The global Gan–Gross–Prasad Conjecture for Fourier–Jacobi periods on unitary groups

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Integral representations of automorphic L-functions

- F number field, $\mathbb{A}$ its adele ring,
- G reductive group over F, $[G] = G(F)\backslash G(\mathbb{A})$,
- $H \leq G$ algebraic subgroup, $[H] = H(F)\backslash H(\mathbb{A})$,
- π an irreducible unitary cuspidal automorphic representation of G,
- $r : {}^L G \to \mathrm{GL}(V)$.

Goal : Find relations when $\varphi \in \pi$ varies

$$\underbrace{\int_{[H]} \varphi(h)dh}_{\text{period side}} \longleftrightarrow \underbrace{L(s, \pi, r)}_{L\text{-function side}}$$

Applications :

- Analytic properties, poles,
- Functional equation,
- Special values.

Example I : Tate's Thesis

Let χ be an automorphic character of $\mathbb{A}^\times$. For $\Phi \in \mathcal{S}(\mathbb{A})$, set

$$\Theta(h, \Phi) = \sum_{x \in F^\times} \Phi(hx), \quad h \in \mathbb{A}^\times.$$

For $\Re(s) > 1$, consider

$$Z(\chi, \Phi, s) = \int_{F^\times \backslash \mathbb{A}^\times} \chi(h)\Theta(h, \Phi)|h|^s dh.$$

Theorem (Tate)

Assume that $\Phi = \otimes_v \Phi_v$. For S a sufficiently large finite set of places of F

$$Z(\chi, \Phi, s) = L(s, \chi) \prod_{v \in S} \frac{Z_v(\chi_v, \Phi_v, s)}{L_v(s, \chi_v)}.$$

Example II : Rankin–Selberg theory

Let $G = \mathrm{GL}_n \times \mathrm{GL}_n$, $H = \mathrm{GL}_n \overset{\Delta}{\hookrightarrow} G$. For $\Phi \in \mathcal{S}(\mathbb{A}^n)$, set

$$\Theta(h, \Phi) = \sum_{x \in F^n \setminus \{0\}} \Phi({}^t h x), \quad h \in H(\mathbb{A}).$$

Let π be a cuspidal automorphic rep. of G. For $\varphi \in \pi$ consider

$$\mathcal{P}_H(\varphi, \Phi, s) = \int_{[H]} \varphi(h) \Theta(h, \Phi) |\det h|^s dh, \quad (\Re(s) > 1).$$

Theorem (Jacquet, Piatetski–Shapiro, Shalika)

Assume that $\varphi = \otimes_v \varphi_v$, $\Phi = \otimes \Phi_v$. *Then*

$$\mathcal{P}_H(\varphi, \Phi, s) = L(s, \pi) \prod_{v \in S} \frac{Z_v(\varphi_v, \Phi_v, s)}{L_v(s, \pi_v)}.$$

Moreover, for generic s

$$\mathcal{P}_H(\cdot, \cdot, s) \neq 0 \iff L(s, \pi) \neq 0.$$

Global Gan–Gross–Prasad conjecture for unitary groups

Goal : generalize these integral representations to $\mathrm{U}(V) \subset \mathrm{U}(V) \times \mathrm{U}(V)$.
Some data :

- E/F quadratic extension of number fields.
- $(V, \langle\cdot,\cdot\rangle)$ a skew-Hermitian space over E/F of dimension n.
- $G_V = \mathrm{U}(V) \times \mathrm{U}(V)$, $H_V = \mathrm{U}(V)$.

We first need a period $\mathcal{P}_{H_V}$ and an L-function.

The Weil representation

Set :

- ψ a non-trivial unitary character of $F\backslash\mathbb{A}$,
- η the quadratic character of $F^\times\backslash\mathbb{A}^\times$ associated to E/F by global class field theory,
- μ a character of $E^\times\backslash\mathbb{A}_E^\times$ such that $\mu_{|F^\times\backslash\mathbb{A}^\times} = \eta$.

By a classical construction of Weil and Kudla, one can associate to (ψ, μ) an automorphic representation $\omega_{\psi,\mu}$ of $\mathrm{U}(V)(\mathbb{A})$. This is the **Weil representation**. It is realized on $\mathcal{S}(\mathbb{A}^n)$.

Fourier Jacobi periods

For $\phi \in \omega_{\psi,\mu} = \mathcal{S}(\mathbb{A}^n)$, set

$$\theta(h,\phi) = \sum_{x \in F^n} (\omega_{\psi,\mu}(h)\phi)(x), \quad h \in H_V(\mathbb{A}).$$

Let π be a cuspidal representation of $G_V(\mathbb{A})$. Let $\varphi \in \pi$. The Fourier–Jacobi period is

$$\mathcal{P}_{H_V}(\varphi,\phi) = \int_{[H_V]} \varphi(h)\theta(h,\phi)dh.$$

Remark : If $E = F \times F$, then $V = F^n \times F^n$, $\mathrm{U}(V) = \mathrm{GL}_n$ and

$$\theta(h,\phi) = \mu(h)|\det h|^{\frac{1}{2}} \sum_{x \in F^n} \phi({}^t hx).$$

Base-change and L-functions

There exists $\mathrm{BC} : {}^L\mathrm{U}(V) \to {}^L\mathrm{GL}_{n,E}$. By functoriality, there should exist a mapping from automorphic rep. of $\mathrm{U}(V)$ to automorphic rep. of $\mathrm{GL}_{n,E}$.

Theorem (Mok; Kaletha, Minguez, Shin, White)

For every cuspidal rep. π of $\mathrm{U}(V)$ the base-change $\mathrm{BC}(\pi)$ exists.

If π is a cuspidal rep. of G_V, we have a completed Rankin–Selberg L-function $L(s, \mathrm{BC}(\pi))$. We want to vary V but keep the L-function fixed.

Definition

Let V_1, V_2 be two skew-Hermitian spaces. Let π_1 be a cuspidal rep. of G_{V_1}, π_2 be a cuspidal rep. of G_{V_2}. We say that π_1 and π_2 are in the same ***L*-packet** if $\mathrm{BC}(\pi_1) = \mathrm{BC}(\pi_2)$.

GGP for Fourier–Jacobi periods

Theorem (B., Lu, Xue)

Let π be an irreducible unitary cuspidal rep. of G_V. Assume that $\mathrm{BC}(\pi)$ is generic. TFAE

1. $L(\frac{1}{2}, \mathrm{BC}(\pi) \otimes \mu) \neq 0$,
2. *There exist V' and π' a cuspidal rep of $G_{V'}$ in the same L-packet than π such that*

$$(\mathcal{P}_{H_{V'}})_{|\pi' \otimes \omega_{\psi,\mu}} \neq 0.$$

- By local GGP (Gan, Ichino; Xue), the pair (V', π') is unique.
- This was known under local conditions by works of Xue.
- In the Bessel case, this was proved by Beuzart-Plessis, Chaudouard, Zydor.

Local Fourier–Jacobi periods

Write $\pi = \otimes_v \pi_v$, $\omega_{\psi,\mu} = \otimes_v \omega_v$, $\varphi = \otimes_v \varphi_v$, $\phi = \otimes_v \phi_v$, $\langle \cdot, \cdot \rangle_\pi = \prod_v \langle \cdot, \cdot \rangle_v$, $\langle \cdot, \cdot \rangle_\omega = \prod \langle \cdot, \cdot \rangle_v$, $dh = \prod dh_v$. Let v be a place of F. The local Fourier–Jacobi period is

$$\mathcal{P}_{H_V,v}(\varphi_v, \phi_v) = \int_{H_V(F_v)} \langle \pi_v(h_v)\varphi_v, \varphi_v \rangle_v \langle \omega_v(h_v)\phi_v, \phi_v \rangle_v dh_v.$$

- If π_v is tempered, this is absolutely convergent.
- Multiplicity one result (Aizenbud–Gourevitch–Rallis–Schiffmann, Sun–Zhu) :

$$\dim \mathrm{Hom}_{H_V(F_v)}(\pi_v \otimes \omega_v, \mathbb{C}) \leq 1.$$

- Non-vanishing (Xue) :

$$\mathcal{P}_{H_V,v} \neq 0 \iff \dim \mathrm{Hom}_{H_V(F_v)}(\pi_v \otimes \omega_v, \mathbb{C}) = 1.$$

Ichino–Ikeda factorization

Theorem (B., Lu, Xue, 2024)

Let π *be an irreducible unitary cuspidal rep. of* G_V*. Assume that* $\mathrm{BC}(\pi)$ *is generic and that for all* v π_v *is tempered. Then*

$$|\mathcal{P}_{H_V}(\varphi,\phi)|^2 = 2^{-\beta}\Delta \frac{L(\frac{1}{2},\mathrm{BC}(\pi)\otimes\mu)}{L(1,\pi,\mathrm{Ad})} \times \prod_{v\in S} \mathcal{P}_{H_V,v}(\varphi_v,\phi_v)\Delta_v^{-1}\frac{L(1,\pi_v,\mathrm{Ad})}{L(\frac{1}{2},\mathrm{BC}(\pi_v)\otimes\mu_v)}$$

where

- $\Delta = \prod_{i=1}^n L(i,\eta^i)$, $\Delta_v = \prod_{i=1}^n L(i,\eta_v^i)$,
- $\beta \in \mathbb{N}$ *is the number of isobaric components of* $\mathrm{BC}(\pi)$,
- $L(s,\pi,\mathrm{Ad})$ *is defined using* BC *and the Asai L-function.*

The Ramanujan conjecture predicts that if $\mathrm{BC}(\pi)$ is generic then all the π_v are tempered.

Non-vanishing of Fourier–Jacobi periods

Corollary

Let π be an irreducible unitary cuspidal rep. of G_V. Assume that $\mathrm{BC}(\pi)$ is generic and that for all v π_v is tempered. Then

$$(\mathcal{P}_{H_V})_{|\pi\otimes\omega_{\psi,\mu}} \neq 0 \iff \begin{cases} L(\frac{1}{2}, \mathrm{BC}(\pi)\otimes\mu) \neq 0, \\ \text{for all } v,\ \dim \mathrm{Hom}_{H_V(F_v)}(\pi_v \otimes \omega_v, \mathbb{C}) = 1. \end{cases}$$

Automorphic forms on Jacobi groups

- The Fourier–Jacobi period $\mathcal{P}_{H_V}$ is a priori not the integral of a cuspidal automorphic form because of $\theta(h, \phi)$ and $\omega_{\psi,\mu}$.
- Let $\mathbb{H}(V) = V \times F$ be the Heisenberg group of V. Set $J(V) = \mathrm{U}(V) \ltimes \mathbb{H}(V)$: this is the **Jacobi group** of V. Let $V' = V \oplus^{\perp} (Ee \oplus Ee^*)$ where $\langle e, e^* \rangle = 1$. Then

$$J(V) = \begin{pmatrix} 1 & * & * \\ & \mathrm{U}(V) & * \\ & & 1 \end{pmatrix} \subset \mathrm{U}(V').$$

  The Weil representation $\omega_{\psi,\mu}$ extends to $J(V)$, and so does $\theta(\cdot, \phi)$.
- Set $\varphi^J(g_1, j) = \varphi(g_1, j)\theta(j, \phi)$: this is a cuspidal automorphic form on $\mathrm{U}(V) \times J(V)$. We have

$$\mathcal{P}_{H_V}(\varphi, \phi) = \int_{[H_V]} \varphi^J(h) dh.$$

- Liu proposed the prove the GGP conjecture by using a (comparison of) relative trace formulae involving $\theta(\cdot, \phi)$ on $H_V \backslash G_V / H_V$. It is better understood as $H_V \backslash \mathrm{U}(V) \times J(V) / H_V$.

Thank you !