \subset I$ their transitions:

$$(10.3) \quad r_{\pi,J,w}^I : \mathcal{M}_{\pi,w}^I \rightarrow \mathcal{M}_{\pi,w}^J, \quad r_{\pi,J,w}^I(\xi) = \xi^J.$$

We recall the transitivity of the constant terms [10, Prop. 6.1]:

Lemma 10.1. *Let $\eta \in \mathcal{M}_\pi$ and $w \in \mathcal{W}$. Then for all $J \subset I$ one has*

$$(\eta_w^I)^J = \eta_w^J.$$

The transitivity of the constant term maps then reflects in

$$(10.4) \quad r_{\pi,J,w}^I \circ r_{\pi,I,w} = r_{\pi,J,w} \quad (J \subset I).$$

Recall that $r_{\pi,I,w}$ is a sum of at most $|W_j|$ -many partial isometries by the Maass-Selberg relations in Theorem 9.6. Hence we obtain

$$(10.5) \quad \|r_{\pi,I,w}\| \leq |W_j|.$$

Proposition 10.2. *Let $I \subset S$ and $w \in \mathcal{W}$. The operator field*

$$(\text{id}_{\mathcal{H}_\pi} \otimes r_{\pi,I,w})_{\pi \in \widehat{G}} : \mathcal{H}_\pi \otimes \mathcal{M}_\pi \rightarrow \mathcal{H}_\pi \otimes \mathcal{M}_{\pi,w}^I$$

is measurable and induces a G -equivariant continuous map

$$R_{I,w} : L^2(Z) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{M}_\pi \, d\mu(\pi) \rightarrow L^2(Z_{I,w}) \simeq \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{M}_{\pi,w}^I \, d\mu(\pi)$$

Moreover

$$(10.6) \quad \|R_{I,w}\| \leq |W_j|.$$

We call $R_{I,w}$ the spectral Radon transform at (I, w) .

Proof. Since the $r_{\pi,I,w}$ reflect the pointwise convergent asymptotics of matrix coefficients, the operator field is measurable. With the upper bound in (10.5) we then obtain that $R_{I,w}$ is defined and continuous with norm bound (10.6). By definition $R_{I,w}$ is then G -equivariant, completing the proof. \square

With (10.3) we obtain spectrally defined Radon transforms:

$$(10.7) \quad R_{J,w}^I : L^2(Z_{I,w}) \rightarrow L^2(Z_{J,w}) \quad (J \subset I)$$

which then by (10.4) satisfy

$$(10.8) \quad R_{J,w} = R_{J,w}^I \circ R_{I,w} \quad (J \subset I)$$

Putting the data of the various (I, w) together, we arrive at the (full) spectral Radon transform

$$R = \oplus_{I,w} R_{I,w} : L^2(Z) \rightarrow \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} L^2(Z_{I,w}).$$

10.3. Characterization of the twisted discrete spectrum. Next we want to define $L^2(Z)_{\text{td}}$ rigorously in terms of the spectral Radon transforms. Set

$$(10.9) \quad \mathcal{M}_{\pi,\text{td}} = \{ \xi \in \mathcal{M}_{\pi} \mid \exists \chi \in \widehat{A}_{Z,E} \forall v \in V_{\pi}^{\infty} : m_{v,\xi} \in L^2(\widehat{Z}, \chi)_{\text{d}} \}$$

and likewise we define $\mathcal{M}_{\pi,w,\text{td}}^I$ for $w \in \mathcal{W}$ and $I \subset S$.

Then

$$(10.10) \quad L^2(Z)_{\text{td}} := \bigcap_{\substack{w \in \mathcal{W} \\ I \subset S}} \ker R_{I,w}.$$

defines a closed subspace G -invariant subspace of $L^2(Z)$.

Next we need a reformulation of the characterization of the twisted discrete series from [29, Sect. 8] in the more suitable language of constant terms [10, Th. 5.12], namely:

Lemma 10.3. *Let $\eta \in \mathcal{M}_{\pi}$. Then the following are equivalent:*

- (1) $\eta \in \mathcal{M}_{\pi,\text{td}}$.
- (2) $\eta_w^I = 0$ for all $w \in \mathcal{W}$ and $I \subsetneq S$.

With the characterization in Lemma 10.3 we arrive at:

Proposition 10.4. *We have*

$$(10.11) \quad L^2(Z)_{\text{td}} \simeq \int_{\widehat{G}}^{\oplus} \mathcal{H}_{\pi} \otimes \mathcal{M}_{\pi,\text{td}} \, d\mu(\pi).$$

In particular $L^2(Z)_{\text{td}} \subset L^2(Z)$ is invariant under the normalized right regular representation \mathcal{R} of $A_{Z,E}$.

Proof. Both assertions follow from Lemma 10.3 and the involved definitions (10.9) and (10.10) \square

Since $L^2(Z)_{\text{td}}$ is $A_{Z,E}$ -invariant we obtain from (10.11) a rigorous definition of (10.2) with $L^2(\widehat{Z}, \chi)_{\text{d}}$ equal to the χ -spectral part of $L^2(Z)_{\text{td}}$ under \mathcal{R} .

10.4. Restriction to the twisted discrete spectrum. Applying the preceding theory with $L^2(Z)$ replaced by $L^2(Z_{I,w})$ we obtain orthogonal projections

$$\text{pr}_{I,w,\text{td}} : L^2(Z_{I,w}) \rightarrow L^2(Z_{I,w})_{\text{td}}$$

and define $R_{I,w} := \text{pr}_{I,w,\text{td}} \circ R_{I,w}$. Note that

$$R_{I,w} : L^2(Z) \rightarrow L^2(Z_{I,w})$$

is a continuous G -equivariant map. The *restricted spectral Radon transform* is then defined to be

$$R = \bigoplus_{I,S} R_{I,w} : L^2(Z) \rightarrow \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} L^2(Z_{I,w})_{\text{td}}.$$

11. BERNSTEIN MORPHISMS

We define the *Bernstein morphism* B as the Hilbert space adjoint R^* of the restricted spectral Radon transform R . With $B_{I,w} := R_{I,w}^*$ we then have

$$B : \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} L^2(Z_{I,w})_{\text{td}} \rightarrow L^2(Z), \quad (f_{I,w})_{I,w} \mapsto \sum_{I,w} B_{I,w}(f_{I,w}).$$

The main result of this section then is:

Theorem 11.1 (Plancherel Theorem – Bernstein decomposition). *The Bernstein morphism is a continuous surjective G -equivariant linear map. Moreover, B is isospectral, that is, image and source have Plancherel measure in the same measure class.*

After some technical preparations we give the proof of Theorem 11.1. Then, after applying the material on open P -orbits developed in Section 5 we derive in Theorem 11.9 a refined Bernstein decomposition, which agrees with the partition $\mathcal{W} = \coprod_{c \in C_I} \coprod_{t \in F_{I,c}} \mathbf{m}_{c,t}(\mathcal{W}_{I,c})$ from (5.18).

Finally, by adding up the refined Bernstein decompositions for the various G -orbits in $\underline{Z}(\mathbb{R})$ we obtain in Theorem 11.11 the statement for $L^2(\underline{Z}(\mathbb{R}))$ which is in full analogy to the p -adic statement of Sakellaridis-Venkatesh [42, Cor. 11.6.2].

11.1. Proof of Theorem 11.1. Denote by $\mathcal{P}(S)$ the power set of S . With regard to $\eta \in \mathcal{M}_\pi$ we call a pair $(I, w) \in \mathcal{P}(S) \times \mathcal{W}$ admissible provided that $\eta_w^I \neq 0$. Finally we call an η -admissible pair (I, w) *optimal* provided that the cardinality $|I|$ is minimal, i.e. we have $\eta_w^J = 0$ for all $w' \in \mathcal{W}$ and $J \subsetneq I$. Notice that, by definition, for every $\eta \neq 0$ there exists an η -optimal pair (I, w) .

The embedding theory of tempered representations into twisted discrete series from [29, Sect. 9] then comes down to:

Theorem 11.2. *Let $0 \neq \eta \in \mathcal{M}_\pi$ and (I, w) be an η -optimal pair. Then $\eta_w^I \in \mathcal{M}_{\pi,w,\text{td}}^I$.*

Proof. Let (I, w) be η -optimal. Applying a base point shift we may assume that $w = \mathbf{1}$. According to Lemma 10.3 applied to Z_I we need to show that $(w_I \cdot \eta^I)^J = 0$ for all $w_I \in \mathcal{W}_I$ and $J \subsetneq I$. Let $\mathbf{m}(w_I) = w \in \mathcal{W}$. By the consistency relations (7.4) we have $w_I \cdot \eta_I = \eta_w^I$. Thus, by the transitivity of the constant term we have

$$(w_I \cdot \eta^I)^J = \eta_w^J = 0$$

by the minimality of $|I|$. The theorem follows. □

Let us denote for each $[\pi] \in \widehat{G}$ and each $I \subset S$, $w \in \mathcal{W}$ by $\xi \mapsto \xi_{\text{td}}$ the orthogonal projection $\mathcal{M}_{\pi,w}^I \rightarrow \mathcal{M}_{\pi,w,\text{td}}^I$.

With that we define a linear map between finite dimensional Hilbert spaces by

$$r_\pi = \bigoplus_{I \subset S} r_{\pi,I,w,\text{td}} : \mathcal{M}_\pi \rightarrow \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} \mathcal{M}_{\pi,w,\text{td}}^I, \quad \eta \mapsto (\eta_{w,\text{td}}^I)_{I,w}$$

with $\eta_{w,\text{td}}^I := (\eta_w^I)_{\text{td}}$.

Remark 11.3. Since $\xi \mapsto \xi_{\text{td}}$ is A_I -equivariant, we have the orthogonal decomposition $\mathcal{M}_{\pi,w,\text{td}}^I = \bigoplus_{\mu \in \rho + ia_I^*} \mathcal{M}_{\pi,w,\text{td}}^{I,\mu}$. Thus every $\xi \in \mathcal{M}_{\pi,w,\text{td}}^I$ decomposes as $\xi = \sum \xi^\mu$ with $\xi^\mu \in \mathcal{M}_{\pi,w,\text{td}}^{I,\mu}$ for $\mu \in \rho|_{\mathfrak{a}_I} + ia_I^*$ by (7.3).

For any $\lambda \in \mathcal{E}_{\eta^I} \subset \rho|_{\mathfrak{a}_I} + ia_I^*$ (cf. (7.3)) we denote by $r_{\pi,I,w,\text{td},\lambda}$ the map $r_{\pi,I,w,\text{td}}$ followed by orthogonal projection to the λ -coordinate $\mathcal{M}_{\pi,w,\text{td}}^{I,\lambda}$ of $\mathcal{M}_{\pi,w,\text{td}}^I$.

Then Theorem 11.2 yields the technical key Lemma:

Lemma 11.4. *The following assertions hold:*

- (1) r_π is injective.
- (2) For all $I \subset S, w \in \mathcal{W}$ and $\lambda \in \mathcal{E}_\pi$ the map

$$r_{\pi,I,w,\text{td},\lambda} : \mathcal{M}_\pi \rightarrow \mathcal{M}_{\pi,w,\text{td}}^{I,\lambda}, \quad \eta \mapsto \eta_{w,\text{td}}^{I,\lambda}$$

is a surjective partial isometry.

- (3) The assignment $\pi \mapsto r_\pi$ is measurable.

Proof. Let $0 \neq \eta \in \mathcal{M}_\pi$. According to Theorem 11.2 we find an η -optimal pair (I, w) such that $\eta_{w,\text{td}}^I \neq 0$, establishing (1). Having shown (1), assertion (2) is obtained from the Maass-Selberg relations in Theorem 9.6: we replace H by H_w and observe that $\eta \mapsto \eta_w$ establishes an isomorphism of $\mathcal{M}_\pi \rightarrow \mathcal{M}_{\pi,w}$ with $\mathcal{M}_{\pi,w}$ referring to \mathcal{M}_π with H replaced by H_w .

Finally (3) is by the definition of the measurable structures involved (see Section 6 and Proposition 10.2): The family of maps

$$r_{\pi,I,w} : \mathcal{M}_\pi \rightarrow \mathcal{M}_{\pi,w}^I, \quad \eta \mapsto \eta_w^I,$$

as well as the projection to discrete parts $r_{\pi,I,w,\text{td}}$ are measurable. □

We now define

$$b_\pi : \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} \mathcal{M}_{\pi,w,\text{td}}^I \rightarrow \mathcal{M}_\pi$$

to be the adjoint of r_π and note that b_π , being the adjoint of an injective morphism, is surjective. Notice that the Bernstein morphism

$$B : \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}} L^2(Z_{I,w})_{\text{td}} \rightarrow L^2(Z)$$

is defined spectrally by the operator field $(b_\pi)_{\pi \in \text{supp } \mu}$.

Remark 11.5 (Decomposition of B into isometries). For $I \subset S$ and $w \in \mathcal{W}$ we denote by $B_{I,w}$ the restriction of B to $L^2(Z_{I,w})_{\text{td}}$.

We claim that there is an orthogonal decomposition

$$L^2(Z_{I,w})_{\text{td}} = \bigoplus_{u \in \mathcal{W}_j} L^2(Z_{I,w})_{\text{td},u}$$

such that every restriction $B_{I,w,u} := B|_{L^2(Z_{I,w})_{\text{td},u}}$ is an isometry. To construct such a decomposition we choose for every $[\pi] \in \text{supp}(\mu)$ with infinitesimal character $\chi_\pi \in \mathfrak{j}_{\mathbb{C}}^*/\mathcal{W}_{\mathfrak{j}}$ a representative $\lambda_\pi \in \mathfrak{j}_{\mathbb{C}}$, i.e. $\chi_\pi = \mathcal{W}_{\mathfrak{j}} \cdot \lambda_\pi$. Let us denote by

$$P_u([\pi]) : \mathcal{M}_{\pi,w}^I \rightarrow \mathcal{M}_{\pi,w}^{I,(\rho-u \cdot \lambda_\pi)|_{\mathfrak{a}_I}}$$

the orthogonal projection. Our request for the choice $\lambda_\pi \in \chi_\pi$ is then such that the operator field

$$\text{supp}(\mu) \ni [\pi] \mapsto P_u([\pi]) \in \text{End}(\mathcal{M}_{\pi,w}^I)$$

is measurable. With

$$L^2(Z_{I,w})_{\text{td},u} := \int_{\widehat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{M}_{\pi,w}^{I,(\rho-u \cdot \lambda_\pi)|_{\mathfrak{a}_I}} d\mu(\pi)$$

we then obtain an orthogonal decomposition $L^2(Z_{I,w})_{\text{td}} = \bigoplus_{u \in \mathcal{W}_{\mathfrak{j}}} L^2(Z_{I,w})_{\text{td},u}$ for which $B_{I,w,u}$ is an isometry by Lemma 11.4(2).

The final piece of information we need for the proof of Theorem 11.1 is the following elementary result of functional analysis whose proof we omit.

Lemma 11.6. *Let $\mathcal{H} = \int_X^{\oplus} \mathcal{H}_x d\mu(x)$ be a direct integral of Hilbert spaces. Let further $\mathcal{K} = \int_X^{\oplus} \mathcal{K}_x d\mu(x)$ and $\mathcal{L} = \int_X^{\oplus} \mathcal{L}_x d\mu(x)$ be closed decomposable subspaces of \mathcal{H} . Suppose that $\mathcal{K}_x + \mathcal{L}_x \subset \mathcal{H}_x$ is closed for every $x \in X$. Then $\mathcal{K} + \mathcal{L} \subset \mathcal{H}$ is closed.*

Proof of Theorem 11.1. The surjectivity of the b_π together with Theorem 9.5 shows that B is an isospectral G -morphism with dense image. To see that B is surjective we note that B is a sum of isometries each one of which has closed range. Thus B is surjective by Lemma 11.6. □

Remark 11.7. In case $\mathcal{W} = \{\mathbf{1}\}$, i.e. there is only one open P -orbit, the Bernstein decomposition becomes a lot simpler as the summation over \mathcal{W} disappears in the domain of B . We recall that $\mathcal{W} = \{\mathbf{1}\}$ is satisfied for reductive groups $G \simeq G \times G/G$, for complex spherical spaces, and for Riemannian symmetric spaces.

11.2. Refinement of the Bernstein morphisms. In the definition of the Bernstein morphism a certain over-parametrizing takes place in the domain. This will now be remedied via the partition $\mathcal{W} = \coprod_{\mathfrak{c} \in \mathfrak{C}_I} \coprod_{\mathfrak{t} \in \mathfrak{F}_{I,\mathfrak{c}}} \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(\mathcal{W}_{I,\mathfrak{c}})$ from (5.18). We recall the corresponding terminology from Subsection 7.3.

For $\eta \in \mathcal{M}_\pi$, $\mathfrak{c} \in \mathfrak{C}_I$, $\mathfrak{t} \in \mathfrak{F}_{I,\mathfrak{c}}$ we recall the functional $\eta_{\mathfrak{c},\mathfrak{t}} = w(\mathfrak{c},\mathfrak{t}) \cdot \eta$ from Subsection 7.3. Further we set $\eta_{\mathfrak{c},\mathfrak{t}}^I := (\eta_{\mathfrak{c},\mathfrak{t}})^I$ and given $w_{I,\mathfrak{c}} \in \mathcal{W}_{I,\mathfrak{c}}$ we define the functional $(\eta_{\mathfrak{c},\mathfrak{t}}^I)_{w_{I,\mathfrak{c}}} := w_{I,\mathfrak{c}} \cdot \eta_{\mathfrak{c},\mathfrak{t}}^I$. Likewise for $\mu \in \mathfrak{a}_{I,\mathfrak{C}}^*$ we set $(\eta_{\mathfrak{c},\mathfrak{t}}^{I,\mu})_{w_{I,\mathfrak{c}}} := w_{I,\mathfrak{c}} \cdot \eta_{\mathfrak{c},\mathfrak{t}}^{I,\mu}$.

Every $w \in \mathcal{W}$ can be written uniquely as $w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}})$ for $\mathfrak{c} \in \mathfrak{C}_I$, $\mathfrak{t} \in \mathfrak{F}_{I,\mathfrak{c}}$ and $w_{I,\mathfrak{c}} \in \mathcal{W}_{I,\mathfrak{c}}$. In this context we recall from (7.10) the consistency relation

$$(11.1) \quad \eta_w^{I,\lambda} = (\eta_{\mathfrak{c},\mathfrak{t}}^{I,\lambda})_{w_{I,\mathfrak{c}}} \quad (w = \mathfrak{m}_{\mathfrak{c},\mathfrak{t}}(w_{I,\mathfrak{c}}) \in \mathcal{W}, \lambda \in \mathcal{E}_{\eta_I}).$$

Recall $H_{I,\mathfrak{c}} = (H_{w(\mathfrak{c})})_I$ equals $H_{I,\mathfrak{c},\mathfrak{t}} = (H_{w(\mathfrak{c},\mathfrak{t})})_I$. Hence $(\eta_{\mathfrak{c},\mathfrak{t}}^{I,\lambda})_{w_{I,\mathfrak{c}}}$ is fixed by $(H_{I,\mathfrak{c}})_{w_{I,\mathfrak{c}}}$. On the other hand $\eta_w^{I,\lambda}$ is fixed under $(H_w)_I$. We recall from (5.19) that the two groups are in fact equal:

$$(H_w)_I = (H_{I,\mathfrak{c}})_{w_{I,\mathfrak{c}}}.$$

In this context it is worth to record the following extension of [10, Th. 5.12]:

Proposition 11.8. *The following assertions are equivalent for $\eta \in \mathcal{M}_\pi$:*

- (1) $\eta \in \mathcal{M}_{\pi, \text{td}}$.
- (2) For all $I \subsetneq S$ and $\mathfrak{c} \in C_I, \mathfrak{t} \in F_{I, \mathfrak{c}}$ one has $\eta_{\mathfrak{c}, \mathfrak{t}}^I = 0$.

Proof. Let $w \in \mathcal{W}$ and write it as $w = \mathfrak{m}_{\mathfrak{c}, \mathfrak{t}}(w_{I, \mathfrak{c}})$. We recall (7.4) which asserts that $\eta_w^I = w_{I, \mathfrak{c}} \cdot \eta_{\mathfrak{c}, \mathfrak{t}}^I$. In particular $\eta_w^I = 0$ if and only if $\eta_{\mathfrak{c}, \mathfrak{t}}^I = 0$ and the proposition follows from Lemma 10.3. □

For $\mathfrak{c} \in C_I$ and $\mathfrak{t} \in F_{I, \mathfrak{c}}$ we set $Z_{I, \mathfrak{c}, \mathfrak{t}} = Z_{I, w(\mathfrak{c}, \mathfrak{t})}$ and note that $Z_{I, \mathfrak{c}, \mathfrak{t}} = G/H_{I, \mathfrak{c}}$ is independent of $\mathfrak{t} \in F_{I, \mathfrak{c}}$ by Lemma 5.16. The following is then a refined version of the Bernstein decomposition, taking the fine partition (5.18) of \mathcal{W} into account.

Theorem 11.9 (Plancherel Theorem – Bernstein decomposition refined). *The restricted Bernstein morphism*

$$B_{\text{res}} : \bigoplus_{I \subset S} \bigoplus_{\mathfrak{c} \in C_I} \bigoplus_{\mathfrak{t} \in F_{I, \mathfrak{c}}} L^2(Z_{I, \mathfrak{c}, \mathfrak{t}})_{\text{td}} \rightarrow L^2(Z)$$

is surjective.

Proof. Given the proof of Theorem 11.1 this comes down to the fact that the map

$$\tilde{r}_\pi : \mathcal{M}_\pi \rightarrow \bigoplus_{I \subset S} \bigoplus_{\mathfrak{c} \in C_I} \bigoplus_{\mathfrak{t} \in F_{I, \mathfrak{c}}} \mathcal{M}_{\pi, w(\mathfrak{c}, \mathfrak{t}), \text{td}}^I, \quad \eta \mapsto (\eta_{\mathfrak{c}, \mathfrak{t}}^I)_{I, \mathfrak{c}, \mathfrak{t}}$$

obtained from r_π by restricting the target remains injective. Now we recall the proof of Lemma 11.4(1) and let $0 \neq \eta \in \mathcal{M}_\pi$ with $\eta_{w, \text{td}}^I \neq 0$ for an η -optimal pair (I, w) . In particular, $\eta_{w, \text{td}}^I \neq 0$. Let $w = \mathfrak{m}_{\mathfrak{c}, \mathfrak{t}}(w_{I, \mathfrak{c}})$ for $w_{I, \mathfrak{c}} \in \mathcal{W}_{I, \mathfrak{c}}$ and $\mathfrak{t} \in F_{I, \mathfrak{c}}$. Then the consistency relation (11.1) yields $\eta_{w, \text{td}}^I = (w_{I, \mathfrak{c}} \cdot \eta_{\mathfrak{c}, \mathfrak{t}}^I)_{\text{td}}$ and thus $\eta_{\mathfrak{c}, \mathfrak{t}}^I \neq 0$, establishing the injectivity of \tilde{r}_π . The theorem follows. □

11.3. Bernstein decomposition for $L^2(\underline{Z}(\mathbb{R}))$. Recall that $Z = G/H$ is only one G -orbit of $\underline{Z}(\mathbb{R})$. To obtain the Bernstein decomposition of $L^2(\underline{Z}(\mathbb{R}))$ we just need to add the data of the various G -orbits in $\underline{Z}(\mathbb{R})$. We recall $W_{\mathbb{R}} = (P \backslash \underline{Z}(\mathbb{R}))_{\text{open}} \simeq F_{\mathbb{R}}/F_M$ and choose representatives $\mathcal{W}_{\mathbb{R}} \subset G$ for $W_{\mathbb{R}}$ as we did with \mathcal{W} for W . For $w \in \mathcal{W}_{\mathbb{R}}$ we set $Z_{I, w} := G/(H_w)_I$ with $(H_w)_I$ the real points of the \mathbb{R} -algebraic group $(\underline{H}_w)_I$. Notice that the G -orbit decomposition of $\underline{Z}(\mathbb{R})$ yields a natural partition of $W_{\mathbb{R}}$ by selecting for a given G -orbit in $\underline{Z}(\mathbb{R})$ the open P -orbits it contains. Summing up the Bernstein morphism of all G -orbits then yields a G -morphism:

$$B_{\mathbb{R}} : \bigoplus_{I \subset S} \bigoplus_{w \in \mathcal{W}_{\mathbb{R}}} L^2(Z_{I, w})_{\text{td}} \rightarrow L^2(\underline{Z}(\mathbb{R})).$$

We then obtain from Theorem 11.1:

Theorem 11.10 (Plancherel Theorem for $L^2(Z(\mathbb{R}))$ – Bernstein decomposition). *The Bernstein morphism $B_{\mathbb{R}}$ is a continuous surjective isospectral G -equivariant linear map.*

Recall from the beginning of Section 5 that $W_{I, \mathbb{R}} = (P \backslash \underline{Z}_I(\mathbb{R}))_{\text{open}}$ and $W_{\mathbb{R}}$ are canonically isomorphic. In particular we obtain a generalization of (5.18) to

$$\mathcal{W}_{\mathbb{R}} = \mathcal{W}_{I, \mathbb{R}} = \prod_{\mathfrak{c} \in C_{I, \mathbb{R}}} \prod_{\mathfrak{t} \in F_{I, \mathfrak{c}}} \mathfrak{m}_{\mathfrak{c}, \mathfrak{t}}(\mathcal{W}_{I, \mathfrak{c}})$$

with $C_{I, \mathbb{R}} := \{G \cdot \hat{z}_{w, I} \mid w \in W_{\mathbb{R}}\}$ etc.

The finer results in Theorem 11.9 then yield the refined restricted Bernstein morphism

$$(11.2) \quad B_{\mathbb{R},\text{res}} : \bigoplus_{I \subset S} L^2(\mathbb{Z}_I(\mathbb{R}))_{\text{td}} \rightarrow L^2(\mathbb{Z}(\mathbb{R}))$$

with the same properties as in Theorem 11.9:

Theorem 11.11 (Plancherel Theorem for $L^2(Z(\mathbb{R}))$ – Bernstein decomposition refined). *The restriction $B_{\mathbb{R},\text{res}}$ of the Bernstein morphism $B_{\mathbb{R}}$ is a continuous surjective isospectral G -equivariant linear map.*

12. ELLIPTIC ELEMENTS AND DISCRETE SERIES

As a consequence of the Bernstein decomposition in Theorem 11.1 we obtain in Theorem 12.1 a general criterion for the existence of a discrete spectrum in $L^2(G/H)$ for a unimodular real spherical space G/H . The main additional tool is a theorem of [20], by which the wave front set of the left regular representation of a unimodular homogeneous space G/H is determined as the closure of $\text{Ad}(G)\mathfrak{h}^\perp$.

12.1. Existence of discrete spectrum. As usual, we call an element $X \in \mathfrak{g}$ semisimple provided $\text{ad } X$ is a semisimple operator. Equivalently, $X \in \mathfrak{g}$ is semisimple if and only if its centralizer $\mathfrak{z}_{\mathfrak{g}}(X)$ is a reductive subalgebra.

An element $X \in \mathfrak{g}_{\mathbb{C}}$ is called *elliptic* if $\text{ad } X$ is semisimple with purely imaginary eigenvalues. If $E \subset \mathfrak{g}_{\mathbb{C}}$ we denote by E_{ell} the subset of E consisting of elliptic elements. More generally we call an element $X \in \mathfrak{g}_{\mathbb{C}}$ *weakly elliptic* if $\text{spec}(\text{ad } X) \subset i\mathbb{R}$ and denote by $E_{w\text{-ell}}$ the corresponding subset of $E \subset \mathfrak{g}_{\mathbb{C}}$.

Theorem 12.1. *Let $Z = G/H$ be a unimodular real spherical space. Suppose that $\text{int } \mathfrak{h}_{w\text{-ell}}^\perp \neq \emptyset$. Then $H = \widehat{H}$ is reductive and $L^2(Z)_{\text{d}} \neq \{0\}$.*

Here $\text{int } \mathfrak{h}_{w\text{-ell}}^\perp$ refers to the interior of $\mathfrak{h}_{w\text{-ell}}^\perp$, in the vector space topology of \mathfrak{h}^\perp . The proof is given in the course of the next two subsections.

Remark 12.2. In case $Z = G$ is a reductive group or more generally $Z = G/H$ is a symmetric space, then Theorem 12.1 comes down to the existence theorems of Harish-Chandra [16] and Flensted-Jensen [12] about discrete series. It is due to Harish-Chandra that $L^2(G)_{\text{d}} \neq \emptyset$ if \mathfrak{g} admits a compact Cartan subalgebra. Flensted-Jensen generalized that to symmetric spaces by showing $L^2(G/H)_{\text{d}} \neq \emptyset$ if there exists a compact abelian subspace $\mathfrak{t} \subset \mathfrak{g} \cap \mathfrak{h}^\perp$ with $\dim \mathfrak{t} = \text{rank } \underline{G}/\underline{H}$.

Remark 12.3. For the twisted discrete series an appropriate generalization of Theorem 12.1 reads

$$(12.1) \quad \text{int } \widehat{\mathfrak{h}}_{w\text{-ell}}^\perp \neq \emptyset \quad \Rightarrow \quad (\forall \chi \in \widehat{A}_{Z,E}) \quad L^2(G/\widehat{H}, \chi)_{\text{d}} \neq \emptyset$$

and will presumably follow from results on wavefront sets of induced representations more general than what is obtained in [20].

12.2. The geometry of elliptic elements. To prepare the way for the proof of Theorem 12.1 we establish some foundational material on elliptic elements in \mathfrak{h}^\perp , and show that if the weakly elliptic elements in \mathfrak{h}^\perp have non-empty interior, then \mathfrak{h} is reductive in \mathfrak{g} .

We consider the H -module $\mathfrak{h}^\perp \subset \mathfrak{g}$ and recall the canonical isomorphism $(\mathfrak{g}/\mathfrak{h})^* \simeq \mathfrak{h}^\perp$. In the sequel we view $\mathfrak{a}_Z \simeq \mathfrak{a}_H^\perp$ as a subspace of \mathfrak{a} and likewise we view $\mathfrak{m}_Z = \mathfrak{m}/\mathfrak{m}_H \simeq \mathfrak{m}_H^\perp$ as a subspace of \mathfrak{m} .

Lemma 12.4. $(\mathfrak{l} \cap \mathfrak{h})^\perp = \mathfrak{h}^\perp \oplus \mathfrak{u}$.

Proof. Clearly $\mathfrak{h}^\perp + \mathfrak{u} \subset (\mathfrak{l} \cap \mathfrak{h})^\perp$. Moreover $\mathfrak{h}^\perp \cap \mathfrak{u} = \{0\}$ because $\kappa(\mathfrak{u}, \mathfrak{q}) = \{0\}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$. The lemma now follows from $\dim \mathfrak{h} = \dim(\mathfrak{l} \cap \mathfrak{h}) + \dim \mathfrak{u}$. \square

Let $T_0 : (\mathfrak{l} \cap \mathfrak{h})^\perp \rightarrow \mathfrak{u}$ be minus the projection along \mathfrak{h}^\perp . It follows that

$$(12.2) \quad (\mathfrak{a}_Z + \mathfrak{m}_Z)^0 := \{X + T_0(X) : X \in \mathfrak{a}_Z + \mathfrak{m}_Z\} \subset \mathfrak{h}^\perp.$$

Similarly we set $\mathfrak{b}^0 := \{X + T_0(X) : X \in \mathfrak{b}\}$ for $\mathfrak{b} \subset \mathfrak{a}_Z + \mathfrak{m}_Z$ a subspace.

Lemma 12.5 is motivated by [24, Th. 5.4 and Cor. 7.2] and [40, Th. 5 and Th. 6].

Lemma 12.5. *Let $Z = G/H$ be a real spherical space for which there exists an $X_0 \in \mathfrak{a}_Z \cap \mathfrak{h}^\perp$ such that $\alpha(X_0) < 0$ for all $\alpha \in \Sigma_{\mathfrak{u}}$. Then the canonical map*

$$\Phi : H \times (\mathfrak{a}_Z + \mathfrak{m}_Z)^0 \rightarrow \mathfrak{h}^\perp, \quad (h, X) \mapsto \text{Ad}(h)X$$

is generically submersive.

Proof. We first note that $(\mathfrak{a}_Z + \mathfrak{m}_Z)^0 + [\mathfrak{h}, X] \subset \mathfrak{h}^\perp$ for all $X \in (\mathfrak{a}_Z + \mathfrak{m}_Z)^0$, and that Φ is generically submersive if and only if there is equality for some $X \in (\mathfrak{a}_Z + \mathfrak{m}_Z)^0$. We will show that

$$(12.3) \quad (\mathfrak{a}_Z + \mathfrak{m}_Z)^0 + [\mathfrak{h}, X_0] = \mathfrak{h}^\perp.$$

For $t > 0$ we set $a_t := \exp(tX_0)$. By conjugation (12.3) is then equivalent to

$$(12.4) \quad (\mathfrak{a}_Z + \mathfrak{m}_Z)_t^0 + [\mathfrak{h}_t, X_0] = \mathfrak{h}_t^\perp,$$

where $(\mathfrak{a}_Z + \mathfrak{m}_Z)_t^0 := \text{Ad}(a_t)(\mathfrak{a}_Z + \mathfrak{m}_Z)^0$ and $\mathfrak{h}_t := \text{Ad}(a_t)\mathfrak{h}$. Now note that by (12.2) we have for $t \rightarrow \infty$ that $(\mathfrak{a}_Z + \mathfrak{m}_Z)_t^0 \rightarrow \mathfrak{a}_Z + \mathfrak{m}_Z$ in the Grassmannian of subspaces. Moreover $\mathfrak{h}_t \rightarrow \mathfrak{h}_\theta = \mathfrak{l} \cap \mathfrak{h} + \bar{\mathfrak{u}}$ by (2.21).

On the other hand $(\mathfrak{h}_\theta)^\perp = \mathfrak{a}_Z + \mathfrak{m}_Z + \bar{\mathfrak{u}}$. As $[X_0, \bar{\mathfrak{u}}] = \bar{\mathfrak{u}}$ we obtain

$$\mathfrak{a}_Z + \mathfrak{m}_Z + [\mathfrak{h}_\theta, X_0] = (\mathfrak{h}_\theta)^\perp,$$

that is, (12.4) holds in the limit. Hence it holds for t sufficiently large. \square

In analogy to [25, Sect. 3] we call $Z = G/H$ *non-degenerate* provided that an element X_0 as in Lemma 12.5 exists, and *degenerate* otherwise. Flag varieties $Z = G/P$ with P a parabolic subgroup of G are degenerate. But in many cases Z is non-degenerate as Example 12.6 shows.

Example 12.6 (cf. [25, Lemma 3.1]). Every quasi-affine real spherical space is non-degenerate. Indeed, the constructive proof of the local structure theorem, see [28, Section 2.1], yields an $X_0 \in \mathfrak{a}_Z \cap \mathfrak{h}^\perp$ such that $\mathfrak{l} = \mathfrak{z}_{\mathfrak{g}}(X_0)$. Moreover, this element can be chosen such that $\alpha(X_0) < 0$ for all roots $\alpha \in \Sigma_{\mathfrak{u}}$.

Lemma 12.7 was communicated to us by B. Harris.

Lemma 12.7. *Let \underline{G} be an algebraic group defined over \mathbb{R} and $\underline{H} \subset \underline{G}$ be an algebraic subgroup defined over \mathbb{R} as well. Suppose that $Z = G/H$ is unimodular. Then Z is quasi-affine, i.e. $\underline{Z} = \underline{G}/\underline{H}$ is a quasi-affine variety.*

Proof. Clearly Z is unimodular if and only if \underline{Z} is unimodular. We assume first that \underline{H} is connected and treat the general case at the end. We recall the following transitivity result, see [45, Lemma 1.1] and [6, Th.4]: If there is a tower $\underline{H} \subset \underline{H}_1 \subset \underline{G}$ of subgroups, such that $\underline{H}_1/\underline{H}$ and $\underline{G}/\underline{H}_1$ are both quasi-affine, then

$\underline{G}/\underline{H}$ is quasi-affine. Now for $d := \dim \underline{H}$ and X_1, \dots, X_d a basis of \mathfrak{h} consider $v_1 := X_1 \wedge \dots \wedge X_d \in \bigwedge^d \mathfrak{g}_{\mathbb{C}}$. As \underline{H} is supposed to be unimodular and connected, we see that \underline{H} fixes v_1 . Let \underline{H}_1 be the stabilizer of v_1 in \underline{G} . Then

$$\underline{G}/\underline{H}_1 \rightarrow \bigwedge^d \mathfrak{g}_{\mathbb{C}}, \quad g\underline{H}_1 \mapsto g \cdot v_1$$

is injective and exhibits $\underline{G}/\underline{H}_1$ as quasi-affine. Moreover, as $\underline{H} \subset \underline{H}_1$ is normal, $\underline{H}_1/\underline{H}$ is affine and the transitivity result of above applies. This shows the lemma for $\underline{H} = \underline{H}_0$ connected. As $F := \underline{H}/\underline{H}_0$ is finite and acts freely on $\underline{Z}_0 = \underline{G}/\underline{H}_0$ the quotient $\underline{Z} = \underline{G}/\underline{H} \simeq \underline{Z}_0/F$ is geometric and quasi-affine as well (average polynomial function over F). \square

It is interesting to record the following (cf. [25, Th.3.2]):

Lemma 12.8. *Let $Z = G/H$ be a non-degenerate real spherical space. Then the set $\mathfrak{h}_{\text{ss}}^\perp$ of semisimple elements in \mathfrak{h}^\perp has non-empty Zariski-open interior in \mathfrak{h}^\perp .*

Proof. Since \mathfrak{g}_{ss} has Zariski-open interior in \mathfrak{g} , it suffices to check that there is a non-empty open set of semisimple elements in \mathfrak{h}^\perp . Now X_0 is semisimple and for all elements $X_1 \in \mathfrak{a}_Z + \mathfrak{m}_Z$ sufficiently close to X_0 we have in addition that $X_1 + \mathfrak{u} = \text{Ad}(U)X_1$ by [28, Lemma 2.6]. In view of (12.2) this implies that all elements $X_1 + T_0(X_1)$ are semisimple and belong to $(\mathfrak{a}_Z + \mathfrak{m}_Z)^0$. With Lemma 12.5 we conclude the proof. \square

Corollary 12.9. *Let $Z = G/H$ be a non-degenerate real spherical space and $E \subset \mathfrak{h}^\perp$. Then the following are equivalent:*

- (1) $\text{int } E_{\text{ell}} \neq \emptyset$.
- (2) $\text{int } E_{\text{w-ell}} \neq \emptyset$.

Lemma 12.10. *The following assertions hold:*

- (1) $[\text{Ad}(\underline{H})(\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0]_{\text{w-ell}} = \text{Ad}(\underline{H})((\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0 \cap \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) + i\mathfrak{a}_Z^0 + \mathfrak{m}_Z^0)$.
- (2) *Suppose that Z is non-degenerate and assume that $\text{int } \mathfrak{h}_{\text{w-ell}}^\perp \neq \emptyset$. Then*

$$\text{int } (\mathfrak{h}^\perp \cap \text{Ad}(\underline{H})(\mathfrak{z}(\mathfrak{g}) + i\mathfrak{a}_Z^0 + \mathfrak{m}_Z^0)) \neq \emptyset.$$

Proof. For (1) we first observe that it suffices to show

$$[(\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0]_{\text{w-ell}} = (\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0 \cap \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) + i\mathfrak{a}_Z^0 + \mathfrak{m}_Z^0$$

Let $Y \in (\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}$ and $X = Y + T_0(Y) \in (\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0$ as in (12.2). Then

$$\text{spec}(\text{ad } X) = \text{spec}(\text{ad } Y).$$

Hence X is weakly elliptic if and only if $Y \in (\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}} \cap \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) + i\mathfrak{a}_Z + \mathfrak{m}_Z$, that is, if and only if $X \in (\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0 \cap \mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) + i\mathfrak{a}_Z^0 + \mathfrak{m}_Z^0$.

For (2) we note that $\text{Ad}(\underline{H})(\mathfrak{a}_Z + \mathfrak{m}_Z)_{\mathbb{C}}^0$ is defined over \mathbb{R} and Zariski dense in $\mathfrak{h}_{\mathbb{C}}^\perp$ as a consequence of Lemma 12.5. Now (2) follows from (1). \square

Recall the edge $\mathfrak{a}_{Z,E} \subset \mathfrak{a}_Z$ and $\mathfrak{a}_{Z,E} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ with $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ the normalizer of \mathfrak{h} in \mathfrak{g} .

Lemma 12.11. *Let Z be a non-degenerate real spherical space. If $\mathfrak{a}_{Z,E} \neq \{0\}$, then $\text{int } \mathfrak{h}_{\text{w-ell}}^\perp = \emptyset$.*

Proof. Let $\mathfrak{a}_Z = \mathfrak{a}_{Z,E} \oplus \mathfrak{a}_{Z,S}$ be the orthogonal decomposition. Recall $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathfrak{a}_{Z,E}$ with $[\mathfrak{a}_{Z,E}, \mathfrak{h}] \subset \mathfrak{h}$. Define $\mathfrak{a}_{Z,E}^0 \subset \mathfrak{h}^\perp$ as below (12.2). Then since $\mathfrak{a}_{Z,E}^0 \cap \widehat{\mathfrak{h}}^\perp = \{0\}$ we obtain by dimension count

$$(12.5) \quad \mathfrak{h}^\perp = \widehat{\mathfrak{h}}^\perp \oplus \mathfrak{a}_{Z,E}^0.$$

Next we claim

$$(12.6) \quad \text{Ad}(h)X - X \in \widehat{\mathfrak{h}}_{\mathbb{C}}^\perp \quad (h \in \underline{H}, X \in \mathfrak{a}_{Z,E}^0).$$

In fact, as \underline{H} is connected it suffices to show that $\kappa(e^{\text{ad } Y} X, U) = \kappa(X, U)$ for all $Y \in \mathfrak{h}_{\mathbb{C}}$ and $U \in \mathfrak{a}_{Z,E}$. By the invariance of the form κ this is then implied by $e^{-\text{ad } Y} U \in U + \mathfrak{h}_{\mathbb{C}}$ as $[\mathfrak{a}_{Z,E}, \mathfrak{h}_{\mathbb{C}}] \subset \mathfrak{h}_{\mathbb{C}}$.

Suppose $\text{int } \mathfrak{h}_{\text{w-ell}}^\perp \neq \emptyset$. According to Lemma 12.10 we thus find some subset $\mathcal{O} \subset i\mathfrak{a}_{Z,E}^0 + i\mathfrak{a}_{Z,S}^0 + \mathfrak{m}_Z^0$ such that $\text{Ad}(\underline{H})\mathcal{O} \cap \mathfrak{h}^\perp$ is open and non-empty.

Let $X = iX_1 + iX_2 + Y \in \mathcal{O}$ with $X_1 \in \mathfrak{a}_{Z,E}^0, X_2 \in \mathfrak{a}_{Z,S}^0, Y \in \mathfrak{m}_Z^0$ and let $h \in \underline{H}$ be such that $\text{Ad}(h)X \in \mathfrak{h}^\perp$. With (12.6) we get

$$\text{Ad}(h)X = iX_1 + \underbrace{(\text{Ad}(h)(iX_1) - iX_1)}_{\in \widehat{\mathfrak{h}}_{\mathbb{C}}^\perp} + \underbrace{\text{Ad}(h)(iX_2 + Y)}_{\in \widehat{\mathfrak{h}}_{\mathbb{C}}^\perp} \in (i\mathfrak{a}_{Z,E}^0 + \widehat{\mathfrak{h}}_{\mathbb{C}}^\perp) \cap \mathfrak{h}^\perp.$$

From (12.5) we then deduce $X_1 = 0$. Hence $\mathcal{O} \subset i\mathfrak{a}_{Z,S}^0 + \mathfrak{m}_Z$. Now as $\mathfrak{a}_{Z,E}^0 \neq \{0\}$ we have

$$\dim \mathfrak{h}/\mathfrak{l} \cap \mathfrak{h} + \dim \mathfrak{a}_{Z,S} + \dim \mathfrak{m}_Z < \dim \mathfrak{h}^\perp = \dim \mathfrak{g}/\mathfrak{h}$$

and therefore $\text{Ad}(\underline{H})(\mathfrak{a}_{Z,S}^0 + \mathfrak{m}_Z)_{\mathbb{C}} \subset \mathfrak{h}_{\mathbb{C}}^\perp$ has empty interior, a contradiction. This concludes the proof. \square

Proposition 12.12. *Let $Z = G/H$ be a unimodular real spherical space. Suppose that $\text{int } \mathfrak{h}_{\text{w-ell}}^\perp \neq \emptyset$ where the interior is taken in \mathfrak{h}^\perp . Then \mathfrak{h} is reductive in \mathfrak{g} .*

Proof. First we note that Z is non-degenerate as Z is requested to be unimodular (see Lemma 12.7 and Example 12.6). We argue by contradiction and assume that \mathfrak{h} is not reductive. Then [26, Cor. 9.10] implies that $\mathfrak{a}_{Z,E} \neq \{0\}$. Now the assertion follows from Lemma 12.11. \square

Corollary 12.13. *Let \mathfrak{h} be a real spherical unimodular subalgebra and $I \subsetneq S$. Then $\text{int } ((\mathfrak{h}_I^\perp)_{\text{w-ell}}) = \emptyset$.*

Proof. Since \mathfrak{h}_I is a proper deformation of \mathfrak{h} it cannot be reductive in \mathfrak{g} . Hence the assertion follows from Proposition 12.12. \square

12.3. Proof of Theorem 12.1.

Proof. The first assertion, $H = \widehat{H}$ reductive in G , repeats Proposition 12.12. In particular $L^2(Z)_{\text{td}} = L^2(Z)_{\text{d}}$.

We recall that to every unitary representation (π, E) of G one attaches a wavefront set $\text{WF}(\pi)$ which is an $\text{Ad}(G)$ -invariant closed cone in $\mathfrak{g}^* \simeq \mathfrak{g}$. If $Z = G/H$ is a unimodular homogeneous space, then the wavefront set of the left regular representation of G on $L^2(G/H)$ was determined in [20, Thm 2.1] as

$$(12.7) \quad \text{WF}(L^2(G/H)) = \text{cl}(\text{Ad}(G)\mathfrak{h}^\perp)$$

with cl referring to the closure.

For the second assertion we compare wavefront sets of unitary G -representations. Recall that unitary representations with disintegration in the same measure class have the same wavefront sets. Hence we obtain from Theorem 11.1 that

$$(12.8) \quad \text{WF}(L^2(Z)) \subset \text{WF}(L^2(Z)_d) \cup \bigcup_{I \subsetneq S} \bigcup_{c \in C_I} \text{WF}(L^2(Z_{I,c})).$$

On the other hand, we obtain from (12.7) that

$$(12.9) \quad \text{WF}(L^2(Z_{I,c})) = \text{cl}(\text{Ad}(G)\mathfrak{h}_{I,c}^\perp) \quad (I \subset S, c \in C_I).$$

Let $Y := \text{Ad}(G)\mathfrak{h}^\perp \subset \mathfrak{g}$ and observe that Y is the image of the algebraic map

$$\Phi : G \times \mathfrak{h}^\perp \rightarrow \mathfrak{g}, \quad (g, X) \mapsto \text{Ad}(g)X.$$

In particular, it follows that $\dim \text{cl}(Y) \setminus Y < \dim Y$. Likewise we have for $Y_{I,c} = \text{Ad}(G)\mathfrak{h}_{I,c}^\perp$ that $\dim \text{cl}(Y_{I,c}) \setminus Y_{I,c} < \dim Y_{I,c}$. By assumption and Corollary 12.9 the elliptic elements Y_{ell} have non-empty interior in Y . Since $\dim \text{cl}(Y) \setminus Y < \dim Y$ we also obtain that Y_{ell} has non empty interior $\text{int}_{\text{cl}(Y)}(Y_{\text{ell}})$ in $\text{cl}(Y)$. On the other hand it follows from Corollary 12.13 that $Y_{I,c,\text{ell}}$ has no interior in $Y_{I,c}$ when $I \neq S$. From $\dim \text{cl}(Y_{I,c}) \setminus Y_{I,c} < \dim Y_{I,c}$ we thus infer that $(\text{cl}(Y_{I,c}))_{\text{ell}}$ has empty interior in $\text{cl}(Y_{I,c})$.

From (12.7) and (12.8) we obtain

$$\emptyset \neq \text{int}_{\text{cl}(Y)}(Y_{\text{ell}}) \subset \text{WF}(L^2(Z)_d) \cup \bigcup_{I \subsetneq S, c} [\text{int}_{\text{cl}(Y)}(Y_{\text{ell}}) \cap \text{WF}(L^2(Z_{I,c}))],$$

and since $Y_{I,c} \subset \text{cl}(Y)$ it follows from (12.9) that

$$\text{int}_{\text{cl}(Y)}(Y_{\text{ell}}) \cap \text{WF}(L^2(Z_{I,c})) \subset \text{int}_{\text{cl}(Y_{I,c})}(\text{cl}(Y_{I,c})_{\text{ell}}) = \emptyset$$

for all $I \neq S$ and c . Hence $L^2(Z)_d \neq \emptyset$. □

12.4. An example.

Example 12.14. We now give two examples of series of non-symmetric real spherical spaces $Z = G/H$ for which $\text{int } \mathfrak{h}_{\text{ell}}^\perp \neq \emptyset$.

(a) Let $Z = G/H = \text{SO}(n, n+1)/\text{GL}(n, \mathbb{R})$ for $n \geq 2$. We realize $\mathfrak{g} = \mathfrak{so}(n, n+1)$ as matrices of the form

$$X = \begin{pmatrix} A & B & v \\ C & -A^T & w \\ -w^T & -v^T & 0 \end{pmatrix}$$

with $v, w \in \mathbb{R}^n$, $A, B, C \in \text{Mat}_{n \times n}(\mathbb{R})$ subject to $B^T, C^T = -B, -C$. Then \mathfrak{h} consists of the matrices $X \in \mathfrak{g}$ with $B, C, v, w = 0$. First we consider the case where $n = 2m$ is even. For $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ we let $D_{\mathbf{t}} = \text{diag}(D_{t_1}, \dots, D_{t_m}) \in \text{Mat}_{n \times n}(\mathbb{R})$ with $D_{t_i} = \begin{pmatrix} 0 & t_i \\ -t_i & 0 \end{pmatrix}$. Further for $\mathbf{s} \in \mathbb{R}^m$ we set $v_{\mathbf{s}} = (s_1, s_1, s_2, s_2, \dots, s_m, s_m)^T \in \mathbb{R}^n$. Now consider the n -dimensional non-abelian subspace

$$\mathfrak{t}^0 := \left\{ \begin{pmatrix} 0 & D_{\mathbf{t}} & v_{\mathbf{s}} \\ -D_{\mathbf{t}} & 0 & v_{\mathbf{s}} \\ -v_{\mathbf{s}}^T & -v_{\mathbf{s}}^T & 0 \end{pmatrix} \mid \mathbf{s}, \mathbf{t} \in \mathbb{R}^m \right\} \subset \mathfrak{h}_{\text{ell}}^\perp.$$

It is then easy to see that the H -stabilizer of a generic element $X \in \mathfrak{t}^0$ is trivial with $[\mathfrak{h}, X] + \mathfrak{t}^0 = \mathfrak{h}^\perp$. Thus the polar map $H \times \mathfrak{t}^0 \rightarrow \mathfrak{h}^\perp$ is generically dominant and therefore $\text{int } \mathfrak{h}_{\text{ell}}^\perp \neq \emptyset$. For $n = 2m + 1$ odd we modify \mathfrak{t}^0 as follows. We consider

$D_{\mathbf{t}}$ now as $n \times n$ -matrix via the left upper corner embedding. For $\mathbf{s} \in \mathbb{R}^{m+1}$ we further set $v_{\mathbf{s}} = (s_1, s_1, \dots, s_m, s_m, s_{m+1}) \in \mathbb{R}^n$ and define

$$\mathfrak{t}^0 := \left\{ \begin{pmatrix} 0 & D_{\mathbf{t}} & v_{\mathbf{s}} \\ -D_{\mathbf{t}} & 0 & v_{\mathbf{s}} \\ -v_{\mathbf{s}}^T & -v_{\mathbf{s}}^T & 0 \end{pmatrix} \mid \mathbf{s} \in \mathbb{R}^{m+1}, \mathbf{t} \in \mathbb{R}^m \right\} \subset \mathfrak{h}_{\text{ell}}^{\perp}.$$

We now complete the arguments as in the even case.

(b) Next we consider the cases $Z = G/H = \text{SU}(n, n + 1)/\text{Sp}(2n, \mathbb{R})$ for $n \geq 2$. Here $\mathfrak{g} = \mathfrak{su}(n, n + 1)$ is realized as the trace-free matrices of the form

$$X = \begin{pmatrix} A & B & v \\ C & -A^* & w \\ -w^* & -v^* & d \end{pmatrix}$$

with $v, w \in \mathbb{C}^n$, $A, B, C \in \text{Mat}_{n \times n}(\mathbb{C})$ subject to $B^*, C^* = -B, -C$, and $d \in i\mathbb{R}$. Further we realize $\mathfrak{h} \simeq \mathfrak{sp}(2n, \mathbb{R})$ as the subalgebra

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid A \in \text{Mat}_{n \times n}(\mathbb{R}), B, C \in i \text{Mat}_{n \times n}(\mathbb{R}), v = w = 0, d = 0\}$$

For $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{C}^n$ we let $E_{\mathbf{t}} = \text{diag}(t_1, \dots, t_n) \in \text{Mat}_{n \times n}(\mathbb{C})$ and consider

$$\mathfrak{t}^0 := \left\{ X = \begin{pmatrix} E_{i\mathbf{t}} & 0 & \mathbf{s} \\ 0 & E_{i\mathbf{t}} & \mathbf{s} \\ -\mathbf{s}^T & -\mathbf{s}^T & d \end{pmatrix} \mid \mathbf{s}, \mathbf{t} \in \mathbb{R}^n, \text{tr}(X) = 0 \right\} \subset \mathfrak{h}_{\text{ell}}^{\perp}.$$

Now proceed as in (a) and obtain that $\text{int } \mathfrak{h}_{\text{ell}}^{\perp} \neq \emptyset$.

Corollary 12.15. *For $Z = \text{SU}(n, n + 1, \mathbb{R})/\text{Sp}(2n, \mathbb{R})$ and $Z = \text{SO}(n, n + 1)/\text{GL}(n, \mathbb{R})$, $n \geq 2$, we have $L^2(Z)_{\mathfrak{d}} \neq \emptyset$.*

Proof. In Example 12.14 we have shown $\text{int } \mathfrak{h}_{\text{ell}}^{\perp} \neq \emptyset$. Apply Theorem 12.1. □

13. MOMENT MAPS AND ELLIPTIC GEOMETRY

We expect that Theorem 12.1 gives in fact an equivalence: $L^2(Z)_{\mathfrak{d}} \neq \{0\}$ if and only if $\text{int } \mathfrak{h}_{\text{w-ell}}^{\perp} \neq \emptyset$. This section is devoted to Theorem 13.1, which gives a geometric version of this expected equivalence.

Theorem 13.1. *Let Z be a non-degenerate real spherical space with a strictly convex compression cone, i.e. $\mathfrak{a}_{Z,E} = \{0\}$. Then the following statements are equivalent:*

- (1) $\text{cl}(\text{Ad}(G)\mathfrak{h}^{\perp}) = \bigcup_{I \subsetneq S} \bigcup_{\mathfrak{c} \in \mathfrak{C}_I} \text{Ad}(G)\mathfrak{h}_{I,\mathfrak{c}}^{\perp}$.
- (2) $\text{int } \mathfrak{h}_{\text{ell}}^{\perp} = \emptyset$.

Remark 13.2.

- (a) From Corollary 12.13 we obtain that

$$\text{int}_{\text{cl}(\text{Ad}(G)\mathfrak{h}^{\perp})} [\text{Ad}(G)\mathfrak{h}_{I,\mathfrak{c}}^{\perp}]_{\text{ell}} = \emptyset$$

for all $I \subsetneq S$ (see also the proof of Theorem 12.1). Hence we get (1) \Rightarrow (2), which is the geometric equivalent of Theorem 12.1.

(b) Note that (2) is equivalent to $\text{int } \mathfrak{h}_{\text{w-ell}}^{\perp} = \emptyset$ by the assumption of non-degeneracy (see Corollary 12.9).

- (c) For fixed $I \subset S$ we recall

$$\{\mathfrak{h}_{I,\mathfrak{c}} : \mathfrak{c} \in \mathfrak{C}_I\} = \{(\mathfrak{h}_w)_I : w \in \mathcal{W}\}.$$

The goal of this section is to prove Theorem 13.1. The proof is obtained via new insights on the geometry of the moment map of the Hamiltonian G -action on the co-tangent bundle T^*Z .

13.1. The moment map. In this subsection $Z = G/H$ is a general algebraic homogeneous space attached to a reductive group $G = \underline{G}(\mathbb{R})$ and an algebraic subgroup $H = \underline{H}(\mathbb{R})$.

In the sequel we identify \mathfrak{g}^* with \mathfrak{g} via our non-degenerate $\text{Ad}(G)$ -invariant form κ . In this sense we also have $(\mathfrak{g}/\mathfrak{h})^* \simeq \mathfrak{h}^\perp \subset \mathfrak{g}$ and we can view the co-tangent bundle T^*Z of Z as $T^*Z = G \times_H \mathfrak{h}^\perp$. Recall that the G -action on T^*Z is Hamiltonian with corresponding G -equivariant moment map given by

$$m : T^*Z \rightarrow \mathfrak{g}, \quad [g, X] \mapsto \text{Ad}(g)X.$$

Now for $X \in \mathfrak{h}^\perp$ the stabilizer in G of $\xi := [\mathbf{1}, X] \in T^*Z$ is $G_\xi = Z_H(X)$ whereas the stabilizer of $X = m(\xi) \in \mathfrak{g}$ is $G_{m(\xi)} = Z_G(X)$. It is then a general fact about the geometry of moment maps (see [14, p. 190]) that for the Lie algebras of $Z_H(X)$ and $Z_G(X)$ one has

$$(13.1) \quad \mathfrak{z}_\mathfrak{h}(X) \triangleleft \mathfrak{z}_\mathfrak{g}(X) \quad (X \in \mathfrak{h}^\perp).$$

Let us call an element $X \in \mathfrak{h}^\perp$ *generic*, provided that $\dim \mathfrak{z}_\mathfrak{h}(X)$ is minimal. Then it follows from [14, Th. 26.5] that

$$(13.2) \quad \mathfrak{z}_\mathfrak{g}(X)/\mathfrak{z}_\mathfrak{h}(X) \text{ is abelian for } X \in \mathfrak{h}^\perp \text{ generic.}$$

A somewhat sharper version of (13.2) is:

Lemma 13.3 ([24, Satz 8.1]). *Assume that $Z = \underline{G}/\underline{H}$ is an algebraic homogeneous space defined over \mathbb{R} attached to a connected reductive group \underline{G} . Then for X in a dense open subset of \mathfrak{h}^\perp one has*

- (1) $Z_H(X) \triangleleft Z_G(X)$.
- (2) $Z_G(X)/Z_H(X)$ is a torus.

In particular, $Z_G(X)/Z_H(X)$ is an abelian reductive Lie group.

13.2. Ellipticity relative to Z . Moment map geometry suggests notions of ellipticity and weak ellipticity of elements $X \in \mathfrak{h}^\perp$ which are more intrinsic to Z .

Let us call an element $X \in \mathfrak{h}^\perp$ *weakly Z -elliptic* provided that $Z_G(X)/Z_H(X)$ is compact. A weakly Z -elliptic element $X \in \mathfrak{h}^\perp$ will be called *Z -elliptic* if in addition X is semisimple.

Lemma 13.4. *Let X be a generic weakly Z -elliptic element and let $(Z_G(X))_0 = L_X \ltimes U_X$ be a Levi-decomposition with L_X reductive. Let $L_{H,X} := L_X \cap H$. Then $(Z_H(X))_0 = L_{H,X} \ltimes U_X$ and there exists a compact torus T_X in the center $Z(L_X)$ of L_X such that $L_X = L_{H,X}T_X$ and $\mathfrak{l}_X = \mathfrak{l}_{H,X} \oplus \mathfrak{t}_X$ orthogonal. Moreover,*

$$(13.3) \quad X \in \mathfrak{t}_X + \mathfrak{u}_X$$

and $X \in \mathfrak{t}_X$ if X is semisimple. In particular,

- (1) *Every generic weakly Z -elliptic element is weakly elliptic.*
- (2) *Every generic Z -elliptic element is elliptic.*

Proof. Let $G_1 := (Z_G(X))_0$ and $G_2 := (Z_H(X))_0$. Then by (13.1) and (13.2), $G_2 \triangleleft G_1$ is a normal subgroup such that G_1/G_2 is compact, connected and abelian, i.e. a compact torus. Furthermore $G_3 := G_2U_X$ is a closed normal subgroup such that $G_1/G_3 = L_X/L_X \cap G_3$ is a compact torus. This implies that $L_X = (G_3 \cap L_X)T_X$

with T_X an (infinitesimally) complementing compact torus in the center of L_X . It then follows that $G_2 \cap L_X = G_3 \cap L_X$, as there are no algebraic morphisms of a reductive group to a unipotent group. Now the compactness of G_1/G_2 implies that $U_X \subset G_2$ as well. Furthermore, since \mathfrak{l}_X and $\mathfrak{l}_{X,H}$ are both algebraic Lie algebras we see that $\mathfrak{t}_X = \mathfrak{l}_{H,X}^\perp$ is the orthogonal complement.

Finally we decompose $X \in \mathfrak{h}^\perp \cap \mathfrak{z}_{\mathfrak{g}}(X)$ as $X = X_0 + X_1$ with $X_0 \in \mathfrak{l}_X$ and $X_1 \in \mathfrak{u}_X$. Then $X \in \mathfrak{h}^\perp$ implies that $X_0 \in \mathfrak{t}_X$, that is (13.3). If we further observe X is semisimple if and only if $\mathfrak{z}_{\mathfrak{g}}(X) = \mathfrak{l}_X$ is reductive, then we see that the remaining statements of the lemma are consequences of (13.3). \square

Remark 13.5. Notice that $X = 0$ is semisimple and elliptic but not weakly Z -elliptic unless $Z = G/H$ is compact. To see an example of a generic weakly Z -elliptic element which is not Z -elliptic, i.e. not semisimple, consider $H = N$ for an \mathbb{R} -split group G . Then $\mathfrak{h}^\perp = \mathfrak{a} + \mathfrak{n}$. Now for a regular nilpotent element $X \in \mathfrak{n}$ we have $Z_G(X) = Z_N(X)$ and thus X is generic and weakly Z -elliptic.

Our notion of non-degeneracy for real spherical spaces now generalizes to all algebraic homogeneous spaces $Z = G/H$ as follows. We call $Z = G/H$ *non-degenerate* provided that $m(T^*Z)$ contains a Zariski dense open set of semisimple elements. We recall from [25, Sect. 3] that all quasi-affine homogeneous spaces are non-degenerate.

Proposition 13.6. *Let $Z = G/H$ be a non-degenerate homogeneous space. Then the following assertions are equivalent:*

- (1) $\text{int } \mathfrak{h}_{\text{ell}}^\perp \neq \emptyset$.
- (2) $\text{int } \mathfrak{h}_{Z-\text{ell}}^\perp \neq \emptyset$.

Proof. Here we prove (1) \Rightarrow (2), as the converse implication follows immediately from Lemma 13.4(2) (in fact without assuming non-degeneracy).

Since Z is non-degenerate, the image $m(\alpha)$ is semisimple for $\alpha = [g, X]$ in a dense open subset of $T^*Z = G \times_H \mathfrak{h}^\perp$. For those α , the centralizer $\underline{L}(\alpha) := Z_G(m(\alpha))$ of $m(\alpha)$ is a Levi subgroup of \underline{G} which is defined over \mathbb{R} . Since there are only finitely many conjugacy classes of such subgroups, there is a dense open subset \mathcal{T} of T^*Z such that for each $\alpha \in \mathcal{T}$, $\underline{L}(\alpha)$ is a Levi subgroup of \underline{G} and the G -conjugacy class of its real points $L(\alpha)$ is locally constant on \mathcal{T} .

Let \mathcal{T}_0 be a connected component of \mathcal{T} , and let $\alpha_0 \in \mathcal{T}_0$ and $\underline{L} = \underline{L}(\alpha_0)$. Then $L(\alpha)$ is G -conjugate to L for all $\alpha \in \mathcal{T}_0$. Moreover, \mathcal{T}_0 is a Hamiltonian G -manifold with moment map $m|_{\mathcal{T}_0} : \mathcal{T}_0 \rightarrow \mathfrak{g}$.

Set $\mathcal{T}_{00} := m_{\mathcal{T}_0}^{-1}(\mathfrak{l})$. Then it follows from the Cross Section Theorem (cf. [15, Th. 2.4.1]) that \mathcal{T}_{00} is a Hamiltonian L -manifold with moment map $m|_{\mathcal{T}_{00}} : \mathcal{T}_{00} \rightarrow \mathfrak{l}$ the restriction of m to \mathcal{T}_{00} .

Note that $Z_G(m(\alpha_0)) = L(\alpha_0) = L$. In particular $m(\alpha_0) \in \mathfrak{z}(\mathfrak{l})$ is regular. As $m(\mathcal{T}_{00}) \subset \mathfrak{l}$ we thus find an open neighborhood U_0 of α_0 in \mathcal{T}_{00} such that $L(\alpha) = Z_G(m(\alpha)) \subset L$ for all $\alpha \in U_0$. On the other hand we know that $L(\alpha)$ is conjugate to L . Thus in fact $L(\alpha) = L$ for $\alpha \in U_0$. Hence by passing to a dense open subset of \mathcal{T}_{00} we may assume that $L(\alpha) = L$ for all $\alpha \in \mathcal{T}_{00}$. Since $G_\alpha \subset G_{m(\alpha)} = L$ we then have $G_\alpha = L_\alpha$ with L_α the stabilizer of $\alpha \in \mathcal{T}_{00}$ in L .

Let $\mathfrak{c} \subset \mathfrak{l}$ be the \mathbb{R} -span of $m(\mathcal{T}_{00})$. We claim that

$$(13.4) \quad \mathfrak{c} \subset \mathfrak{z}(\mathfrak{l})$$

with $\mathfrak{z}(\mathfrak{l})$ the center of \mathfrak{l} . In fact, we have just seen that $G_{m(\alpha)} = L$ for all $\alpha \in \mathcal{T}_{00}$. Thus $m(\alpha) \in \mathfrak{z}(\mathfrak{l})$ for all $\alpha \in \mathcal{T}_{00}$.

Next we recall the basic equivariant property for the derivative of the moment map [14, eq. (26.2)]:

$$(13.5) \quad \kappa(dm(\alpha)(v), X) = \Omega_\alpha(\tilde{X}_\alpha, v) \quad (\alpha \in \mathcal{T}_{00}, v \in T_\alpha \mathcal{T}_{00}, X \in \mathfrak{l})$$

where Ω is the symplectic form on \mathcal{T}_{00} , \tilde{X} is the vector field on \mathcal{T}_{00} associated to X and $T_\alpha \mathcal{T}_{00}$ is the tangent space at α . Let $\mathfrak{g}_\alpha = \mathfrak{l}_\alpha$ be the Lie algebra of the stabilizer $G_\alpha = L_\alpha$ of $\alpha \in \mathcal{T}_{00}$. We claim that $\mathfrak{c}^{\perp\mathfrak{l}} \subset \mathfrak{g}_\alpha$. To see that we first note that $dm(\alpha)(v) \in \mathfrak{c}$ by the definition of \mathfrak{c} . Hence we derive from (13.5) that $\Omega_\alpha(\tilde{X}_\alpha, v) = 0$ for all v if $X \perp \mathfrak{c}$. Since Ω is non-degenerate one obtains $\tilde{X}_\alpha = 0$. Hence $\mathfrak{c}^{\perp\mathfrak{l}}$ acts with vanishing vector fields on \mathcal{T}_{00} and thus $\mathfrak{c}^{\perp\mathfrak{l}} \subset \mathfrak{g}_\alpha = \mathfrak{l}_\alpha$.

Notice that the claim implies in particular that the L -action on \mathcal{T}_{00} factors through the group $C := L/\langle \exp(\mathfrak{c}^{\perp\mathfrak{l}}) \rangle$ with Lie algebra \mathfrak{c} .

On the other hand, by passing to a further open dense subset of \mathcal{T}_{00} we may assume that $C_\alpha = G_{m(\alpha)}/G_\alpha = L/G_\alpha$ is a real form of a complex torus for all $\alpha \in \mathcal{T}_{00}$, see Lemma 13.3. Notice that C_α is a quotient of \mathfrak{c} and likewise the Lie algebra \mathfrak{c}_α of C_α is a quotient of \mathfrak{c} .

Via the non-degenerate form κ we realize \mathfrak{c}_α as a subalgebra of $\mathfrak{c} \subset \mathfrak{l}$ and note that C_α is compact if and only if \mathfrak{c}_α consists of elliptic elements. Further $m(\alpha) \in \mathfrak{c}_\alpha$.

From $\mathcal{T}_0 = G \cdot \mathcal{T}_{00}$ we obtain that for all ξ in a dense open subset of \mathcal{T}_0 it holds true that $m(\mathcal{T}_0)$ consists of elliptic elements if $G_{m(\xi)}/G_\xi$ is a compact torus.

Finally, every $\alpha \in T^*Z$ is in the G -orbit of an element $\xi = [\mathbf{1}, X]$ with $X \in \mathfrak{h}^\perp$ for which we recall $G_{m(\xi)}/G_\xi = Z_G(X)/Z_H(X)$. Now the implication (1) \Rightarrow (2) follows from $m([\mathbf{1}, X]) = X$. □

13.3. The logarithmic tangent bundle. Let $Z \hookrightarrow \widehat{Z}$ be a compactification corresponding to a complete fan \mathcal{F} as in Section 3. In particular we recall that \widehat{Z} was constructed as the closure of Z in the smooth toroidal compactification $\widehat{\underline{Z}}(\mathbb{R})$ of $\underline{Z}(\mathbb{R})$ attached to \mathcal{F} .

According to [26, Cor. 12.3], there is a unique G -equivariant morphism $\phi : \widehat{\underline{Z}}(\mathbb{R}) \rightarrow \text{Gr}(\mathfrak{g})$ into the Grassmannian of \mathfrak{g} with $\phi(z_0) = \mathfrak{h}^\perp$. Let $\mathcal{E} \rightarrow \text{Gr}(\mathfrak{g})$ be the tautological vector bundle. Then the *logarithmic cotangent bundle of $\widehat{\underline{Z}}(\mathbb{R})$* is defined by $T^{\text{log}}\widehat{\underline{Z}}(\mathbb{R}) := \phi^*\mathcal{E}$. Concretely

$$T^{\text{log}}\widehat{\underline{Z}}(\mathbb{R}) = \{(z, X) \in \widehat{\underline{Z}}(\mathbb{R}) \times \mathfrak{g} \mid X \in \phi(z)\}.$$

Then $T^{\text{log}}\widehat{\underline{Z}}(\mathbb{R})$ is a smooth G -manifold containing $T^*Z(\mathbb{R})$ as a dense open subset. It comes with a projection to the first factor

$$p : T^{\text{log}}\widehat{\underline{Z}}(\mathbb{R}) \rightarrow \widehat{\underline{Z}}(\mathbb{R}), \quad (z, X) \mapsto z$$

making it into a vector bundle. On the other hand, the second projection

$$m : T^{\text{log}}\widehat{\underline{Z}}(\mathbb{R}) \rightarrow \mathfrak{g}, \quad (z, X) \mapsto X$$

is called the *logarithmic moment map* since it restricts to the moment map on T^*Z . Since $\widehat{\underline{Z}}(\mathbb{R})$ is compact, the logarithmic moment map is proper in the Hausdorff topology.

Next we recall from Section 3 that each cone $\mathcal{C} \in \mathcal{F}$ corresponds to a \underline{G} -orbit $\widehat{\underline{Z}}_{\mathcal{C}} = \underline{G} \cdot \widehat{z}_{\mathcal{C}} \subset \widehat{\underline{Z}}$. We have defined $A_{\mathcal{C}} \subset A_Z$ to be the subtorus with Lie algebra

$\mathfrak{a}_{\mathcal{C}} = \text{span}_{\mathbb{R}} \mathcal{C}$. Moreover for $I = I(\mathcal{C})$ the set of spherical roots vanishing on \mathcal{C} , we have $\mathfrak{a}_{\mathcal{C}} \subset \mathfrak{a}_I$ and $\widehat{\mathfrak{h}}_{\mathcal{C}} = \mathfrak{h}_I + \mathfrak{a}_{\mathcal{C}}$. Also recall $\widehat{Z}_{\mathcal{C}} \simeq \underline{G}/\underline{A}_{\mathcal{C}}\underline{H}_I = \underline{Z}_I/\underline{A}_{\mathcal{C}}$. Next we recall from Remark 3.6(c) that

$$(13.6) \quad \widehat{Z} \cap \widehat{Z}_{\mathcal{C}}(\mathbb{R}) = \bigcup_{w \in \mathcal{W}} G \cdot \widehat{z}_{w, \mathcal{C}}.$$

Set $T^{\text{log}} := p^{-1}(\widehat{Z})$. For all $\mathcal{C} \in \mathcal{F}$ we define $T_{\mathcal{C}}^{\text{log}} := p^{-1}(\widehat{Z}_{\mathcal{C}} \cap \widehat{Z})$ and note that $T^{\text{log}} = \coprod_{\mathcal{C} \in \mathcal{F}} T_{\mathcal{C}}^{\text{log}}$. Furthermore, for $I \subset S$ we put $T_I^{\text{log}} := \bigcup_{\substack{\mathcal{C} \in \mathcal{F} \\ I = I(\mathcal{C})}} T_{\mathcal{C}}^{\text{log}}$. Since $m(\widehat{z}_{w, \mathcal{C}}) = \mathfrak{h}_{w, \mathcal{C}} = (\mathfrak{h}_w)_I$ for all $\mathcal{C} \in \mathcal{F}$ with $I(\mathcal{C}) = I$ we obtain with Remark 13.2(c) and (13.6) that

$$(13.7) \quad m(T_I^{\text{log}}) = \bigcup_{c \in C_I} \text{Ad}(G)\mathfrak{h}_{I, c}^{\perp}.$$

13.4. Proof of Theorem 13.1. As mentioned in Remark 13.2(a) we only need to show (2) \Rightarrow (1). Let $\alpha \in T^*Z$ be generic. Then $m(\alpha)$ is not elliptic by assumption. Hence the torus $A_{\alpha} := G_{m(\alpha)}/G_{\alpha}$ is not compact and therefore contains a 1-parameter subgroup $\mu : \mathbb{R}^{\times} \hookrightarrow \underline{A}_{\alpha}(\mathbb{R})$. Consider the orbit $A_{\alpha} \cdot \alpha \subset T^*Z$. Since its projection into Z is closed (being a flat) also $A_{\alpha} \cdot \alpha$ is closed in T^*Z . The limit $\alpha_0 := \lim_{t \rightarrow 0^+} \mu(t)\alpha$ exists in \widehat{Z} since m is proper. Since $\alpha_0 \notin T^*Z$ we have $\alpha_0 \in T_I^{\text{log}}$ for some $I \neq S$ (here we used that the compression cone is strictly convex which implies that $T_S^{\text{log}} = T^*Z$.) Hence

$$m(\alpha) = \text{Ad}(\mu(t))(m(\alpha)) = m\left(\lim_{t \rightarrow 0^+} \mu(t)\alpha\right) = m(\alpha_0) \in m(T_I^{\text{log}}) = \bigcup_{c \in C_I} \text{Ad}(G)\mathfrak{h}_{I, c}^{\perp}$$

by (13.7). Thus we obtain for $\alpha \in T^*Z$ generic that

$$(13.8) \quad m(\alpha) \in \bigcup_{I \subsetneq S} \bigcup_{c \in C_I} \text{Ad}(G)\mathfrak{h}_{I, c}^{\perp}.$$

Since the right hand side in (13.8) consists of all proper deformations of $\text{Ad}(G)\mathfrak{h}^{\perp}$, hence is closed in $\text{cl}(\text{Ad}(G)\mathfrak{h}^{\perp})$, we obtain (1) from (13.8) and the density of the generic elements. \square

14. HARISH-CHANDRA’S GROUP CASE

In this section we apply the results of this paper to derive Harish-Chandra’s formula for the Plancherel measure for a real reductive group [19]. The Plancherel measure contains naturally the formal degrees of discrete series representations of various inducing data. The formal degrees were computed by Harish-Chandra in [16]. The explicit knowledge of the formal degree is treated as a black box in what follows.

We are considering a real reductive group G' together with its both-sided symmetries $G = G' \times G'$, by which G' gets identified with $Z = G/H$ where $H = \text{diag}(G') \subset G$ is the diagonal subgroup. Let us recall that the topological assumption on G' is that $G' = \underline{G}'(\mathbb{R})$ for a reductive algebraic group \underline{G}' which is assumed to be connected. If $P' = M'A'N' \subset G'$ is a minimal parabolic subgroup of G' and \overline{P}' is its opposite, then we obtain with $P = P' \times \overline{P}' \subset G$ a minimal parabolic subgroup of G with $PH \subset G$ open and dense as consequence of the Bruhat decomposition. In particular $\mathcal{W} = \{\mathbf{1}\}$.

Next note that $\mathfrak{a} = \mathfrak{a}' \times \mathfrak{a}'$, $\mathfrak{a}_H = \text{diag}(\mathfrak{a}')$ and $\mathfrak{a}_Z = \mathfrak{a}_H^{\perp}$ is the anti-diagonal

$$\mathfrak{a}_Z = \{(X, -X) \mid X \in \mathfrak{a}'\}.$$

The assignment

$$\mathfrak{a}' \rightarrow \mathfrak{a}_Z, \quad X \mapsto \frac{1}{2}(X, -X)$$

gives a natural identification. If we denote by $\Sigma' = \Sigma(\mathfrak{a}', \mathfrak{g}') \subset (\mathfrak{a}')^* \setminus \{0\}$ the (possibly reduced) root system for the pair $(\mathfrak{a}', \mathfrak{g}')$, and further by $\Phi' \subset \Sigma'$ the set of simple roots determined by the positive roots $\Sigma'(\mathfrak{a}', \mathfrak{n}')$, then the set of spherical roots $S \subset \mathfrak{a}'_Z^*$ naturally identifies with Φ' .

14.1. The abstract Plancherel theorem for $L^2(Z)$. Here we specialize the abstract Plancherel theory of Section 6.3 to the case at hand. Recall that

$$(L, L^2(Z)) \simeq \left(\int_{\widehat{G}} \pi \otimes \text{id} \, d\mu(\pi), \int_{\widehat{G}} \mathcal{H}_\pi \otimes \mathcal{M}_\pi \, d\mu(\pi) \right)$$

with $\mathcal{M}_\pi \subset (\mathcal{H}_\pi^{-\infty})^H$.

Now any $\pi \in \widehat{G}$ has the form $\pi = \pi_1 \otimes \pi_2$ with $\pi_i \in \widehat{G}'$. Further, since $\mathcal{H}_{\pi_i}^\infty$ is a nuclear Fréchet space (as a consequence of Harish-Chandra’s admissibility theorem) we have $\mathcal{H}_\pi^\infty = \mathcal{H}_{\pi_1}^\infty \widehat{\otimes} \mathcal{H}_{\pi_2}^\infty \simeq \text{Hom}(\mathcal{H}_{\pi_1}^{-\infty}, \mathcal{H}_{\pi_2}^\infty)$ together with $\mathcal{H}_\pi^{-\infty} = \mathcal{H}_{\pi_1}^{-\infty} \widehat{\otimes} \mathcal{H}_{\pi_2}^{-\infty} \simeq \text{Hom}(\mathcal{H}_{\pi_1}^\infty, \mathcal{H}_{\pi_2}^{-\infty})$. Thus

$$(\mathcal{H}_\pi^{-\infty})^H \simeq \text{Hom}_{G'}(\mathcal{H}_{\pi_1}^\infty, \mathcal{H}_{\pi_2}^{-\infty}).$$

We then claim

$$(14.1) \quad \dim(\mathcal{H}_\pi^{-\infty})^H \leq 1$$

and

$$(14.2) \quad (\mathcal{H}_\pi^{-\infty})^H \neq \{0\} \iff \pi_2 \simeq \bar{\pi}_1$$

with $\bar{\pi}_1$ the dual representation of π_1 .

We first show “ \implies ” of (14.2) and assume that $(\mathcal{H}_\pi^{-\infty})^H \neq \{0\}$. This means that $\text{Hom}_{G'}(\mathcal{H}_{\pi_1}^\infty, \mathcal{H}_{\pi_2}^{-\infty}) \neq \{0\}$. On the level of Harish-Chandra modules this yields $\text{Hom}_{\mathfrak{g}'}(V_{\pi_1}, V_{\bar{\pi}_2}) \neq \{0\}$ and thus $\pi_2 \simeq \bar{\pi}_1$. The same reasoning also shows (14.1).

To see the converse in (14.2), we first supply some useful notation. Given a Hilbert space \mathcal{H} we denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert space of Hilbert-Schmidt operators and note that $\mathcal{B}_2(\mathcal{H}) \simeq \mathcal{H} \widehat{\otimes} \overline{\mathcal{H}}$ with $\widehat{\otimes}$ the tensor product in the category of Hilbert space and $\overline{\mathcal{H}}$ the dual to \mathcal{H} . Further we denote by $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$ the space of trace-class operators.

Given a unitary representation (π, \mathcal{H}_π) of G' , we set $\mathcal{H}_\Pi = \mathcal{B}_2(\mathcal{H}_\pi)$ and obtain a unitary representation (Π, \mathcal{H}_Π) of $G = G' \times G'$ by

$$\Pi(g'_1, g'_2)T = \pi(g'_1) \circ T \circ \pi(g'_2)^{-1} \quad (g'_1, g'_2 \in G', T \in \mathcal{H}_\Pi = \mathcal{B}_2(\mathcal{H}_\pi)).$$

Notice that $\Pi \simeq \pi \otimes \bar{\pi}$ under the isomorphism $\mathcal{B}_2(\mathcal{H}_\pi) \simeq \mathcal{H}_\pi \widehat{\otimes} \overline{\mathcal{H}_\pi}$, and that the HS-norm on $\mathcal{B}_2(\mathcal{H}_\pi)$ does not depend on the positive scaling class of the Hilbert norm which defines the Hilbertian structure of \mathcal{H}_π .

Let us assume from now on that (π, \mathcal{H}_π) is irreducible. We remind that Harish-Chandra’s basic admissibility theorem implies

$$\mathcal{H}_\Pi^\infty \subset \mathcal{B}_1(\mathcal{H}_\pi).$$

Together with (14.1) we thus obtain that

$$(\mathcal{H}_\Pi^{-\infty})^H = \mathbb{C} \operatorname{tr}_\pi$$

with tr_π denoting the restriction of the trace on $\mathcal{B}_1(\mathcal{H}_\pi)$ to \mathcal{H}_Π^∞ . In particular, this completes the proof of (14.2).

From (14.2) we then deduce

$$\operatorname{supp} \mu \subset \{[\Pi] \mid [\pi] \in \widehat{G'}\} \simeq \widehat{G'}$$

and

$$\mathcal{M}_\Pi = \mathbb{C} \operatorname{tr}_\pi \quad ([\pi] \in \operatorname{supp} \mu).$$

As the Hilbert-Schmidt norm on $\mathcal{H}_\Pi = \mathcal{B}_2(\mathcal{H}_\pi)$ is independent of the particular G' -invariant Hilbert norm on \mathcal{H}_π we obtain a natural Hilbert space structure on the one-dimensional space \mathcal{M}_Π by the request $\|\operatorname{tr}_\pi\| = 1$. Then the natural left right representation $L = L' \otimes R'$ of $G = G' \times G'$ on $L^2(Z)$ decomposes as

$$(L' \otimes R', L^2(Z))_{G' \times G'} \simeq \left(\int_{\widehat{G'}}^\oplus \Pi \, d\mu(\pi), \int_{\widehat{G'}}^\oplus \mathcal{H}_\Pi \, d\mu(\pi) \right).$$

14.2. The Plancherel theorem for $L^2(Z_I)_{\text{td}}$. We recall from Theorem 11.1 the Bernstein decomposition

$$L^2(Z) = \sum_{I \subset S} B_I(L^2(Z_I)_{\text{td}}).$$

For $I \subset S \simeq \Phi'$ we obtain a standard parabolic $P'_I = M'_I A'_I N'_I \supset P'$ and the deformation H_I of H as

$$H_I = \operatorname{diag}(M'_I A'_I)(\overline{N'_I} \times N'_I),$$

with

$$\widehat{H}_I = \operatorname{diag}(M'_I)(A'_I \overline{N'_I} \times A'_I N'_I).$$

Next we describe $L^2(Z_I)_{\text{td}}$. As in Subsection 10.1 we decompose every $f \in L^2(Z_I)$ as an A_I -Fourier integral

$$f = \int_{ia_I^*} f_\lambda \, d\lambda$$

where $f_\lambda \in L^2(\widehat{Z}_I, \lambda)$ is given by

$$f_\lambda(g) = \int_{A_I} a^{-\rho-\lambda} f(gaH_I) \, da \quad (g \in G)$$

If we denote by

$$\xi_\lambda : L^2(\widehat{Z}_I, \lambda)^\infty \rightarrow \mathbb{C}, \quad f \mapsto f(\mathbf{1})$$

the evaluation at $\mathbf{1}$, and write L_λ for the left regular representation of G on $L^2(\widehat{Z}_I, \lambda)$, then we can rewrite the Fourier-inversion in terms of spherical characters (as in Remark 6.4)

$$(14.3) \quad f(z_{0,I}) = \int_{ia_I^*} \xi_\lambda(L_{-\lambda}(f)\xi_{-\lambda}) \, d\lambda$$

with $L_{-\lambda} = \overline{L_\lambda}$ the dual representation and $\xi_{-\lambda} = \overline{\xi_\lambda}$. Next note that we have by induction in stages

$$L^2(\widehat{Z}_I, -\lambda) = \operatorname{Ind}_{\widehat{H}_I}^G(\lambda) \simeq \operatorname{Ind}_{P'_I \times P'_I}^{G' \times G'}(L^2(M'_I) \otimes \lambda).$$

Thus $L^2(\widehat{Z}_I, -\lambda)_d$ is induced from the discrete series of M'_I .

In more detail, let $(\sigma, \mathcal{H}_\sigma)$ be a discrete series representation of M'_I and $\lambda \in i(\mathfrak{a}')^*$. Then we denote by $\text{Ind}_{P'_I}^{G'}(\sigma \otimes \lambda)$ the Hilbert space of measurable functions $f : G' \rightarrow \mathcal{H}_\sigma$ with the transformation property

$$f(g'm'_I a'_I \overline{n'_I}) = \sigma(m'_I)^{-1} (a'_I)^{-\lambda + \rho'} f(g') \quad (g' \in G', m'_I a'_I \overline{n'_I} \in \overline{P'_I})$$

and endowed with the inner product (of which the convergence is an extra assumption)

$$\langle f_1, f_2 \rangle = \int_{K'} \langle f_1(k'), f_2(k') \rangle_\sigma dk'$$

where $K' \subset G'$ is a maximal compact subgroup of G' with $\mathfrak{k}' \perp \mathfrak{a}'$. The left regular representation of G' on $\mathcal{H}_{\sigma, \lambda} := \text{Ind}_{P'_I}^{G'}(\sigma \otimes \lambda)$ is then unitary and denoted by $\pi_{\sigma, \lambda} = \text{ind}_{P'_I}^{G'}(\sigma \otimes \lambda)$. Let us denote by $d(\sigma)$ the formal degree of the discrete series representation of M'_I (with respect to a chosen Haar measure dm'_I), i.e. the positive number for which we have

$$(14.4) \quad d(\sigma) \int_{M'_I} \langle \sigma(m'_I)u, u' \rangle \overline{\langle \sigma(m'_I)v, v' \rangle} dm'_I = \langle u, v \rangle \overline{\langle u', v' \rangle}$$

for all $v, v', u, u' \in \mathcal{H}_\sigma$.

We now define a $G' \times G'$ -equivariant linear map

$$\Phi_{\sigma, \lambda} : \text{Ind}_{P'_I}^{G'}(\sigma \otimes \lambda) \widehat{\otimes} \text{Ind}_{P'_I}^{G'}(\overline{\sigma} \otimes (-\lambda)) \rightarrow L^2(\widehat{Z}_I, -\lambda)_d$$

by

$$\Phi_{\sigma, \lambda}(f_1 \otimes f_2)(g'_1, g'_2) := (f_1(g'_1), f_2(g'_2))_\sigma,$$

with $(\cdot, \cdot)_\sigma$ referring here to the natural bilinear pairing of σ with its dual representation $\overline{\sigma}$. The square integrability of the image follows from the fact that the norm for $f \in L^2(\widehat{Z}_I, -\lambda)$ can be computed by means of the Haar measures on K' and M'_I (with the latter properly normalized) as

$$\|f\|_{L^2(\widehat{Z}_I, -\lambda)}^2 = \int_{K'} \int_{K'} \int_{M'_I} |f(k'_1 m'_I, k'_2)|^2 dm'_I dk'_1 dk'_2.$$

In fact, with (14.4) this calculation shows that $d(\sigma)^{1/2} \Phi_{\sigma, \lambda}$ is isometric.

With the operator $\sum_\sigma \int \Phi_{\sigma, \lambda} d_\sigma \lambda$ we thus obtain a unitary G -equivalence

$$(14.5) \quad L^2(Z_I)_{\text{td}} \underset{G' \times G'}{\simeq} \widehat{\bigoplus_{\sigma \in \widehat{M}'_{I, \text{disc}}} \int_{i(\mathfrak{a}'_I)^*} \text{Ind}_{P'_I}^{G'}(\sigma \otimes \lambda) \widehat{\otimes} \text{Ind}_{P'_I}^{G'}(\overline{\sigma} \otimes (-\lambda)) d_\sigma \lambda}$$

where

$$d_\sigma \lambda = d(\sigma) d\lambda$$

with $d\lambda$ the Lebesgue-measure on the Euclidean space $i(\mathfrak{a}'_I)^*$, suitably normalized.

For any $I \subset S$ we now denote by $\mu^{I, \text{td}}$ the restriction of the Plancherel measure μ to the closed subspace $\text{im } B_I \subset L^2(Z)$.

From Theorem 9.5 we obtain from the uniqueness of the measure class of the Plancherel measure for $L^2(Z_I)$ that:

- $\text{supp } \mu^{I, \text{td}} = \{[\Pi_{\sigma, \lambda}] \mid [\pi_{\sigma, \lambda}] \in \widehat{G}', \sigma \in \widehat{M}'_{I, \text{disc}}, \lambda \in i(\mathfrak{a}'_I)^*\}$,
- $\text{ind}_{P'_I}^{G'}(\overline{\sigma} \otimes (-\lambda))$ is isomorphic to $\pi_{\sigma, \lambda}^* = \text{ind}_{P'_I}^{G'}(\sigma \otimes \lambda)^*$ for $\mu^{I, \text{td}}$ -almost all parameters (σ, λ) .

Next we move to the subtle point on how to identify $\text{Ind}_{P'_I}^{G'}(\bar{\sigma} \otimes (-\lambda))$ with the dual representation of $\text{Ind}_{\overline{P}'_I}^{G'}(\sigma \otimes \lambda)$. For that we first remark that the pairing

$$(14.6) \quad \text{Ind}_{\overline{P}'_I}^{G'}(\sigma \otimes \lambda) \times \text{Ind}_{P'_I}^{G'}(\bar{\sigma} \otimes (-\lambda)) \rightarrow \mathbb{C}, \quad (f_1, f_2) \mapsto \int_{K'} (f_1(k'), f_2(k'))_{\sigma} dk'$$

is G' -equivariant. Thus the dual representation of $\pi_{\sigma, \lambda} = \text{ind}_{\overline{P}'_I}^{G'}(\sigma \otimes \lambda)$ is unitarily equivalent to $\pi_{\bar{\sigma}, -\lambda} = \text{ind}_{P'_I}^{G'}(\bar{\sigma} \otimes (-\lambda))$.

Next, we consider the long intertwining operator

$$(14.7) \quad \mathcal{A}_{\sigma, \lambda} : \text{Ind}_{\overline{P}'_I}^{G'}(\bar{\sigma} \otimes (-\lambda)) \rightarrow \text{Ind}_{P'_I}^{G'}(\bar{\sigma} \otimes (-\lambda))$$

$$(14.8) \quad \mathcal{A}_{\sigma, \lambda}(f)(g') = \int_{N'_I} f(g'n'_I) dn'_I \quad (g' \in G').$$

Clearly, $\mathcal{A}_{\sigma, \lambda}(f)$ is defined near $g' = \mathbf{1}$ for functions f with compact support in the non-compact picture, i.e. $\text{supp } f \subset \Omega \overline{P}'_I$ for $\Omega \subset N'_I$ compact. By standard techniques of meromorphic continuation in the λ -variable, summarized in Remark 14.1, we obtain that $\mathcal{A}_{\sigma, \lambda}$ is defined for generic $\lambda \in i(\mathfrak{a}'_I)^*$.

Remark 14.1. Let us briefly recall the basic constructions leading to the definition of $\mathcal{A}_{\sigma, \lambda}$ in terms of meromorphic continuation (originally obtained in [23]). In the first step one embeds the irreducible representation $\bar{\sigma}$ of M'_I into a minimal principal series representation of M'_I via the Casselman subrepresentation theorem. In formulae, we consider $\bar{\sigma}$ as a subrepresentation of $\text{ind}_{M'_I \cap \overline{P}'_I}^{M'_I}(\bar{\sigma}_{M'} \otimes \lambda_0)$, where $\bar{\sigma}_{M'} \in \widehat{M}'$ and $\lambda_0 \in (\mathfrak{a}' \cap \mathfrak{m}'_I)_{\mathbb{C}}^*$. Via induction in stages we then obtain that $\pi_{\bar{\sigma}, -\lambda}$ is a subrepresentation of the minimal principal series $\text{ind}_{\overline{P}'_I}^{G'}(\bar{\sigma}_{M'} \otimes \mu)$ where $\mu = \lambda_0 - \lambda$. It is important to note that $\mu|_{\mathfrak{a}'_I} = -\lambda$ for this initial parameter μ . In the sequel $\bar{\sigma}_{M'} \in \widehat{M}'$ will be fixed, but we will allow the parameter $\mu \in (\mathfrak{a}'_{\mathbb{C}})^*$ to vary. For $\text{Re } \mu$ in a certain open cone this then leads to an intertwining operator

$$\mathcal{A}(\mu) : \text{Ind}_{\overline{P}'_I}^{G'}(\bar{\sigma}_{M'} \otimes \mu) \rightarrow \text{Ind}_{(\overline{P}'_I \cap M'_I)A'_I N'_I}^{G'}(\bar{\sigma}_{M'} \otimes \mu)$$

given by absolutely convergent integrals as in (14.8).

In the second step, via Gindikin-Karpelevic change of variable (i.e. by using a minimal string of parabolics in the terminology of [23, Sect. 4]), one obtains that the intertwining operator is a product of rank one intertwiners $\mathcal{A}_{\alpha}(\mu)$ attached to indivisible roots $\alpha \in \Sigma(\mathfrak{a}', \mathfrak{n}'_I)$. For these rank one operators one has well known explicit formulae which show that they admit a meromorphic continuation via Bernstein's p^{λ} . In this regard it is important to note that the μ -dependence of $\mathcal{R}_{\alpha}(\mu)$ is in fact only a dependence on $\mu_{\alpha} = \mu(\alpha^{\vee}) \in \mathbb{C}$. Moreover, regardless of $\sigma_{M'}$, the operator $\mathcal{A}_{\alpha}(\mu)$ is defined and invertible provided that $\mu_{\alpha} \notin \frac{1}{N}\mathbb{Z}$ for an $N \in \mathbb{N}$ only depending on G' , see [32, Prop. B.1] which was based on [44, Th. 1.1].

If we now use that the roots α do not vanish identically on \mathfrak{a}'_I , we obtain $\mathcal{A}_{\sigma, \lambda}$, as in (14.7), is defined and invertible for generic $\lambda \in i(\mathfrak{a}'_I)^*$. In more precision we define $\mathcal{A}_{\sigma, \lambda}$ as the restriction of $\mathcal{A}(\mu)$ to the subrepresentation $\text{Ind}_{\overline{P}'_I}^{G'}(\bar{\sigma} \otimes (-\lambda))$.

The operator $\mathcal{A}_{\sigma,\lambda}$ is G' -equivariant and continuous, and hence we obtain from Schur's Lemma that

$$(14.9) \quad \mathcal{A}_{\sigma,\lambda}^* \circ \mathcal{A}_{\sigma,\lambda} = \tau(\sigma, \lambda) \text{id}$$

for a number $\tau(\sigma, \lambda) \in [0, \infty]$ which is positive for generic $\lambda \in i(\mathfrak{a}'_I)^*$. Here $\mathcal{A}_{\sigma,\lambda}^*$ is the Hilbert adjoint to $\mathcal{A}_{\sigma,\lambda}$. This implies in particular for all $f \in \mathcal{H}_{\bar{\sigma},-\lambda} = \text{Ind}_{P'_I}^{G'}(\bar{\sigma} \otimes (-\lambda))$ the following norm identity

$$(14.10) \quad \|\mathcal{A}_{\sigma,\lambda} f\|^2 = \tau(\sigma, \lambda) \|f\|^2.$$

Remark 14.2. The numbers $\tau(\sigma, \lambda)$ are computable via rank one reduction, see Remark 14.1.

Recall that $\mathcal{B}_2(\mathcal{H}_{\sigma,\lambda}) \simeq \mathcal{H}_{\sigma,\lambda} \otimes \overline{\mathcal{H}_{\sigma,\lambda}}$ and from (14.6) that $\mathcal{H}_{\bar{\sigma},-\lambda} = \overline{\mathcal{H}_{\sigma,\lambda}}$ is the natural (isometric) dual of $\mathcal{H}_{\sigma,\lambda}$. By combining (14.5) and (14.10) we thus obtain that the operator

$$\sum_{\sigma} \int \Phi_{\sigma,\lambda} \circ (\text{id}_{\mathcal{H}_{\sigma,\lambda}} \otimes \mathcal{A}_{\sigma,\lambda}) \mu(\sigma, \lambda) d\lambda$$

provides a unitary G -equivalence

$$(14.11) \quad L^2(Z_I)_{\text{td}} \underset{G' \times G'}{\simeq} \widehat{\bigoplus_{\sigma \in \overline{M}'_{I \text{disc}}} \int_{i(\mathfrak{a}'_I)^*}^{\oplus} \mathcal{B}_2(\mathcal{H}_{\sigma,\lambda}) \mu(\sigma, \lambda) d\lambda}$$

where

$$(14.12) \quad \mu(\sigma, \lambda) := \frac{d(\sigma)}{\tau(\sigma, \lambda)}.$$

Next we want to keep track of the implied isomorphism in (14.11) with more suitable language. For that we define a one-dimensional Hilbert space $\mathbb{C}_{\sigma,\lambda} = \mathbb{C}\xi_{\sigma,\lambda} \subset (\mathcal{B}_2(\mathcal{H}_{\sigma,\lambda})^{-\infty})^{H_I}$ with $\|\xi_{\sigma,\lambda}\| = 1$ and where $\xi_{\sigma,\lambda}$ is defined by

$$(14.13) \quad \xi_{\sigma,\lambda}(f_1 \otimes f_2) = (f_1(e), \mathcal{A}_{\sigma,\lambda}(f_2)(e))_{\sigma} \quad (f_1 \in \mathcal{H}_{\sigma,\lambda}^{\infty}, f_2 \in \mathcal{H}_{\bar{\sigma},-\lambda}^{\infty}).$$

In this regard we note for $g = (g'_1, g'_2) \in G$ that

$$(14.14) \quad \Phi_{\sigma,\lambda}(f_1 \otimes \mathcal{A}_{\sigma,\lambda}(f_2))(g) = \xi_{\sigma,\lambda}(\Pi_{\sigma,\lambda}(g^{-1})(f_1 \otimes f_2))$$

so that with the extended notation

$$(14.15) \quad L^2(Z_I)_{\text{td}} \underset{G' \times G'}{\simeq} \widehat{\bigoplus_{\sigma \in \overline{M}'_{I \text{disc}}} \int_{i(\mathfrak{a}'_I)^*}^{\oplus} \mathcal{B}_2(\mathcal{H}_{\sigma,\lambda}) \otimes \mathbb{C}_{\sigma,\lambda} \mu(\sigma, \lambda) d\lambda}.$$

We keep track also of the isomorphism from right to left. In view of (14.3) and the orthogonality relations for the discrete series, this isomorphism is the inverse of the Fourier transform

$$F_{\text{td}} \mapsto \overline{\Pi}_{\sigma,\lambda}(F) \bar{\xi}_{\sigma,\lambda} \in \mathcal{B}_2(\mathcal{H}_{\sigma,\lambda})^{\infty}$$

for $F \in C_c^{\infty}(Z_I)$, see (14.3) and Remark 6.4. Here F_{td} refers to the orthogonal projection of $F \in C_c^{\infty}(Z_I) \subset L^2(Z_I)$ to $L^2(Z_I)_{\text{td}}$.

14.2.1. *Grouping into irreducibles.* The $G = G' \times G'$ -representation in (14.15) is not multiplicity free as different $\Pi_{\sigma,\lambda}$ can yield equivalent representations. These equivalences are induced by Weyl group orbits. In more precision let W' be the Weyl group of Σ' . Then

$$W'_I := \{w|_{\mathfrak{a}_I} \mid w \in W', w(\mathfrak{a}'_I) = \mathfrak{a}'_I\}$$

gives rise to a subquotient of W' and finite subgroup of the orthogonal group of \mathfrak{a}'_I .

Remark 14.3 (Structure of W'_I). In general we are not aware of a criterion for subsets $I \subset \Phi' = S$ which characterizes those for which W'_I is a reflection group. Nevertheless we can describe a fundamental domain for the action of W'_I as a union of simplicial cones as follows.

For $\alpha \in \Phi'$ we denote by $s_\alpha \in W'$ the corresponding simple reflection and recall that

$$W'(I) := \langle s_\alpha \mid \alpha \in I \rangle$$

is naturally a reflection group on

$$\mathfrak{a}'(I) := \text{span}_{\mathbb{R}}\{\alpha^\vee \mid \alpha \in I\}$$

with simple roots given by I . Note that $\mathfrak{a}' = \mathfrak{a}'(I) \oplus \mathfrak{a}'_I$ is an orthogonal decomposition. Next we recall the set $D'_I \subset W'$ of distinguished representatives for $W'/W'(I)$, namely with

$$D'_I := \{w \in W' \mid w(I) \subset (\Sigma')^+\}$$

we obtain a bijection

$$D'_I \rightarrow W'/W'(I), \quad w \mapsto [w] = wW'(I)$$

with w the unique minimal length representative of $[w]$.

For $I, J \subset S$ set

$$W'(I, J) = \{w \in W' \mid w(J) = I\}.$$

We claim that the map

$$R : W'(I, I) \rightarrow W'_I, \quad w \mapsto w|_{\mathfrak{a}'_I}$$

is an isomorphism of groups. Let us first show that R is defined. In fact, if $w \in W'(I, I)$, then $w(I) = I$ implies that w preserves $\mathfrak{a}'(I)$ and hence its orthogonal complement \mathfrak{a}'_I . Hence R is defined. Let us show now that R is injective and assume $w|_{\mathfrak{a}'_I} = \text{id}$. In particular w fixes the face

$$(\mathfrak{a}'_I)^- := \{X \in \mathfrak{a}'_I \mid (\forall \alpha \in S \setminus I) \alpha(X) \leq 0\}$$

of $(\mathfrak{a}')^-$, the closure of the Weyl chamber $(\mathfrak{a}')^{--}$. Hence Chevalley's Lemma implies that $w \in W'(I)$, thus $w = \mathbf{1}$ as $w(I) = I$. Finally we show that R is surjective. Let $w \in W'$ such that $w(\mathfrak{a}'_I) = \mathfrak{a}'_I$. Hence $w(\mathfrak{a}'(I)) = \mathfrak{a}'(I)$. From the description of $W'/W'(I) \simeq D'_I$ we find $w_1 \in W'(I)$ such that $ww_1(I) \subset (\Sigma')^+$. Note that $ww_1(\mathfrak{a}'_I) = ww_1(\mathfrak{a}'_I)$ holds as well. Hence $ww_1(I) \subset (\Sigma')^+ \cap \text{span}_{\mathbb{R}} I = W'(I) \cdot I$. In particular we find $w_2 \in W'(I)$ such that $w_2ww_1(I) = I$. Hence $w_2ww_1 \in W'(I, I)$. Since $w_1|_{\mathfrak{a}'_I} = w_2|_{\mathfrak{a}'_I} = \text{id}$, the surjectivity of R follows.

Recall that subsets I, J are called associated provided that $W'(I, J) \neq \emptyset$. This defines an equivalence relation $I \sim J$ among subsets of S . Next we record the tiling

$$(14.16) \quad \mathfrak{a}'_I = \bigcup_{J \sim I} \bigcup_{w \in W'(I, J)} w(\mathfrak{a}'_J)^-$$

meaning that the union of interiors $\coprod_{J \sim I} \coprod_{w \in W'(I, J)} w(\mathfrak{a}'_J)^{-}$ is disjoint. Now note that $W'_I \simeq W'(I, I)$ acts on each $W'(I, J)$ from the left. We pick for each orbit $[w] = W'_I w \subset W'(I, J)$ a representative w (of minimal length). Then the cone

$$(14.17) \quad C'_I := \bigcup_{J \sim I} \bigcup_{[w] \in W'_I \backslash W'(I, J)} w(\mathfrak{a}'_J)^{-}$$

is a fundamental domain for the action of W'_I on \mathfrak{a}'_I .

Let C_I^* be a fundamental domain for the dual action of W'_I on $(\mathfrak{a}'_I)^*$, constructed as in (14.17). Let $w \in W'_I$. Then for $\lambda \in i(\mathfrak{a}'_I)^*$ we define $\lambda_w = w \cdot \lambda = \lambda(w^{-1} \cdot)$. Likewise one defines σ_w by $\sigma_w(m'_I) = \sigma(w^{-1}m'_I w)$ where we tacitly allowed ourselves to identify $w \in W'_I \simeq (N_K(\mathfrak{a}'_I) \cap N_K(\mathfrak{a}'))/M'$ with a lift to K which normalizes M'_I .

Lemma 14.4. *Let $w \in W'_I$. Then for generic $\lambda \in i(\mathfrak{a}'_I)^*$, the representation $\pi_{\sigma, \lambda}$ is equivalent to $\pi_{\sigma_w, \lambda_w}$.*

Proof. We use the intertwining operator

$$(14.18) \quad \mathcal{A}_w : \text{Ind}_{P'_I}^{G'}(\sigma \otimes \lambda) \rightarrow \text{Ind}_{w^{-1}P'_I w}^{G'}(\sigma \otimes \lambda)$$

$$(14.19) \quad \mathcal{A}_w(f)(g') = \int_{w^{-1}\overline{N}'_I w / w^{-1}\overline{N}'_I w \cap \overline{N}'_I} f(g'x) \, dx \quad (g' \in G')$$

which is, as a product of rank one intertwiners, generically defined by Remark 14.1. The desired equivalence of $\pi_{\sigma, \lambda}$ and $\pi_{\sigma_w, \lambda_w}$ is then obtained by composing \mathcal{A}_w with the right shift by w , i.e.

$$(14.20) \quad \mathcal{R}_w(f)(g') := (\mathcal{A}_w f)(g'w)$$

yields the desired equivalence between $\pi_{\sigma, \lambda}$ and $\pi_{\sigma_w, \lambda_w}$. □

Lemma 14.5 is a generic form of the Langlands Disjointness Theorem (see [22, Th. 14.90]) for which we provide an elementary proof.

Lemma 14.5. *Let $\sigma, \sigma' \in \widehat{M}'_{I, \text{disc}}$ and $\lambda, \lambda' \in iC_I^*$ be such that the unitary representations $\pi_{\sigma, \lambda}$ and $\pi_{\sigma', \lambda'}$ are equivalent. Then for generic λ one has $\lambda = \lambda'$ and $\sigma = \sigma'$.*

Proof. We first show that $\lambda = \lambda'$ for generic λ . For that we consider the infinitesimal characters $\pi_{\sigma, \lambda}$. For the discrete series σ a fairly elementary and short proof that their infinitesimal characters are real is given in [32]. Now, from the standard formulae for infinitesimal characters of induced representations, see [22, Prop. 8.22], we deduce from $\pi_{\sigma, \lambda} \simeq \pi_{\sigma', \lambda'}$ that

$$(14.21) \quad W'_{j'} \cdot (\mu_\sigma + \lambda) = W'_{j'} \cdot (\mu_{\sigma'} + \lambda').$$

Here $j' = \mathfrak{a}' + \mathfrak{t}'$ is a Cartan subalgebra of \mathfrak{g}' which inflates the maximal split torus $\mathfrak{a}' \subset \mathfrak{g}'$ by a maximal torus $\mathfrak{t}' \subset \mathfrak{m}'$. Further, $\mu_\sigma, \mu_{\sigma'} \in (i\mathfrak{t}' + \mathfrak{a}' \cap \mathfrak{m}'_I)^*$ are representatives of the infinitesimal character for σ , resp. σ' . Note that $W'_{j'}$ leaves the real form $j'_\mathbb{R} := \mathfrak{a}' + i\mathfrak{t}'$ of $j'_\mathbb{C}$ invariant. Hence comparing the imaginary parts (with respect to $j'_\mathbb{R}$) in (14.21) yields for generic $\lambda, \lambda' \in iC_I^*$ that $\lambda = \lambda'$.

Finally we show that σ is equivalent to σ' . Let F be a finite dimensional representation of G' with strictly dominant highest weight Λ and highest weight vector

fixed by M'_I . Hence $\Lambda \in (\mathfrak{a}'_I)^*$. The translation functor moves for λ generic the representations $\pi_{\sigma,\lambda}$ to $\pi_{\sigma,\lambda+\Lambda}$ and $\pi_{\sigma',\lambda}$ to $\pi_{\sigma',\lambda+\Lambda}$, see [47, proof of Lemma 10.2.7].

We conclude that $\pi_{\sigma,\lambda}$ is equivalent to $\pi_{\sigma',\lambda}$ also for a parameter λ with $\text{Re } \lambda$ sufficiently dominant. This allows us to apply Langlands' Lemma [37, Lemma 3.12] for the asymptotics of $\langle \pi_{\sigma,\lambda}(m'_I a'_t) f_1, f_2 \rangle$ for $f_1, f_2 \in \mathcal{H}_{\sigma,\lambda}^\infty$, $m'_I \in M'_I$ and $a'_t = \exp(tX')$ for $X' \in (\mathfrak{a}'_I)^{--}$:

$$\lim_{t \rightarrow \infty} (a'_t)^{\lambda - \rho'} \langle \pi_{\sigma,\lambda}(a'_t m'_I) f_1, f_2 \rangle = \langle \sigma(m'_I)[f_1(\mathbf{1})], \mathcal{A}_{\sigma,\lambda}(f_2)(\mathbf{1}) \rangle_\sigma,$$

see also (14.26). Notice that with $f_1, f_2 \in \mathcal{H}_{\sigma,\lambda}^\infty$ the vectors $f_1(\mathbf{1}), \mathcal{A}(f_2)(\mathbf{1})$ run over all pairs of smooth vectors in V_σ^∞ . Likewise holds for σ' and we obtain that the unitary representations σ and σ' feature the same (smooth) matrix coefficients.

Now we recall the Gelfand-Naimark-Segal construction which asserts for an irreducible unitary representation π of a locally compact group G on a Hilbert space \mathcal{H} that one can recover π by one matrix coefficient $g \mapsto \langle \pi(g)v, v \rangle$ for $v \in \mathcal{H}$, $v \neq 0$. Consequently σ and σ' are equivalent, concluding the proof of the lemma. \square

Remark 14.6. Let us stress that the only property of discrete series used in the preceding proof of Lemma 14.5 was that infinitesimal characters are real.

By applying Lemma 14.4 and Lemma 14.5 to the disintegration formula (14.15) we obtain the grouping in inequivalent irreducibles, i.e. the Plancherel formula for $L^2(Z_I)_{\text{td}}$:

$$(14.22) \quad L^2(Z_I)_{\text{td}} \underset{G' \times G'}{\simeq} \sum_{\sigma \in M'_{I,\text{disc}}} \int_{iC_I^*} \mathcal{B}_2(\mathcal{H}_{\sigma,\lambda}) \otimes \mathcal{M}_{\sigma,\lambda}^I \mu(\sigma, \lambda) d\lambda,$$

where

$$\mathcal{M}_{\sigma,\lambda}^I := \mathcal{M}_{\Pi_{\sigma,\lambda}}^I = (\mathcal{B}_2(\mathcal{H}_{\sigma,\lambda})^{-\infty})_{\text{temp}}^{H_I}$$

is the multiplicity space. Moreover, for generic λ we have also seen that

$$(14.23) \quad \mathcal{M}_{\sigma,\lambda}^I \simeq \bigoplus_{w \in W'_I} \mathbb{C}_{\sigma_w, \lambda_w}$$

as \mathfrak{a}_I -module. In particular, we obtain that

$$(14.24) \quad \text{spec}_{\mathfrak{a}'_I} \mathcal{M}_{\sigma,\lambda}^I = \rho' |_{\mathfrak{a}'_I} - W'_I \cdot \lambda.$$

14.3. The Maass-Selberg relations. From Theorem 9.5 we obtain that the multiplicity space $\mathcal{M}_{\sigma,\lambda}^I$ is endowed with the Hilbert space structure induced from the one dimensional space $\mathcal{M}_{\sigma,\lambda} := \mathcal{M}_{\Pi_{\sigma,\lambda}} = \mathbb{C} \text{tr}_{\pi_{\sigma,\lambda}}$.

Set $\eta_{\sigma,\lambda} := \text{tr}_{\pi_{\sigma,\lambda}} \in \mathcal{M}_{\sigma,\lambda}$ and recall from (9.3) the orthogonal decomposition

$$\eta_{\sigma,\lambda}^I = \sum_{\xi \in (\rho - W_I \chi) |_{\mathfrak{a}_I}} \eta_{\sigma,\lambda}^{I,\xi}$$

with χ the infinitesimal character of $\Pi_{\sigma,\lambda}$.

Upon our identification of \mathfrak{a}_I with \mathfrak{a}'_I we obtain for λ generic from (14.24) that $\eta_{\sigma,\lambda}^{I,\xi} \neq 0$ if and only if $\xi \in \rho' + W'_I \cdot \lambda$ and accordingly

$$\eta_{\sigma,\lambda}^I = \sum_{w \in W'_I} \eta_{\sigma,\lambda}^{I,\rho' - w\lambda}.$$

Further, our Maass-Selberg relations in Theorem 9.6 give

$$(14.25) \quad 1 = \|\eta_{\sigma,\lambda}\| = \|\eta_{\sigma,\lambda}^{I,\xi}\|_{\mathcal{M}_{\sigma,\lambda}^I}$$

for any ξ with $\eta_{\sigma,\lambda}^{I,\xi} \neq 0$.

In order to proceed we need an elementary result on the asymptotics of the matrix coefficient

$$\eta(\Pi_{\sigma,\lambda}(g)(f_1 \otimes \langle \cdot, f_2 \rangle)) = \langle \pi_{\sigma,\lambda}(g_1^{-1})f_1, \pi_{\sigma,\lambda}(g_2^{-1})f_2 \rangle \quad (f_1, f_2 \in \mathcal{H}_{\sigma,\lambda})$$

for $g = a = (\sqrt{a'}, \sqrt{a'}^{-1}) \in A_I^{-}$ with $a' \in (A_I')^{-}$. In other words we are interested in the asymptotics of

$$a' \mapsto \langle \pi_{\sigma,\lambda}((a')^{-1})f_1, f_2 \rangle$$

for $a' = a'_t = \exp(tX')$ with $X' \in (\mathfrak{a}'_I)^{-}$ and $t \rightarrow \infty$. Then we have the following variant, observed in [32], of [37, Lemma 3.12].

Lemma 14.7. *Let $\lambda \in i\mathfrak{a}'_I^*$ and suppose that $f_1, f_2 \in \mathcal{H}_{\sigma,\lambda}^\infty$ are such that $\text{supp } f_i \subset \Omega \overline{P}'_I$ for some $\Omega \subset N'_I$ compact. Then*

$$(14.26) \quad \lim_{t \rightarrow \infty} a_t^{\lambda - \rho} \langle \pi_{\sigma,\lambda}((a'_t)^{-1})f_1, f_2 \rangle = \langle f_1(\mathbf{1}), \mathcal{A}_{\sigma,\lambda}(f_2)(\mathbf{1}) \rangle_\sigma$$

Proof. We use the non-compact model for $\pi_{\sigma,\lambda}$ and realize f_1, f_2 as σ -valued functions on N_I :

$$\langle \pi_{\sigma,\lambda}((a'_t)^{-1})f_1, f_2 \rangle = (a'_t)^{-\lambda + \rho'} \int_{N'_I} \langle f_1(a'_t n'_I (a'_t)^{-1}), f_2(n'_I) \rangle_\sigma \, dn'_I.$$

Observe that

$$a'_t \Omega (a'_t)^{-1} \xrightarrow{t \rightarrow \infty} \{\mathbf{1}\}$$

for all $\Omega \subset N'_I$ compact. By the compactness of supports we are allowed to interchange limit and integral and the asserted formula follows. \square

The Maass-Selberg relations (14.25) then yield the following key-identity:

Lemma 14.8. *For generic $\lambda \in iC_I^*$ we have $\xi_{\sigma,\lambda} = \eta_{\sigma,\lambda}^{I,\rho' - \lambda}$ together with $\mathbb{C}_{\sigma,\lambda} \subset \mathcal{M}_{\sigma,\lambda}^I$ as Hilbert spaces.*

Proof. First note that $\xi_{\sigma,\lambda}$ and $\eta_{\sigma,\lambda}^{I,\rho' - \lambda}$ have to be multiples of each other as they have the same \mathfrak{a}_I -weight. Let us show that this multiple is indeed 1 by computing the asymptotics of the matrix coefficient: Recall that for $a = (\sqrt{a'}, \sqrt{a'}^{-1}) \in A_I^{-}$ with $a' \in (A_I')^{-}$ we have

$$\eta(\Pi_{\sigma,\lambda}(a)(f_1 \otimes \langle \cdot, f_2 \rangle)) = \langle \pi_{\sigma,\lambda}((a')^{-1})f_1, f_2 \rangle.$$

Now for f_1, f_2 as in Lemma 14.7 we obtained in (14.26)

$$\langle \pi_{\sigma,\lambda}((a')^{-1})f_1, f_2 \rangle \sim (a')^{\rho' - \lambda} \langle f_1(e), \mathcal{A}(f_2)(e) \rangle_\sigma.$$

Comparing with (14.13) we then get indeed that $\xi_{\sigma,\lambda} = \eta_{\sigma,\lambda}^{I,\rho' - \lambda}$.

Finally, as $\|\xi_{\sigma,\lambda}\| = 1$ we obtain from the Maass-Selberg relations (14.25) that $\mathbb{C}_{\sigma,\lambda} \subset \mathcal{M}_{\sigma,\lambda}^I$ as Hilbert spaces. This completes the proof of the lemma. \square

14.4. **The Plancherel theorem for $L^2(Z)$.** From the fact that source and target of the Bernstein morphism have equivalent Plancherel measures we obtain

$$(14.27) \quad \text{supp } \mu = \bigcup_{I \subset S} \text{supp } \mu^{I, \text{td}}$$

with

$$\text{supp } \mu^{I, \text{td}} = \{[\Pi_{\sigma, \lambda}] \in \widehat{G} \mid \lambda \in iC_I^*, \sigma \in \widehat{M}'_{I, \text{disc}}\}$$

In the union (14.27) a certain overcounting takes place, which will be taken care of in Lemma 14.9:

Lemma 14.9. *Let $I, J \subset S$. Then the following assertions hold:*

- (1) *If I and J are associated, i.e. there exists a $w \in W'$ such that $w(I) = J$, then $\text{supp } \mu^{I, \text{td}} = \text{supp } \mu^{J, \text{td}}$.*
- (2) *Otherwise $\text{supp } \mu^{I, \text{td}} \cap \text{supp } \mu^{J, \text{td}}$ has μ -measure zero.*

Proof. (1) Basic intertwining theory (assuming no particular knowledge on the discrete spectrum) as used above implies that

$$\text{spec } L^2(Z_I)_{\text{td}} = \text{spec } L^2(Z_J)_{\text{td}} \subset \widehat{G}$$

if I and J are associated.

(2) As the infinitesimal characters for the discrete series of M'_I and M'_J are real (see [32]), we obtain that the infinitesimal characters of the induced representations in $L^2(\widehat{Z}_I, \lambda_I)_{\text{d}}$ and $L^2(\widehat{Z}_J, \lambda_J)_{\text{d}}$ for generic $\lambda_I, \lambda_J \in i\mathfrak{a}_J^*$ are different if I and J are not associated, see (14.21) and the text following it. □

We are now ready to phrase the Plancherel theorem of Harish-Chandra in terms of the Bernstein morphism. For this let

$$\mathcal{H}_I := \sum_{\sigma \in \widehat{M}'_{I, \text{disc}}} \int_{iC_I^*} \mathcal{B}_2(\mathcal{H}_{\sigma, \lambda}) \otimes \mathbb{C}\xi_{\sigma, \lambda} \mu(\sigma, \lambda) d\lambda,$$

viewed as a subspace of $L^2(Z_I)_{\text{td}}$ as in (14.22).

Let B'_I be the restriction of B_I to \mathcal{H}_I . Select a family \mathcal{I} of representatives of subsets of S modulo association and set

$$B' := \bigoplus_{I \in \mathcal{I}} B'_I.$$

Theorem 14.10. *The map*

$$B' : \bigoplus_{I \in \mathcal{I}} \mathcal{H}_I \rightarrow L^2(Z)$$

is a bijective isometry, hence the inverse of a Plancherel isomorphism. In particular we obtain the explicit Parseval-formula:

$$(14.28) \quad \|f\|_{L^2(Z)}^2 = \sum_{I \in \mathcal{I}} \sum_{\sigma \in \widehat{M}'_{I, \text{disc}}} \int_{iC_I^*} \|\pi_{\sigma, \lambda}(f)\|_{\text{HS}}^2 \mu(\sigma, \lambda) d\lambda$$

for all $f \in C_c^\infty(Z)$.

Proof. By Lemma 14.9 both sides have the same support in \widehat{G} and moreover have multiplicity one. Next B'_I is isometric by Lemma 14.8 and the spectral definition of the Bernstein morphism (compare also to Remark 11.5). Since for different $I \neq J \in \mathcal{I}$ the spectral supports are disjoint by Lemma 14.9, the images of the various B'_I are orthogonal. The theorem follows. \square

To obtain the original Parseval formula of Harish-Chandra in its standard form we unwind (14.28) via $i\mathfrak{a}'_I = W'_I \cdot iC'_I$ and average over association classes

$$(14.29) \quad \|f\|_{L^2(Z)}^2 = \sum_{I \subset S} \frac{1}{|[I]| \cdot |W'_I|} \sum_{\sigma \in M'_{I, \text{disc}}} \int_{i(\mathfrak{a}'_I)^*} \|\pi_{\sigma, \lambda}(f)\|_{\text{HS}}^2 \mu(\sigma, \lambda) d\lambda$$

where $[I]$ is the equivalence class of $I \subset S$ under association.

Remark 14.11. Regarding the knowledge about representations of the discrete series, let us stress that in the above derivation of the Plancherel formula for a real reductive group we only used the results of [32] on the infinitesimal characters of discrete series. These are valid for general real spherical spaces and when specialized to the group case comparably soft and elementary opposed to the usage of the difficult classification of the discrete series by Harish-Chandra.

As byproduct of his classification of the discrete series Harish-Chandra obtained the following beautiful geometric characterization of the discrete spectrum

$$(14.30) \quad L^2(G)_d \neq \emptyset \iff \mathfrak{g} \text{ contains a compact Cartan subalgebra.}$$

Let us emphasize once more that we obtained “ \Leftarrow ” in this paper in the full generality of real spherical spaces, see Theorem 12.1.

For a general real spherical space a description of the (twisted) discrete spectrum in terms of parameters is currently out of reach. Therefore, regarding the discrete spectrum of a real spherical space, the emphasis is to obtain “ \Rightarrow ” of (14.30) in general. Now for the group case, there is an economic way to obtain that: one first characterizes the discrete spectrum as cusp forms and then relates cusp forms to orbital integrals, see the account of Wallach [46, Ch. 7]. This idea, as well as all other known methods for the group, fails to generalize to a real spherical space.

Finally, Harish-Chandra determined with the parameters of the discrete series also their formal degrees. In the group case we saw that there is a canonical normalization of the one dimensional space of H -invariant functionals $\mathcal{M}_\pi = \mathbb{C} \text{tr}_\pi$, namely by the trace. Now for a general real spherical space the space of H -invariant functionals $\mathcal{M}_{\pi, \text{td}}$ for a (twisted) discrete series is no longer one-dimensional nor is it clear whether there is a canonical normalization of the inner product on $\mathcal{M}_{\pi, \text{td}}$. The only known general result beyond the group case is the case of holomorphic discrete series on a symmetric space [31].

15. THE PLANCHEREL FORMULA FOR SYMMETRIC SPACES

In this section we apply the Bernstein decomposition to symmetric spaces and derive the Plancherel formula of Delorme [9] and van den Ban-Schlichtkrull [3]. The account is rather parallel to the group case. The only needed extra tool is the description of a generic basis of H -invariant distribution vectors for induced representations in terms of open H -orbits on real flag varieties G/P , see [1], [7].

For this section $Z = G/H$ is symmetric and we use the notation and results from Subsection 5.2.

15.1. Normalization of discrete series. This small paragraph is valid for a general unimodular spherical space $Z = G/H$. Let $[\pi] \in \widehat{G}$ and (π, \mathcal{H}) be a unitary model of $[\pi]$. We write $\mathcal{M}_{\pi,d} \subset (\mathcal{H}^{-\infty})^H$ for the subspace of those η for which $m_{v,\eta} \in L^2(Z)$ for all $v \in \mathcal{H}^\infty$. We define an inner product on $\mathcal{M}_{\pi,d}$ by the request that the Schur-Weyl orthogonality relations hold true:

$$(15.1) \quad \int_Z m_{v,\eta}(z) \overline{m_{v',\eta'}(z)} dz = \langle v, v' \rangle_{\mathcal{H}} \langle \eta, \eta' \rangle_{\mathcal{M}_{\pi,d}}$$

Notice that the norm on $\mathcal{M}_{\pi,d}$ depends on the unitary norm of \mathcal{H} which is only unique up to positive scalar.

Remark 15.1. Given a pair of normalizations of $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{M}_{\pi,d}}$ one obtains a notion of formal degree $d(\pi)$ analogous to (14.4) by requiring

$$d(\pi) \int_Z m_{v,\eta}(z) \overline{m_{v',\eta'}(z)} dz = \langle v, v' \rangle_{\mathcal{H}} \langle \eta, \eta' \rangle_{\mathcal{M}_{\pi,d}}.$$

The normalization of $\langle \cdot, \cdot \rangle_{\mathcal{M}_{\pi,d}}$ by (15.1) therefore amounts to setting $d(\pi) = 1$. Without a canonical normalization of $\langle \cdot, \cdot \rangle_{\mathcal{M}_{\pi,d}}$ this is the best we can offer.

15.2. The Plancherel formula for $L^2(Z_I)_{\text{td}}$. Recall that $H_I = (M_I \cap H) \overline{U_I}$ is contained in $\overline{P_I} = M_I \overline{U_I}$ with $\overline{P_I}/H_I \simeq M_I/M_I \cap H$. Hence $L^2(Z_I)$ is parabolically induced from $L^2(M_I/M_I \cap H)$, and hence we obtain

$$(15.2) \quad L^2(Z_I)_{\text{td}} \simeq \sum_{\sigma \in \widehat{M_I}} \int_{i\mathfrak{a}_I^*}^{\oplus} \mathcal{H}_{\sigma,\lambda} \otimes \mathcal{M}_{\sigma,d} d\lambda.$$

Here $\mathcal{H}_{\sigma,\lambda} = \text{Ind}_{\overline{P_I}}^G(\sigma \otimes \lambda)$ and $\mathcal{M}_{\sigma,d}$ is the space of $M_I \cap H$ -invariant functionals on $\mathcal{H}_\sigma^\infty$, which are square integrable for the symmetric space $M_I/M_I \cap H$, as defined in Subsection 15.1 for G/H and $\pi = \sigma$.

The space $\mathcal{H}_{\sigma,\lambda} \otimes \mathcal{M}_{\sigma,d}$ embeds into $L^2(\widehat{Z_I}, -\lambda)_d$ isometrically (by our normalization of the discrete series) by

$$\Phi_{\sigma,\lambda} : \mathcal{H}_{\sigma,\lambda} \otimes \mathcal{M}_{\sigma,d} \rightarrow L^2(\widehat{Z_I}, -\lambda)_d$$

defined by linear extension and completion of

$$\Phi_{\sigma,\lambda}(f \otimes \zeta)(gH_I) := \zeta(f(g)) \quad (f \in \mathcal{H}_{\sigma,\lambda}^\infty, \zeta \in \mathcal{M}_{\sigma,d}, g \in G).$$

For $\zeta \in \mathcal{M}_{\sigma,d}$ let us also define an H_I -invariant functional $\xi_{\sigma,\lambda,\zeta}$ on $\mathcal{H}_{\sigma,\lambda}^\infty$ via

$$(15.3) \quad \xi_{\sigma,\lambda,\zeta}(f) := \zeta(f(e)) \quad (f \in \mathcal{H}_{\sigma,\lambda}^\infty)$$

and record

$$\Phi_{\sigma,\lambda}(f \otimes \zeta) = \xi_{\sigma,\lambda,\zeta}(\pi_{\sigma,\lambda}(g^{-1})f).$$

Recall the little Weyl group $W = W_Z$ of the restricted roots system $\Sigma(\mathfrak{g}, \mathfrak{a}_Z)$ from Subsection 5.2.1.

The decomposition (15.2) is not yet the Plancherel formula for $L^2(Z_I)_{\text{td}}$, since it is not a grouping into irreducibles as different $\pi_{\sigma,\lambda}$ may yield equivalent representations. Similar to the group case this possibility is governed by the subquotient

$$W_I = \{w|_{\mathfrak{a}_I} \mid w \in W, w(\mathfrak{a}_I) = \mathfrak{a}_I\}$$

of $W = W_Z$ and a cone $C_I^* \subset \mathfrak{a}_I^*$ as fundamental domain for the dual action of W_I (see Remark 14.3). As in the group case we identify elements $w \in W_I$ with lifts to K which normalize M_I .

As in Lemma 14.4 and (14.20) we obtain for every $w \in \mathbb{W}_I$, $\sigma \in \widehat{M}_I$ and generic $\lambda \in i\mathfrak{a}_I^*$ a G -intertwiner

$$\mathcal{R}_w : \mathcal{H}_{\sigma,\lambda} \rightarrow \mathcal{H}_{\sigma_w,\lambda_w}$$

with σ_w and λ_w defined as before. Next in full analogy to Lemma 14.5 we obtain:

Lemma 15.2. *For $\lambda, \lambda' \in iC_I^*$ generic and σ, σ' in the discrete series of $L^2(M_I/M_I \cap H)$ (i.e. both $\mathcal{M}_{\sigma,d}$ and $\mathcal{M}_{\sigma',d}$ are non-zero) one has*

$$\pi_{\sigma,\lambda} \simeq \pi_{\sigma',\lambda'} \iff \lambda = \lambda' \text{ and } \sigma \simeq \sigma'.$$

Proof. We recall that the proof of Lemma 14.5 only requires that the infinitesimal character of the inducing data σ and σ' are real, see Remark 14.6. By [32] this is the case in the current situation as well.

Next we recall that the root system $\Sigma_Z = \Sigma(\mathfrak{g}, \mathfrak{a}_Z) \subset \mathfrak{a}_Z^*$ is obtained from the root system $\Sigma(\mathfrak{g}_\mathbb{C}, \mathfrak{j}_\mathbb{C}) \subset \mathfrak{j}_\mathbb{R}^*$ as the non-vanishing restrictions. In particular, the faces \mathfrak{a}_I^- of \mathfrak{a}_Z^- are contained in the faces $\mathfrak{j}_\mathbb{R}^-$ with respect to our lined up positive systems. This allows us now to argue as in (14.21) and conclude that $\lambda = \lambda'$.

The rest of the argument is then fully analogous. □

By grouping equivalent representations in (15.2) we then obtain the Plancherel formula

$$(15.4) \quad L^2(Z_I)_{\text{td}} \simeq \sum_{\sigma \in \widehat{M}_I} \int_{iC_I^*}^{\oplus} \mathcal{H}_{\sigma,\lambda} \otimes \mathcal{M}_{\sigma,\lambda}^I \, d\lambda$$

with generic multiplicity space $\mathcal{M}_{\sigma,\lambda}^I = \mathcal{M}_{\sigma,\lambda,\text{td}}^I$ of dimension

$$(15.5) \quad \dim \mathcal{M}_{\sigma,\lambda}^I = |\mathbb{W}_I| \cdot \dim \mathcal{M}_{\sigma,d}.$$

For $w \in \mathbb{W}_I$ let us denote by $\mathcal{M}_{\sigma_w,d}$ the space $\mathcal{M}_{\sigma,d}$ with $M_I \cap H$ replaced by $w(M_I \cap H)w^{-1} = M_I \cap H_w$ and σ replaced by σ_w . Since $L^2(M_I/M_I \cap H)_d \simeq L^2(M_I/M_I \cap H_w)_d$ we infer that $\mathcal{M}_{\sigma,d}$ and $\mathcal{M}_{\sigma_w,d}$ are canonically isomorphic. Now for each $w \in \mathbb{W}_I$ and $\zeta \in \mathcal{M}_{\sigma_w,d}$ we can define an H_I -invariant functional of \mathfrak{a}_I -weight $\rho - \lambda_w$ via

$$\xi_{\sigma_w,\lambda_w,\zeta} : \mathcal{H}_{\sigma,\lambda}^\infty \rightarrow \mathbb{C}, \quad f \mapsto \zeta((\mathcal{R}_w f)(\mathbf{1})).$$

This functional yields an embedding of $\mathcal{H}_{\sigma,\lambda}^\infty$ into $L^2(Z_I)_{\text{td}}$, i.e. $\xi_{\sigma_w,\lambda_w,\zeta} \in \mathcal{M}_{\sigma,\lambda,\text{td}}^{I,\rho-\lambda_w}$. Moreover, by varying ζ we obtain for each $w \in \mathbb{W}_I$ a linear injection

$$(15.6) \quad \mathcal{M}_{\sigma_w,d} \rightarrow \mathcal{M}_{\sigma,\lambda,\text{td}}^{I,\rho-\lambda_w}, \quad \zeta \mapsto \xi_{\sigma_w,\lambda_w,\zeta}.$$

We now count dimensions. With $\mathcal{M}_{\sigma,\lambda}^I = \bigoplus_{\mu \in \rho + i\mathfrak{a}_I^*} \mathcal{M}_{\sigma,\lambda,\text{td}}^{I,\mu}$ and (15.5) we obtain for generic λ that the inclusion (15.6) is an isomorphism, that is

$$(15.7) \quad \mathcal{M}_{\sigma,\lambda,\text{td}}^{I,\rho-\lambda_w} = \{\xi_{\sigma_w,\lambda_w,\zeta} \mid \zeta \in \mathcal{M}_{\sigma_w,d}\}, \quad (w \in \mathbb{W}_I).$$

15.3. Support of the Plancherel measure. Previously we defined for $\sigma \in \widehat{M}_I$ and $w \in \mathbb{W}_I$ the multiplicity space $\mathcal{M}_{\sigma_w,d}$. We now also need a notion for every $w \in \mathcal{W}$. For $w \in \mathcal{W}$ we write $\mathcal{M}_{\sigma_w,d}$ for the space of $M_I \cap H_w$ -invariant functionals on $\mathcal{H}_\sigma^\infty$, which are square integrable for the symmetric space $M_I/M_I \cap H_w \simeq w^{-1}M_I w/w^{-1}M_I w \cap H$.

Then, by the isospectrality of the Bernstein morphism we obtain that

$$(15.8) \quad \text{supp } \mu = \left\{ [\pi_{\sigma,\lambda}] \in \widehat{G} \mid \begin{array}{l} I \subset S, \lambda \in iC_I^*, \\ \sigma \in \widehat{M}_I \text{ s.t. } \exists w \in \mathcal{W} : \mathcal{M}_{\sigma,w,d} \neq \{0\} \end{array} \right\}.$$

Let us introduce the notion that $\sigma \in \widehat{M}_I$ is *cuspidal* provided $\mathcal{M}_{\sigma,w,d} \neq \{0\}$ for some $w \in \mathcal{W}$.

15.4. Generic dimension of multiplicity spaces. To abbreviate matters let us set $\mathcal{M}_{\sigma,\lambda} = \mathcal{M}_{\pi_{\sigma,\lambda}}$ for $[\pi_{\sigma,\lambda}] \in \text{supp } \mu$. The next goal is to obtain a precise description of $\mathcal{M}_{\sigma,\lambda}$ for generic λ . This is related to the geometry of open $H \times \overline{P}_I$ -double cosets in G which we recall from Section 5.2.3. From Lemma 5.7 there is an action of $W(I)$ on \mathcal{W} with identifications $(P_I \backslash Z)_{\text{open}} \simeq W(I) \backslash \mathcal{W}$ and $(P \backslash Z_I)_{\text{open}} \simeq W(I) / W(I) \cap W_H$.

For what to come we need to interpret the quotient $W(I) \backslash \mathcal{W}$ in terms of the geometric decomposition $\mathcal{W} = \coprod_{\mathbf{c} \in C_I} \coprod_{\mathbf{t} \in F_{I,\mathbf{c}}} \mathbf{m}_{\mathbf{c},\mathbf{t}}(\mathcal{W}_{I,\mathbf{c}})$ from (5.18).

Lemma 15.3. *With regard to $\mathcal{W} = \coprod_{\mathbf{c} \in C_I} \coprod_{\mathbf{t} \in F_{I,\mathbf{c}}} \mathbf{m}_{\mathbf{c},\mathbf{t}}(\mathcal{W}_{I,\mathbf{c}})$ the action of $W(I)$ on \mathcal{W} acts on each subset $\mathbf{m}_{\mathbf{c},\mathbf{t}}(\mathcal{W}_{I,\mathbf{c}}) \subset \mathcal{W}$ transitively and induces a natural bijection*

$$(15.9) \quad W(I) \backslash \mathcal{W} \simeq \coprod_{\mathbf{c} \in C_I} F_{I,\mathbf{c}}$$

Proof. Let us fix \mathbf{c}, \mathbf{t} , and to save notation, assume first $\mathbf{c} = \mathbf{t} = \mathbf{1}$. Then $Z_{I,\mathbf{c},\mathbf{t}} = Z_I$ and $\mathcal{W}_I = \mathcal{W}_{I,\mathbf{1}}$. Lemma 5.7(2) implies that $W(I)$ acts transitively on $\mathcal{W}_I \simeq W_I = (P \backslash Z_I)_{\text{open}}$. We claim that this holds for every $Z_{I,\mathbf{c}} \simeq Z_{I,\mathbf{c},\mathbf{t}}$, i.e. $W(I)$ acts transitively on $(P \backslash Z_{I,\mathbf{c},\mathbf{t}})_{\text{open}}$. To see that we recall the identifications $W \simeq W \simeq F_M \backslash F_{\mathbb{R}}$ with $F_{\mathbb{R}}$ the 2-torsion subgroup of $\underline{A}_Z(\mathbb{R})$. Further we need the splitting $F_{\mathbb{R}} = F_{I,\mathbb{R}} \times F_{I,\mathbb{R}}^{\perp}$ derived from (5.12). Now the $W(I)$ -orbits on $\mathcal{W} \simeq F_M \backslash F_{\mathbb{R}}$ correspond exactly to the $F_{I,\mathbb{R}}^{\perp}$ -orbits on $F_M \backslash F_{\mathbb{R}}$. Now the claim follows from the definition of $Z_{I,\mathbf{c},\mathbf{t}}$ and the fact that $(P \backslash Z_{I,\mathbf{c},\mathbf{t}})_{\text{open}} \simeq \mathcal{W}_{I,\mathbf{c}}$ is mapped under $\mathbf{m}_{\mathbf{c},\mathbf{t}}$ in a $W(I)$ -equivariant way into \mathcal{W} , see Lemma 5.8 applied to $Z_{\mathbf{c},\mathbf{t}} = G/H_{w(\mathbf{c},\mathbf{t})} \simeq Z = G/H$.

The reasoning above implies further that the $W(I)$ -action on \mathcal{W} respects with regard to the decomposition $\mathcal{W} = \coprod_{\mathbf{c} \in C_I} \coprod_{\mathbf{t} \in F_{I,\mathbf{c}}} \mathbf{m}_{\mathbf{c},\mathbf{t}}(\mathcal{W}_{I,\mathbf{c}})$ the disjoint union and is trivial on the fibers $F_{I,\mathbf{c}}$. The lemma follows. \square

15.4.1. The description of $\mathcal{M}_{\sigma,\lambda}$. We wish to relate H -invariant functionals on the induced representation $\mathcal{H}_{\sigma,\lambda} = \text{Ind}_{\overline{P}_I}^G(\sigma \otimes \lambda)$ with regard to the open $H \times \overline{P}_I$ -double cosets in G . Recall from (5.10) the bijection

$$W(I) \backslash \mathcal{W} \rightarrow (H \backslash G / \overline{P}_I)_{\text{open}}, \quad W(I)w \mapsto Hw^{-\theta} \overline{P}_I.$$

Now we define for each $[w] = W(I)w \in W(I) \backslash \mathcal{W}$ a subspace

$$\mathcal{H}_{\sigma,\lambda}^{\infty}[w] = \{f \in \mathcal{H}_{\sigma,\lambda}^{\infty} \mid \text{supp } f \subset Hw^{-\theta} \overline{P}_I\}$$

and for each $\eta \in (\mathcal{H}_{\sigma,\lambda}^{-\infty})^H$ we define the restrictions

$$\eta[w] := \eta|_{\mathcal{H}_{\sigma,\lambda}^{\infty}[w]}.$$

These functionals have now a straightforward description. Notice that $\eta[w]$ only depends on the double coset $Hw^{-\theta} \overline{P}_I$. This allows us to replace $W(I) \backslash \mathcal{W}$ by $W(I) \backslash W$ and since elements $w \in W$ have representatives in K have $w^{-\theta} = w^{-1}$ for $w \in W$.

Let now $w \in W$. Notice that the H -stabilizer of the point $w^{-1}\overline{P_I} \in G/\overline{P_I}$ is given by the (symmetric) subgroup $H \cap w^{-1}M_I w$ of $w^{-1}M_I w$. Allowing a slight conflict with previous notation we let σ_w be the representation of $w^{-1}M_I w$ induced from the group isomorphism $M_I \simeq w^{-1}M_I w$.

Frobenius reciprocity then associates to each η and $[w]$ a unique distribution vector

$$\zeta_\eta[w] \in (\mathcal{H}_{\sigma_w}^{-\infty})^{H \cap w^{-1}M_I w}$$

such that

$$\eta[w](f) = \int_{H/H \cap w^{-1}M_I w} \zeta_\eta[w](f(hw^{-1})) dh (H \cap w^{-1}M_I w) \quad (f \in \mathcal{H}_{\sigma,\lambda}^\infty[w]).$$

For each $[w] \in W(I) \backslash W$ we pick with $w \in W$ a representative. As $W(I)$ normalizes M_I via inner automorphisms, it follows that σ_w depends only on $[w]$, up to equivalence. Set

$$(15.10) \quad V(\sigma) := \bigoplus_{[w] \in W(I) \backslash W} (\mathcal{H}_{\sigma_w}^{-\infty})^{H \cap w^{-1}M_I w}$$

and consider then the evaluation map

$$(15.11) \quad \text{ev}_{\sigma,\lambda} : (\mathcal{H}_{\sigma,\lambda}^{-\infty})^H \rightarrow V(\sigma), \quad \eta \mapsto (\zeta_\eta[w])_{w \in W(I) \backslash W}$$

This map is a bijection for generic λ by [1, Thm. 5.10] for the case of $P_I = Q$ and [7, Thm. 3] in general. Sometimes it is useful to indicate the choice of the parabolic $\overline{P_I}$ above $M_I A_I$ which was used in the definition of the induced representation $\mathcal{H}_{\sigma,\lambda} = \text{Ind}_{\overline{P_I}}^G(\sigma \otimes \lambda)$. Then we write $\text{ev}_{\overline{P_I},\sigma,\lambda}$ instead of $\text{ev}_{\sigma,\lambda}$. Further, for λ generic we recall the standard notation of [1] and [7]:

$$(15.12) \quad j(\overline{P_I}, \sigma, \lambda, \zeta) := \text{ev}_{\overline{P_I},\sigma,\lambda}^{-1}(\zeta) \in (\mathcal{H}_{\sigma,\lambda}^{-\infty})^H \quad (\zeta \in V(\sigma)).$$

Next we define a subspace of $V(\sigma)$ by

$$V(\sigma)_2 := \bigoplus_{[w] \in W(I) \backslash W} \mathcal{M}_{\sigma_w,d}$$

with $\mathcal{M}_{\sigma_w,d} \subset (\mathcal{H}_{\sigma_w}^{-\infty})^{H \cap w^{-1}M_I w}$ referring to $\mathcal{M}_{\sigma,d} \subset (\mathcal{H}_{\sigma}^{-\infty})^{H \cap M_I}$ for M_I replaced by $w^{-1}M_I w$.

In the sequel we assume that $\lambda \in \mathfrak{ia}_I^*$ is generic, i.e. $j(\overline{P_I}, \sigma, \lambda, \zeta)$ is defined (the obstruction is a countable set of hyperplanes) and the representation $\pi_{\sigma,\lambda}$ is a generic member in $\text{supp}^{I,\text{td}} \mu \subset \text{supp} \mu$, see Subsection 10.2. Recall that our request is that σ is cuspidal, as defined in Section 15.3.

The main result of this subsection then is:

Theorem 15.4. *Let σ be cuspidal. Then for Lebesgue-almost all $\lambda \in \mathfrak{ia}_I^*$ the image of $\mathcal{M}_{\sigma,\lambda}$ by $\text{ev}_{\sigma,\lambda}$ is $V(\sigma)_2$, i.e.*

$$\text{ev}_{\sigma,\lambda} : \mathcal{M}_{\sigma,\lambda} \rightarrow V(\sigma)_2$$

is a bijection. In particular we have

$$(15.13) \quad \dim \mathcal{M}_{\sigma,\lambda} = \sum_{[w] \in W(I) \backslash W} \dim \mathcal{M}_{\sigma_w,d}.$$

The proof of this theorem will be prepared by several lemmas. The first lemma is valid for a general unimodular real spherical space $Z = G/H$ with Plancherel measure μ . In the sequel we consider $L^2(Z)$ as a unitary module for $G \times A_{Z,E}$ and recall that the twisted discrete spectrum $L^2(Z)_{\text{td}} \subset L^2(Z)$ is a $G \times A_{Z,E}$ -invariant subspace. Define the *essentially continuous spectrum* by $L^2(Z)_{\text{ec}} := L^2(Z)_{\text{td}}^\perp$. We write μ^{td} and μ^{ec} for the Plancherel measures of $L^2(Z)_{\text{td}}$ and $L^2(Z)_{\text{ec}}$.

Lemma 15.5. *Let $Z = G/H$ be a unimodular real spherical space with Plancherel measure μ . Then*

$$\mu^{\text{ec}}(\text{supp } \mu^{\text{td}}) = 0,$$

i.e. the Plancherel supports of $L^2(Z)_{\text{td}}$ and $L^2(Z)_{\text{ec}}$ do μ -almost not interfere.

Proof. The proof goes by comparing the infinitesimal characters of the representations occurring in μ^{td} and μ^{ec} . For that we recall that the map

$$\Phi : \widehat{G} \rightarrow \mathfrak{j}_{\mathbb{C}}^*/\mathbb{W}_j, \quad \pi \mapsto \chi_\pi$$

is continuous. Next the Bernstein decomposition of $L^2(Z)$ implies that μ^{ec} is equivalent to $\sum_{w \in \mathbb{W}} \sum_{I \subsetneq S} \mu^{I,w,\text{td}}$ with $\mu^{I,w,\text{td}}$ the Plancherel measure of $L^2(Z_{I,w})_{\text{td}}$. In this regard we note moreover that $\mu^{I,w,\text{td}}$ is built up by the Lebesgue measure on $i\mathfrak{a}_I^*$ and counting measure over each fiber $\lambda \in i\mathfrak{a}_I^*$. Now the main result of [32] asserts that for each pair I, w there is a \mathbb{W}_j -invariant lattice $\Lambda = \Lambda(I, w) \subset \mathfrak{j}_{\mathbb{R}}^*$ such that

$$\Phi(\text{supp } \mu^{I,w,\text{td}}) \subset \left[\bigcup_{s \in \mathbb{W}_j} (\Lambda + i \text{Ad}(s)\mathfrak{a}_I^*) \right] / \mathbb{W}_j.$$

Now the continuity of Φ and the aforementioned structure of the various $\mu^{I,w,\text{td}}$ with regard to Lebesgue measures imply the lemma. \square

A further important ingredient in the proof of Theorem 15.4 is the long intertwiner, which we also used in treatment of the group case (14.7):

$$\begin{aligned} \mathcal{A}_{\sigma,\lambda} &: \text{Ind}_{P_I}^G(\sigma \otimes \lambda) \rightarrow \text{Ind}_{\overline{P_I}}^G(\sigma \otimes \lambda) \\ \mathcal{A}_{\sigma,\lambda}(u)(g) &= \int_{\overline{N_I}} u(g\overline{n_I}) \, d\overline{n_I} \quad (g \in G) \end{aligned}$$

which is defined near $g = \mathbf{1}$ for all u with $\text{supp } u \subset \Omega P_I$ for $\Omega \subset \overline{N_I}$ compact, and for general u and g by meromorphic continuation with respect to λ .

We wish to compute the asymptotics of $m_{v,\eta}$ for $\eta \in \mathcal{M}_{\pi,\sigma}$ for certain test vectors $v \in \mathcal{H}_{\sigma,\lambda}^\infty$. In more precision, let $u \in \text{Ind}_{P_I}^G(\sigma \otimes \lambda)^\infty$ with $\text{supp } u \subset \Omega P_I$ with Ω as above. Our test vectors v are then given by $v = \mathcal{A}_{\sigma,\lambda}(u)$. Now with

$$(15.14) \quad \tilde{\eta} = \eta \circ \mathcal{A}_{\sigma,\lambda}$$

we obtain the tautological identity

$$m_{v,\eta}(g) = m_{u,\tilde{\eta}}(g).$$

The advantage of using the opposite representative $\text{Ind}_{\overline{P_I}}^G(\sigma \otimes \lambda)$ for $[\pi_{\sigma,\lambda}]$ is that it allows us to compute the asymptotics of $m_{u,\tilde{\eta}}(a)$ for $a = a_t = \exp(tX) \in A_I^-$ on rays to infinity. In more precision, we have the following symmetric space analogue of Lemma 14.7. Let

$$(15.15) \quad \tilde{\zeta} = \text{ev}_{P_I,\sigma,\lambda}(\tilde{\eta}) \in V(\sigma).$$

Lemma 15.6. *With the notation introduced above we have for all $m_I \in M_I$:*

$$(15.16) \quad \lim_{t \rightarrow \infty} a_t^{\lambda - \rho} m_{u, \tilde{\eta}}(m_I a_t) = \tilde{\zeta}_{[\mathbf{1}]}(\sigma(m_I^{-1})(\mathcal{A}_{\sigma, \lambda}(u)(\mathbf{1}))) = \tilde{\zeta}_{[\mathbf{1}]}(\sigma(m_I^{-1})(v(\mathbf{1})))$$

for the $[\mathbf{1}]$ -component $\tilde{\zeta}_{[\mathbf{1}]} \in (\mathcal{H}_{\sigma}^{-\infty})^{M_I \cap H}$ of $\tilde{\zeta}$.

Proof. It is sufficient to prove the assertion for $m_I = \mathbf{1}$. Next, since $a_t^{-1} \Omega a_t \rightarrow \{\mathbf{1}\}$ we may assume in addition that $\text{supp } u \subset \Omega P_I \cap H P_I$. Hence

$$m_{u, \tilde{\eta}}(a_t) = m_{u, \tilde{\eta}[\mathbf{1}]}(a_t) = (\pi_{\sigma, \lambda}(a_t)(\tilde{\eta}[\mathbf{1}])(u))$$

by the support condition of u . It is then easy to verify (see the proof of [8, Lemma 16]) that

$$\lim_{t \rightarrow \infty} a_t^{\lambda - \rho} (\pi_{\sigma, \lambda}(a_t)(\tilde{\eta}[\mathbf{1}])) = \tilde{\zeta}_{[\mathbf{1}]} \cdot d\bar{n}_I$$

as a distribution. The lemma follows. □

Note that for generic λ the intertwiner $\mathcal{A}_{\sigma, \lambda}$ induces a natural linear isomorphism

$$b_{\sigma, \lambda} : V(\sigma) \rightarrow V(\sigma),$$

defined by

$$b_{\sigma, \lambda}(\xi) = \text{ev}_{P_I, \sigma, \lambda} (j(\overline{P_I}, \sigma, \lambda, \xi) \circ \mathcal{A}_{\sigma, \lambda}) .$$

In this regard we recall from [8, Th. 2]:

Lemma 15.7. *For generic λ one has*

$$(15.17) \quad b_{\sigma, \lambda}(V(\sigma)_2) = V(\sigma)_2 .$$

Proof of Theorem 15.4. Let $\eta \in (\mathcal{H}_{\sigma, \lambda}^{-\infty})^H$ and $\zeta = \zeta_{\eta} = \text{ev}_{\sigma, \lambda}(\eta) \in V(\sigma)$. The task is to show that $\zeta \in V(\sigma)_2$ if and only if $\eta \in \mathcal{M}_{\sigma, \lambda}$. Recall that $\zeta = (\zeta_{[w]})_{[w] \in W(I) \setminus W}$ is a tuple in accordance with the definition of $V(\sigma)$ in (15.10).

Assume first that $\eta \in \mathcal{M}_{\sigma, \lambda}$. The proof goes by comparing two different expressions for the constant term m_{v, η_I} for certain test vectors $v \in \mathcal{H}_{\sigma, \lambda}^{\infty}$. According to Lemma 15.5 applied to $Z = Z_I$ we may assume that $\mathcal{M}_{\sigma, \lambda}^I = \mathcal{M}_{\sigma, \lambda, \text{td}}^I$.

Hence $\eta^{I, \rho - \lambda} \in \mathcal{M}_{\sigma, \lambda}^{I, \rho - \lambda} = \mathcal{M}_{\sigma, \lambda, \text{td}}^{I, \rho - \lambda}$. By (15.7) we then have $\eta^{I, \rho - \lambda} = \xi_{\sigma, \lambda, \zeta'}$ for some $\zeta' \in \mathcal{M}_{\sigma, \text{d}}$, that is,

$$(15.18) \quad \eta^{I, \rho - \lambda}(\pi(m_I)v) = \zeta'(\sigma(m_I^{-1})(v(\mathbf{1}))) \quad (v \in \mathcal{H}_{\sigma, \lambda}^{\infty}, m_I \in M_I) .$$

On the other hand we can compute the asymptotics via Lemma 15.6. Comparing (15.16) with (15.18) and using Theorem 7.1 yields

$$\zeta'(\sigma(m_I^{-1})(v(\mathbf{1}))) = \tilde{\zeta}_{[\mathbf{1}]}(\sigma(m_I^{-1})(v(\mathbf{1}))) \quad (m_I \in M_I)$$

for our test vectors $v = \mathcal{A}(u)$. Thus we have

$$(15.19) \quad m_{\zeta', v(\mathbf{1})} = m_{\tilde{\zeta}_{[\mathbf{1}]}, v(\mathbf{1})}$$

as functions on $M_I/M_I \cap H$. We claim that $\tilde{\zeta}_{[\mathbf{1}]} \in \mathcal{M}_{\sigma, \text{d}}$ and $\tilde{\zeta}_{[\mathbf{1}]} = \zeta'$. To see that we first observe that there exists at least one v with $v(\mathbf{1}) \neq 0$. This is because $v(\mathbf{1}) \neq 0$ translates into $\int_{\overline{N_I}} u(\bar{n}_I) d\bar{n}_I \neq 0$ which can obviously be achieved for one of our test vectors u . Now recall that $\zeta' \in \mathcal{M}_{\sigma, \text{d}}$. Hence (15.19) implies that $\tilde{\zeta}_{[\mathbf{1}]} \in \mathcal{M}_{\sigma, \text{d}}$, since for $\tilde{\zeta}_{[\mathbf{1}]}$ to yield an embedding into $L^2(M_I/M_I \cap H)$ only one non-zero matrix coefficient $m_{\tilde{\zeta}_{[\mathbf{1}]}, v(\mathbf{1})}$ has to be square integrable. With that $\zeta' = \tilde{\zeta}_{[\mathbf{1}]}$

follows from the orthogonality relations (15.1) and (15.19): For $\zeta_0 = \zeta' - \tilde{\zeta}_{[1]}$ we have

$$0 = \|m_{\zeta_0, v(\mathbf{1})}\|_{L^2(M_I/M_I \cap H_I)}^2 = \|v(\mathbf{1})\|_{\mathcal{H}_\sigma}^2 \|\zeta_0\|_{\mathcal{M}_{\sigma, d}}^2.$$

Next we let $w \in W \simeq \mathcal{W}$ vary. Analogous reasoning via transport of structure $Z \rightarrow Z_w$ yields that $\tilde{\zeta}_{[w]} \in \mathcal{M}_{\sigma_w, d}$. Thus we arrive at

$$(15.20) \quad \tilde{\zeta} := (\tilde{\zeta}_{[w]})_{[w] \in W(I) \setminus \mathcal{W}} \in V(\sigma)_2.$$

Now observe that $b_{\sigma, \lambda}(\zeta) = \tilde{\zeta} \in V(\sigma)_2$ in view of (15.14), (15.15), and (15.20), and from Lemma 15.7 it then follows that $\zeta \in V(\sigma)_2$, i.e. we have shown the implication $\text{ev}_{\overline{P}_I, \sigma, \lambda}(\mathcal{M}_{\sigma, \lambda}) \subset V(\sigma)_2$ of the theorem.

To complete the proof of the theorem we remain with the converse inclusion $\text{ev}_{\overline{P}_I, \sigma, \lambda}(\mathcal{M}_{\sigma, \lambda}) \supset V(\sigma)_2$. For that let $\zeta = (\zeta_{[w]})_{[w] \in W(I) \setminus \mathcal{W}} \in V(\sigma)_2$. Forming wave packets via $\eta = j(\overline{P}_I, \sigma, \lambda, \zeta)$ for varying λ , we finally deduce with [8, Thm. 4] that $j(\overline{P}_I, \sigma, \lambda, \zeta)$ contributes to the L^2 -spectrum of Z . Hence $\eta = j(\overline{P}_I, \sigma, \lambda, \zeta) \in \mathcal{M}_{\sigma, \lambda}$ for Lebesgue almost all λ , completing the proof of the theorem. \square

In the course of the proof of Theorem 15.4 we have shown the following identity:

Lemma 15.8. *Let λ be generic and $\eta \in \mathcal{M}_{\sigma, \lambda}$ such that $\eta = j(\overline{P}_I, \sigma, \lambda, \zeta)$ for some $\zeta \in V(\sigma)_2$. Then $\tilde{\eta} = \eta \circ \mathcal{A}_{\sigma, \lambda}$ is of the form $\tilde{\eta} = j(P_I, \sigma, \lambda, \tilde{\zeta})$ for a unique $\tilde{\zeta} \in V(\sigma)_2$ and*

$$(15.21) \quad \eta^{I, \rho - \lambda} = \xi_{\sigma, \lambda, \tilde{\zeta}_{[1]}}$$

with $\tilde{\zeta}_{[1]} = \tilde{\eta}[\mathbf{1}][\mathbf{1}] \in \mathcal{M}_{\sigma, d}$ and $\xi_{\sigma, \lambda, \tilde{\zeta}_{[1]}}$ defined as in (15.3).

Let us now transport the structure from $Z = G/H$ to $Z_w = G/H_w$ for $w \in \mathcal{W}$ and write j_w and $V(\sigma)_w$ for the j -map (15.12) and multiplicity space for Z_w . Note that $V(\sigma)_w \simeq V(\sigma)$ by permutation of coordinates.

Then $\eta = j(\overline{P}_I, \sigma, \lambda, \zeta)$ for $\zeta = (\zeta_{[u]})_{[u] \in W(I) \setminus \mathcal{W}} \in V(\sigma)$ will be moved to η_w which then can be written as $\eta_w = j_w(\overline{P}_I, \sigma, \lambda, \zeta^w)$ for some $\zeta^w = (\zeta_{[u]}^w)_{[u] \in W(I) \setminus \mathcal{W}} \in V(\sigma)_w$. By the construction of j -maps which relates invariant functionals to open H -orbits we then obtain from $\eta_w = \eta \circ w^{-1}$ the transition relations

$$(15.22) \quad \zeta_{[1]}^w = \zeta_{[w]}.$$

Theorem 15.9. *For generic $\lambda \in iC_I^*$ and $\pi = \pi_{\sigma, \lambda}$ for $\sigma \in \widehat{M}_I$ cuspidal the map*

$$(15.23) \quad \mathcal{M}_\pi \rightarrow \bigoplus_{[w] \in W(I) \setminus \mathcal{W}} \mathcal{M}_{\pi, w, \text{td}}^{I, \rho - \lambda}, \quad \eta \mapsto (\eta_w^{I, \rho - \lambda})_{[w] \in W(I) \setminus \mathcal{W}}$$

is a bijective isometry.

Proof. First note that both target and source have the same dimension by Theorem 15.4 and equation (15.7) applied to all spaces Z_w via transport of structure. Now Lemma 15.8 together with (15.22) implies that the map is bijective. Finally that the map is an isometry follows from the Maass-Selberg relations from Theorem 9.6 – for that we use $W(I) \setminus \mathcal{W} \simeq \bigcup_{c \in c_I} F_{I, c}$ from Lemma 15.3. \square

15.5. The Plancherel formula. As in the group case we select now with $\mathcal{I} \subset S$ a subset of representatives for the association classes. Let us describe in terms of (15.23) the inner product on the multiplicity space \mathcal{M}_π for $[\pi] = [\pi_{\sigma,\lambda}]$, where σ is cuspidal with respect to M_I and $I \in \mathcal{I}$.

For that observe that the map

$$\mathcal{M}_{\sigma,d} \rightarrow \mathcal{M}_\pi^{I,\rho-\lambda}, \quad \zeta \mapsto \xi_{\sigma,\lambda,\zeta}$$

is a linear isometry by (15.7) if we request that the Plancherel measure for $L^2(Z_I)_{\text{td}}$ is the Lebesgue measure $d\lambda$ times the counting measure of the discrete series, i.e. we request the normalization (15.4).

Via Theorem 15.9 we can now normalize the Plancherel measure μ such that we have an isometric isomorphism:

$$\bigoplus_{[w] \in \mathbf{W}(I) \backslash \mathcal{W}} \mathcal{M}_{\pi,w,\text{td}}^{I,\rho-\lambda} \simeq \bigoplus_{[w] \in \mathbf{W}(I) \backslash \mathcal{W}} \mathcal{M}_{\sigma,w,d}$$

where $\mathcal{M}_{\sigma,w,d}$ refers to the $M_I \cap H_w$ -invariant square integrable functionals of the symmetric space $M_I/M_I \cap H_w \simeq w^{-1}M_I w/w^{-1}M_I w \cap H$. We now define

$$\mathcal{H}_I = \bigoplus_{[w] \in \mathbf{W}(I) \backslash \mathcal{W}} \sum_{\sigma \in \bar{M}_I} \int_{iC_I^*}^{\oplus} \mathcal{H}_{\sigma,\lambda} \otimes \mathcal{M}_{\sigma,w,d} \, d\lambda$$

considered as a subspace of $L^2(Z_I)_{\text{td}}$. Let B'_I be the restriction of B to \mathcal{H}_I . Then with Theorem 15.9 we obtain with the same reasoning as in the group case the Plancherel formula for symmetric spaces:

Theorem 15.10 (Plancherel formula for symmetric spaces). *Let $Z = G/H$ be a symmetric space and let its Plancherel measure be normalized by unit asymptotics. Then*

$$B' = \bigoplus_{I \in \mathcal{I}} B'_I : \bigoplus_{I \in \mathcal{I}} \mathcal{H}_I \rightarrow L^2(Z)$$

is a bijective G -equivariant isometry and is the inverse of the Fourier transform.

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