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On the spectral theorem of Langlands

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A R T I C L E I N F O

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We show that the Hilbert subspace of $L^2(G(F)\backslash G(\mathbb{A}))$ generated by wave packets of Eisenstein series built from discrete series is the whole space. Together with the work of Lapid [17], it achieves a proof of the spectral theorem of R.P. Langlands ([16], [19]) based on the work of J. Bernstein and E. Lapid [6] on the meromorphic continuation of Eisenstein series built from discrete data. I use truncation on compact sets as J. Arthur did for the local trace formula in [2].

1. Introduction

Let G be a connected reductive group over a number field F with ring of adeles $\mathbb{A}$. We fix a minimal parabolic subgroup P_0 of G defined over F with a Levi decomposition $P_0 = M_0U_0$ over F. Denote by $\mathcal{P}_{st}$ the set of standard parabolic subgroups of G (i.e., those containing P_0) that are defined over F. Any $P \in \mathcal{P}_{st}$ admits a unique Levi decomposition $P = M_PU_P$ where M_P contains M_0. Set $\mathfrak{a}_P^* = X^*(P) \otimes_{\mathbb{Z}} \mathbb{R} = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$ where $X^*(\cdot)$ denotes the lattice of characters defined over F. Denote the dual vector space by $\mathfrak{a}_P$.

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Fix a maximal compact subgroup K of $G(\mathbb{A})$ that is in a "good position" with respect to M_0. We let $H_P : G(\mathbb{A}) \to \mathfrak{a}_P$ be the left-$U(\mathbb{A})$ and right-K invariant surjective map on $G(\mathbb{A})$ such that

$$e^{<\chi, H_P(m)>} = |\chi(m)|, m \in M_P(\mathbb{A}), \chi \in X^*(M_P).$$

The surjective H_P admits a canonical section whose image is a connected Lie group, A_P^∞, contained in the center of $M_P(\mathbb{A})$.

Denote by $\mathcal{A}_P(G)$ the space of automorphic forms on $U(\mathbb{A})P(F)\backslash G(\mathbb{A})$ and by $\mathcal{A}_P^2$, the space of "square integrable" elements ϕ of $\mathcal{A}_P(G)$ which satisfy in particular:

$$\|\phi\|_P^2 := \int_{A_P^\infty P(F)U(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|^2 dg < \infty.$$

For any $\phi \in \mathcal{A}_P(G)$ and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ set $\phi_\lambda(g) = \phi(g)e^{<\lambda, H_P(g)>}$. We have $\phi_\lambda \in \mathcal{A}_P(G)$.

Consider the Eisenstein series defined by:

$$E_P(g, \phi, \lambda) = \sum_{\gamma \in P(F)\backslash G(F)} \phi_\lambda(\gamma g), g \in G(\mathbb{A}).$$

The series converges absolutely for $Re(\lambda)$ sufficiently regular in the positive Weyl chamber of $\mathfrak{a}_P^*$.

If P, Q are standard parabolic subgroups of G, let $W(Q\backslash G/P)$ be a set of representatives of minimal length of $Q\backslash G/P$ in the Weyl group W of G relative to M_0 with the order given by P_0. Let $W(P|Q)$ be the set of $w \in W(Q\backslash G/P)$ such that $wM_Pw^{-1} = M_Q$. For any $w \in W(P|Q)$, the intertwining operator $M(w, \lambda) : \mathcal{A}_P \to \mathcal{A}_Q$ is defined by

$$[M(w, \lambda)\phi]_{w\lambda}(g) = \int_{(wU_P(\mathbb{A})w^{-1} \cap U_Q(\mathbb{A}))\backslash U_Q(\mathbb{A})} \phi_\lambda(w^{-1}ug)du.$$

The integral converges in particular for $Re(\lambda)$ sufficiently regular in the positive Weyl chamber of $\mathfrak{a}_P^*$.

J. Bernstein and E. Lapid (cf. [6]) gave a short proof of the following results due to R.P. Langlands when $\phi \in \mathcal{A}_P^2$ (cf. [16], [19] Chapter IV):

> Let $P, Q \in \mathcal{P}_{st}$. When $\phi \in \mathcal{A}_P(G)$:
> 1) The Eisenstein series $E_P(\phi, \lambda)$, which is absolutely convergent and holomorphic for $Re(\lambda)$ sufficiently regular in the positive Weyl chamber of a_P^*, extends to a meromorphic function $\lambda \mapsto E_P(\phi, \lambda)$ on $\mathfrak{a}_{P,\mathbb{C}}^*$. Whenever regular, $E_P(\phi, \lambda) \in \mathcal{A}(G)$. (1.1)
> 2) For any $w \in W(P|Q)$, the map $\lambda \mapsto M(w, \lambda)\phi$, taking values in a finite-dimensional subspace of $\mathcal{A}_Q$, admits a meromorphic continuation to $\mathfrak{a}_{P,\mathbb{C}}^*$ which is holomorphic around $i\mathfrak{a}_P^*$. Moreover $M(w, \lambda)$ is unitary for $\lambda \in i\mathfrak{a}_P^*$.

When ϕ is cuspidal, the theorem was proved by Langlands ([16]) and he described the discrete part of $L^2(G(F)\backslash G(\mathbb{A}))$ in terms of residues of Eisenstein series for ϕ cuspidal and used it to extend the theorem to the case where ϕ is square-integrable. When ϕ is an element $\mathcal{A}^2_P(G)$, E. Lapid ([17]) has given a short proof of:

$$\text{For } \phi \in \mathcal{A}^2_P \text{ the map } \lambda \mapsto E_P(\phi, \lambda) \text{ is analytic on } i\mathfrak{a}^*_P, \tag{1.2}$$

by studying the truncated inner product of Eisenstein series which is also due to R.P. Langlands (cf. [16], [19]) for cuspidal data and was extended by Arthur ([3]).

Let P be a standard parabolic subgroup of G. Let $\mathcal{W}_P$ be the space of compactly supported smooth functions on $i\mathfrak{a}^*_P$ taking values in a finite dimensional subspace of $\mathcal{A}^2_P$. For $\phi \in \mathcal{W}_P$, write:

$$\|\phi\|^2_* = \int_{i\mathfrak{a}^*_P} \|\phi(\lambda)\|^2_P d\lambda, \tag{1.3}$$

and let

$$\Theta_{P,\phi}(g) = \int_{i\mathfrak{a}^*_P} E_P(g, \phi(\lambda), \lambda) d\lambda,$$

which we call wave packets of Eisenstein series. Let $L^2_{disc}(A^\infty_M M(F)\backslash M(\mathbb{A}))$ be the Hilbert sum of irreducible $M(\mathbb{A})$-subrepresentations of $L^2(A^\infty_M M(F)\backslash M(\mathbb{A}))$.

If P is a standard parabolic subgroup of G, let $|\mathcal{P}(M_P)|$ be equal to the number of parabolic subgroups having M_P as Levi factor.

Recall (cf. (1.1)) that $M(w, \lambda)$ is well defined and holomorphic for λ around $i\mathfrak{a}^*_P$ and unitary for $\lambda \in i\mathfrak{a}^*_P$. Consider the space $\mathcal{L}$ consisting of families of measurable functions $F_P : i\mathfrak{a}^*_P \to Ind^{G(\mathbb{A})}_{P(\mathbb{A})} L^2_{disc}(A^\infty_M M(F)\backslash M(\mathbb{A}))$ where P describes the set of standard parabolic subgroups of G such that:

$$\|(F_P)\|^2 = \sum_{P \in \mathcal{P}_{st}} |\mathcal{P}(M_P)|^{-1} \|F_P\|^2_* < \infty$$

and

$$F_Q(w\lambda) = M(w, \lambda) F_P(\lambda), w \in W(P|Q), \lambda \in i\mathfrak{a}^*_P, \tag{1.4}$$

the right hand side being well defined due to the properties of $M(w, \lambda)$.

Let $\mathcal{L}'$ be the subspace of $\mathcal{L}$ consisting of those families such that $F_P \in \mathcal{W}_P$ for all P.

We recall the statement of Theorem 2 of [17], also due to Langlands ([16], [19]). The short proof of Lapid is based on a direct proof of the asymptotic formula of the truncated inner product of Eisenstein series with square integrable data, independent of [16], [3], simply using the results of [6].

The map $\mathcal{E}$ from $\mathcal{L}'$ to $L^2(G(F)\backslash G(\mathbb{A}))$

$$(F_P) \mapsto \sum_{P \in \mathcal{P}_{st}} |\mathcal{P}(M_P)|^{-1} \Theta_{P,F_P} \tag{1.5}$$

extends to an isometry $\overline{\mathcal{E}}$ from $\mathcal{L}$ to $L^2(G(F)\backslash G(\mathbb{A}))$.

The main result of the present article, which achieves a new proof of the Spectral Theorem of R.P. Langlands ([16], [19]), is:

$$\text{The map } \overline{\mathcal{E}} \text{ is onto, i.e. it is an isometric isomorphism from } \mathcal{L} \text{ to } L^2(G(F)\backslash G(\mathbb{A})). \tag{1.6}$$

One uses the notion of temperedness of automorphic forms introduced by J. Franke [14] (cf. also [20] section 4.4). We show that this notion of temperedness is equivalent to the notion of temperedness introduced before by Joseph Bernstein in [5].

We introduce the notion of weak constant term of tempered automorphic forms. We prove that for bounded sets of unitary parameters, the Eisenstein series are uniformly tempered. One uses for this that the growth of an automorphic form is controlled by the exponents of its constant terms and that the constant terms of Eisenstein series are explicit.

The wave packets are in the Harish-Chandra Schwartz space (cf. [18]): this is due to J. Franke [14], section 5.3, Proposition 2 (2) but his proof rests on the work of Langlands. We give here a selfcontained proof (cf. Proposition 5.4) which uses the general scheme of Harish-Chandra's study of wave packets in the Schwartz space in the real case (cf. [15], see also [4]).

From [5], one knows that only tempered automorphic forms can contribute to the spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. Then, one shows that if the image of $\overline{\mathcal{E}}$ is not the full space, there would exist a tempered automorphic form orthogonal to these wave packets: the proof, by a measurability argument, is similar to what we did for real symmetric spaces (cf. [10], [13]).

One can compute an explicit asymptotic formula for the truncated inner product of this form with an Eisenstein series. Actually, here, we use "true truncation" i.e. truncation on compact sets as in [2] and the weak constant term. This uses partitions of $G(F)\backslash G(\mathbb{A})$ depending on the truncation parameter (cf. [1]).

By a process of limit, as in [13], one computes explicitly the scalar product of this form with a wave packet of Eisenstein series.

Then one shows that it implies that this form is zero. A contradiction which shows that $\overline{\mathcal{E}}$ is onto.

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2. Notation

We introduce the notation for functions f, g defined on a set X with values in $\mathbb{R}^+$:

$$f(x) << g(x), x \in X$$

if there exists $C > 0$ such that $f(x) < Cg(x), x \in X$. We will denote this also $f << g$.

Let us denote

$$f(x) \sim g(x), x \in X$$

if $f << g$ and $g << f$. In that case we will say that the functions are equivalent.

If moreover f and g take values greater or equal to 1, we write:

$$f(x) \approx g(x)$$

if there exists $N > 0$ such that

$$g(x)^{1/N} << f(x) << g(x)^N, x \in X.$$

If V is a real vector space, we denote by V^* its dual, by $V_{\mathbb{C}}$ its complexification and if $v \in V_{\mathbb{C}}$ we write $v = Rev + \sqrt{-1}Imv$ where $Rev, Imv \in V$.

Let F be a number field and $\mathbb{A}$ be its adele ring. If G is a linear algebraic group defined over F, we denote its unipotent radical by N_G. Let $X^*(G)$ be the group of characters of G defined over F. Let $\mathfrak{a}_G = Hom_{\mathbb{Z}}(X^*(G), \mathbb{R})$ and $\mathfrak{a}_G^* = X^*(G) \otimes_{\mathbb{Z}} \mathbb{R}$. We have a canonical paring $< . >$ between $\mathfrak{a}_G$ and $\mathfrak{a}_G^*$ and a canonical morphism:

$$H_G : G(\mathbb{A}) \to \mathfrak{a}_G$$

such that for all $\chi \in X^*(G)$ and $g \in G(\mathbb{A})$, $< \chi, H_G(g) >= log|\chi(g)|$.

Let $G(\mathbb{A})^1$ be the kernel of H_G.

From now on we assume that G is reductive and connected. Let P_0 be a parabolic subgroup of G defined over F and minimal for this property. Let M_0 be a Levi factor of P_0. We will denote $\mathfrak{a}_{M_0}$ by $\mathfrak{a}_0$.

We have the notion of standard and semi-standard parabolic subgroup of G. Let K be a good maximal compact subgroup of $G(\mathbb{A})$ in good position with respect to M_0.

If P is a semistandard parabolic subgroup of G we extend the map H_P to a map

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_P$$

in such a way that $H_P(pk) = H_P(p)$ for $p \in P(\mathbb{A}), k \in K$.

We have a Levi decomposition $P = M_P N_P$, where M_P is the Levi factor of P containing M_0. Let A_P be the maximal split torus of the center of M_P and $A_0 = A_{M_0}$.

Let $G_{\mathbb{Q}}$ be the restriction of scalar from F to $\mathbb{Q}$ of G. We denote by $A_P(\mathbb{R})$ the group of real points of the maximal split torus of the center of $M_{P,\mathbb{Q}}$ and by A_P^∞ the neutral component of this real Lie group. The map H_P induces an isomorphism between the neutral component A_P^∞ and $\mathfrak{a}_P$. The inverse map will be denoted *exp* or exp_P.

We define:

$$[G]_P = M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A}), [G] = [G]_G.$$

The map H_P goes down to a map $[G]_P \to \mathfrak{a}_P$.

The inverse image of 0 by this map is denoted $[G]_P^1$ and one has $[G]_P = A_P^\infty [G]_P^1$.

If $P \subset Q$ are semistandard parabolic subgroups of G, we have the usual decomposition

$$\mathfrak{a}_P = \mathfrak{a}_P^Q \oplus \mathfrak{a}_Q.$$

This allows to view elements of $\mathfrak{a}_Q^*$ as linear forms on $\mathfrak{a}_P$ which are zero on $\mathfrak{a}_P^Q$.

Let Ad_P^Q adjoint action of M_P on the Lie algebra of $M_Q \cap N_P$. Let ρ_P^Q be the element of $\mathfrak{a}_P^{Q,*}$ such that for every $m \in M_P(\mathbb{A})$:

$$|det(Ad_P^Q(m)| = exp(< 2\rho_P^Q, H_P(m) >).$$

If $Q = G$ we omit Q from the notation and we write ρ for ρ_{P_0}.

If P is a standard parabolic subgroup of G, let Δ_0^P be the set of simple roots of A_0 in $M_P \cap P_0$ and $\Delta_P \subset \mathfrak{a}_P^*$ be the set of restriction to $\mathfrak{a}_P$ of the elements of $\Delta_{P_0} \setminus \Delta_0^P$. If $Q \subset P$, one defines also Δ_P^Q as the set of restrictions to $\mathfrak{a}_P$ of elements of $\Delta_0^Q \setminus \Delta_0^P$. One has also the set of simple coroots $\check{\Delta}_P^Q \subset \mathfrak{a}_P^Q$. By duality we get simple weights $\hat{\Delta}_P^Q$ denoted $\varpi_\alpha, \alpha \in \Delta_P^Q$. If $Q = G$ we omit Q from the notation.

We denote by $\mathfrak{a}_0^+$ the closed Weyl chamber and by $\mathfrak{a}_P^+$ (resp. $\mathfrak{a}_P^{++}$) the set of $X \in \mathfrak{a}_P$ such that $\alpha(X) \geqslant 0$ (resp. > 0) for $\alpha \in \Delta_P$.

If P is a standard parabolic subgroup of G, we say that $\nu \in \mathfrak{a}_{P,\mathbb{C}}^*$ is subunitary (resp. strictly subunitary) if $Re\nu(X) \leqslant 0$ for all $X \in \mathfrak{a}_P^+$ (resp. if $Re\nu = \sum_{\alpha \in \Delta_P} x_\alpha \alpha$ where $x_\alpha < 0$ for all α). If $\nu \in \mathfrak{a}_P^*$, it is viewed as a linear form on $\mathfrak{a}_0$ which is zero on $\mathfrak{a}_0^P$. Then $\nu \in \mathfrak{a}_{P,\mathbb{C}}^*$ is subunitary if and only if one has:

$$Re\nu(X) \leqslant 0, X \in \mathfrak{a}_0^+. \tag{2.1}$$

This follows from the fact that $\alpha \in \Delta_P$ is proportional to $\sum_{\beta \in \Sigma_0^+, \beta_{|\mathfrak{a}_P} = \alpha} \beta$ where Σ_0^+ is the set of positive roots. In fact this sum is invariant by the Weyl group generated by the reflections around roots which are 0 on $\mathfrak{a}_P$.

Let W be the Weyl group of (G, A_0) and choose a scalar product on $\mathfrak{a}_0^G$ invariant by W.

This determines a Lebesgue measure on $\mathfrak{a}_P^G$ and $\mathfrak{a}_P^{G*}$ for every standard parabolic subgroup P. (2.2)

If P, Q are standard parabolic subgroups of G, let $W(Q\backslash G/P)$ be a set of representatives of $Q\backslash G/P$ in W of minimal length. If $s \in W(Q\backslash G/P)$ the subgroup $M_P \cap s^{-1}M_Q s$ is the Levi factor of a standard parabolic subgroup P_s contained in P and $M_Q \cap sM_P s^{-1}$ is the Levi factor of a standard parabolic subgroup Q_s contained in Q. Let $W(P,Q)$ be the set of $s \in W(Q\backslash G/P)$ such that $sM_Ps^{-1} \subset M_Q$. Let $W(P|Q)$ be the set of $s \in W(Q\backslash G/P)$ such that $sM_Ps^{-1} = M_Q$. Hence:

$$W(P|Q) = W(P,Q) \cap W(Q,P)^{-1}.$$

By a Siegel domain $\mathfrak{s}_P$ for $[G]_P$, we mean a subset of $G(\mathbb{A})$ of the form:

$$\mathfrak{s}_P = \Omega_0\{expX|X \in \mathfrak{a}_0, \alpha(X+T) \geqslant 0, \forall\alpha \in \Delta_0^P\}K \tag{2.3}$$

where Ω_0 is a compact of $P_0(\mathbb{A})^1$, $T \in \mathfrak{a}_0$, such that $G(\mathbb{A}) = M_P(F)N_P(\mathbb{A})\mathfrak{s}_P$. Let

$$\mathfrak{a}_0^{+,P} := \{X \in \mathfrak{a}_0 | \alpha(X) \geqslant 0, \forall\alpha \in \Delta_0^P\}.$$

Let us show

Any Siegel set $\mathfrak{s}_P$ is contained in $\Omega_{N_P}\{expX|X \in \mathfrak{a}_0^{+,P}\}\Omega$ where Ω is a compact subset of $G(\mathbb{A})$ and Ω_{N_P} is a compact subset of $N_P(\mathbb{A})$. (2.4)

There is a compact subset, in fact reduced to a single element, $expT$, $\Omega_1 \subset A_0^\infty$ such that

$$\{expX|X \in \mathfrak{a}_0, \alpha(X+T) \geqslant 0, \forall\alpha \in \Delta_0^P\} \subset \{\exp X|X \in \mathfrak{a}_0^{+,P}\}\Omega_1.$$

The compact set Ω_0 is a subset of $\Omega_{N_P}\Omega_{P_0\cap M_P}$, where Ω_{N_P} (resp. $\Omega_{P_0\cap M_P}$) is a compact subset of $N_P(\mathbb{A})$ (resp. $(P_0 \cap M_P)(\mathbb{A})$). Then the conjugate by $exp - X$, $X \in \mathfrak{a}_0^{+,P}$ of $\Omega_{P_0\cap M_P}$ remains in a compact set, Ω_2, when X varies in $\mathfrak{a}_0^{+,P}$. The compact subset $\Omega = \Omega_2\Omega_1 K$ satisfies the required property.

Let Ξ (resp. σ_L) be the function on $G(\mathbb{A})$ (resp. on $[G]$) introduced by Lapid in [18], beginning of section 2 (resp. after the proof of Lemma 2.1 and denoted σ there). From [18] section 2 (9), one has:

There exists $d \in \mathbb{N}$ such that:

$$\int_{P_0(F)\backslash\mathfrak{s}_G} \Xi^2(g)\sigma_L(g)^{-d}dg < \infty. \tag{2.5}$$

If f is a function on $G(\mathbb{A})$ with values in $\mathbb{R}^+$, we denote $f_{[G]_P}$ the function on $[G]_P$ defined by

$$f_{[G]_P}(g) = inf_{\gamma \in M_P(F)N_P(\mathbb{A})} f(\gamma g), g \in G(\mathbb{A}). \tag{2.6}$$

We fix a norm $\|.\|$ on $G(\mathbb{A})$ as in [7], Appendix A.1. From [8] 2.4.1.2, one has:

$$\|g\| \approx \|g\|_{[G]_P}, g \in \mathfrak{s}_P. \tag{2.7}$$

Let us define:

$$\sigma(g) = 1 + log(\|g\|), g \in G(\mathbb{A}).$$

From (2.7), one deduces:

$$\sigma_{[G]_P}(g) \sim \sigma(g) \sim 1 + \|H_0(g)\|, g \in \mathfrak{s}_P, \tag{2.8}$$

where $H_0 := H_{P_0}$, the last relation being a consequence of the properties of $\mathfrak{s}_P$ (cf. (2.3)). We normalize the measures as in [19], I.1.13.

If X is a topological space, let $C(X)$ be the space of complex valued continuous functions on X. Let Ω be a compact subset of $G[\mathbb{A}]$ and $[\Omega]$ its image in $[G]$. Let B be a symmetric bounded neighborhood of 1 in $G(\mathbb{A})$ (a ball) and let $\Xi^{[G]_P}(x) = (vol_{[G]_P} xB)^{-1/2}, g \in [G]_P$.

The equivalence class of the function $\Xi^{[G]}$ on $[G]$ does not depend of the choice of B. (2.9)

2.1 Lemma. *One has:*

(i)

$$\Xi^{[G]}(g) \sim e^{\rho(H_0(g))} \sim \Xi(g), g \in \mathfrak{s}_G.$$

(ii)

$$\sigma_{[G]}(g) \sim 1 + \|H_0(g)\| \sim \sigma_L(g), g \in \mathfrak{s}_G$$

Proof. The first relation of (i) follows from [8], 2.4.2.3. and the second follows easily from the definition of Ξ. The first relation of (ii) follows from (2.8) and the second From [18], 2, after the proof of Lemma 2.1. $\square$

One deduces from (2.5) and from the preceding lemma:

$$\int_{[G]} (1 + \sigma_{[G]}(x))^{-d_G} \Xi^{[G]}(x)^2 dx < \infty. \tag{2.10}$$

Let $\mathbb{A}_f$ be the ring of finite adeles of F. Let $C^\infty(G(\mathbb{A})$ be the spaces of functions which are right invariant by a compact open subgroup of $G(\mathbb{A}_f)$, say J, and which are C^∞ on $G(\mathbb{A})/J$ which is a smooth real differentiable manifold.

Let G_∞ be the product of $G(F_v)$ where v describes the Archimedean places of F and let $U(\mathfrak{g}_\infty)$ be the enveloping algebra of the Lie algebra $\mathfrak{g}_\infty$ of this real Lie group. We have similar definition for subgroups of G defined over F.

One has the Harish-Chandra Schwartz space $\mathcal{C}([G])$, denoted $\mathcal{S}([G])$ in [18], Corollary 2.6. From Lemma 2.1, it can be defined as the space of functions in $C^\infty([G])$ such that for all $n \in \mathbb{N}$ and $u \in U(\mathfrak{g}_\infty)$:

$$|(R_u\phi)(x)| << \sigma_{[G]}^{-n}(x)\Xi^{[G]}(x), x \in [G],$$

where R denotes the right regular representation of $U(\mathfrak{g}_\infty)$.

3. Tempered automorphic forms

3.1. Definition of temperedness

The space of automorphic forms on $[G] = G(F)\backslash G(\mathbb{A})$, $\mathcal{A}(G)$ is defined as in [19], I.2.17. In particular they are K-finite.

If $\phi \in \mathcal{A}(G)$, it has uniform moderate growth on $G(\mathbb{A})$ (cf. [19] end of I.2.17, Lemma I.2.17 and Lemma I.2.5).

It means that there exists $r > 0$ such that for all $u \in U(\mathfrak{g}_\infty)$:

$$|R_u\phi(g)| << \|g\|^r, g \in G(\mathbb{A}).$$

Let Ω be a compact subset of $G(\mathbb{A})$. Then one sees easily that this implies that for all $u \in U(\mathfrak{g}_\infty)$, one has

$$|R_uR_\omega\phi(g)| << \|g\|^r, g \in G(\mathbb{A}), \omega \in \Omega. \tag{3.1}$$

If P is a standard parabolic subgroup of G, the space of automorphic forms on $[G]_P = N_P(\mathbb{A})M(F)\backslash G(\mathbb{A})$ denoted $A(N_P(\mathbb{A})M(F)\backslash G(\mathbb{A}))$ in [19] will be denoted $\mathcal{A}_P(G)$. The constant term along P (cf. [19], I.2.6), ϕ_P, of an element ϕ of $\mathcal{A}(G)$ is an element of $\mathcal{A}_P(G)$. Similarly if Q is a standard parabolic subgroup contained in P and $\phi \in \mathcal{A}_P(G)$, ϕ_Q is a well defined element of $\mathcal{A}_Q(G)$. Let $\mathcal{A}_P^n(G)$ be the space of elements $\phi \in \mathcal{A}_P(G)$ such that:

$$\phi(expXg) = e^{\rho_P(X)}\phi(g), g \in G(\mathbb{A}), X \in \mathfrak{a}_P.$$

If $\phi \in \mathcal{A}_P(G)$, and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we define

$$\phi_\lambda(g) = e^{\lambda(H_P(g))}\phi(g), g \in G(\mathbb{A}).$$

We view $S(\mathfrak{a}_P^*)$ as the space of polynomial functions on $\mathfrak{a}_P$ and $S(\mathfrak{a}_P^*) \otimes \mathcal{A}_P^n(G)$ as a space of functions on $G(\mathbb{A})$ by setting $(p \otimes \phi)(g) = p(H_P(g))\phi(g)$. If $\phi \in \mathcal{A}_P(G)$, one can write it uniquely as:

$$\phi(g) = \sum_{\lambda \in \mathcal{E}_P(\phi)} e^{\lambda(H_P(g))}(\phi_{0,\lambda})(g), \tag{3.2}$$

where $\mathcal{E}_P(\phi) \subset \mathfrak{a}_{P,\mathbb{C}}^*$, $\phi_{0,\lambda}$ is a non zero element of $S(\mathfrak{a}_P^*) \otimes \mathcal{A}_P^n(G)$. The set $\mathcal{E}_P(\phi)$ is called the set of exponents of ϕ. We define also $\mathcal{A}_P^2(G)$ as the subspace of elements ϕ of $\mathcal{A}_P^n(G)$ such that:

$$\|\phi\|_P^2 := \int\limits_{A_P^\infty M_P(F) N_P(\mathbb{A}) \backslash G(\mathbb{A})} |f(x)|^2 dx.$$

We will need the following variant of [19], Lemma I.2.10:

3.1 Lemma. *Let $P = MU$ be a standard parabolic subgroup of G. Let $\mathbb{A}_f$ be the ring of finite adeles of F. Let K_f' be a compact open subgroup of $G(\mathbb{A}_f)$ and $u \in U(\mathfrak{g}_\infty)$, $c > 0$ and $t > 0$. Let $n \in \mathbb{N}$.*

Then there exists two finite subsets $\{u_i | i = 1, \ldots N\} \subset U(\mathfrak{g}_\infty)$ and $\{c_i | i = 1, \ldots, N\} \subset \mathbb{R}^{+*}$ such that the following property is satisfied:

Let ϕ be a smooth function (see [19] section I.2.5 for the definition) on $U_0(F)\backslash G(\mathbb{A})$, right invariant by K_f'. Let $r > 0$ and $\lambda \in \mathfrak{a}_{M_0}^*$. Suppose that for all $a \in A_G^\infty$, $g \in G(\mathbb{A})^1 \cap \mathfrak{s}_G$, $i \in \{1, \ldots, N\}$, we have the inequality:

$$|R_{u_i}\phi(ag)| \leqslant c_i \|a\|^r e^{\lambda(H_{P_0}(g))}$$

(resp.

$$|R_{u_i}\phi(ag)| \leqslant c_i(1 + \|H_G(a)\|)^r e^{\lambda(H_{P_0}(g))}(1 + \|H_{P_0}(g)\|)^N.)$$

Then for all $a \in A_G^\infty, g \in G(\mathbb{A})^1 \cap \mathfrak{s}_G$, one has the inequality:

$$|R_u(\phi - \phi_P)(ag)| \leqslant c\|a\|^r e^{\lambda - t\beta_P(H_{P_0}(g))}$$

(resp.

$$|R_u(\phi - \phi_P)(ag)| \leqslant c(1 + \|H_G(a)\|)^r e^{\lambda - t\beta_P(H_{P_0}(g))}(1 + \|H_{P_0}(g)\|)^N)$$

where

$$\beta_P(X) = \inf_{\alpha \in \Delta_0 \backslash \Delta_0^P} \alpha(X), X \in \mathfrak{a}_0.$$

Proof. The statement and the proof of [19], Lemma I.2.10 remains true if one changes P to a standard parabolic subgroup and, in the conclusion, one changes α to β_P. This gives the first statement. If one replaces $m_{P_0}^{\lambda}(g)$ in [19] in the hypothesis by $m_{P_0}^{\lambda}(g)(1+\|log(m_{P_0}(g))\|)^n$ for some n, one can replace $m_{P_0}(g)^{\lambda-t\alpha}$ by $m_{P_0}(g)^{\lambda-t\beta_P}(1+\|log(m_{P_0}(g))\|)^n$ in the conclusion: one has to use that $(1+\|(logm_{P_0}(g))\|)^n$ is $U_0(\mathbb{A})$ invariant after (6) in the proof.

One can also replace in the statement $\|a\|^r$ by $(1+\|loga\|)^r$ for $a \in A_G$.

Altogether this gives the second statement. $\square$

3.2 Lemma. *Let P be a standard parabolic subgroup of G. Let $d > 0$. Let $\phi \in \mathcal{A}_P(G)$. The following conditions are equivalent:*

a)

$$|\phi(x)| << \Xi^{[G]_P}(x)\sigma_{[G]_P}(x)^d, x \in [G]_P.$$

b) For all Siegel domains $\mathfrak{s}_P$, one has:

$$|\phi(g)| << e^{\rho(H_0(g))}(1+\|H_0(g)\|)^d, g \in \mathfrak{s}_P.$$

c) For every compact subset Ω of $G(\mathbb{A})$, one has:

$$|\phi(expX\omega)| << e^{\rho(X)}(1+\|X\|)^d, \omega \in \Omega, X \in \mathfrak{a}_0^{+,P}.$$

Proof. a) is equivalent to b) follows from Lemma 2.1 and (2.9).

To prove c) implies b), we choose (cf. (2.4)), a compact subset Ω of $G(\mathbb{A})$ and a compact subset, Ω_{N_P} of $N_P(\mathbb{A})$ such that $\mathfrak{s}_P \subset \Omega_{N_P} A_0^{\infty,+,P}\Omega$, where $A_0^{\infty,+,P} = \{expX | X \in \mathfrak{a}_0^{+,P}\}$. One has for $g \in \mathfrak{s}_P$:

$$\|X - H_0(g)\| << 1, g = \omega_{N_P} expX\omega, X \in \mathfrak{a}_0^{+,P}, \omega_{N_P} \in \Omega_{N_P}, \omega \in \Omega. \tag{3.3}$$

Then c) implies b) follows.

Similarly b) implies c). $\square$

3.3 Definition. Let us define the space of tempered automorphic forms on $[G]_P$, $\mathcal{A}_P^{temp}(G)$, as the space of automorphic forms satisfying, as well as its derivatives by elements of $U(\mathfrak{g}_\infty)$, the equivalent properties a), b), c) of the preceding Lemma for some d.

3.4 Remark. This notion was introduced by J. Franke in [14] (cf. also [20], section 4.4 where the space of tempered form is denoted $\mathcal{A}_{log}(G)$).

Let us recall some facts from [5]. With the notation there, one can take $dm_X(x) = (\Xi^{[G]})(x)^2 dx$ where dx is the $G(\mathbb{A})$-invariant measure on $X = [G]$, as it follows from the proof of [5] Lemma 3.3(ii). From (2.10) and the Criterion in [5] Section 3.3 it follows that the weight $\sigma_{[G]}^{-d}$ on $[G]$ is summable. In this context, J. Bernstein introduced in [5], the notion of $[G]$-temperedness for smooth functions on $[G]$:

A smooth function on $[G]$ is said $[G]$-tempered if there exists $d > 0$ such that for every of its derivatives by elements of $U(\mathfrak{g}_\infty)$, ϕ verifies:

$$(1 + \sigma_{[G]})^{-d}\phi \in L^2([G]).$$

3.5 Proposition. *For $\phi \in \mathcal{A}(G)$ the following conditions are equivalent:*

(i) ϕ is tempered.

(ii) ϕ is $[G]$-tempered.

Proof. (i) implies (ii) follows from the definition of temperedness, especially condition a) in Lemma 3.2 and (2.10).

(ii) Let ϕ be $[G]$-tempered, i.e. there exists $d > 0$ such that all its derivatives by elements of $U(\mathfrak{g}_\infty)$ are in $L^2([G], (1 + \sigma_{[G]})^{-d}dx)$.

Let us use the notation of [5], Lemma-Definition 3.3. From the proof of this Lemma, one can take $dm_X(x) = \nu(x)dx$ where dx is the $G(\mathbb{A})$-invariant measure on $X = [G]$ and $\nu = (\Xi^{[G]})^2$, as it follows from the proof of [5] Lemma-Definition 3.3 (ii).

Let $k \geqslant dimG$ and f be a continuously k-times differentiable function on $G(\mathbb{A})$. Fix a basis $d_1, \dots, d_r$ of the space $U(\mathfrak{g}_\infty)^k$ of elements of $U(\mathfrak{g}_\infty)$ of degree $\leqslant k$ and define:

$$Q(f) = \sum_i |d_i f|^2.$$

Let J be a compact open subgroup of $G(\mathbb{A}_f)$. Let $C([G])^{k,J}$ be the space of continuously k-times differentiable and fixed by J. One has from the Key Lemma of [5], p. 686:

$$|f(x)|^2 << \int_{xB} Q(f)dm_X = \int_{xB} Q(f)\nu(y)dy, x \in [G], f \in C([G])^{k,J}.$$

Let $w = \sigma_{[G]}^d$. Then

$$|f(x)|^2 << \int_{xB} Q(f)\nu w w^{-1}dy, x \in [G].$$

We use now that w is a weight, in the sense of [5], Definition 3.1, as well as ν to get:

$$|f(x)|^2 << \nu(x)w(x) \int_{xB} Q(f)w^{-1}dx \leqslant \nu(x)w(x) \int_{[G]} Q(f)w^{-1}dx, x \in [G].$$

As our hypothesis implies that $\int_{[G]} Q(f)w^{-1}dx < \infty$, this finishes the proof of the Lemma. $\square$

3.2. Characterization of temperedness and definition of the weak constant term

3.6 Proposition. *Let $\phi \in \mathcal{A}(G)$. It is in $\mathcal{A}_G^{temp}(G)$ if and only if the exponents of its constant term ϕ_P along any standard parabolic subgroup P of G are subunitary.*

Proof. Let us show that the condition is necessary. Let $\phi \in \mathcal{A}^{temp}(G)$. Let $X \in \mathfrak{a}_G$. Then there exists d such that for all $g \in [G]$:

$$|\phi(gexptX)| << (1+t)^d, t > 0.$$

It follows from [11], Proposition A.2.1, that the exponents of ϕ restricted to $\mathfrak{a}_G$ are unitary. Let P be a standard parabolic subgroup of G. Let $X \in \mathfrak{a}_P^{++} \subset \mathfrak{a}_0^+$, where $\mathfrak{a}_P^{++}$ is the relative interior of $\mathfrak{a}_P^+$. Let $g \in G(\mathbb{A}), t \in \mathbb{R}$. From the property c) of the definition of temperedness, applied to $\Omega = g$, one gets:

$$|\phi(exptXg)| << e^{\rho(tX)}(1+t)^d, t > 0. \tag{3.4}$$

Due to (3.1), one can apply Lemma 3.1 to the right translate by g of ϕ. Write $X = X_G + X^G$ with $X_G \in \mathfrak{a}_G, X^G \in \mathfrak{a}^G$. Applying the second statement of Lemma 3.1 for the parameter t large and with g equals to $exptX^G \in \mathfrak{s}_G \cap G(\mathbb{A})^1$, a equals to $exptX_G$, one gets, for $k > 0$,

$$|\phi(exptXg) - \phi_P(exptXg)| << e^{\rho(tX)}e^{-kt}, t > 0.$$

Together with (3.4) this implies that the exponential polynomial in t, $\phi_P(exptXg)$ satisfies:

$$|\phi_P(exptXg)| << e^{\rho(tX)}(1+t)^d, t > 0.$$

There is a dense open set O in $\mathfrak{a}_P^{++}$ such that different exponents of ϕ_P take different values on any element of O. We use the notation of (3.2). Then [11], Proposition A.2.1 gives:

If $\phi_{P,0,\lambda}(g) \neq 0$:

$$Re\lambda(X) \leqslant 0, \lambda \in \mathcal{E}_P(\phi), X \in O.$$

As $\phi_{P,0,\lambda}$ is not identically zero, one has $Re\lambda(X) \leqslant 0$ for $X \in O$, hence also for $X \in \mathfrak{a}_P^+$ by density. This achieves the proof of (i).

The sufficiency of the condition follows from [19], Lemma I.4.I. □

3.7 Definition. Let $\phi \in \mathcal{A}^{temp}(G)$ and let P be a standard parabolic subgroup of G. We define the weak constant term of ϕ, denoted ϕ_P^w as the sum of the terms for ϕ_P in (3.2) corresponding to unitary exponents. It is an element of $\mathcal{A}_P^{temp}(G)$ from the preceding proposition applied to M_P (see below Lemma 3.8 for a detailed proof). Let $\phi_P^- = \phi_P - \phi_P^w$ and let us denote $\mathcal{E}_P^w(\phi)$ (resp. $\mathcal{E}_P^-(\phi)$) the exponents of ϕ_P^w (resp. ϕ_P^-).

3.3. Transitivity of the weak constant term

Let $Q \subset P$ be standard parabolic subgroups of G. If ϕ is a function on $G(\mathbb{A})$ and $k \in K$, we define a function on $M_Q(\mathbb{A})$ by:

$$\phi^{k,M_Q}(m_Q) = e^{-\rho_Q(H_Q(m_Q))}\phi(m_Q k), m_Q \in M_Q(\mathbb{A}) \tag{3.5}$$

where $\rho_Q \in \mathfrak{a}_Q$ is the restriction of ρ to $\mathfrak{a}_Q$, which can be extended to $\mathfrak{a}_0$ by zero on $\mathfrak{a}_0^Q$. One has the following immediate properties, by coming back to the definitions:

If $\phi \in \mathcal{A}(G)$ one has:

$$\phi_Q = (\phi_P)_Q, (\phi_Q)^{M_Q,k} = [((\phi_P)^{M_P,k})_{Q \cap M_P}]^{M_Q,1}, k \in K. \tag{3.6}$$

Notice that the function in bracket is a function on $M_P(\mathbb{A})$, hence the upper index ^{M_Q,1} indicates that we multiply by $e^{-\rho_{Q\cap M_P}(H_Q(m))}$ the restriction of this function to $M_Q(\mathbb{A})$.

3.8 Lemma. *Let $Q \subset P$ be as above.*

(i) If $\phi \in \mathcal{A}_P^{temp}(G)$ the exponents of ϕ_Q are subunitary and one can define ϕ_Q^w as the sum of the terms of ϕ_Q corresponding to unitary exponents.

If $\phi \in \mathcal{A}^{temp}(G)$ one has:

(ii) ϕ_P^w is in $\mathcal{A}_P^{temp}(G)$.

(iii)

$$\phi_Q^w = (\phi_P^w)_Q^w.$$

(iv)

$$(\phi_Q^w)^{M_Q,k} = [((\phi_P^w)^{M_P,k})_{Q\cap M_P}^w]^{M_Q,1}, k \in K.$$

Proof. (i) is proved as Proposition 3.6 (i).

(ii) From (3.6) and Proposition 3.6 applied to ϕ, one sees that the exponents of $(\phi_P^w)^{k,M_P}$ are subunitary. Hence by this Proposition applied to M_P, one sees: $\phi_P^{w,M_P,k} \in \mathcal{A}^{temp}(M_P)$. Let Ω_{M_P} be a compact subset of $M_P(\mathbb{A})$. Using K-finiteness, this gives that there exists $d \in \mathbb{N}$ such that:

$$|\phi_P(expX\omega k)| << e^{\rho(X)}(1 + \|X\|)^d, X \in \mathfrak{a}_0^{+,P}, \omega \in \Omega_{M_P}, k \in K.$$

Every compact subset of $G(\mathbb{A})$ is contained in a set of the form $N_P(\mathbb{A})\Omega' K$ where Ω' is a compact subset of $M_P(\mathbb{A})$. Hence, recalling the definition of $\mathcal{A}_P^{temp}(G)$ (cf. Definition 3.3), the preceding estimate achieves the proof of (ii).

(iii) Write $\phi_P = \phi_P^w + \phi_P^-$. Then none of the exponents of $(\phi_P^-)_Q$ is unitary. Hence as $\phi_Q = (\phi_P)_Q$ we get (iii).

(iv) follows from the second assertion of (3.6). $\square$

3.4. A characterization of elements of square integrable automorphic forms

It follows from [19], Lemmas I.4.1 and I.4.11 that:

$$\mathcal{A}^2(G) \subset \mathcal{C}(G).$$

From this, (2.10) and Proposition 3.5, one has:

For all $\phi \in \mathcal{A}^{temp}(G)$ and $\psi \in \mathcal{A}^2(G)$, for all $X \in \mathfrak{a}_G$, the integral

$$\int_{[G]^1} \phi(g_1 expX)\overline{\psi}(g_1 expX) dg_1$$

is absolutely convergent. It is denoted $(\phi, \psi)_G^X$. Moreover $X \mapsto (\phi, \psi)_G^X$ is an exponential polynomial in X. One defines similarly for $\phi \in \mathcal{A}_P^2(G), \psi \in \mathcal{A}_P^{temp}(G)$, an exponential polynomial, $p_P(\phi, \psi)$ on $\mathfrak{a}_P$ by $p_P(\phi, \psi)(X) = (\phi, \psi)_P^X$, $X \in \mathfrak{a}_P$, where: (3.7)

$$(\phi, \psi)_P^X = e^{-2\rho_P(X)} \int_{M(\mathbb{A})^1 \times K} \phi(exp_P X m^1 k)\overline{\psi(exp_P X m^1 k)} dm^1 dk.$$

We denote by $\mathcal{A}^{temp,c}(G)$ the space of $\psi \in \mathcal{A}^{temp}(G)$ such that for all $\phi \in \mathcal{A}^2(G)$, the polynomial $p_G(\phi, \psi)$ is zero. We define similarly $\mathcal{A}_P^{temp,c}(G)$.

Then one has a direct sum: $\mathcal{A}_P^{temp,c}(G) \oplus \mathcal{A}_P^2(G)$ and one can define, for $\phi = \phi_1 + \phi_2 \in \mathcal{A}_P^{temp,c}(G) \oplus \mathcal{A}_P^2(G)$ and $\psi \in \mathcal{A}_P^{temp}(G)$, an exponential polynomial denoted $p_P(\phi, \psi)$ equal to $p_P(\phi_2, \psi)$.

3.9 Remark. If ψ is moreover in $S(\mathfrak{a}_P^*) \otimes \mathcal{A}_P^n(G)$, $p(\phi, \psi)$ is a polynomial.

With these definitions one has:

3.10 Lemma. *(i) Let Q be a standard parabolic subgroup of G and let $\phi \in \mathcal{A}_Q^{temp}(G) \cap \mathcal{A}_Q^n(G)$ (resp. $\phi \in \mathcal{A}^{temp}(G)$) such that $\phi_P^w = 0$ for any standard parabolic subgroup of G with $P \subset Q$, $Q \neq P$, P standard then $\phi \in \mathcal{A}_Q^2(G)$ (resp. ϕ is a linear combination of products of exponential polynomials on $\mathfrak{a}_G$ with elements of $\mathcal{A}^2(G)$).*

(ii) If $\phi \in \mathcal{A}^{temp}(G)$ and $\phi_Q^w \in \mathcal{A}_Q^{temp,c}(G)$ for all standard parabolic subgroup Q of G, then $\phi = 0$.

Proof. (i) We first prove the result for $Q = G$ and $\phi \in \mathcal{A}_Q^{temp}(G) \cap \mathcal{A}_Q^n(G)$. Let us show that for any standard parabolic subgroup of G, $P \neq G$, the exponents of ϕ_P are strictly subunitary. Let ν be such an exponent. From the hypothesis it is subunitary but not unitary. If it is not strictly subunitary, there exists $\alpha \in \Delta_P$ such that:

$$Re\nu = \sum_{\beta \in \Delta_P \setminus \{\alpha\}} x_\beta \beta, x_\beta \leqslant 0.$$

Let Q be the maximal parabolic subgroup of G, containing P, such that $\Delta_Q = \{\alpha_{|\mathfrak{a}_Q}\}$. Then $\nu_{|\mathfrak{a}_Q}$ is an exponent of ϕ_Q. But it is clear that it is unitary, hence ϕ_Q^w is non zero which contradicts our hypothesis. Hence ν is strictly subunitary. Then (i) for $Q = G$ and $\phi \in \mathcal{A}_Q^{temp}(G) \cap \mathcal{A}_Q^n(G)$ follows from Lemma I.4.11 of [19]. For $\phi \in \mathcal{A}_Q^{temp}(G) \cap \mathcal{A}_Q^n(G)$, (i) follows from Lemma 3.8 (iv) and what we have just proved for M_Q instead of G.

If $\phi \in \mathcal{A}^{temp}(G)$, it is equal to a linear combination of products of exponential polynomials on $\mathfrak{a}_G$ with elements of $\mathcal{A}_Q^{temp}(G) \cap \mathcal{A}_Q^n(G)$ to which one can apply what we have just proved. (i) follows.

Let us prove (ii) by induction on the dimension on $\mathfrak{a}_0^G$. If it is zero the claim is clear. Suppose now $dim\mathfrak{a}_0^G > 0$. By applying the induction hypothesis to M_P for a strict standard parabolic subgroup P of G and Lemma 3.8, one sees that $\phi_P^w = 0$. Hence by (i), ϕ is a linear combination of products of exponential polynomials on $\mathfrak{a}_G$ with elements of $\mathcal{A}^2(G)$. As ϕ is in $\mathcal{A}^{temp,c}(G)$, one deduces from this that $\phi = 0$. $\square$

4. Uniform temperedness of Eisenstein series

4.1. Exponents of Eisenstein series

Let P be a standard parabolic subgroup of G. Let $\phi \in \mathcal{A}_P^2(G)$. Let $E_P(., \phi, \lambda)$ be the Eisenstein series (cf. [6], (2.1)). Let P, Q be two standard parabolic subgroups of G and $w \in W(P|Q)$. One has the operators $M(w, \lambda) : \mathcal{A}_P^2(G) \to \mathcal{A}_Q^2(G)$ meromorphic in $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ when restricted to a finite dimensional space of $\mathcal{A}_P^2(G)$ (cf. [6], after (2.1) and Theorem 2.3 (3).

One can define $E_P^Q(., \phi, \lambda)$ which is, when defined, in $\mathcal{A}_Q(G)$ and is characterized by:

$$E_P^Q(., \phi, \lambda)^{M_Q,k} = E_{P\cap M_Q}(., \phi^{M_Q,k}, \lambda), k \in K. \tag{4.1}$$

They are analytic on the imaginary axis: this follows from [17], Proposition 1, as the Working Hypothesis of this article follows from [6], Theorem 2.3 and Corollary 8.6. We recall the formula for generic λ (cf. [17], Proposition 4, with the notation there)

$$(E_P(., \phi, \lambda))_Q = \sum_{s \in W(Q\backslash G/P)} E_{Q_s}^Q(., M(s, \lambda)\phi_{P_s}, s\lambda). \tag{4.2}$$

The exponents of $E_P(., \phi, \lambda)_Q$ are given by [17], equation (13). Moreover they are subunitary by [17] Lemma 6 for λ unitary. Hence by Proposition 3.6, $E_P(., \phi, \lambda)$ is tempered for $\phi \in \mathcal{A}_P^2(G), \lambda \in i\mathfrak{a}_P^*$ and the weak constant term is given by:

$$(E_P(., \phi, \lambda))_Q^w = \sum_{s \in W(P,Q)} E_{Q_s}^Q(., M(s, \lambda)\phi, s\lambda) \tag{4.3}$$

which is holomorphic in a neighborhood of $i\mathfrak{a}_P^*$. The same is true for $E_P(., \phi, \lambda)_Q^- = E_P(., \phi, \lambda)_Q - E_P(., \phi, \lambda)_Q^w$ whose exponents are contained in

$$\mathcal{E}_Q^-(\lambda) = \cup_{s \in W(Q\backslash\ G/P)\backslash W(P,Q)} \{s(\mathcal{E}_{P_s}(\phi) + \lambda)_{|\mathfrak{a}_Q}\}.$$

Hence, by analyticity, this inclusion holds for all λ in $i\mathfrak{a}_P^*$. This implies:

For $\lambda \in i\mathfrak{a}_P^*$, $X \mapsto E_P(expXg, \phi, \lambda)_Q^-, X \in \mathfrak{a}_Q$ is an exponential polynomial with exponents in a set $\tilde{\mathcal{E}}_Q^-(\lambda)$, where $\tilde{\mathcal{E}}_Q^-(\lambda)$ is built from $\mathcal{E}_Q^-(\lambda)$ by some repetitions, the multiplicities depending on the multiplicities of the exponents of ϕ. Moreover the real parts of the exponents above do not depend on $\lambda \in i\mathfrak{a}_P^*$. (4.4)

4.2. Uniform temperedness of Eisenstein series

Let Λ be a compact subset of $i\mathfrak{a}_P^*$.

Let $\mu \in \mathfrak{a}_0^{G,+}$, $n \in \mathbb{N}$. Let $\mathcal{F}_{\Lambda,\mu,n}$ be the space of functions F on $G(\mathbb{A}) \times \Lambda$ which satisfy for every compact subset Ω of $G(\mathbb{A})$ and $u \in U(\mathfrak{g}_\infty)$:

$$\begin{gathered} |R_u F(expX\omega, \lambda)| << (1 + \|X\|)^n e^{\mu(X)}, X \in \mathfrak{a}_0^+, \omega \in \Omega, \lambda \in \Lambda, \\ F(expXg, \lambda) = e^{\lambda(X)} F(g, \lambda), X \in \mathfrak{a}_G, g \in G(\mathbb{A}), \lambda \in \Lambda. \end{gathered} \tag{4.5}$$

4.1 Proposition. *Let $E : [G] \times \Lambda \to \mathbb{C}$ be defined by: $E(g, \lambda) = E_P(g, \phi, \lambda)$ for $\phi \in \mathcal{A}_P^2(G)$. Then there exists $n \in \mathbb{N}$ such that*

$$E \in \mathcal{F}_{\Lambda,\rho,n}.$$

Proof. We will need the fact that lemma 1.4.1 of [19] holds uniformly for a set of automorphic forms which is bounded in a space of functions with given moderate growth and whose constant terms uniformly satisfy the assumption of that lemma. This is easy to see from the proof given in [19]. This applies to Eisenstein series from [6], Corollary 8.6, from the holomorphy of Eisenstein series on the imaginary axis and from (4.1), (4.2). □

Let us show:

$$\text{If } P \neq G,\ E_P(., \phi, \lambda) \in \mathcal{A}^{temp,c}(G) \text{ for all } \phi \in \mathcal{A}_P^2(G), \lambda \in i\mathfrak{a}_P^*. \tag{4.6}$$

Let $X \in \mathfrak{a}_G$ and us look to

$$I(\lambda) = \int\limits_{[G]^1} E_P(xexpX, \phi, \lambda)\overline{\psi}(xexpX)dx, \lambda \in i\mathfrak{a}_P^*$$

for $\psi \in \mathcal{A}^2(G)$. From the uniform temperedness of Eisenstein series, it is a continuous function in λ. Let z be an element of the center $Z([\mathfrak{g}_\infty, \mathfrak{g}_\infty])$ of the enveloping algebra of $[\mathfrak{g}_\infty, \mathfrak{g}_\infty]$. Let us assume that z^* is in the cofinite dimensional ideal of $Z([\mathfrak{g}_\infty, \mathfrak{g}_\infty])$ which annihilates ψ. One can assume that ϕ is $Z(\mathfrak{g}_\infty)$ eigen and let p_z be the polynomial on $i\mathfrak{a}_P^*$ such that $R_z E_P(., \phi, \lambda) = p_z(\lambda) E_P(., \phi, \lambda), \lambda \in i\mathfrak{a}_P^*$. Then on one hand:

$$\int_{[G]^1} R_z E_P(xexpX, \phi, \lambda)\overline{\psi}(xexpX)dx = p_z(\lambda)I(\lambda), \lambda \in i\mathfrak{a}_P^*$$

and on the other hand, taking adjoint, this integral is zero.

Moreover by the cofinite dimension of the annihilator in $Z([\mathfrak{g}_\infty, \mathfrak{g}_\infty])$ of ψ, there exists z as above such that p_z in non identically zero. Then it follows, by continuity and density, that $I(\lambda)$ is identically zero. This proves our claim.

5. Wave packets

5.1. Difference of a tempered automorphic form with its weak constant term

Let Q be a parabolic subgroup of G. For $\delta > 0$, we define:

$$\mathfrak{a}_{Q,\delta}^{G,+} = \{X \in \mathfrak{a}_Q^{G,+} | \alpha(X) \geqslant \delta \|X\|, \forall \alpha \in \Delta_Q\}.$$

5.1 Lemma. *Let $\phi \in \mathcal{A}^{temp}(G)$. Let Ω be a compact subset of $G(\mathbb{A})$. Let $\delta > 0$. Then there exists $\varepsilon > 0$ and $d \in \mathbb{N}$ such that:*

$$|(\phi - \phi_Q^w)(n_Q expX expY\omega)| << (1 + \|X\|)^d e^{\rho(X)} e^{\rho(Y) - \varepsilon\|Y\|},$$
$$n_Q \in N_Q(\mathbb{A}), X \in \mathfrak{a}_0^+, Y \in \mathfrak{a}_{Q,\delta}^{G,+}, \omega \in \Omega.$$

Proof. Using that conjugation by $exp - X$ and $exp - Y$ contracts $N_Q(\mathbb{A})$ and as $N_Q(F)\backslash N_Q(\mathbb{A})$ is compact, possibly changing Ω, one is reduced to prove a similar claim, but without n_Q.

Let $S_{Q,\delta}$ be the intersection of the unit sphere of $\mathfrak{a}_Q$ with $\mathfrak{a}_{Q,\delta}^{G,+}$. It is compact. Let us look to the family of exponential polynomials in $t \in \mathbb{R}$:

$$p_{Y,X,\omega}(t) := \phi_Q(expXexptY\omega) - \phi_Q^w(expXexptY\omega), X \in \mathfrak{a}_0^+, Y \in S_{Q,\delta}, \omega \in \Omega.$$

On one hand, from the definition of temperedness of ϕ and Lemma 3.1 one gets that there exists $d \in \mathbb{N}$ such that:

$$|\phi_Q(expXexptY\omega)| << (1 + \|X\|)^d e^{\rho(X)} (1 + t)^d e^{t\rho(Y)}, X \in \mathfrak{a}_0^+, Y \in S_{Q,\delta}, \omega \in \Omega, t > 0.$$

On the other hand the temperedness of ϕ_Q^w (cf. Lemma 3.8) and the definition of the temperedness of ϕ_Q^w implies a similar bound for $\phi_Q^w(expXexptY\omega)$. Hence by difference it follows that there exists $d \in \mathbb{N}$ such that:

$$|p_{Y,X,\omega}(t)| << (1 + \|X\|)^d e^{\rho(X)}(1+t)^d e^{t\rho(Y)}, X \in \mathfrak{a}_0^+, Y \in S_{Q,\delta}, \omega \in \Omega, t > 0. \quad (5.1)$$

Moreover the exponents of these exponential polynomials are equal to $\mu(Y)+\rho(Y)$ where μ is an exponent of ϕ_Q which is not imaginary. Hence its real part is equal to $\sum_{\alpha\in\Delta_Q} c_\alpha \alpha$ with $Rec_\alpha \leqslant 0$, with at least one Rec_α non zero. Hence there exists $\varepsilon' > 0$ such that for $Y \in S_{Q,\delta}$, $Re\mu(Y) < -\varepsilon'$.

By applying (5.1) to Ω' such that Ω' contains $\{exptY||t| < \varepsilon'', Y \in S_{Q,\delta}\}\Omega$, one gets that the modulus of these polynomials restricted to the interval $[-\varepsilon'', \varepsilon'']$ is bounded by a constant times $(1 + \|X\|)^d e^{\rho(X)}$. Applying Lemma 3 of [17] to the polynomials $[(1 + \|X\|)^d e^{\rho(X)}]^{-1} p_{Y,X,\omega}$, one gets:

$$|(\phi_Q - \phi_Q^w)(n_Q expXexpY\omega)| << (1 + \|X\|)^d e^{\rho(X)} e^{\rho(Y)-\varepsilon\|Y\|},$$
$$n_Q \in N_Q(\mathbb{A}), X \in \mathfrak{a}_0^+, Y \in \mathfrak{a}_{Q,\delta}^{G,+}, \omega \in \Omega.$$

Then one uses Lemma 3.1 to have a similar bound for $|\phi - \phi_Q|$. By addition, this gives the Lemma. □

5.2 Lemma. *Let $\phi \in \mathcal{A}^{temp}(G)$. Let $\delta > 0$ and $\mathfrak{a}_{0,Q,\delta}^{G+} := \{X \in \mathfrak{a}_0^{G+} | \alpha(X) \geqslant \delta\|X\|, \forall \alpha \in \Delta_{P_0} \setminus \Delta_{P_0}^Q\}$. Let Ω be a compact subset of $G(\mathbb{A})$.*

Let $\phi \in \mathcal{A}^{temp}(G)$. There exists $\varepsilon > 0$ such that

$$|\phi(n_Q expX\omega) - \phi_Q^w(n_Q expX\omega)| << e^{\rho(X)-\varepsilon\|X\|}, n_Q \in N_Q(\mathbb{A}), X \in \mathfrak{a}_{0,Q,\delta}^{G+}, \omega \in \Omega.$$

Proof. For $X \in \mathfrak{a}_{0,Q,\delta}^{G+}$, let Y be the element of $\mathfrak{a}_Q^G$ such that $\alpha(Y) = \alpha(X), \alpha \in \Delta_{P_0} \setminus \Delta_{P_0}^Q$. Then, looking at coordinates in $\mathfrak{a}_Q^G$, one sees that there exists $\delta_1 > 0$ such that $\delta_1\|Y\| \leqslant \delta\|X\|$. Hence $Y \in \mathfrak{a}_{Q,\delta_1}^+$. Moreover as $X \in \mathfrak{a}_{0,Q,\delta}^{G+}$, $X' = X - Y$ is in $\mathfrak{a}_0^{G+}$. One gets the required estimate by using the preceding lemma with X' instead of X as:

$$\|X'\| << \|X\| + \|Y\| << \|X\|$$

and if $\alpha \in \Delta_{P_0} \setminus \Delta_{P_0}^Q$,

$$\delta\|X\| \leqslant \alpha(Y) << \|Y\|. \quad \square$$

5.3 Lemma. *Let Ω be a compact subset of $G(\mathbb{A})$ and let $\delta > 0$. Let Λ be a bounded subset of $i\mathfrak{a}_P^*$ and $\phi \in \mathcal{A}_P^2(G)$. There exists $\varepsilon > 0$ such that:*

$$|E_P(n_Q expX\omega, \phi, \lambda) - E_P(n_Q expX\omega, \phi, \lambda)_Q^w| << e^{\rho(X)-\varepsilon\|X\|},$$
$$n_Q \in N_Q(\mathbb{A}), X \in \mathfrak{a}_{0,Q,\delta}^{G+}, \omega \in \Omega, \lambda \in \Lambda.$$

Proof. The proof is similar to the proof of the preceding lemma. One has to prove an analogous of Lemma 5.1 for Eisenstein series by using Proposition 4.1, that the real part of the exponents of $E_P(., \phi, \lambda)_Q$ do not depend of $\lambda \in i\mathfrak{a}_P^*$ and the expression of the weak constant term of Eisenstein series (cf. (4.3)). $\square$

5.2. Wave packets in the Schwartz space

5.4 Proposition. *Let a be a smooth compactly supported function on $i\mathfrak{a}_P^*$ and $\phi \in \mathcal{A}_P^2(G)$. Then the wave packet*

$$E_a := \int_{i\mathfrak{a}_P^*} a(\lambda) E_P(., \phi, \lambda) d\lambda$$

is in the Schwartz space $\mathcal{C}([G])$.

5.5 Remark. As already said in the introduction, this is due to Franke, [14], section 5.3, Proposition 2 (2). His proof rests on the main result of [3] for which Lapid in [17] has given a proof independent of [16]. We give below a more selfcontained proof.

Proof. We proceed by induction on the dimension of $\mathfrak{a}_0^G$. The case where $dim {\mathfrak{a}_0}^G = 0$ is immediate by classical Fourier analysis on $\mathfrak{a}_G$: the classical Fourier transform of a compactly supported function on $\mathbb{R}^n$ is in the Schwartz space.

Now we assume $dim\mathfrak{a}_0^G > 0$. Let S^+ be the intersection of the unit sphere of $\mathfrak{a}_0^G$ with ${\mathfrak{a}_0^{G,+}}$. Let X_0 in S^+. Let Q be the standard parabolic subgroup of G such that $X_0 \in \mathfrak{a}_Q^{++}$. As $X_0 \in S^+$, Q is not equal to G. Let $\beta_Q(X) := \inf_{\alpha \in \Delta_{P_0} \setminus \Delta_{P_0}^Q} \alpha(X), X \in \mathfrak{a}_0$. Then $\beta_Q(X_0) > 0$. We choose a neighborhood S_0 of X_0 in S^+ such that

$$\beta_Q(X) \geqslant \beta_Q(X_0)/2, X \in S_0.$$

Let $\delta = \beta_Q(X_0)/2$. Then

$$S_0 \subset \mathfrak{a}_{0,Q,\delta}^{G,+}.$$

Let Λ be the support of a. We use the notation of Proposition 4.1. Let $E(., \lambda) := E_P(., \phi, \lambda)$. Then $E(., \lambda)$ is the sum of 2 terms: $E(., \lambda) - E(., \lambda)_Q^w$ and $E(., \lambda)_Q^w$. Let us show that, for all $k \in \mathbb{N}$ one has:

$$\begin{aligned} &| \int_{i\mathfrak{a}_P^*} a(\lambda) F(expX_G exptX\omega, \lambda) d\lambda| << (1 + \|X_G\|)^{-k} (1 + t)^{-k} e^{t\rho(X)}, \\ &t > 0, X_G \in \mathfrak{a}_G, X \in S_0, \omega \in \Omega, \lambda \in \Lambda, \end{aligned} \tag{5.2}$$

when F is any of these two families of functions. The case where $F = E_Q^w$ follows from the induction hypothesis, using the formula for the weak constant term of Eisenstein series (cf. (4.3), (4.1)) and the fact that in this formula $M(w,\lambda)$ is analytic in λ (cf. e.g. [17], after the Working Hypothesis). Let us treat the case where $F = E - E_Q^w$. One knows from Lemma 5.3 that there exists $\varepsilon > 0$ such that:

$$|E_P(expX_G exptX\omega, \phi, \lambda) - E_P(expX_G exptX\omega, \phi, \lambda)_Q^w| << e^{t\rho(X)-\varepsilon t},$$
$$X_G \in \mathfrak{a}_G, X \in S_0, \omega \in \Omega, \lambda \in \Lambda.$$

By multiplying by a and integrating on $i\mathfrak{a}_P$, we get (5.2) for $F = E - E_Q^w$ and $k = 0$. One applies this to successive partial derivatives of a with respect to elements of $\mathfrak{a}_G$. Then using that E_Q^w transforms under $\mathfrak{a}_G$ by λ and applying integration by part one gets the result for all k. One can do the same for $R_u E$, $u \in U(\mathfrak{g}_\infty)$.

As a finite number of S_0 covers S^+ this achieves to prove the proposition. □

6. An isometry

We recall the statement of Theorem 2 of [17].

Let $\mathcal{P}_{st}$ be the set of standard parabolic subgroups of G. Let P be a standard parabolic subgroup of G. Let $\mathcal{W}_P$ be the space of compactly supported smooth functions on $i\mathfrak{a}_P^*$ taking values in a finite dimensional subspace of $\mathcal{A}_P^2$. Write:

$$\|\phi\|_*^2 = \int_{i\mathfrak{a}_P^*} \|\phi(\lambda)\|_P^2 d\lambda. \tag{6.1}$$

For $\phi \in \mathcal{W}_P$, let

$$\Theta_{P,\phi}(g) = \int_{i\mathfrak{a}_P^*} E_P(g, \phi(\lambda), \lambda) d\lambda.$$

Let $L^2_{disc}(A_M^\infty M(F)\backslash M(\mathbb{A}))$ be the Hilbert sum of irreducible $M(\mathbb{A})$-subrepresentations of $L^2(A_M^\infty M(F)\backslash M(\mathbb{A}))$.

If P is a standard parabolic subgroup of G, let $|\mathcal{P}(M_P)|$ be equal to the number of parabolic subgroups having M_P as Levi factor. Consider the space $\mathcal{L}$ consisting of families of measurable functions $F_P : i\mathfrak{a}_P^* \to Ind_{P(\mathbb{A})}^{G(\mathbb{A})} L^2_{disc}(A_M^\infty M(F)\backslash M(\mathbb{A}))$ where P describes the set of standard parabolic subgroups of G such that:

$$\|(F_P)\|^2 = \sum_{P \in \mathcal{P}_{st}} |\mathcal{P}(M_P)|^{-1} \|F_P\|_*^2 < \infty$$

and

$$F_Q(w\lambda) = M(w,\lambda) F_P(\lambda), w \in W(P|Q), \lambda \in i\mathfrak{a}_P^*. \tag{6.2}$$

Let $\mathcal{L}'$ be the subspace of $\mathcal{L}$ consisting of those families such that $F_P \in \mathcal{W}_P$ for all P.

6.1 Theorem. *(cf. Lapid, [17], Theorem 2, for a short proof) The map $\mathcal{E}$ from $\mathcal{L}'$ to $L^2(G(F)\backslash G(\mathbb{A}))$*

$$(F_P) \mapsto \sum_{P\in\mathcal{P}_{st}} |\mathcal{P}(M_P)|^{-1}\Theta_{P,F_P}$$

extends to an isometry $\overline{\mathcal{E}}$ from $\mathcal{L}$ to $L^2(G(F)\backslash G(\mathbb{A}))$.

6.2 Lemma. *We take the notation of Proposition 5.4. In particular P is fixed. Then E_a is in the image of $\mathcal{E}$.*

Proof. For this one has to define a family in $\mathcal{L}'$ whose image by $\mathcal{E}$ is a non zero multiple of E_a. Let ψ be the map on $i\mathfrak{a}_P^*$ with values in $\mathcal{A}_P^2(G)$ defined by:

$$\psi(\lambda) = a(\lambda)\phi.$$

$$F_Q(\lambda) = \sum_{s\in W(P|Q)} M(s, s^{-1}\lambda)\psi(s^{-1}\lambda).$$

It is an easy consequence of the product formula for intertwining operators (cf. [6], Theorem 2.3 (5)) that the family (F_Q) satisfies (6.2). Moreover it is in $\mathcal{L}'$ as the intertwining operators are analytic on the imaginary axis (cf. [6] Remark 1.3). Then the functional equation for Eisenstein series (cf. [6] Theorem 1.3 (3)), implies that the image of (F_Q) by $\mathcal{E}$ is a non zero multiple of E_a. $\square$

7. Truncated inner product

If Q is a semistandard parabolic subgroup of G, let:

$$\theta_Q(\lambda) = \prod_{\alpha\in\Delta_Q} \lambda(\check{\alpha}), \lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*.$$

Let L_Q be the cocompact lattice of $\mathfrak{a}_Q^G$ generated by $\check{\Delta}_Q$ and let $C_Q = vol(\mathfrak{a}_Q^G/L_Q)$.

We fix a Siegel domain as in (2.3) associated to a compact set $\Omega_0 \subset P_0^1(\mathbb{A})$ and to $T_0 \in \mathfrak{a}_0$ that we might choose in $-\mathfrak{a}_0^+$. We can choose $\Omega_0 = \Omega_{N_0}\Omega_{M_0^1}$ where Ω_{N_0} (resp. $\Omega_{M_0^1}$) is a compact subset of $N_0(\mathbb{A})$ (resp. $M_0(\mathbb{A})^1$) such that

$$N_0(\mathbb{A}) = N_0(F)\Omega_{N_0}, M_0(\mathbb{A})^1 = M_0(F)\Omega_{M_0^1}. \tag{7.1}$$

If C is a subset of $\mathfrak{a}_0$ we define $M_0(C) = \{m \in M_0(\mathbb{A}) \cap G(\mathbb{A})^1 | H_0(m) \in C\}$ which is right invariant by $M_0(\mathbb{A})^1$. We take T dominant and regular in $\mathfrak{a}_0^G$. We let $d_{P_0}(T) = \inf_{\alpha\in\Delta_{P_0}} \alpha(T)$ and if Q is a standard parabolic subgroup of G, T_Q is the orthogonal projection of T on $\mathfrak{a}_Q$.

If Q is a standard parabolic subgroup of G, we define the convex set C_T^Q of $\mathfrak{a}_0^G$ by

$$C_T^Q = \{X \in \mathfrak{a}_0^G | \alpha(X - T_0) \geqslant 0, \forall \alpha \in \Delta_0^Q, \varpi(X - T) \leqslant 0, \forall \varpi \in \hat{\Delta}_0^Q, \beta(X - T) > 0, \\ \forall \beta \in \Delta_Q\}.$$

Notice that C_T^G is compact.

Let $T_{M_Q} = T - T_Q, T_{0,M_Q} = T_0 - T_{0,Q}$. Let $C_{T_{M_Q}}^{M_Q} \subset \mathfrak{a}_0^{M_Q} \subset \mathfrak{a}_0^G$ be defined as C_T^Q with $T_{M_Q} = T - T_Q$ instead of T and T_{0,M_Q} instead of T_0. Let $\mathfrak{a}_Q^{G,++}(T) = T_Q + \mathfrak{a}_Q^{G,++}$. We have:

$$C_T^Q = C_{T_{M_Q}}^{M_Q} + \mathfrak{a}_Q^{G,++}(T). \tag{7.2}$$

We define

$$\mathfrak{C}_T^G = G(F)\Omega_{N_0} M_0(C_T^G)K \subset [G]$$

which is compact. Using (7.1), one has:

$$\mathfrak{C}_T^G = G(F)N_0(\mathbb{A})M_0(C_T^G)K.$$

Replacing N_0 by $N_0 \cap M_Q$ and G by M_Q we define $\mathfrak{C}_T^{M_Q} \subset [M_Q]$ by:

$$\mathfrak{C}_{T_{M_Q}}^{M_Q} = M_Q(F)(N_0 \cap M_Q)(\mathbb{A})M_0(C_T^{M_Q})(K \cap M_Q(\mathbb{A})), \tag{7.3}$$

which is independent of the choice of Ω_0.

We define

$$\mathfrak{C}_T^Q = Q(F)N_0(\mathbb{A})M_0(C_T^Q)K \subset Q(F)\backslash G(\mathbb{A})^1. \tag{7.4}$$

Then $\mathfrak{C}_T^Q$ is $N_Q(\mathbb{A})$ invariant as

$$N_Q(\mathbb{A})Q(F)N_0(\mathbb{A}) = Q(F)N_Q(\mathbb{A})N_0(\mathbb{A}) = Q(F)N_0(\mathbb{A}).$$

As $N_0(\mathbb{A}) = N_Q(\mathbb{A})(N_0 \cap M_Q)(\mathbb{A})$ one has from (7.2):

$$\mathfrak{C}_T^Q = N_Q(\mathbb{A})exp(\mathfrak{a}_Q^{G,++}(T))\mathfrak{C}_{T_{M_Q}}^{M_Q} K \subset Q(F)\backslash G(\mathbb{A})^1. \tag{7.5}$$

We say that a strictly P_0-dominant $T \in \mathfrak{a}_0^G$ is sufficiently regular if there exists a sufficiently large $d > 0$ with $d_{P_0}(T) \geqslant d$. We have the following result due to Arthur ([1], Lemma 6.4).

$$\begin{array}{l}\text{Let } T \text{ be sufficiently regular.}\\ \text{(i) For each standard parabolic subgroup } Q \text{ of } G\text{, viewing } \mathfrak{C}_T^Q \text{ as a subset of}\\ Q(F)\backslash G(\mathbb{A})\text{, the projection to } [G] \text{ is injective on this set. Its image is still}\\ \text{denoted } \mathfrak{C}_T^Q.\\ \text{(ii) The } \mathfrak{C}_T^Q \text{ form a partition of } [G]^1.\end{array} \tag{7.6}$$

For a compactly supported function f on $\mathfrak{C}_T^Q$ we have, using (7.5):

$$\int_{\mathfrak{C}_T^Q} f(x)dx = \int_{(N_Q(F)\backslash N_Q(\mathbb{A}))\times\mathfrak{a}_Q^{G,++}(T)\times\mathfrak{C}_{T_{M_Q}}^{M_Q}\times K} f(n_Q expXm_Q^1k)e^{-2\rho_Q(X)}dn_QdXdm_Q^1dk, \tag{7.7}$$

as follows from the integration formula on $G(\mathbb{A})$ related to the decomposition $G(\mathbb{A})^1 = N_Q(\mathbb{A})exp\mathfrak{a}_Q^G M_Q(\mathbb{A})^1K$. Here dm_Q^1 is the measure on $[M_Q]^1$.

Let

$$\mathfrak{a}_{0,-}^G = \{X \in \mathfrak{a}_0^G | \omega(X) \leqslant 0, \forall\omega \in \hat{\Delta}_0\}$$

be the cone generated by the negative coroots and

$$\mathfrak{a}_{Q,-}^G(T) = \{X \in \mathfrak{a}_Q^G | \omega(X - T) \leqslant 0, \forall\omega \in \hat{\Delta}_Q\} = T_Q + \mathfrak{a}_Q \cap \mathfrak{a}_{0,-}^G.$$

Let p be an exponential polynomial with unitary exponents on $\mathfrak{a}_Q$ and $Z \in \mathfrak{a}_G$. If $\mu \in \mathfrak{a}_{Q,\mathbb{C}}^*$ has its real part strictly Q-dominant, the integral:

$$\int_{\mathfrak{a}_{Q,-}^G(T)} e^{\mu(X+Z)}p(X)dX$$

is convergent and has a meromorphic continuation in μ. When it is defined, its value in $\lambda \in i\mathfrak{a}_{Q,\mathbb{C}}^*$ is denoted:

$$\int^*_{Z+\mathfrak{a}_{Q,-}^G(T)} e^{\lambda(X)}p(X)dX.$$

Notice that

$$\int^*_{\mathfrak{a}_{Q,-}^G(0)} e^{\lambda(X)}dX = C_Q\theta_Q^{-1}(\lambda). \tag{7.8}$$

We use the notation defined just before Remark 3.9 in order to define $p_Q(\phi, \Psi)(X)$. We define for $\phi \in \mathcal{A}_Q^2(G) \oplus \mathcal{A}_Q^{temp,c}(G)$, $\lambda \in \mathfrak{a}_{Q,\mathbb{C}}^*$ and $\Psi \in \mathcal{A}_Q^{temp}(G)$, $Z \in \mathfrak{a}_G$:

$$r_Q^T(\phi_\lambda, \Psi)^Z = \int^{*}_{Z+\mathfrak{a}^G_{Q,-}(T)} e^{\lambda(X)} p_Q(\phi, \Psi)(X) dX. \tag{7.9}$$

Let $Z \in \mathfrak{a}_G$. If p is a polynomial on $\mathfrak{a}_Q$, let p^Z be the polynomial on $\mathfrak{a}_Q^G$ defined by:

$$p^Z(X) = p(X + Z), X \in \mathfrak{a}_Q^G$$

and $p^Z(\partial)$ its Fourier transform viewed as a differential operator on $\mathfrak{a}_Q^{G,*}$. More precisely, let us define the Fourier transform $\mathcal{F}\phi$ of $\phi \in C_c^\infty(\mathfrak{a}_Q^G)$ by

$$\mathcal{F}\phi(\lambda) = \int_{\mathfrak{a}_Q^G} \phi(X) e^{<\lambda, X>} dX, \lambda \in i\mathfrak{a}_Q^{G*}, \tag{7.10}$$

where the measure on $\mathfrak{a}_Q^G$ has been chosen in (2.2). Then $p(\partial)$ is characterized by:

$$\mathcal{F}(p\phi) = p(\partial)(\mathcal{F}\phi). \tag{7.11}$$

Recall that C_Q has been defined in the beginning of this section. Taking into account the definition of $\Psi_{0,\nu}$ (cf. (3.2)), Remark 3.9 shows that $p_Q(\phi, \Psi_{0,\mu})^Z$ is a polynomial. Using (7.8) and (7.11), one sees:

$$r_Q^T(\phi_\lambda, \Psi)^Z = C_Q \sum_{\mu \in \mathcal{E}_Q(\Psi)} [p_Q(\phi, \Psi_{0,\mu})^Z(\partial) e^{<., T_Q>} \theta_Q^{-1}](\lambda - \mu), \tag{7.12}$$

where $\Psi_{0,\mu}$ is defined as in (3.2).

One can define $r_T(\psi, \Psi)$ where ψ is a linear combination of ϕ_λ. If Φ is a function on $G(\mathbb{A})$ and $Z \in \mathfrak{a}_G$, one defines a function on $G(\mathbb{A})^1$ by:

$$\Phi^Z(g^1) = \Phi(g^1 expZ), g^1 \in G(\mathbb{A})^1.$$

7.1 Theorem. *Let $T \in \mathfrak{a}_0^G$ be sufficiently regular, $Z \in \mathfrak{a}_G$ and $\lambda \in i\mathfrak{a}_P^*$. Let Φ be an element of $\mathcal{A}^{temp}(G)$ and $\phi \in \mathcal{A}_P^2(G)$. We denote by $E(., \lambda)$ the function $E_P(., \phi, \lambda)$. Let:*

$$\Omega_{P_0}^T(E(\lambda), \Phi)^Z := \int_{\mathfrak{C}_T^G} E(x, \lambda)^Z \overline{\Phi}^Z(x) dx,$$

$$\omega_{P_0}^T(E(\lambda), \Phi)^Z := \sum_{Q \in \mathcal{P}_{st}} r_Q^T(E(\lambda)_Q^w, \Phi_Q^w)^Z.$$

(i) Let $\mathcal{H}^c$ be the subset of $\lambda \in i\mathfrak{a}_P^*$ where the summands of $\omega_{P_0}^T(E(\lambda, \Phi)$ are analytic for all Z. From (7.12), this set contains the complementary set of a finite union of hyperplanes. The function $\omega_{P_0}^T(E(\lambda), \Phi)^Z$ on $\mathcal{H}^c$ extends to an analytic function on $i\mathfrak{a}_P^*$ denoted in the same way.

(ii) Let $\delta > 0$. Let Λ be a bounded set of $i\mathfrak{a}_P^*$. There exists $k \in \mathbb{N}$, $\varepsilon > 0$ such that the difference

$$\Delta_{P_0}^T(E(\lambda), \Phi)^Z := \Omega_{P_0}^T(E(\lambda), \Phi)^Z - \omega_{P_0}^T(E(\lambda), \Phi)^Z$$

is an $O(e^{-\varepsilon\|T)\|}(1 + \|Z\|^k)$ for $\lambda \in \Lambda$, for T such that $d_{P_0}(T) \geqslant \delta\|T\|$, $Z \in \mathfrak{a}_G$.

The proof is by induction on $dim\mathfrak{a}_0^G$. The statement is clear for $dim\mathfrak{a}_0^G = 0$. We suppose that the Theorem is true for all groups G' with $dim\mathfrak{a}_0^{G'} < dim\mathfrak{a}_0^G$.

7.2 Lemma. *Let $k_0, \delta > 0$. Then if Λ is a bounded subset of $\mathcal{H}^c$, there exists $C > 0$, $k \in \mathbb{N}$, $\varepsilon > 0$ such that*

$$|\Delta_{P_0}^{T+S}(E(\lambda), \Phi)^Z - \Delta_{P_0}^T(E(\lambda), \Phi)^Z| \leqslant Ce^{-\varepsilon\|T\|}(1 + \|Z\|)^k,$$

for $\lambda \in \Lambda, Z \in \mathfrak{a}_G$, for T, S strictly P_0-dominant such that $d_{P_0}(T) \geqslant \delta\|T\|$, $\|S\| \leqslant k_0\|T\|$, $\|T_0\| \leqslant \|T\|$.

Proof. Let us define:

$$C_{T+S,T}^Q = C_{T+S}^G \cap C_T^Q, \tag{7.13}$$

and

$$\mathfrak{C}_{T+S,T}^Q = G(F)N_0(\mathbb{A})M_0(C_{T+S,T}^Q)K$$

From (7.6), these subsets of $[G]$ are disjoints. Moreover from (7.6), they cover $\mathfrak{C}_{T+S}^G$.

Let us show that, for T, S as in the Lemma, there exists $\delta_1 > 0$ such that:

$$\alpha(X) \geqslant \delta_1\|X\|, X \in C_{T+S,T}^Q, \alpha \in \Delta_0 \setminus \Delta_0^Q. \tag{7.14}$$

Let $X \in C_{T+S,T}^Q$ and $\alpha \in \Delta_0 \setminus \Delta_0^Q$. The definition of C_T^Q shows in particular that $X = T - X' + Y$ where $X' = \sum_{\beta \in \Delta_0^Q} d_\beta\check{\beta}$ with $d_\beta > 0$ and $Y \in \mathfrak{a}_Q^+$. Since $\alpha(\check{\beta}) \leqslant 0$ for each $\beta \in \Delta_0^Q$, from the properties of simple roots, one has

$$\alpha(X) \geqslant \alpha(T) \geqslant \delta\|T\|. \tag{7.15}$$

Let us show:

$$\|X - T_0\| \leqslant \|T + S\|, X \in C_{T+S}^G. \tag{7.16}$$

As $X \in C_{T+S}^G$, $X - T_0 = (T + S) - Y'$ where Y' is a linear combination with nonnegative coefficients of coroots. Moreover $T + S$ and $X - T_0$ are in $\mathfrak{a}_0^+$. Hence

$$(X - T_0, X - T_0) \leqslant (X - T_0, T + S) \leqslant (T + S, T + S)$$

which proves our claim. Hence

$$\|X\| \leqslant \|T\| + \|S\| + \|T_0\| \leqslant (2 + k_0)\|T\|, X \in C_T^Q$$

and

$$\alpha(X) \geqslant \delta_1 \|X\|$$

where $\delta_1 = (2 + k_0)^{-1}\delta$. This proves (7.14).

From (7.15) one gets:

$$\|X\| >> \|T\|, X \in C_T^Q$$

if $Q \neq G$.

Hence from Lemma 5.2, one gets:

Let Ω be a compact subset of $G(\mathbb{A})$. There exists $\varepsilon > 0, k \in \mathbb{N}$ such that

$$\begin{gathered} |(\Phi - \Phi_Q^w)^Z (n_Q expX\omega)| << e^{\rho(X) - \varepsilon\|T\|}(1 + \|Z\|)^k, \\ n_Q \in N_Q(\mathbb{A}), X \in C^Q_{T+S,T}, \omega \in \Omega, Z \in \mathfrak{a}_G \end{gathered} \tag{7.17}$$

and T, S as in the Lemma.

Similarly one gets from Lemma 5.3 that there exists $\varepsilon > 0$ and $k \in \mathbb{N}$ such that:

$$\begin{gathered} |(E_P(n_Q expX\omega, \phi, \lambda)^Z - E_P(n_Q expX\omega, \phi, \lambda)^Z_{Q^w}| << e^{\rho(X) - \varepsilon\|T\|}(1 + \|Z\|)^k, \\ n_Q \in N_Q(\mathbb{A}), X \in C^Q_{T+S,T}, \omega \in \Omega, \lambda \in \Lambda, Z \in \mathfrak{a}_G. \end{gathered} \tag{7.18}$$

One has:

$$\text{The volume of } \mathfrak{C}_T^G \text{ and } \mathfrak{C}^Q_{T+S,T} \text{ is bounded by the volume of } [G]^1. \tag{7.19}$$

Let us introduce:

$$I_Q(T, \lambda)^Z := \int\limits_{\mathfrak{C}^Q_{T+S,T}} E(x, \lambda)^Z \overline{\Phi}^Z(x) dx,\ I_Q^w(T, \lambda)^Z := \int\limits_{\mathfrak{C}^Q_{T+S,T}} E(x, \lambda)_Q^{w,Z} \overline{\Phi}_Q^{w,Z}(x) dx.$$

Notice that:

$$I_G(T, \lambda)^Z = I_G^w(T, \lambda)^Z = \Omega^T_{P_0}(E(\lambda)^Z, \Phi^Z),\ \sum_{Q \in \mathcal{P}_{st}} I_Q(T, \lambda)^Z = \Omega^{T+S}_{P_0}(E(\lambda)^Z, \Phi^Z). \tag{7.20}$$

For $C > 0, k \in \mathbb{N}$ and $\varepsilon > 0$ let us consider the function of T and Z:

$$Ce^{-\varepsilon\|T\|}(1 + \|Z\|)^k. \tag{7.21}$$

It follows from (7.17), (7.18), as well as the tempered estimate for Φ and the uniform estimate for Eisenstein series (cf. Proposition 4.1) that:

The difference of

$$|I_Q(T, \lambda)^Z - I_Q^w(T, \lambda)^Z| \tag{7.22}$$

is bounded for λ, T, S as in the Lemma, by a function of type (7.21).

Let us define

$$\mathfrak{a}_Q^{G,++}(T + S, T) := \{T_Q + Y | Y \in \mathfrak{a}_Q^{G,++}, \varpi_\alpha(Y - S) \leqslant 0, \forall \alpha \in \Delta_Q\} \subset \mathfrak{a}_Q^G.$$

Let us show

$$C_{T+S,T}^Q = \mathfrak{a}_Q^{G,++}(T + S, T) + C_{T_{M_Q}}^{M_Q}. \tag{7.23}$$

Let $X \in C_T^{M_Q}$ and $T_Q + Y \in \mathfrak{a}_Q^{G,++}(T + S, T)$. Let us show that $X + T_Q + Y$ is an element of $C_{T+S,T}^Q$. In view of (7.2), the only thing to prove is that it is an element C_{T+S}^G. One has:

$$X + T_Q + Y - S - T = Y - S + X - T_{M_Q}.$$

Let $\alpha \in \Delta_0 \setminus \Delta_0^Q$. Then $\varpi_\alpha(Y - S + X - T_{M_Q}) = \varpi_\alpha(Y - S)$ which is less than or equal to 0, by the definition of $\mathfrak{a}_Q^{G,++}(T + S, T)$. Let $\alpha \in \Delta_0^Q$. The difference $Y - S_Q$ is a linear combination with coefficients less or equal to zero of elements of $\check{\Delta}_Q$ hence of $\check{\Delta}_0$. The same is true for $Y - S = Y - S_Q - S_{M_Q}$. Hence $\varpi_\alpha(Y - S) \leqslant 0$. The definition of $C_T^{M_Q}$ shows that $\varpi_\alpha(X - T_{M_Q}) \leqslant 0$. Hence $\varpi_\alpha(Y - S + X - T_{M_Q}) = \varpi_\alpha(Y - S) \leqslant 0, \alpha \in \Delta_0^Q$. This achieves to prove $X + T_Q + Y \in C_{T+S,T}^G$ as wanted. Hence

$$\mathfrak{a}_Q^{G,++}(T + S, T) + C_T^{M_Q} \subset C_{T+S,T}^Q. \tag{7.24}$$

The reciprocal inclusion follows easily from (7.2) and of the definition of $C_{T+S,T}^Q$. This achieves to prove (7.23).

We use that Φ_Q^w and $E_P(x, \phi, \lambda)_Q^w$ are left $N_Q(\mathbb{A})$-invariant and that the volume of $N_Q(F)\backslash N_Q(\mathbb{A})$ is equal to 1. If P is a parabolic subgroup of G with Levi factor M_P and $k \in K$, we have defined (cf. section 3.3) for any function ϕ on $G(\mathbb{A})$, the function $\phi^{M_P,k}$ on $M_P(\mathbb{A})$ by:

$$\phi^{M_P,k}(m) = e^{-\rho(H_P(m))}\phi(mk), m \in M_P(\mathbb{A}).$$

Thus, using (7.7) and (3.5), we get:

$$I_Q^w(T,\lambda)^Z = \int_{\mathfrak{C}^Q_{T+S,T}} E(x,\lambda)_Q^{w,Z}\overline{\Phi}_Q^{w,Z}(x)dx =$$

$$\int_{\mathfrak{a}_Q^{G,++}(T+S,T)\times\mathfrak{C}_T^{M_Q}\times K} E(expXm_Q^1k,\lambda)_Q^{w,Z}\overline{\Phi}_Q^{w,Z}(expXm_Q^1k)e^{-2\rho_Q(X)}dXdm_Q^1dk$$

$$= \int_{\mathfrak{a}_Q^{G,++}(T+S,T)\times K} \Omega^T_{P_0\cap M_Q}([E(\lambda)_Q^w]^{M_Q,k},[\Phi_Q^w]^{M_Q,k})^{X+Z}dm_Q^1dXdk.$$

Recall that by induction hypothesis, Theorem 7.1 is true for M_Q if $Q \neq G$. Taking into account (7.19) and the previous equality, one sees, using K-finiteness, that the difference of the preceding expression with the same expression, where $\Omega^T_{P_0\cap M_Q}$ is replaced by $\omega^T_{P_0\cap M_Q}$, denoted $J_Q(T,\lambda)^Z$, is bounded by a function of type (7.21).

One has:

$$J_Q(T,\lambda)^Z = \int_{\mathfrak{a}_Q^{G,++}(T+S,T)\times K} \omega^T_{P_0\cap M_Q}([E(\lambda)_Q^w]^{M_Q,k},[\Phi_Q^w]^{M_Q,k})^{X+Z}dXdk$$

$$= \int_{\mathfrak{a}_Q^{G,++}(T+S,T)\times K} \sum_{R_1\in\mathcal{P}_{st}(M_Q)} r^T_{R_1}(([E(\lambda)_Q^w]^{M_Q,k})^w_{R_1},([\Phi_Q^w]^{M_Q,k})^w_{R_1})^{X+Z}dXdk.$$

If R_1 is a standard parabolic subgroup of M_Q, let P_1 be the standard parabolic subgroup of G contained in Q with $P_1\cap M_Q = R_1$. Using Lemma 3.8 (iv), the definition (7.9), for M_Q and R_1, and integrating over K, one sees:

$$J_Q(T,\lambda)^Z = \sum_{P_1\in\mathcal{P}_{st}(G),P_1\subset Q} \int^*_{\mathfrak{a}_Q^{G,++}(T+S,T)+\mathfrak{a}^{M_Q}_{P_1\cap M_Q,-}(T)} p_{P_1}(E(\lambda)^w_{P_1},\Phi^w_{P_1})(X+Z)dX.$$

We observe that $J_G(T,\lambda)^Z = \omega^T_{P_0}(E(\lambda)^Z,\Phi^Z)$ and one has seen that $I_G(T,\lambda)^Z = \Omega^T_{P_0}(E(\lambda)^Z,\Phi^Z)$. One writes:

$$\Omega^T_{P_0}(E(\lambda)^Z,\Phi^Z) = \Delta^T_{P_0}(E(\lambda)^Z,\Phi^Z) + \omega^T_{P_0}(E(\lambda)^Z,\Phi^Z).$$

Using what we have just proved and (7.20), and (7.22), we get:

The modulus of the difference

$$\begin{aligned}&\Omega_{P_0}^{T+S}(E(\lambda)^Z,\Phi^Z)-\Delta_{P_0}^T(E(\lambda)^Z,\Phi^Z)\\&=\Delta_{P_0}^{T+S}(E(\lambda)^Z,\Phi^Z)+\omega_{P_0}^{T+S}(E(\lambda)^Z,\Phi^Z)-\Delta_{P_0}^T(E(\lambda)^Z,\Phi^Z)\end{aligned}\tag{7.25}$$

with $J(T,\lambda)^Z = \sum_{Q\in\mathcal{P}_{st}(G)} J_Q(T,\lambda)^Z$ is bounded by a function of type (7.21).

Thus it is enough, to finish the proof of the Lemma, to prove:

$$J(T,\lambda)^Z=\omega_{P_0}^{T+S}(E(\lambda)^Z,\Phi^Z).$$

Using the expression of $J_Q(T,\lambda)^Z$ above and interverting the sum over Q and P_1, one sees that:

$$\begin{aligned}&J(T,\lambda)^Z\\&=\sum_{P_1\in\mathcal{P}_{st}(G),Q\in\mathcal{P}_{st}(G),P_1\subset Q}\ \int^{*}_{\mathfrak{a}_Q^{G,++}(T+S,T)+\mathfrak{a}^{M_Q}_{P_1\cap M_Q,-}(T)+Z} e^{-2\rho_{P_1}(X)}p_{P_1}(E(\lambda)^w_{P_1},\Phi_{P_1})(X)dX.\end{aligned}$$

Let $\mathfrak{a}^{M_Q}_{P_1\cap M_Q,--}$ be the interior in $\mathfrak{a}^Q$ of $\mathfrak{a}^{M_Q}_{P_1\cap M_Q,-}$ and let $\mathfrak{a}_Q^{G,+}(T+S,T)$ be the closure of $\mathfrak{a}_Q^{G,++}(T+S,T)$ in $\mathfrak{a}_Q$. Let us show:

The union

$$\cup_{Q\in\mathcal{P}_{st},P_1\subset Q}\mathfrak{a}_Q^{G,+}(T+S,T)+\mathfrak{a}^{M_Q}_{P_1\cap M_Q,--}(T_{M_Q})\tag{7.26}$$

is disjoint and is a partition of $\mathfrak{a}^G_{P_1,-}(T+S)$.

Let us consider the projection of $\mathfrak{a}_{P_1}$ on the closed convex cone $\mathfrak{a}^G_{P_1,-}$. By translating, one sees, using e.g. [9] Corollary 1.4, that, if $X\in\mathfrak{a}^G_{P_1,-}(T+S)$, there exists a unique standard parabolic subgroup of G, Q with $P_1\subset Q$ such that $X=X'+Y$, $X'\in\mathfrak{a}^{M_Q}_{P_1\cap M_Q,--}(T_{M_Q}), Y\in\mathfrak{a}_Q^{G,+}(T)$. As $X\in\mathfrak{a}^G_{P_1,-}(T+S)$, one has $Y\in\mathfrak{a}_Q^{G,+}(T+S,T)$. Hence the union in (7.26) contains $\mathfrak{a}^G_{P_1,-}(T+S)$ and is disjoint.

Reciprocally let us prove that for $P_1\subset Q$:

$$\mathfrak{a}^{M_Q}_{P_1\cap M_Q,--}(T_{M_Q})+\mathfrak{a}_Q^{G,+}(T+S,T)\subset\mathfrak{a}^G_{P_1,-}(T+S).$$

To see this, by translation, it is enough to prove that if $X\in\mathfrak{a}^{M_Q}_{P_1\cap M_Q,--}(T_{M_Q}), Y\in\mathfrak{a}^G_{Q,-}$ one has $X+Y\in\mathfrak{a}^G_{P_1,-}$ which is clear by convexity. This proves (7.26).

Neglecting sets of measure zero, this implies that the sum $J(T,\lambda)$ is equal to $\omega_{P_0}^{T+S}(E(\lambda),\Phi)^Z$. This achieves to prove the lemma. □

7.3 Remark. A similar decomposition than (7.26) without replacing $\mathfrak{a}_Q^{G,++}$, $\mathfrak{a}_{P_1\cap M_Q,-}^{M_Q}$ by $\mathfrak{a}_Q^{G,+}$, $\mathfrak{a}_{P_1\cap M_Q,--}^{M_Q}$ respectively and that is a direct consequence of the Langlands combinatorial lemma.

We will give below a proof of Theorem 7.1. It is done using first the argument of [2], Lemma 9.2 and second using wave packets as in [13] Lemma 3 and end of proof of Proposition 1 (see also the end of the proof of Theorem 1 in [12]).

One fixes $\delta > 0$ and one writes $lim_{T\xrightarrow{\delta}\infty}$ to describe the limit when $\|T\|$ tends to infinity verifying $d_{P_0}(T) \geqslant \delta\|T\|$. One deduces from the preceding Lemma, as in ([2], Lemma 9.2) that the limit

$$\Delta_{P_0}^{\infty}(E(\lambda),\Phi)^Z = lim_{T\xrightarrow{\delta}\infty}\Delta_{P_0}^{T}(E(\lambda),\Phi)^Z$$

exists uniformly for λ in any compact subset of $\mathcal{H}^c$ and if Λ is a bounded set in $\mathcal{H}^c$, there exists $C,\varepsilon > 0$, $k \in \mathbb{N}$ such that for $\lambda \in \Lambda$ and T such that $d_{P_0}(T) \geqslant \delta\|T\|$, $Z \in \mathfrak{a}_G$, one has:

$$|\Delta_{P_0}^{\infty}(E(\lambda),\Phi)^Z - \Delta_{P_0}^{T}(E(\lambda),\Phi)^Z| \leqslant Ce^{-\varepsilon\|T\|}(1+\|Z\|)^k. \tag{7.27}$$

We prepare some Lemmas to prove that $\Delta_{P_0}^{\infty}(E(\lambda),\Phi)^Z$ is identically zero on $\mathcal{H}^c$. Using Proposition 5.4, we define a distribution $T_{\Phi,Z}$ on $i\mathfrak{a}_P^*$ by:

$$T_{\Phi,Z}(a) = \int\limits_{[G]^1} E_a(x)^Z\overline{\Phi}^Z(x)dx, a \in C_c^{\infty}(i\mathfrak{a}_P^*),$$

where E_a is the wave packet $\int_{i\mathfrak{a}_P^*} a(\lambda)E(\lambda)d\lambda$.

7.4 Lemma. *The support $\mathcal{S}$ of $T_{\Phi,Z}$ is a finite set.*

Proof. For $\lambda \in i\mathfrak{a}_P^*$, the center $Z(\mathfrak{g}_\infty)$ of $U(\mathfrak{g}_\infty)$ acts on $E(\lambda)$ by a character denoted χ_λ and Φ is annihilated by an ideal I of $Z(\mathfrak{g}_\infty)$ of finite codimension. Let us compute in two ways:

$$A := \int\limits_{[G]^1} (zE_a(x))^Z\overline{\Phi}^Z(x)dx, z \in Z([\mathfrak{g}_\infty,\mathfrak{g}_\infty]) \subset Z(\mathfrak{g}_\infty), z^* \in I,$$

where z^* is the adjoint of z.

On one hand, looking to the action of z on $E(\lambda)$ and differentiating under the integral defining E_a we get:

$$A = T_{\Phi,Z}(p(z)a),$$

where $p(z)(\lambda) = \chi_\lambda(z)$, which is a polynomial in λ. On the other hand:

$$A = \int_{[G]^1} (E_a(x))^Z \overline{z^*\Phi}^Z(x)dx = 0.$$

From the equality above, if $z^* \in I$ the distribution $p(z)T_{\Phi,Z}$ is equal to zero. Let $I^* = \{z^*|z \in I\}$. As I is finite codimensional, the set of $\lambda \in i\mathfrak{a}_P^*$ such that $I^* \subset \ker \chi_\lambda$ is a finite set $\mathcal{F}$. Hence if $\lambda \notin \mathcal{F}$, there exists $z \in I^*$ such that $p(z)(\lambda) \neq 0$. Hence $T_{\Phi,Z}$ restricted to a neighborhood of λ is zero. Hence $\mathcal{S} \subset \mathcal{F}$. $\square$

7.5 Lemma. *If $a \in C_c^\infty(i\mathfrak{a}_P^*)$ has its support in the complimentary set of $\mathcal{S}$, one has:*

$$\lim_{T \xrightarrow{\delta} \infty} \int_{i\mathfrak{a}_P^*} a(\lambda)\Omega_{P_0}^T(E(\lambda), \Phi)^Z d\lambda = 0.$$

Proof. From Fubini theorem and Lebesgue dominated convergence the limit is equal to $T_{\Phi,Z}(a)$, which is equal to zero by the preceding lemma. $\square$

7.6 Lemma. *If $a \in C_c^\infty(i\mathfrak{a}_P^*)$ has its support in $\mathcal{H}^c$ one has:*

$$\lim_{T \xrightarrow{\delta} \infty} \int_{i\mathfrak{a}_P^*} a(\lambda)\omega_{P_0}^T(E(\lambda), \Phi)^Z d\lambda = 0.$$

Proof. This follows from the definition of $\omega_{P_0}^T(E(\lambda), \Phi)^Z$, (7.12) and from the fact that the Fourier transform of a C_c^∞ function on $\mathbb{R}^n$ is rapidly decreasing. $\square$

7.7 Lemma. *If $\mathcal{S}^c$ is the complimentary set of $\mathcal{S}$ in $i\mathfrak{a}_P^*$ one has:*

$$\Delta_{P_0}^\infty(E(\lambda), \Phi)^Z = 0, \lambda \in \mathcal{H}^c \cap \mathcal{S}^c.$$

Proof. From the two preceding lemmas one has for all in $C_c^\infty(i\mathfrak{a}_P^*)$ with support in the intersection $\mathcal{H}^c \cap \mathcal{S}^c$:

$$\int_{i\mathfrak{a}_P^*} a(\lambda)\Delta_{P_0}^\infty(E(\lambda), \Phi)^Z d\lambda = 0.$$

This implies the Lemma. $\square$

Proof. Let us finish the proof of the Theorem 7.1. The vanishing property of the preceding Lemma together with (7.27) shows that the bound of the theorem is true for λ in a bounded subset of $\mathcal{H}^c \cap \mathcal{S}^c$. Recall that $\Omega_{P_0}^T(E(\lambda), \Phi)^Z$ is analytic in λ. Hence, for any λ in $i\mathfrak{a}_P^*$ and any compact neighborhood of λ, V, $\omega_{P_0}^T(E(\lambda), \Phi)^Z$ is bounded on $V \cap \mathcal{H}^c \cap \mathcal{S}^c$. But this meromorphic function has only possible poles along hyperplanes. It follows that it is analytic on $i\mathfrak{a}_P^*$. This proves the first part of the Theorem. The second part follows from (7.27) by continuity and density. $\square$

Let $\Lambda \in \mathfrak{a}^*_{M_P}$ be strictly P-dominant. If Q is a parabolic subgroup of G with Levi factor M_Q one will denote by ψ^Λ_Q the characteristic function of:

$$C^\Lambda_Q = \{X \in \mathfrak{a}^G_Q | \omega_\alpha(X)\Lambda(\check{\alpha}) > 0, \forall \alpha \in \Delta_Q\}$$

that we look as a tempered measure on $\mathfrak{a}^G_P$ by our choice of Haar measures. Let β^Λ_Q be the number of elements $\check{\alpha}$ of $\check{\Delta}_Q$ such that $\Lambda(\check{\alpha}) < 0$. Then one has the following proposition, whose proof is analogous to Proposition 2 in [13], using (4.2) and (7.12). One has to remark that only terms with $s \in W(Q|P)$ due to Lemma 4.6.

7.8 Proposition. *Using the notation of Theorem 7.1, the analytic function* $\lambda \mapsto \omega^T_{P_0}(E(\lambda, \Phi)^Z$ *is equal, as a distribution on* $i\mathfrak{a}^*_P$*, to the sum:*

(a) on $Q \in \mathcal{P}_{st}$

(b) on $s \in W(Q|P)$

(c) on $\mu \in \mathcal{E}_Q(\Phi^w_Q)$ *of:*

$$C_Q[(p_Q(M(s^{-1}, \lambda)\phi, \Phi^w_{Q,0,\mu})^Z \circ s^{-1})(\partial)((-1)^{\beta^\Lambda_{Q^s}} \mathcal{F}(\psi^\Lambda_{Q^s, T_{Q^s}})](\lambda - s\mu),$$

where $Q^s = sQs^{-1}$, $T_{Q^s} = sT_Q$ *and* $\psi^\Lambda_{Q^s,T_{Q^s}}$ *is the characteristic function of the translate of* $C^\Lambda_{Q^s}$, $C^\Lambda_{Q^s} - T_{Q^s}$ *and* $\mathcal{F}$ *indicates that we take the Fourier transform.*

Proof. First $\omega(\lambda) := \omega^T_{P_0}(E(\lambda), \Phi)$ is analytic on $i\mathfrak{a}^*_P$ from Theorem 7.1 (i). Then, $\omega(\lambda)$ is the limit when t to 0^+ of $\omega(\lambda + t\Lambda)$ in the sense of distributions. Then one uses [12], Lemma 11, for each term of the sum defining $\omega(\lambda + t\Lambda)$. The Lemma follows. $\square$

The following theorem is the main result of this article.

7.9 Theorem. *The image of the map* $\overline{\mathcal{E}}$ *of Theorem 6.1 is equal to* $L^2(G(F)\backslash G(\mathbb{A}))$*.*

We start with a preliminary remark. From [5], end of section 3.5, automorphic forms which may contribute to the spectrum (see below for a precise meaning) are $[G]$-tempered. Hence they are tempered, by Proposition 3.5.

7.10 Lemma. *If the image of* $\overline{\mathcal{E}}$ *is not equal to* $L^2(G(F)\backslash G(\mathbb{A}))$*, there exists a non zero tempered automorphic form* Φ*, transforming under a unitary character* ν_G *of* $\mathfrak{a}_G$ *and orthogonal to all the wave packets* E_a *of Proposition 5.4, when* P*,* ϕ *and* a *vary.*

Proof. The proof is similar to [10], Lemma 11. Let $\mathcal{H}$ be the orthogonal to the image of $\mathcal{E}$, which is assumed to be non zero. One considers the decomposition of this representation of $G(\mathbb{A})$ into an Hilbert integral of multiple of irreducible representations:

$$\mathcal{H} = \int_{\hat{G}} \mathcal{H}_\pi d\mu(\pi).$$

The restriction ξ of the Dirac measure at the neutral element to the space $\mathcal{H}^\infty$ of C^∞ vectors, disintegrates:

$$\xi = \int_{\hat{G}} \xi_\pi d\mu(\pi),$$

where $\xi_\pi \in (\mathcal{H}_\pi^{-\infty})^{G(F)}$, i.e. is a $G(F)$-invariant distribution vector on $\mathcal{H}_\pi$.

Let

$$v = \int_{\hat{G}} v_\pi d\mu(\pi) \in \mathcal{H}^\infty.$$

We assume that it is K-finite and non zero.

Let (g_n) be a dense sequence in $G(\mathbb{A})$. For $p, q, n \in \mathbb{N}$, let:

$$X_{p,q,n} = \{\pi \in \hat{G} |\, | < \xi_\pi, \pi(g_n)v_\pi > | \leqslant p\Xi(g_n)(1 + \sigma_{[G]}(g_n))^q\}$$

For all $g \in G(\mathbb{A})$, the map $\pi \mapsto < \xi_\pi, \pi(g)v_\pi >$ is μ-measurable. Hence all $X_{p,q,n}$ are measurable as well as $X_{p,q} = \cap_{n\in\mathbb{N}} X_{p,q,n}$. Moreover, from our preliminary remark, just after the statement of the theorem, $\cup_{p,q\in\mathbb{N}} X_{p,q}$ is equal to $\hat{G}$ up to a set of μ-measure 0. Let $X^0_{p,q}$ be the set of elements π of $X_{p,q}$ such that $g \mapsto < \xi_\pi, \pi(g)v_\pi >$ is non identically zero. As v is non zero, one can find p, q such that the set $X^0_{p,q}$ is of non zero measure. Let χ be the characteristic function of $X^0_{p,q}$. Then one has for any $\theta \in L^\infty(\hat{G}, \mu)$, going back to the definition:

$$f_\theta := \int_{\hat{G}} \chi(\pi)\theta(\pi)v_\pi d\mu(\pi) \in \mathcal{H}^\infty.$$

Hence by using the disintegration of ξ, viewing f_θ as a function on $G(\mathbb{A})$, one has:

$$f_\theta(g) = \int_{\hat{G}} \chi(\pi)\theta(\pi) < \xi_\pi, \pi(g)v_\pi > d\mu(\pi), g \in G(\mathbb{A}).$$

Let us show that the map $(\pi, g) \mapsto < \xi_\pi, \pi(g)v_\pi >$ is measurable. Let $g = g_\infty g_f$ where $g \in G(\mathbb{A}_\infty)$ and $g_f \in G(\mathbb{A}_f)$. As v is smooth the map is locally constant in g_f. One easily reduces to $g_f = 1$ and look to the dependence on (π, g_∞) only. Then one uses the argument given in [10], p. 96 which uses step functions.

Using (2.10), one can apply Fubini's theorem to

$$\int_{[G]} E_a(x)f_\theta(x)dx = \int_{X^0_{p,q}} \theta(\pi) \int_{[G]} E_a(x)\overline{< \xi_\pi, \pi(x)v_\pi >}dxd\mu(\pi).$$

This has to be zero for all θ. Hence for almost all π in $X^0_{p,q}$ one has:

$$\int_{[G]} E_a(x) < \xi_\pi, \pi(x)v_\pi > dx = 0$$

for a given E_a. Using a separability argument, one finds an element π_0 of $X^0_{p,q} \subset \hat{G}$ such that it is true for all E_a. One takes $\Phi =< \xi_{\pi_0}, \pi_0(x)v_{\pi_0} >$. $\square$

Let $a = a_1 \otimes a_2$ where $a_1 \in C_c^\infty((i\mathfrak{a}_P^G)^*)$ and $a_2 \in C_c^\infty(i\mathfrak{a}_G^*))$. Let $\nu_G \in i\mathfrak{a}_G^*$ which describes the action of $\mathfrak{a}_G$ on π_0. Then, using Fourier inversion formula for $i\mathfrak{a}_G^*$, one has:

$$\int_{G(F)\backslash G(\mathbb{A})} E_a(x)\overline{\Phi(x)}dx = a_2(\nu_G) \int_{G(F)\backslash G(\mathbb{A})^1} E_{a_1}(x)\overline{\Phi(x)}dx$$

where $E_{a_1} = \int_{i\mathfrak{a}_P^{G,*}} a_1(\lambda)E_P(x, \phi, \lambda)d\lambda$. We want to compute

$$I = \int_{G(F)\backslash G(\mathbb{A})^1} E_{a_1}(x)\overline{\Phi(x)}dx$$

using the preceding theorem.

7.11 Lemma.

$$I = C_P \sum_{\mu \in \mathcal{E}_P^w(\Phi)} [p_P(\phi, \Phi_{P,0,\mu})^0(\partial)a_1](\mu_{|\mathfrak{a}_P^G}).$$

Proof. We can compute I as limit. Using Lebesgue dominated convergence and Fubini theorems, one can write I as a limit. Let T be strictly P_0-dominant. Then:

$$I = lim_{n\to+\infty} \int_{i\mathfrak{a}_P^{G,*}} a_1(\lambda)\Omega_{P_0}^{nT}(E(\lambda), \Phi)^0 d\lambda.$$

From Theorem 7.1, one can replace Ω by ω.

Then one uses Proposition 7.8 with $Z = 0$. One sees easily that unless $Q^s = P$, the characteristic function of $C_{Q^s}^\Lambda - nT_{Q^s}$ tends to 0 in the sense of tempered distributions. But in this case Q is standard and conjugate to P. Hence $Q = P$ and $s = 1$. Using Proposition 7.8, one computes easily the limit. $\square$

Now we can finish the proof of the theorem. The hypothesis on Φ above shows that the right hand side of the equality of the Lemma is zero for all P, ϕ, a_1, a_2. One concludes, by varying a_1, a_2 and ϕ, that $\Phi_{P,0,\mu} \in \mathcal{A}_P^{temp,c}(G)$ for all P and $\mu \in \mathcal{E}_P^w(\Phi)$. Then, using Lemma 3.10 (ii), one concludes that $\Phi = 0$. A contradiction which finishes the proof. $\square$

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