SUBSTITUTIONS AND RAUZY FRACTALS
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by

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Contents

1. Introduction : Sturmian sequences

Sturmian sequences have many characterizations that make it an interesting object: there is a combinatorial definition, a geometrical one, a definition as the coding of various dynamical systems, and a link with continuous fraction expansion. The aim of this course is to give a generalization of what occurs with Sturmian words and we will be particularly interested by dynamical systems coming from substitution. Before giving a generalization, let us recall the various characterizations of sturmian words.

1.1. Low complexity words. — Sturmian words are the non-ultimately periodic infinite words with the lowest complexity (recall that the complexity is the number of different factors of a given length). The complexity function p verify $p(n) = n + 1$.

1.2. Balanced words. — Sturmian words are also balanced words over an alphabet of two letters such that the density of one letter is rational.

1.3. Best approximations of a line. — Sturmian words can be constructed geometricaly by taking a line in $\mathbb{R}^2$ with a positive and irrational slope. There are several equivalent ways to get a Sturmian words from such line (and every Sturmian words is obtained in each way) : see figure 1.

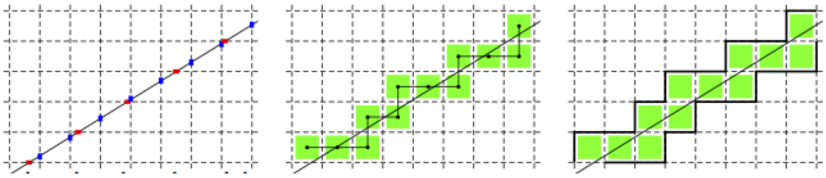
— By considering the sequence of point where we intersect the square grid : we have a two letters whenever we intersect a vertical or a horizontal line.

— By considering the sequence of "pixels" that the line runs through. The next pixel is to the left or to the top : it gives each time two possible letters.

— By considering up border or the down border of the previous sequence of pixels.

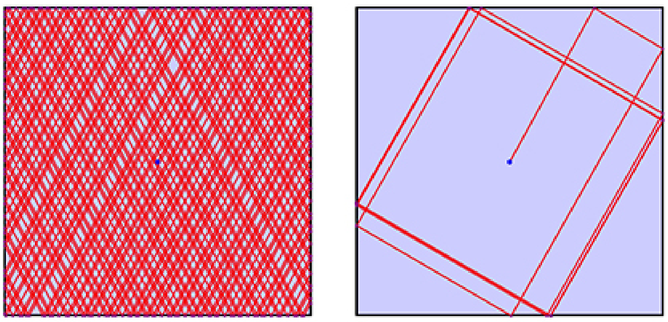
— By considering for each integer $n \in \mathbb{Z}$ the smallest integer $k \in \mathbb{Z}$ such that (n, k) is above the line. If the line has the equation $y = \alpha x + \gamma$ with $\alpha \in (0, \frac{1}{2})$, then we have $k = \lceil \alpha n + \gamma \rceil$. Hence, the corresponding Sturmian sequence is a rotation sequence : $(\lceil \alpha(n+1) + \gamma \rceil - \lceil \alpha n + \gamma \rceil)_{n \in \mathbb{E}}$, $\mathbb{E} = \mathbb{N}$ or $\mathbb{Z}$.

FIGURE 1. Different ways to define a Sturmian word from a line



1.4. Billard on a square. — By unfolding the trajectory of the ball in a billiard, we see that it is equivalent to follow a line in a $\mathbb{Z}^2$-grid.

FIGURE 2. Billiard in a square

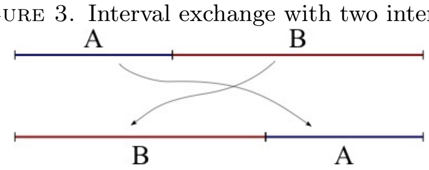


Hence, we get Sturmians word by considering trajectoris of a ball thrown with an irrational angle. Each time we hit a vertical border we get a 1 and each time we hit a horizontal border we get a 0 : it define an infinite or bi-infinite Sturmian word.

1.5. Coding of a translation on the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. — For a given irrational rotation, we can colorate the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ with two colors such that the sequence of colors obtained from the orbit of a point permits to identify uniquely the point. The sequence obtained is a Sturmian sequence (it is a rotation sequence) and every Sturmian sequence is obtained in that way.

1.6. Interval exchange. — A translation on a circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ is equivalent to an interval exchange transformation with two intervals. Indeed, if the two intervals are $[0, 1-\alpha]$ and $[1-\alpha, 1]$, then the corresponding translations are respectively α and $\alpha - 1$ which are equivalent modulo 1. Hence it corresponds to the translation by α on the circle.

FIGURE 3. Interval exchange with two intervals



1.7. Continued fractions. — There are two type of Sturmian words over the alphabet $\{0, 1\}$: words containing factor 00, we call it words of type 0, and words that contains the factor 11 : we call it words of type 1. A word of type 0 doesn't contains 11 as a subword, so every 1 is followed by a 0, and we can rewrite it using the new alphabet $\{0, 10\}$. The word obtained is still a Sturmian word (exercice !), so we can iterate this operation. If the word is of type 1, then we recode using the new alphabet $\{01, 1\}$.

Example 1.1. — *The Fibonnacci word is the fixed point of the Fibonnacci substitution*

$$\begin{array}{ccc} 0 & \mapsto & 01 \\ 1 & \mapsto & 0 \end{array}$$

$$0100101001001010010100100101001001010010....$$

It is a Sturmian word of type 0*, so we can rewrite it :*

$$\underbrace{0}_{0}\underbrace{10}_{1}\underbrace{0}_{0}\underbrace{10}_{1}\underbrace{10}_{1}\underbrace{0}_{0}\underbrace{10}_{1}\underbrace{0}_{0}\underbrace{10}_{1}\underbrace{10}_{1}\underbrace{0}_{0}\underbrace{10}_{1}\underbrace{10}_{1}...$$

Then we get a new Sturmian word which is of type 1*. We can rewrite it in the same way :*

$$\underbrace{01}_{0}\underbrace{01}_{0}\underbrace{1}_{1}\underbrace{01}_{0}\underbrace{01}_{0}\underbrace{1}_{1}\underbrace{01}_{0}\underbrace{10}_{1}...$$

Then we get a new Sturmian word of type 0*.*
(Exercice : we get exactly the Fibonnacci word with an added 0 *in the beginning.)*

Remark 1.2. — *Such rewriting of a word u of type i, $i \in \{0,1\}$, is equivalent to find a new Sturmian word v such that $u = \sigma_i(v)$, where σ_0 and σ_1 are the substitutions:*

$$\sigma_0 : \begin{cases} 0 & \mapsto & 0 \\ 1 & \mapsto & 10 \end{cases} \quad \textit{and} \quad \sigma_1 : \begin{cases} 0 & \mapsto & 01 \\ 1 & \mapsto & 1 \end{cases}.$$

If we iterate the process from a Sturmian word u, we get an longer and longer sequence of desubstitution:

$$u = \sigma_1^{k_1}\sigma_0^{k_2}\sigma_1^{k_3}...\sigma_0^{k_{2n}}(v)$$

for some Sturmian word v. The sequence k_1, k_2, k_3, ... of positive integers that appears is related to the continued fraction expansion of the rotation number of the Sturmian sequence u:

$$\alpha = \cfrac{1}{k_1 + 1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cfrac{1}{k_4 + \cfrac{1}{\ddots}}}}}$$

Remark 1.3. — *The $+1$ after the k_1 comes from the fact that we consider here the rotation number. If we consider the slope p rather than the rotation number α, we get the same continued fraction but without this $+1$. Indeed we have the following link between α and p:*

$$\alpha = \frac{p}{p+1} = \cfrac{1}{1 + \cfrac{1}{p}}.$$

Remark 1.4. — *The continued fraction expansion of a real number $x \in [0,1[$ is given by the following algorithm:*

1. *start with $i = 1$*
2. *$k_i = \left\lfloor \frac{1}{\alpha} \right\rfloor$*
3. *$\alpha \leftarrow \left\{ \frac{1}{\alpha} \right\}$ (i.e. replace α by the fractional part of $1/\alpha$)*
4. *if $\alpha \neq 0$ go to 2 and continue.*

Exercise 1.5. — *Show that this algorithm terminates if and only if α is a rational number.*

1.8. Generalizations. — There are a lots of ways to generalize Sturmian sequences:

- One can study dynamics of billiards with various shapes (polygons, disk, etc...). The dynamics can be very hard to study and a lot of people is working on it (particularly in Marseille).
- One can try to generalize the continued fraction algorithm with different substitutions than σ_0 and σ_1. It's called S-adic systems, and there are still a lot of people working on it, and particularly in Marseille.
- One can study more general interval exchange transformations with more intervals. A lot of work has been done in this topic, and there are very hard problems.
- One can try to generalize approximation of lines to higher dimensions : this works partially, and we will see that we have the equivalence with a domain exchange and with a translation on a torus.
- One can try to consider nice words with a bigger alphabet, like for example fixed points of substitutions. This is what we will do in the following !

2. An interesting example: Tribonnacci

G. Rauzy has generalized what happen for the Sturmian sequences for the following example. Consider the Tribonnacci substitution:

$$s : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$

and consider the infinite fixed point u obtained by iterating the letter a.

$$\begin{array}{rcl} & & a \\ s(a) & = & ab \\ s^2(a) & = & abac \\ s^3(a) & = & abacaba \\ s^4(a) & = & abacabaabacab \\ & \ldots & \end{array}$$

$$u = abacabaabacababacabaabacabacabaabacababa...$$

Remark 2.1. — *The complexity of this word u is $p(n) = 2n + 1$.*

Remark 2.2. — *The length of the word $s^n(a)$ can be computed using the incidence matrix. Indeed, for every word $v \in A^*$, where $A = \{a, b, c\}$, we have*

$$\begin{pmatrix} |s(v)|_a \\ |s(v)|_b \\ |s(v)|_c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} |v|_a \\ |v|_b \\ |v|_c \end{pmatrix},$$

where $|v|_a$ denotes the number of occurrence of the letter a in the word v. If we denote by $\mathrm{Ab}(v) := \begin{pmatrix} |v|_a \\ |v|_b \\ |v|_c \end{pmatrix}$ *(we call it the* ***abelianization*** *of v) and by* $M_s := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *the incidence matrix, we have a shorter writting:*

$$\mathrm{Ab}(s(v)) = M_s \,\mathrm{Ab}(v).$$

This permits to compute the number of occurence of each letters in $s^n(a)$ (hence we get the total length by summing):

$$\mathrm{Ab}(s^n(a)) = M_s^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Exercise 2.3. — *Show that we have the following induction formulae:*

$$s^{n+3}(a) = s^{n+2}(a)s^{n+1}(a)s^n(a).$$

2.1. Construction of a broken line. — In order to have a characterization of the fixed point u of the Tribonnacci substitution with a broken line approximating a line, like for Sturmian words, we define a **broken line** from u. It is the line connecting points of the discrete line:

Definition 2.4. — *We call* ***discrete line*** *the set L of points of $\mathbb{R}^3$:*

$$L = \left\{ \mathrm{Ab}(v) \;\middle|\; v \text{ finite prefix of } u \right\},$$

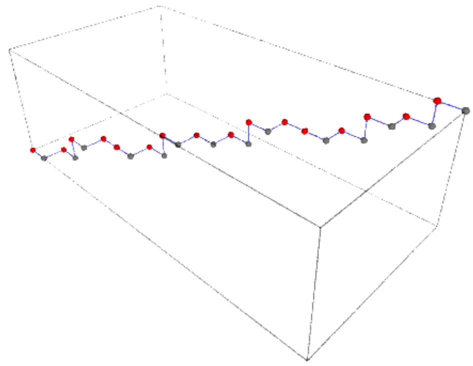
where u is the fixed point of the Tribonnacci substitution and Ab *is the abelianization map defined before.*

In order to have the same phenomenon than for Sturmian words, we would like this discrete line to be near a line. This is the case:

Proposition 2.5. — *The discrete line L is at bounded distance of the eigenspace (which is a line) of the highest eigenvalue of the incidence matrix M_s.*

The idea of the proof will be to use that all the other eigenvalues are less than one in absolute value (Pisot property, see later). This is an important property of the number β.

Figure 4. Begining of the broken line corresponding to the word u
The red points correspond to the set $M_s L$ and the points in grey are the other points of the discrete line L



Remark 2.6. — *The highest eigenvalue of the incidence matrix M_s is a* ***Perron number****. By definition it is an algebraic integer β (i.e. a root of a unitary polynomial with integer coefficients) such that the conjugates (i.e. the others roots of the minimal polynomial) are less than β in absolute value. In particular, we have $\beta > 1$.*

Remark 2.7. — *The highest eigenvalue of the incidence matrix M_s is even a* ***Pisot number****. By definition it is an algebraic integer β such that the conjugates are less than 1 in absolute value. In particular, we have $\beta > 1$.*

Remark 2.8. — *Every Pisot number λ has the property that powers are near integers:*

$$d(\lambda^n, \mathbb{Z}) \xrightarrow[n\to\infty]{} 0.$$

(And the convergence is exponentially fast.) It is a very particular property, since x^n is equidistributed mod 1 for almost every real number $x > 1$.

Conjecture 2.9. — *For $x > 1$, if we have $d(x^n, \mathbb{Z}) \xrightarrow[n\to\infty]{} 0$, then x is a Pisot number.*

The fact that the discrete line stays near a line associated to the matrix M_s comes from the following.

Lemma 2.10. — *We have $M_s L \subseteq L$.*

See the figure 4 for a illustration of this lemma.

Proof. — This is a consequence of the fact that for every word $v \in A^*$, $A = \{a, b, c\}$, we have

$$M_s \operatorname{Ab}(v) = \operatorname{Ab}(s(v)).$$

This comes directly from the definition of M_s.

This gives the wanted inclusion : if $x \in L$, by definition there exists a finite prefix v of the fixed point u such that $x = \mathrm{Ab}(v)$. Then $s(v)$ is also a prefix of u, so $M_s \,\mathrm{Ab}(v) = \mathrm{Ab}(s(v)) \in L$. □

The fact that the highest eigenvalue β of the incidence matrix of M_s is a Pisot number permits to prove the proposition 2.5:

Proof of the proposition 2.5. — The highest eigenvalue β of the incidence matrix M_s of the Tribonnacci substitution s is a Pisot number. Indeed, it is a Perron number by the Perron-Frobenius theorem because $M_s^3 = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} > 0$, and the characteristic polynomial of M_s is $X^3 - X^2 - X - 1$. So if we call α and α' the other roots, then we have

$$\beta + \alpha + \alpha' = 1 \quad \text{and} \quad \beta\alpha\alpha' = 1.$$

And we know that $\beta > 1$ (it is a Perron number), so we have $\alpha + \alpha' = 1 - \beta < 0$ and $\alpha\alpha' = \frac{1}{\beta}$. Moreover, the equality $\beta = 1 + \frac{1}{\beta} + \frac{1}{\beta^2}$ implies that $\beta < 2$. Hence, we have the equalities $-1 < \alpha + \alpha' < 0$ and $0 < \alpha\alpha' < 1$, and this implies that $|\alpha| < 1$ and $|\alpha'| < 1$. In fact, we could prove that α and α' are complex numbers, but we will not need it in the proof.

Using the kernels decomposition lemma ("lemme des noyaux" in french), we have

$$\mathbb{R}^3 = \mathbb{R}V_\beta \oplus F$$

where V_β is an eigenvector for the eigenvalue β, and F is an linear vector subspace of dimension 2 invariant by M_s : $M_s F \subseteq F$. Let's denote by p_β the linear projection onto the line $\mathbb{R}V_\beta$ along F, and p_F the linear projection onto F along the line $\mathbb{R}V_\beta$. What we want to prove is that $p_F(L)$ is bounded: it is equivalent to say that the discrete line L stays at bounded distance of the line $\mathbb{R}V_\beta$. In order to do that, we use the lemma 2.11 (see below) that permits to express the elements of L as a expansion in base M_s:

$$L = \left\{ \sum_{i=0}^{n} M_s^i \epsilon_i \;\middle|\; \epsilon_n \epsilon_{n-1} ... \epsilon_0 \in \mathcal{L} \right\},$$

where $\mathcal{L}$ is a language over the alphabet $\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$. In fact what we need is just the inclusion

$$L = \left\{ \sum_{i=0}^{n} M_s^i \epsilon_i \;\middle|\; \forall i, \epsilon_i \in \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}, n \in \mathbb{N} \right\}.$$

Using this, if $x \in L$, then we take $n \in \mathbb{N}$ and $(\epsilon_i)_{i=0}^n \in \{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}^{n+1}$ such that $x = \sum_{i=0}^n M_s \epsilon_i$. Then we have

$$\|p_F(x)\| \leq \sum_{i=0}^n \|M_s \epsilon_i\| \leq \sum_{i=0}^n C\,|\alpha|^i = \frac{C}{1-|\alpha|},$$

for a constant C and for α the second highest eigenvalue of M_s in modulus. This proves that $p_F(L)$ is bounded, hence L stays at bounded distance of the line $\mathbb{R}V_\beta$. □

It remains to prove that we can decompose the elements of the discrete line L in basis M_s as we used in the proof:

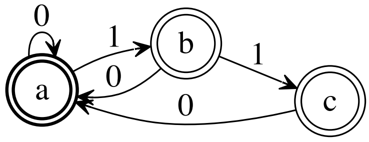
Lemma 2.11. —

$$L = \left\{ \sum_{i=0}^n M_s^i \epsilon_i \;\middle|\; \epsilon_n \epsilon_{n-1} ... \epsilon_0 \in \mathcal{L} \right\},$$

where $\mathcal{L}$ is the language over the alphabet $\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$ defined by the automaton of the figure 5 (where we denote the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ simply by 0 and the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ by 1).

Definition 2.12. — *We call **language of the automaton** of the figure 5 the sets of words obtained by following a path in the graph from the initial state a to a final state (here every state is final). The word is given by the successive labels of the edges followed.*

FIGURE 5. Automaton coming from the Tribonnacci substitution



We will see with more details in the following what is this automaton and why it describes the discrete line L.

3. A criterion for a subshift to be measurably conjugated to a domain exchange and to a translation on a torus

In this section, we describe a general geometric criterion for a subshift to be measurably isomorphic to a translation on a torus. Let us start by introduce some notations.

3.1. Subshift. — We denote by $A^{\mathbb{N}}$ (respectively $A^{\mathbb{Z}}$) the set of infinite (respectively bi-infinite) words over the alphabet A. We denote by $|u|$ the length of a word u, and $|u|_a$ denotes the number of occurrences of the letter a in a word $u \in A^*$. And we denote by

$$\mathrm{Ab}(u) = (|u|_a)_{a \in A} \in \mathbb{N}^A$$

the **abelianisation vector** of a word $u \in A^*$. The canonical basis of $\mathbb{R}^A$ will be denoted by $(e_a)_{a \in A} = (\mathrm{Ab}(a))_{a \in A}$.

The **shift** on infinite words is the application

$$S : \begin{array}{ccc} A^{\mathbb{N}} & \longrightarrow & A^{\mathbb{N}} \\ (u_i)_{i \in \mathbb{N}} & \longmapsto & (u_{i+1})_{i \in \mathbb{N}} \end{array}$$

We can also define the shift on bi-infinite words in an obvious way, and it becomes invertible.

We use the usual metric on $A^{\mathbb{N}}$:

$$d(u, v) = 2^{-n} \text{ where } n \text{ is the length of the maximal common prefix.}$$

The map S is continuous for this metric. Given an infinite word u, the compact set $\overline{S^{\mathbb{N}} u}$ is S-invariant. We call **subshift** generated by u, the dynamical system $(\overline{S^{\mathbb{N}} u}, S)$.

The same can be done for bi-infinite words.

3.2. Discrete line associated to a word. — Let $u \in A^{\mathbb{N}}$ be an infinite word over the alphabet A. Then, the associated **discrete line** is the following subset of $\mathbb{Z}^A$:

$$D_u := \left\{ Ab(v) \in \mathbb{Z}^A \;\middle|\; v \text{ finite prefix of } u \right\}.$$

If $u \in A^{\mathbb{Z}}$ is a bi-infinite word, then the corresponding discrete line is

$$D_u := -D_v \cup D_w,$$

where $u, v \in A^{\mathbb{N}}$ are infinite words such that $u = {}^t v w$

We can partition this discrete line into $d = |A|$ pieces. For every $a \in A$, let

$$D_{u,a} := \left\{ Ab(v) \in \mathbb{Z}^A \;\middle|\; va \text{ finite prefix of } u \right\}.$$

The sets $D_{u,a} + e_a$, $a \in A$, also give a partition of D_u:

$$D_u = \{0\} \cup \bigcup_{a \in A} D_{u,a} + e_a.$$

This partition permits to see the shift S on the word u as a domain exchange E:

$$E : \begin{array}{ccl} D_u & \longrightarrow & D_u \\ x & \longmapsto & x + e_a \text{ for } a \in A \text{ such that } x \in D_{u,a}. \end{array}$$

The same can be done for bi-infinite words.

There is also a property of tiling for this discrete line: we have the following

Proposition 3.1. — *Let Γ_0 be the subgroup of $\mathbb{Z}^A$ generated by $(e_a - e_b)_{a,b\in A}$, and let u be any bi-infinite word on the alphabet A. Then D_u is a fondamental domain for the action of Γ_0 on $\mathbb{Z}^A$. In other words, we have the disjoint union:*

$$D_u + \Gamma_0 = \mathbb{Z}^A.$$

And the translation T by e_a (for any $a \in A$) on $\mathbb{Z}^A/\Gamma_0$ is conjugated to the domain exchange E on D_u by the natural quotient map.

Remark 3.2. — *A **fundamental domain** for the action of a group G on a set X is a part F of X such that we have the disjoint union*

$$X = \bigcup_{g\in G} g.F.$$

This is equivalent to say that the natural map $X \to X/G$ restricted to F is bijective. Exemple : the unit cube $[0,1[^d$ is a fundamental domain for the action of $\mathbb{Z}^d$ on $\mathbb{R}^d$: this cube tile the plane such that to translates are disjoint.

Remark 3.3. — *We have the same for infinite words, but we get a fundamental domain for the action on the half-space $\left\{(x_a)_{a\in A} \in \mathbb{Z}^A \;\middle|\; \sum_{a\in A} x_a \geq 0\right\}$, and a conjugacy with the shift on $S^{\mathbb{N}}u$.*

Proof. — The vectors $(e_a)_{a\in A}$ are equivalent modulo the group Γ_0. Hence, this discrete line is equivalent to $\mathbb{Z}e_a$ for any letter $a \in A$, and this is an obvious fundamental domain of $\mathbb{Z}^A$ under the action of Γ_0. The natural coding of the domain exchange E for the partition $D_u = \bigcup_{a\in A} D_{u,a}$ gives a conjugacy with the shift S on $S^{\mathbb{Z}}u$, and the projection onto the quotient by Γ_0 gives a conjugacy between the domain exchange E on D_u and the translation T on $\mathbb{Z}^A/\Gamma_0$. □

If the discrete line D_u stay near a given line of $\mathbb{R}^A$ (this will be the case for example for periodic points of Pisot substitutions), then we can project onto a hyperplane $\mathcal{P}$ of $\mathbb{R}^A$ (for example the hyperplane of equation $\sum_{a\in A} x_a = 0$) along this line. The projection of $\mathbb{Z}^A$ is dense in the hyperplane for almost lines, but the group Γ_0 becomes a lattice in the hyperplane. If the projection of the discrete line is not so bad, we can expect that the closure gives a tiling of the hyperplane, and that the closure of each piece of the partition of the discrete line doesn't intersect each other. And we can expect that the conjugacy given by the previous proposition becomes a conjugacy of the closures. The figure 6 show the conjugacy given by the proposition 3.1, and what we get if everything go well.

3.3. Geometrical criterion for a subshift to be conjugated to a toral translation. — Here is the main general geometric criterion that permits to know if a subshift is measurably conjugated to a domain exchange and to a toral translation.

FIGURE 6. Commutative diagrams of the conjugacy between the shift S, the domain exchange E and the translation on the quotient T, before and after taking the closure

$$
\begin{array}{ccc}
S^{\mathbb{Z}}u & \xrightarrow{S} & S^{\mathbb{Z}}u \\
\downarrow c & & \downarrow c \\
D_u & \xrightarrow{E} & D_u \\
\downarrow \pi_0 & & \downarrow \pi_0 \\
\mathbb{Z}^A/\Gamma_0 & \xrightarrow{T} & \mathbb{Z}^A/\Gamma_0
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{ccc}
\overline{S^{\mathbb{Z}}u} & \xrightarrow{S} & \overline{S^{\mathbb{Z}}u} \\
\downarrow c & & \downarrow c \\
\overline{\pi(D_u)} & \xrightarrow{E} & \overline{\pi(D_u)} \\
\downarrow \pi_0 & & \downarrow \pi_0 \\
\mathcal{P}/\pi(\Gamma_0) & \xrightarrow{T} & \mathcal{P}/\pi(\Gamma_0)
\end{array}
$$

Theorem 3.4. — *Let $u \in A^{\mathbb{N}}$ be an non-eventually periodic infinite word over an alphabet A, and let π be a projection from $\mathbb{R}^A$ onto a linear hyperplane $\mathcal{P}$. We assume that we have the following:*

— *the restriction of π to $\mathbb{Z}^A$ is injective and has a dense image,*

— *the set $\pi(D_u)$ is bounded,*

— *the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is minimal.*

Then there exists a σ-algebra and a S-invariant measure μ such that the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is an extension of the translation of the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$, where T is the translation by e_a (for any $a \in A$) on the torus $\mathcal{P}/\pi(\Gamma_0)$, where Γ_0 is the group generated by $\{e_a - e_b \mid a, b \in A\}$, and λ is the Lebesgue measure. And this is also a topological semi-conjugacy.

If moreover we have

— *for every $a \in A$, the boundary of $\overline{\pi(D_{u,a})}$ has zero Lebesgue measure,*

— *the sets $\mathring{\overline{\pi(D_{u,a})}}$, $a \in A$, are disjoint.*

Then the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is isomorphic to a domain exchange on $\overline{\pi(D_u)}$ with the Lebesgue measure.

If moreover

— *the sets $\mathring{\overline{\pi(D_u)}} + t$, $t \in \pi(\Gamma_0)$, are disjoint,*

then the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is isomorphic to the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.

In order to prove this theorem, we start by showing that we can prolongate by continuity the map $c : S^{\mathbb{N}}u \to \pi(D_u)$ that gives the conjugacy between $(S^{\mathbb{N}}u, S)$ and $(\pi(D_u), E)$.

Lemma 3.5. — *Let $u \in A^{\mathbb{N}}$ be an infinite word over an alphabet A, and let π be a projection from $\mathbb{R}^A$ onto a hyperplane $\mathcal{P}$. We assume that $\pi(D_u)$ is bounded. Then the map $c : S^{\mathbb{N}}u \to \pi(D_u)$, defined as the inverse of the natural coding of $(\pi(D_u), E)$*

for the partition $D_u = \bigcup_{a\in A} D_{u,a}$*, can be prolongated by continuity at any point of the closure whose orbit is dense in* $\overline{S^{\mathbb{N}}u}$*.*

To prove this lemma, we need the following geometric lemma, saying that we can always translate a bounded set of $\mathbb{R}^d$ in order to have a non empty but arbitrarily small intersection with the initial set.

Lemma 3.6. — *Let* Ω *be a bounded subset of* $\mathbb{R}^d$*. Then, we have*

$$\inf_{t\in\Omega-\Omega} \operatorname{diam}(\Omega\cap(\Omega-t)) = 0.$$

The proof is left as an exercise.

proof of lemma 3.5. — Let $w \in \overline{S^{\mathbb{N}}u}$ having dense orbit in $\overline{S^{\mathbb{N}}u}$ and let $\epsilon > 0$. By lemma 3.6, there exists $t \in \pi(D_u) - \pi(D_u)$ such that $\operatorname{diam}(\pi(D_u)\cap(\pi(D_u)-t)) \leq \epsilon$. Let n_1 and $n_2 \in \mathbb{N}$ such that $c(S^{n_2}u) - c(S^{n_1}u) = t$. We can assume that $n_1 \leq n_2$ up to replace t by $-t$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(S^{n_0}w, u) \leq 2^{-n_2}$. Now, for all $v \in S^{\mathbb{N}}u$ such that $d(w, v) \leq 2^{-(n_0+n_2)}$, we have that $c(S^{n_0+n_1}v) \in D\cap(D-t)$, because $c(S^{n_0+n_2}v) - c(S^{n_0+n_1}v) = t$. Hence, if we let $\eta = 2^{-(n_0+n_2)}$, we have

$$\forall v, v' \in D_u, \left\{\begin{array}{c} d(v,w)\leq\eta \\ \text{and} \\ d(v',w)\leq\eta \end{array}\right\} \Longrightarrow d(c(v), c(v')) = d(c(S^{n_0+n_1}v), c(S^{n_0+n_1}v')) \leq \epsilon.$$

This proves that we can prolongate c by continuity at point w. □

Lemma 3.7. — *Let* $u \in A^{\mathbb{N}}$ *be an infinite word over an alphabet* A*, and let* π *be a projection from* $\mathbb{R}^A$ *onto a hyperplane* $\mathcal{P}$*. We assume that we have the following conditions:*

- *the restriction of the projection* π *to* $\mathbb{Z}^A$ *is injective and has a dense image,*
- *the set* $\pi(D_u)$ *is bounded,*
- *for every* $a \in A$*, the boundary of* $\overline{\pi(D_{u,a})}$ *has zero Lebesgue measure,*
- *the sets* $\mathring{\overline{\pi(D_{u,a})}}$*,* $a \in A$*, are disjoint.*

Then the natural coding cod *of* $(\pi(D_u), E)$ *for the partition* $D_u = \bigcup_{a\in A} D_{u,a}$*, can be prolongated by continuity to a full measure part of the closure. And we have*

$$\forall x \in M, \lim_{\substack{y\to x \\ y\in\pi(D_u)}} c^{-1}(y) = \operatorname{cod}(x).$$

Proof. — Let $\Omega = \overline{\pi(D_u)}$ and $\forall a \in A,\ \Omega_a = \overline{\pi(D_{u,a})}$. We can prolongate the domain exchange E in an obvious way:

$$E' : \begin{array}{ccl} \bigcup_{a\in A}\mathring{\Omega}_a & \longrightarrow & \Omega \\ x & \longmapsto & x + \pi(e_a) \text{ for } a \in A \text{ such that } x \in \mathring{\Omega}_a. \end{array}$$

The part of full Lebesgue measure that we consider is the E'-invariant set

$$M := \bigcap_{n\in\mathbb{N}} E'^{-n}\Omega.$$

Let $\epsilon > 0$ and let $x \in M$. Let $n_0 \in \mathbb{N}_{\geq 1}$ such that $2^{-n_0} \leq \epsilon$. The set

$$M_{n_0} := \bigcap_{n=0}^{n_0} E'^{-n}\Omega$$

is an open set containing x, because E' is continuous and $E'^{-1}\Omega = \bigcup_{a\in A} \overset{\circ}{\Omega}_a + e_a$ is open. Hence there exists $\eta > 0$ such that $B(x,\eta) \subseteq M_{n_0}$. And for every $y \in B(x,\eta)\cap M$, the natural coding of (M, E') for the partition $M = \bigcup_{a\in A} M\cap\Omega_a + \pi(e_a)$ coincides with the coding of x for the n_0 first steps. Hence, cod is continuous on M. We get also the last part of the lemma by observing that if $y \in B(x,\eta)\cap\pi(D_u)$, then the coding of y (which is equal to $c^{-1}(y)$) also coincide with the coding of x for the n_0 first steps. □

In the following, we will denote by E the domain exchange on M.

proof of theorem 3.4. — The lemma 3.5 shows that we can prolongate the map c by continuity : $c : \overline{S^{\mathbb{Z}}u} \to \overline{D_u}$. If we compose c with the natural projection π_0 onto the torus $\mathcal{P}/\pi(\Gamma_0)$, we get a continuous function which is onto, because of the equality $\pi(\Gamma_0) + \overline{\pi(D_u)} = \mathcal{P}$ that comes from $\Gamma_0 + D_u = \mathbb{Z}^A$. Let's consider the σ-algebra that we get from the Borel σ-algebra with the continuous map $\pi_0 \circ c : \overline{S^{\mathbb{N}}u} \to \mathcal{P}/\pi(\Gamma_0)$. A measure μ on this σ-algebra can be defined by $\mu(c^{-1}(A)) = \lambda(A)$ for any borel set A of $\mathcal{P}/\pi(\Gamma_0)$, where λ is the Lebesgue measure. By continuity, this measure μ that we get on $\overline{S^{\mathbb{N}}u}$ is S-invariant, and for this measure the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is semi-conjugated to $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$, where T is the translation by e_a (for any $a \in A$) on the torus.

By lemma 3.7, the natural coding gives a continuous function from M to $\mathrm{cod}(M) \subseteq \overline{S^{\mathbb{N}}u}$. By lemma 3.5, the function c can be prolongated by continuity everywhere, so $c : \mathrm{cod}(M) \to \overline{\pi(D_u)}$ is continuous. Hence we have

$$\forall x \in M,\ x = \lim_{\substack{y\to x\\ y\in\pi(D_u)}} c\circ c^{-1}(y) = c\circ \mathrm{cod}(x).$$

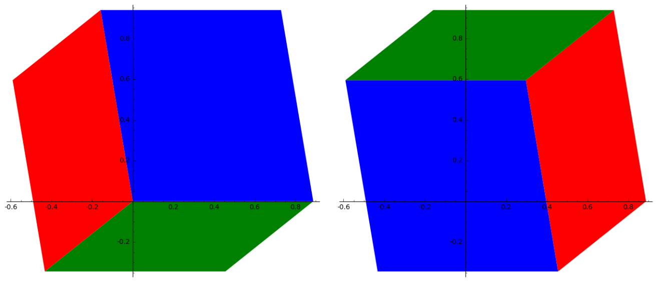
In the same way, we also have $\forall x \in c^{-1}(M),\ \mathrm{cod}\circ c(x) = x$. Hence, $\restriction c\mathrm{cod}(M)$ is a homeomorphism, and we get a topological conjugacy between the dynamical system (M, E) and the dynamical system $(c^{-1}(M), S)$. This gives that the measurable dynamical systems $(\overline{S^{\mathbb{Z}}u}, S, \mu)$ and $(\overline{\pi(D_u)}, E, \lambda)$ are isomorphic.

If we assume moreover that the sets $\overset{\circ}{\overline{\pi(D_u)}} + t$, $t \in \pi(\Gamma_0)$, are disjoint, then the projection $\pi_0 : M \to \mathcal{P}/\pi(\Gamma_0)$ is invertible in a set of full measure. Hence, this gives a topological conjugacy between (N, T) and $(\pi_0^{-1}(N), E)$, where $N \subseteq \pi_0(M)$ is a T-invariant subset of full Lebesgue measure of $\mathcal{P}/\pi(\Gamma_0)$. This gives a measurable conjugacy between the domain exchange $(\overline{\pi(D_u)}, E, \lambda)$ and the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$. □

3.4. An example : generalization of Sturmian sequences. — An example where all works fine is obtained by taking a random line of $\mathbb{R}^d$ with a positive direction vector. We consider the natural $\mathbb{Z}^d$-tiling by hypercubes, and we take the sequence of hyperfaces that intersect the line. Almost surely, this gives a discrete line corresponding to some word u over the alphabet of the d type of hyperfaces. And for such word, we can describe completely the sets $\overline{\pi(D_u)}$ and $\overline{\pi(D_{u,a})}$: the color of a point of the discrete line is given by the direction we have to follow to reach the next point. But this is given by the hyperface where we go out of the cube. Hence, the domains of the domain exchange are the projections of the positive faces of a cube. And after exchange, we get the projection of the other faces of the cube, because they are the faces where we enter into a cube, and it gives in which cube the line was before entering this cube.

The figure 7 shows the domain exchange for a line whose a direction vector is near (0.549734033503, 0.364904263473, 0.995017553314) in $\mathbb{R}^3$.

FIGURE 7. Domain exchange conjugated to a translation on the torus $\mathbb{T}^2$, and also conjugated to the subshift generated by a word corresponding to a discrete approximation of a line of $\mathbb{R}^3$.



Remark 3.8. — *The word appearing in this last example is obtain by a simple algorithm: If the positive direction vector of the line is (v_1, v_2, v_3), and if the line go throw the point (c_1, c_2, c_3) then we have almost surely*

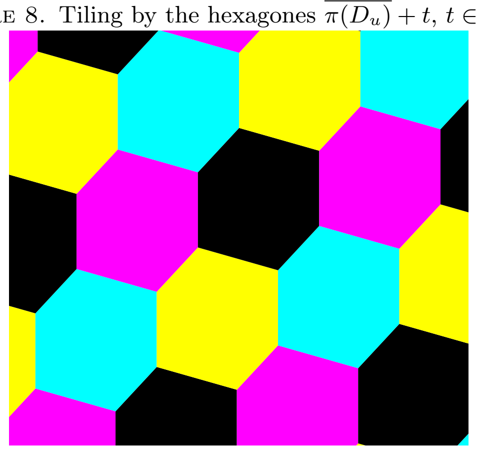
$$\exists (k_1,k_2,k_3) \in \mathbb{Z}^3,\ \forall j \in \{1,2,3\},\ k_j = \left\lfloor v_j \frac{k_{i_n} - c_{i_n}}{v_{i_n}} + c_j \right\rfloor$$

$$\Longrightarrow \quad \exists ! i_{n+1} \in \{1,2,3\},\ \forall j \in \{1,2,3\} \backslash \{i_{n+1}\},\ k_j = \left\lfloor v_j \frac{k_{i_{n+1}} + 1 - c_{i_{n+1}}}{v_{i_{n+1}}} + c_j \right\rfloor .$$

The sequence $(i_n)_{n \in \mathbb{N}}$ define an infinite word, and we get a bi-infinite word by invertibility of this algorithm.

We can check here that every hypothesis of the theorem 3.4 is satisfied:
- — The projection is irrationnal as soon as the direction vector of the line is irrational.
- — The subshift $(\overline{S^{\mathbb{Z}}u}, S)$ is minimal (it is a consequence of the irrationality of the direction vector).
- — The set $\pi(D_u)$ is bounded : it is included in the projection of a cube.
- — The interior of the domains $\overline{\pi(D_{u,a})}$ have disjoint interior : it is the projection of half of the faces of the cube (see figure 7).
- — The boundaries of the domains $\overline{\pi(D_{u,a})}$ have zero Lebesgue measure : it is the projection of the boundaries of the faces of a cube that have dimension $1 < 2$.
- — The translates of the domain $\overline{\pi(D_u)}$ by the group $\pi(\Gamma_0)$ have disjoint interiors : we get a tiling by the projection of a cube. See figure 8.

FIGURE 8. Tiling by the hexagones $\overline{\pi(D_u)} + t$, $t \in \pi(\Gamma_0)$



Hence, the subshift generated by such word given by a line is measurably conjugated to a simple domain exchange that corresponds to projections of faces of a cube. And it is also measurably conjugated to a translation by the projection of one of the vector of the canonical basis, on the torus given by the tiling of the hyperplane by the projection of a cube (which is conjugated to a translation on the torus $\mathbb{T}^d$).

4. Application of the criterion to Tribonnacci

Using the criterion of the previous section (see theorem 3.4), we show that the subshift generated by the Tribonnacci substitution is measurably isomorphic to a domain exchange on the Rauzy fractal:

Proposition 4.1. — *If $u = abacaba...$ is the infinite fixed point of the Tribonnacci substitution*

$$s : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}$$

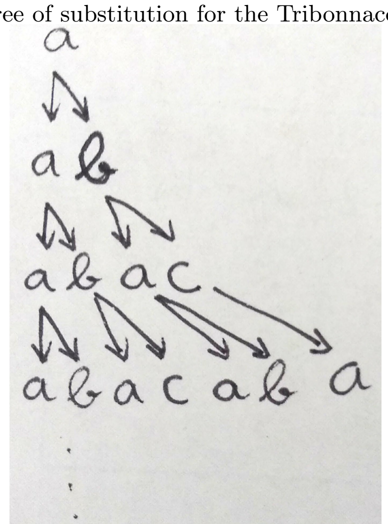
then, the system $(\overline{S^{\mathbb{N}}u}, S)$ is uniquely ergodic for a measure μ, and we have the isomorphism

$$(\overline{S^{\mathbb{N}}u}, S, \mu) \simeq (\overline{\pi(D_u)}, E, \lambda),$$

where $\overline{\pi(D_u)}$ is the Rauzy fractal of s defined in the section 2, E is the domain exchange defined in the same section, and λ is the Lebesgue measure.

In order to apply the criterion, we need to check the hypothesis. For that, we need to prove some properties of the Rauzy fractal. We start by giving a more precise description of the Rauzy fractal and to give a characterization by a set of equations (that we call gIFS equations).

Figure 9. Tree of substitution for the Tribonnacci substitution

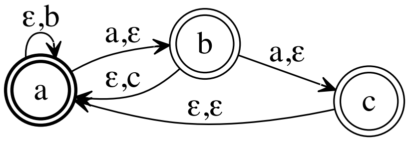


4.1. Representation of the Rauzy fractal using numeration in basis γ. — The figure 9 show the tree of subtitution, starting from the letter a, and iterating the substitution. A path in this tree can be represented by a path in the automaton of the figure 10 called **prefix-suffix automaton**.

The labels on the edges permits to keep track of where we are in the tree. Indeed, if we start with the couple of empty words (ϵ, ϵ) in the state a, and if we do the following calculation for each transition:

$$(u, v) \text{ and state } i \xrightarrow{(x,y)} (s(u)x, ys(v)) \text{ and state } j$$

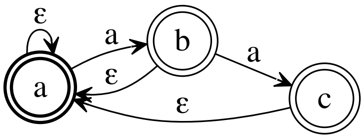
FIGURE 10. Prefix-suffix automaton for the Tribonnacci substitution



We have that $s(uiv) = s(u)xjys(v)$. And if we get the couple (u, v) and finish in the state i after following a path from a of length n, then we have $uiv = s^n(a)$. Hence, the sequence of labels of a path in the automaton corresponds uniquely with a path in the tree. So the number of different paths of length n starting from state a is exactly the length of the word $s^n(a)$

In order to describe the discrete line $D_u = \left\{\mathrm{Ab}(v) \;\middle|\; v \text{ finite prefix of } u\right\}$, we want to take care only of what happens for the prefixes. If we forget the suffixes, we get what we call the **prefix automaton**, that is drawn in picture 11.

FIGURE 11. Prefix automaton for the Tribonnacci substitution

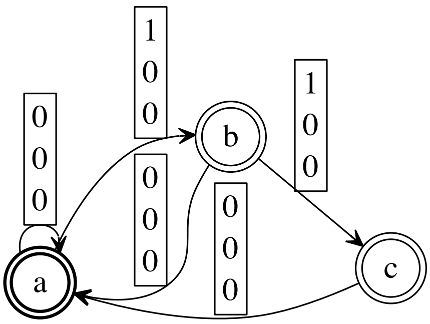


If we take care only of the prefixes, the computation begins

$$u \text{ and state } i \xrightarrow{x} s(u)x \text{ and state } j.$$

And if we take the abelianisation, we get the automaton of the figure 12.

FIGURE 12. Abelianisation of the prefix automaton for the Tribonnacci substitution



We can take the abelianisation of the previous calculation:

$$\mathrm{Ab}(u) \text{ and state } i \xrightarrow{x} \mathrm{Ab}(s(u)) + x \text{ and state } j$$

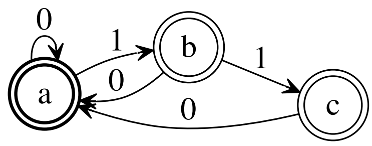
We can choose the projection π along the eigenspace for the greatest eigenvalue of the incidence matrix $M_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ such that the image of π can be identified with $\mathbb{C}$, and with the relation

$$\pi(MV) = \gamma\pi(V),$$

where γ is a complex eigenvalue of M_s (i.e. a root of $X^3 - X^2 - X - 1$). In other words, the restriction of M_s to the hyperplane image of π has the matrix $\begin{pmatrix} \mathrm{Re}(\gamma) & \mathrm{Im}(\gamma) \\ -\mathrm{Im}(\gamma) & \mathrm{Re}(\gamma) \end{pmatrix}$ in a well-chosen basis of $\mathcal{P}$.

If we project the abelianisation using this projection, we get the automaton of the figure 13.

FIGURE 13. Projection of the abelianisation of the prefix automaton for the Tribonnacci substitution
It corresponds to the Dumont-Thomas umeration



This automaton permits to describe the projection of the discrete line:

$$\pi(D_u) = \left\{ \sum_{i=0}^{n} u_i \gamma^{n-i} \;\middle|\; u_0 u_1 ... u_n \text{ sequence of labels of a path from } a \right\}$$

This comes from the projection of the abelianisation of the previous calculation:

$$\pi(\mathrm{Ab}(u)) \text{ and state } i \xrightarrow{x} \pi(\mathrm{Ab}(s(u))) + x \text{ and state } j$$

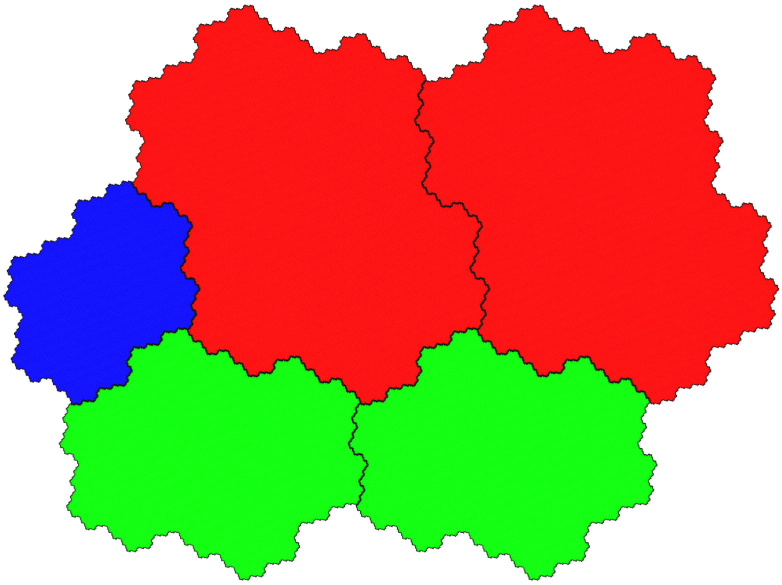
4.2. Description of the Rauzy fractal by gIFS. — If we call L_l the set of sequences of labels in the automaton of the figure 13 that ends in state l, for $l \in \{a, b, c\}$, then we have the following relations

$$\begin{aligned} L_a &= \{\epsilon\} \cup L_a 0 \cup L_b 0 \cup L_c 0 \\ L_b &= L_a 1 \\ L_c &= L_b 1 \end{aligned}$$

and we have $\pi(D_{u,l}) = \left\{ \sum_{i=0}^{n} v_i \gamma^i \;\middle|\; v_0 ... v_n \in L_l \right\}$, for all $l \in \{a, b, c\}$. Hence, if we denote $R_l := \overline{\pi(D_{u,l})}$, for $l \in \{a, b, c\}$ the three domains of the Rauzy fractal, then we have

$$\begin{aligned} R_a &= \gamma R_a \cup \gamma R_b \cup \gamma R_c \\ R_b &= \gamma R_a + 1 \\ R_c &= \gamma R_b + 1 \end{aligned}$$

FIGURE 14. Cutting of each domain of the Rauzy fractal given by the gIFS equations



Such set of equations are called **gIFS equations**. The name gIFS stands for graph iterated functions system. Here there are two functions $x \mapsto \gamma x$ and $x \mapsto \gamma x + 1$.

gIFS are usefull for describing and understanding fractals sets. For example, the triadic Cantor set C satisfies the equation

$$C = \frac{1}{3}C \cup \left(\frac{1}{3}C + \frac{2}{3}\right).$$

And C is the smallest non-empty compact set satisfying this equation.

The set of three equation satisfied by the domains of the Rauzy fractal of the Tribonnacci substitution has also this property: they are the smallest non-empty compact sets satisfying this set of equations.

4.3. Properties of the Rauzy fractal. —

Properties 4.2. — — *The projection π is irrationnal (i.e. $\pi_{\mathbb{Z}^3}$ is injective and has a dense image).*

— *The set $\pi(D_u)$ is bounded.*

— *Each domain of the Rauzy fractal $R_l = \overline{\pi(D_{u,l})}$, $l \in \{a, b, c\}$, has non-empty interior.*

— *The boundary of each domain of the Rauzy fractal has zero Lebesgue measure : $\forall l \in \{a, b, c\}$, $\lambda(\partial R_l) = 0$.*

Proof. — We admit the fact that π is irrational, but we prove it later (it is true as soon as the incidence matrix of the Pisot substitution has an irreducible characteristic polynomial).

We have already shown that the set $\pi(D_u)$ is bounded (see before).

In order to proof that each R_l has non-empty interior, we use the fact that we have

$$\bigcup_{t\in\pi(\Gamma_0)} R + t = \mathbb{C},$$

where $R = \cup_{l\in\{a,b,c\}} R_l$ is the Rauzy fractal. This comes from the fact that D_u is a fundamental domain for the action of $\Gamma_0 :=< e_a - e_b >_{a,b\in A}$ (see before).

Using the theorem of Baire, we deduce that R_u has non-empty interior. Indeed, if R_u has empty interior, then the union would be a countable union of closed sets of empty interior, so it would have empty interior by the theorem of Baire, but it is equal to $\mathbb{C}$.

Remark 4.3. — *It is not enough to prove that the Lebesgue measure is zero. For example, if $(x_n)_{n\in\mathbb{N}}$ is an enumeration of the rational numbers, then the set*

$$[0, 1]\setminus \bigcup_{n\in\mathbb{N}} [x_n, x_n + \frac{1}{2^{n+2}}]$$

has empty interior, but it has a Lebesgue measure greater than $\frac{1}{2}$.

We deduce that each piece R_l has non-empty interior, because $R_a = \gamma(R_a \cup R_b \cup R_c) = \gamma R$, and $R_b = \gamma R_a$ and $R_c = \gamma R_b$.

In order to prove that the boundaries have zero Lebesgue measure, we need the following lemma:

Proposition 4.4. — *The unions in the gIFS equations of the Tribonnacci substitution are disjoint in measure (i.e. the intersection of two parts of the unions have zero Lebesgue measure).*

Proof. — Let $\mu = (\mu_l)_{l\in\{a,b,c\}} = (\lambda(R_l))_{l\in\{a,b,c\}}$. The gIFS equations gives

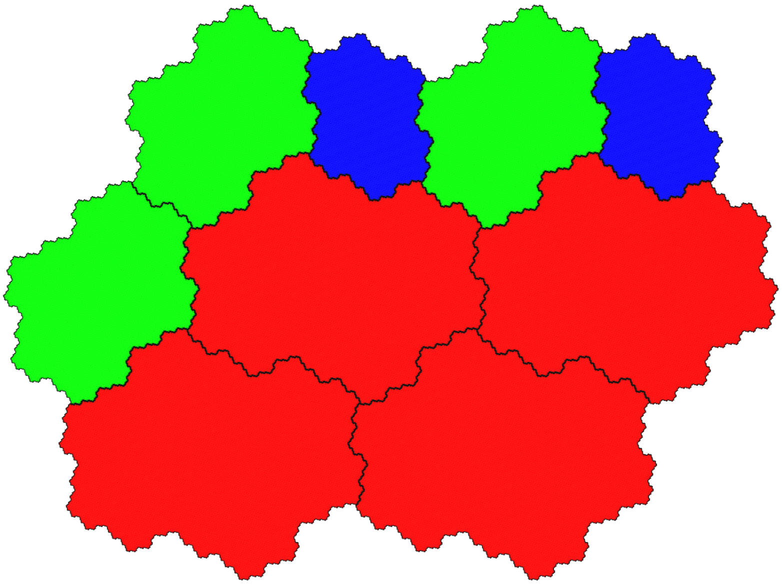
$$\begin{array}{rcl} \mu_a = \lambda(R_a) & \leq & \lambda(\gamma R_a) + \lambda(\gamma R_b) + \lambda(\gamma R_c) = |\gamma|^2 (\mu_a + \mu_b + \mu_c) \\ \mu_b = \lambda(R_b) & = & \lambda(\gamma R_a + 1) = |\gamma|^2 \mu_a \\ \mu_c = \lambda(R_c) & = & \lambda(\gamma R_b + 1) = |\gamma|^2 \mu_b \end{array}$$

Hence, we have the inequality $\mu \leq |\gamma|^2 M_s\mu = \frac{1}{\beta} M_s\mu$ (because $|\gamma|^2 \beta = 1$). But the theorem of Perron-Frobenius (or a generalization) gives us that such inequality is necessarily a equality and μ is an eigenvector for the greatest eigenvalu β: $\beta\mu = M_s\mu$. We deduce that the inequalities of measures are in fact equalities, and the unions are disjoint in measure. □

Now we know that each domain R_l has non empty interior. So for every $l \in \{a, b, c, \}$, we can take a small ball $B(x, \epsilon)$ included in R_l. If we iterate the gIFS equations, we get

$$\begin{aligned} R_a &= \gamma(\gamma R_a \cup \gamma R_b \cup \gamma R_c) \cup \gamma^2 R_a + \gamma \cup \gamma R_b + 1 \\ R_b &= \gamma(\gamma R_a \cup \gamma R_b \cup \gamma R_c) + 1 \\ R_c &= \gamma R_b + 1 \end{aligned}$$

FIGURE 15. Second iteration of the gIFS equations



And unions are still disjoint in measure. We can iterate it again and again, and it gives a tiling of each domain R_l by very small domains $\gamma^n R_i$ for $i \in \{a, b, c\}$, translated.

We can find a little tile included in the ball $B(x, \epsilon)$ up to iterate enough. Then, the disjointness in measure of the union implies that this little tile has a boundary with zero Lebesgue measure. But if it is true for one of the tile R_a, R_b or R_c, then it is true for the three, because $R_b = \gamma R_b + 1$ and $R_c = \gamma R_b + 1$. □

We can apply the criterion (theorem 3.4), and it gives that the subshift (X_u, S) is measurably isomorphic to the domain exchange (R, E, λ).

Figure 16. Third iteration of the gIFS equations

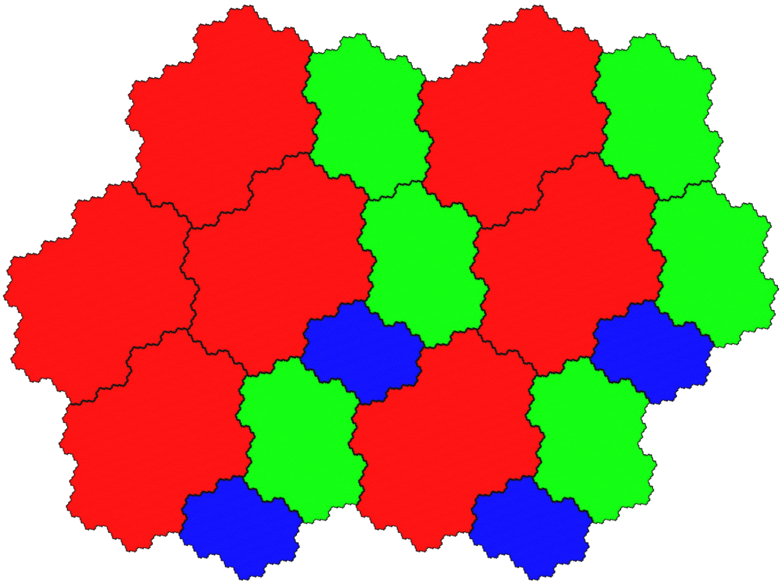


Figure 17. Ninth iteration of the gIFS equations

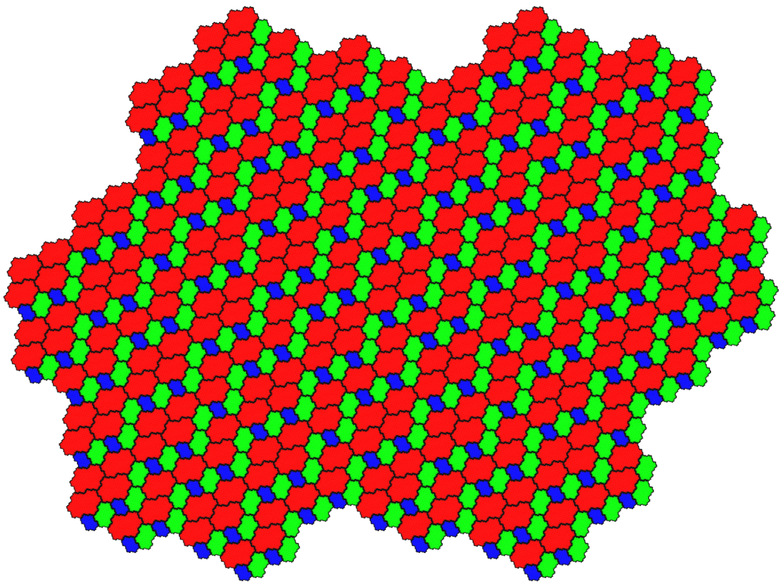
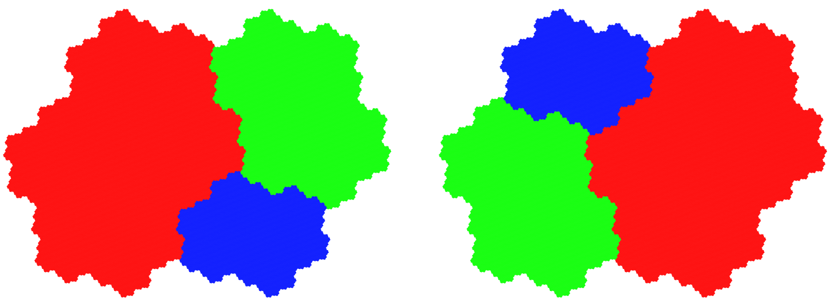


Figure 18. Domain exchange measurably conjugated to the sub-shift for the Tribonnacci substitution

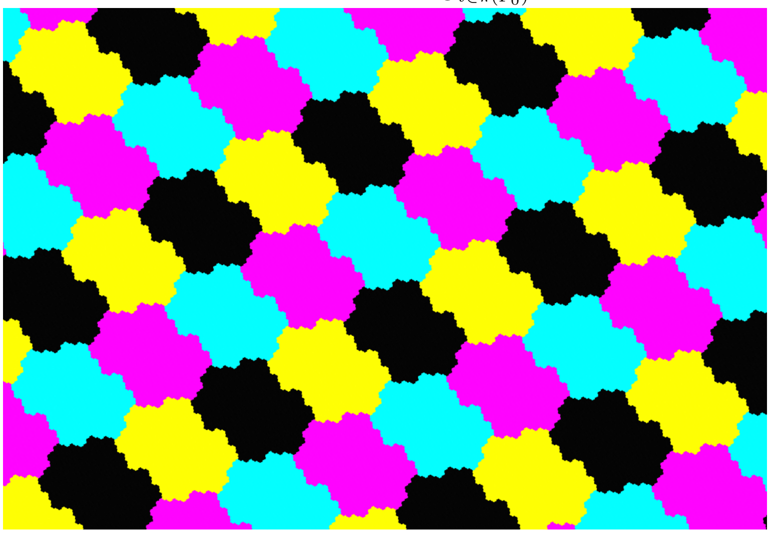


It remains to prove that the union

$$\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathbb{C}$$

is disjoint in measure in order to obtains also the conjugacy with a translation on a torus. It works, but we will do it later.

FIGURE 19. The union $\bigcup_{t\in\pi(\Gamma_0)} R + t$



5. Generalization to substitutions of Pisot unit type

In this section, we generalize what we have done for the Tribonnacci substitution for a large class of substitutions. We show that like for Tribonnacci, we have a Rauzy fractal that is bounded, with non-empty interior, with zero Lebesgue measure boundary, and we will give a simple combinatoric condition to have that the domains of the Rauzy fractal are disjoint in measure.

5.1. Substitutions of Pisot unit type. — Let A be a **alphabet** (i.e. a finite set), and $A^* = \cup_{n=0}^{\infty} A^n$ be the set of finite words over A. A **substitution** s over the alphabet A is a map that associate for each letter of A, a finite word over the alphabet A: $s : A \to A^*$. A substitution can be naturally extended to a word morphism $s : A^* \to A^*$, for the concatenation of words. The **incidence matrix** of the substitution is the matrix $(|s(a)|_b)_{b,a\in A}$, where $|v|_b$ denotes the numbers of occurrences of the letter b in a word v.

We say that a substitution s is of **Pisot type** if the incidence matrix M_s has an eigenvalue $\beta > 1$ of multiplicity 1 and if all the other eigenvalues γ are strictly less than one in modulus: $|\gamma| < 1$. If moreover the incidence matrix has determinant ± 1 (we say that the matrix is **unimodular**), then we say that the substitution is **unimodular**, or of **Pisot unit type**.

This property that all other eigenvalues are less than one in modulus will permits to prove that Rauzy fractal is bounded. And it also implies useful properties:

Proposition 5.1. — *If a substitution s is of Pisot unit type, then s is primitive, and the characteristic polynomial of the incidence matrix M_s is irreducible.*

Proof. — If the characteristic polynomial χ_{M_s} was not irreducible, then we can factorize it in $\mathbb{Z}[X]$: $\chi_{M_s} = PQ$ where $P, Q \in \mathbb{Z}[X]$. Moreover, P and Q are unitary and have ± 1 as constant coefficient (because the constant term of χ_{Ms} is $\det(M_s) = \pm 1$). The product of the roots of P is equal to the constant coefficient ± 1, so if $\deg(P) > 1$, then P has a root of modulus ≥ 1. Idem with Q. But it would contradict the hypothesis, so one of the polynomials P or Q have degree 1 and χ_{M_s} is irreducible.

The primitivity of s is done above: we prove that a matrix with an irreducible polynomial and with non-negative coefiicients is primitive. □

5.2. Graphs and matrices. — In this subsection, we explain how to see a matrix with non-negative coefficients as a graph, or the converse. And we show the following result:

Proposition 5.2. — *Let $M \in M_n(\mathbb{N})$ be a square matrix with non-negative coefficients and with an irreducible characteristic polynomial. Then M is primitive.*

In order to show that, we make a link between matrices and graphs: For a matrix $M \in M_n(\mathbb{N})$, we associate the oriented graph $\mathcal{G}_M$ whose vertices are $\{1, 2, ..., n\}$ and with $m_{i,j}$ edges from i to j. And for an oriented graph $\mathcal{G}$ with multiple edges and allowing loops, we associate the matrix $M_{\mathcal{G}} = (\text{ number of edges from } i \text{ to } j\ )_{i,j} \in M_n(\mathbb{N})$, where n is the number of vertices of $\mathcal{G}$. We have obviously that $M_{\mathcal{G}_M} = M$ and $\mathcal{G}_{M_{\mathcal{G}}} = \mathcal{G}$.

Properties 5.3. — *Let $M \in M_n(\mathbb{N})$. We have*
— *M is irreducible if and only if $\mathcal{G}_M$ is strongly connected.*
— *$M^n = (\text{ number of paths of length } n \text{ from } i \text{ to } j\ )_{i,j}$.*

Lemma 5.4. — *If C is a strongly connected component of $\mathcal{G}$, then the characteristic polynomial of M_C divide the characteristic polynomial of $M_{\mathcal{G}}$.*

Proof. — Let V be the set of vertices of $\mathcal{G}$. Let $A = \{x \in V \mid \text{there exists a edge from } x \text{ to } C\}$, and $B = V \backslash A$. Then, if we relabel the vertices of $\mathcal{G}$ in order to have the vertices in A smaller than the vertices in C, smaller than the vertices of B, the matrix of $\mathcal{G}$ becomes triangular by blocks (and it is conjugate to the matrix before reordering). So the characteristic polynomial of $M_{\mathcal{G}}$ will factorize with a factor for A, a factor for C and a factor for B. We have the wanted result. □

This lemma permits to see that the irreducibility of the characteristic polynomial implies the irreducibility of the matrix. In order to have the primitivity, we need one more property about graphs.

Definition 5.5. — *We call* **period** *of a graph $\mathcal{G}$ the gcd of the length of the loops in $\mathcal{G}$.*

Lemma 5.6. — *An irreducible matrix $M \in M_n(\mathbb{N})$ is primitive if and only if the graph $\mathcal{G}_M$ has a period 1.*

Proof. — If $\mathcal{G}_M$ has a period 1, then we can construct paths of the same length to go from any vertex to any other one. We can do that by taking a path that go throw loops with gcd 1 for each couple of vertices (it is possible because the graph $\mathcal{G}_M$ is strongly connnected). Then, it is a positive version of the Bézout theorem that permits to get the result.

For the converse, remarks that if $\mathcal{G}_M$ has a period $p \geq 2$, then the diagonal of M^{np+1} has only zero coefficients for every $n \in \mathbb{N}_{\geq 1}$, otherwise we have a loop of length $np + 1$. □

Proof of the proposition 5.2. — Using the lemma 5.4 with the graph $\mathcal{G}_M$, we see that the fact that the characteristic polynomial of M is irreducible implies that the graph $\mathcal{G}_{M_s}$ is strongly connnected.

Now, if we assume that the period of the graph $\mathcal{G}_M$ is $p \geq 2$, then we can partition the vertices of $\mathcal{G}$ in p classes: We define a equivalence relation on the vertices: $x \equiv y \iff$ there exists a path of length $\in p\mathbb{Z}$ between x and y. There is at least p classes for this equivalence relations, otherwise there is a loop of length $pn + k$ with $k \notin p\mathbb{Z}$. Hence, the graph $\mathcal{G}_{M^p}$ is not strongly connected, so the characteristic polynomial χ_p of M^p is not irreducible. But the roots of the characteristic polynomial χ of M_s are roots of $\chi_p(X^p)$, so it implies that χ is not irreducible. It contradicts the hypothesis, so the period of $\mathcal{G}_{M_s}$ is 1 and M_s is primitive. □

5.3. Automata and regular languages. — In order to describe the discrete line, we will need an automaton. We define what is an automaton here.

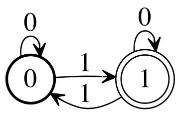
Definition 5.7. — *An* **automaton** *is a quintuplet $\mathcal{A} = (\Sigma, Q, T, I, F)$, where*
- *Σ is an alphabet,*
- *Q is a finite set of states called* **states**,
- *$T \subseteq Q \times \Sigma \times Q$ is the set of* **transitions**,
- *$I \subseteq Q$ is the set of* **initial states**,
- *$F \subseteq Q$ is the set of* **final states**.

We often denote by $p \xrightarrow{t} q$ a transition $(p, t, q) \in T$.

Definition 5.8. — *The language $L_{\mathcal{A}}$ of an automaton $\mathcal{A} = (\Sigma, Q, T, I, F)$ is the set of words over the alphabet Σ labeling a path from I to F in $\mathcal{A}$:*

$$L_{\mathcal{A}} := \left\{ u_1 u_2 ... u_n \in \Sigma^* \;\middle|\; n \in \mathbb{N},\ \exists (q_i)_{i=0}^n \in Q^{n+1}, q_0 \in I, q_n \in F, q_0 \xrightarrow{u_1} q_1 \xrightarrow{u_2} q_2 ... \xrightarrow{u_n} q_n \right\}$$

FIGURE 20. Example of automaton, with initial states $\{0\}$ (in bold), and final states $\{1\}$ (double circle)



Example 5.9. — *The language of the automaton of the figure 20 is the set of words over the alphabet $\{0, 1\}$ with an odd number of* 1.

Definition 5.10. — *A language is* **regular** *if it is the language of an automaton.*

We have the following useful characterization of regular languages:

Proposition 5.11. — *A language L over a alphabet Σ is regular if and only if the set $\left\{ u^{-1}L \;\middle|\; u \in \Sigma^* \right\}$ is finite, where $u^{-1}L := \left\{ v \in \Sigma^* \;\middle|\; uv \in L \right\}$.*

5.4. Description of the discrete line by the Prefix Abelianised Automaton. —

5.4.1. Prefix-suffix automaton. — Like for Tribonnacci, we can represent the discrete line by an automaton.

The **prefix-suffix automaton** is the automaton $\mathcal{A}^{PS}$ whose states are the alphabet A, and with transitions

$$a \xrightarrow{(u,b,v)} b \quad \Longleftrightarrow \quad s(a) = ubv.$$

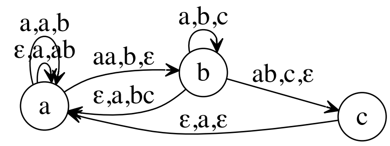
The alphabet of this automaton is $\left\{ (u, b, v) \in A^* \times A \times A^* \;\middle|\; \exists a \in A,\ s(a) = ubv \right\}$, and initial states and final states are not defined.

Example 5.12. — *For the substitution*

$$s : \begin{cases} a \mapsto aab \\ b \mapsto abc \\ c \mapsto a \end{cases},$$

we have the prefix-suffix automaton of the figure 21.

FIGURE 21. Prefix-suffix automaton of the substitution $a \mapsto aab,\ b \mapsto abc,\ c \mapsto a$



Proposition 5.13. — *If we have a path in the prefix-suffix automaton*

$$a_0 \xrightarrow{(u_1,a_1,v_1)} a_1 \xrightarrow{(u_2,a_2,v_2)} a_2 ... \xrightarrow{(u_n,a_n,v_n)} a_n,$$

then we have

$$s^n(a_0) = s^{n-1}(u_1)s^{n-2}(u_2)...s(u_{n-1})u_n a_n v_n s(v_{n-1})...s^{n-2}(v_2)s^{n-1}(v_1).$$

Proof. — By induction. □

Hence, the paths of length n from a in this automaton are in bijection with the positions in the word $s^n(a)$. In particular, the number of paths of length n from a is exactly the length of the word $s^n(a)$.

And the paths from a to b are in bijection with the positions of the letter b in the word $s^n(a)$. In particular, the number of paths of length n from a to b is exactly $|s^n(a)|_b$.

5.4.2. Prefix automaton. — If we keep only the prefixes, we get the prefix automaton: the **prefix automaton** is the automaton $\mathcal{A}^P$ whose states are the alphabet A, and with transitions

$$a \xrightarrow{u} b \quad \Longleftrightarrow \quad \exists v \in A^*,\ s(a) = ubv.$$

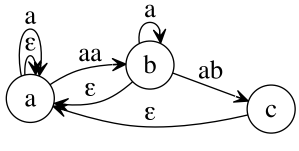
The alphabet of this automaton is $\{u \in A^* \mid \exists a \in A,\ \exists v \in A^*,\ s(a) = ubv\}$, and initial states and final states are not defined.

Example 5.14. — *For the substitution*

$$s : \begin{cases} a \mapsto aab \\ b \mapsto abc \\ c \mapsto a \end{cases},$$

we have the prefix automaton of the figure 22.

FIGURE 22. Prefix automaton of the substitution $a \mapsto aab,\ b \mapsto abc,\ c \mapsto a$



Proposition 5.15. — *If we have a path in the prefix automaton*

$$a_0 \xrightarrow{u_1} a_1 \xrightarrow{u_2} a_2 ... \xrightarrow{u_n} a_n,$$

then we have

$$s^{n-1}(u_1)s^{n-2}(u_2)...s(u_{n-1})u_n a_n \text{ is a prefix of } s^n(a_0).$$

Proof. — A path in the prefix automaton corresponds to a path in the prefix-suffix automaton, so the results comes from what we have done for the prefix-suffix automaton. □

Hence, the paths of length n from a in this automaton are in bijection with the strict prefixes of the word $s^n(a)$. In particular, the number of paths of length n from a is exactly the length of the word $s^n(a)$.

And the paths from a to b are in bijection with the set of prefixes of $s^n(a)$ ending with the letter b. In particular, the number of paths of length n from a to b is exactly $|s^n(a)|_b$.

5.4.3. Abelianised prefix automaton. — Now, if we abelianise the prefixes we get the abelianised prefix automaton: the **abelianised prefix automaton** is the automaton $\mathcal{A}^{AP}$ whose states are the alphabet A, and with transitions

$$a \xrightarrow{x} b \quad \Longleftrightarrow \quad \exists u \in A^*,\ \exists v \in A^*,\ s(a) = ubv \text{ and } x = \mathrm{Ab}(u).$$

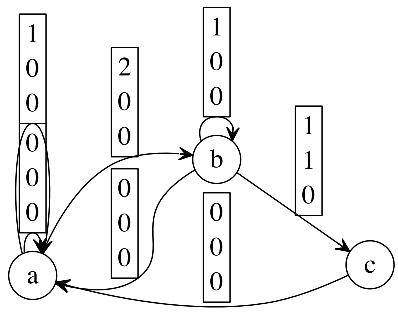
The alphabet of this automaton is $\left\{\mathrm{Ab}(u) \in \mathbb{Z}^A \;\middle|\; u \in A^* \text{ such that } \exists a \in A,\ \exists v \in A^*,\ s(a) = ubv\right\}$, and initial states and final states are not defined.

Example 5.16. — *For the substitution*

$$s : \begin{cases} a \mapsto aab \\ b \mapsto abc \\ c \mapsto a \end{cases},$$

we have the abelianised prefix automaton of the figure 23.

FIGURE 23. Abelianised prefix automaton of the substitution $a \mapsto aab,\ b \mapsto abc,\ c \mapsto a$



Proposition 5.17. — *If we have a path in the prefix automaton*

$$a_0 \xrightarrow{x_1} a_1 \xrightarrow{x_2} a_2 ... \xrightarrow{x_n} a_n,$$

then we have that

$$M^{n-1}x_1 + M^{n-2}x_2 + ... + Mx_{n-1} + x_n$$

is the abelianisation of a strict prefix of the word $s^n(a_0)$*.*

Proof. — Abelianise what we have for the prefix automaton. □

Hence, the paths of length n from a in this automaton are in bijection with the strict prefixes of the word $s^n(a)$. In particular, the number of paths of length n from a is exactly the length of the word $s^n(a)$.

And the paths from a to b are in bijection with the set of prefixes of $s^n(a)$ ending with the letter b. In particular, the number of paths of length n from a to b is exactly $|s^n(a)|_b$.

So we can describe the discrete line:

Proposition 5.18. — *If u is a fixed point of the substitution s starting by the letter a, then we have for every $b \in A$,*

$$D_{u,b} = \left\{ \sum_{i=1}^n M^{n-i} u_i \;\middle|\; n \in \mathbb{N},\ u_1 u_2 ... u_n \in L_{a,b} \right\}$$

where $L_{a,b}$ is the language of the abelianised prefix automaton $\mathcal{A}^{AP}$ with set of initial states $\{a\}$ and set of final states $\{b\}$.

5.5. First properties of the Rauzy fractal. — We assume now that s is a substitution of Pisot unit type. We show that the discrete line stays at bounded distance of a line:

Proposition 5.19. — *If s is a substitution of Pisot unit type, then the set $\pi(D_u)$ is bounded, where $u \in A^{\mathbb{N}}$ is a fixed point and π is a linear projection along the eigenspace for the greatest eigenvalue of M_s.*

Proof. — Let β be the greatest eigenvalue of M_s. The eigenspace E_β is a line, and by the kernel decomposition lemma, there exists a linear vector subspace of $\mathbb{R}^A$ such that

$$\mathbb{R}^A = E_\beta \oplus F,$$

with $M_s F \subseteq F$. (It can be shown that F is the orthogonal of E_β.) Let π be the projection on F along E_β (the choice of any projection along E_β doesn't change the result). Then, we have

$$\pi(D_u) = \left\{ \pi\left(\sum_{i=0}^{|u|} M^{|u|-i} u_i\right) \;\middle|\; u \in L_a \right\},$$

where L_a is the language of the automaton $\mathcal{A}^{AP}$ where we choose $\{a\}$ as initial states and A as final states. Let $N : F \to F$ be the restriction of M_s to F. Then we have

$$\pi(D_u) = \left\{ \sum_{i=0}^{|u|} N^{|u|-i}\pi(u_i) \;\middle|\; u \in L_a \right\}.$$

So for $x = \sum_{i=0}^{|u|} N^{|u|-i}\pi(u_i) \in \pi(D_u)$, we have

$$\|x\| \leq \sum_{i=0}^{|u|} \|N^{|u|-i}\pi(u_i)\| \leq \max_{x \in \Sigma_L} \|\pi(x)\| \sum_{i=0}^{|u|} |||N|||^{|u|-i} \leq \frac{\max_{x\in\Sigma_L} \|\pi(x)\|}{1 - |||N|||},$$

because $|||N||| \leq \max_{\lambda \in sp(N)} |\lambda| < 1$, because the eigenvalues of N are the eigenvalues of M_s different of β, and there are < 1 by hypothesis. Hence, $\pi(D_u)$ is bounded. □

Hence, the Rauzy fractal $\overline{\pi(D_u)}$ is bounded. In order to show more properties of the Rauzy fractal, we need another description by gIFS equations.

5.6. gIFS equations: a characterization of the Rauzy fractal. — Let $u \in A^{\mathbb{N}}$ be a fixed point of s starting with letter a. For every letter $b \in A$, we define $L_{a,b}$ as the language of the abelianised prefix automaton $\mathcal{A}^{AP}$ with initial states $\{a\}$ and final states $\{b\}$. By definition of $L_{a,b}$, we have the following relations for all $b \in A$:

$$L_{a,b} = \Delta_{a,b} \cup \bigcup_{c \xrightarrow{x} b} L_{a,c}x,$$

where $\Delta_{a,b} = \{\epsilon\}$ if $a = b$ and $\Delta_{a,b} = \emptyset$ otherwise.

Let $R_b = \overline{\pi(D_{u,b})}$ be the domain of the Rauzy fractal associated to the letter b. We have $R_b = \overline{\left\{ \sum_{i=0}^{|u|} N^{|u|-i}\pi(u_i) \;\middle|\; u \in L_{a,b} \right\}}$, therefore the previous relations with languages becomes

$$R_b = \bigcup_{c \xrightarrow{x} b} NR_c + \pi(x),$$

for every $b \in A$. (The equations implies that $0 \in R_a$, so we don't need to add it: indeed, we have $NR_a \subseteq R_a$ since a is the first letter of a fixed point.)

We call such equalities a **gIFS equation**. It completely characterize the Rauzy fractal:

Proposition 5.20. — *The sets R_b, $b \in A$ are the smallest non empty compact sets such that for every $b \in A$,*

$$R_b = \bigcup_{c \xrightarrow{x} b} NR_c + \pi(x).$$

Proof. — We already know that R_b, $b \in A$, are compact sets satisfying such gIFS equalities. Let us show the converse: assume that R'_b, $b \in A$ are non-empty compact subsets satisfying the gIFS equation. If R'_a is non-empty, then it contains 0, because we have $NR'_a \subseteq R'_a$ (because u is a fixed point starting by a). So $x \in R'_a \Longrightarrow \forall n \in \mathbb{N}, N^n x \in R'_a$, but we have $N^n x \xrightarrow[n\to\infty]{} 0$.

Then, if we iterate the gIFS equation from the point 0, we get that for every $b \in A$,

$$\pi(D_{u,b}) = \left\{ \sum_{i=0}^{|u|} N^{|u|-i} \pi(u_i) \;\middle|\; u \in L_{a,b} \right\} \subseteq R'_b.$$

Hence, for every $b \in A$ we have $R_b \subseteq R'_b$ and this ends the proof. □

Like for Tribonnacci, we show that the unions in the gIFS equations are disjoint in measure:

Proposition 5.21. — *For every $b \in A$, the union*

$$\bigcup_{c \xrightarrow{x} b} NR_c + \pi(x)$$

is disjoint in measure.

Proof. — Let λ be the Lebesgue measure. We have the inequality

$$\lambda(R_b) \leq \sum_{c \xrightarrow{x} b} \lambda(NR_c) = \sum_{c \in A} |s(c)|_b \, \lambda(NR_c) = |\det(N)| \sum_{c \in A} |s(c)|_b \, \lambda(R_c).$$

And we have $|\det(N)| = \frac{1}{\beta}$, because we have $\det(M_s) = \det(N)\beta = \pm 1$. So, if we let $X = (\lambda(R_b))_{b \in A}$, we have the inequality

$$\beta X \leq M_s X.$$

But by a generalization of the Perron-Frobenius theorem, such inequality implies that it is an equality: $\beta X = M_s X$. So, every inequality $\lambda R_b \leq \sum_{c \xrightarrow{x} b} \lambda(NR_c)$ is an equality, therefore the union is disjoint in measure. □

5.7. Properties of the Rauzy fractal. — We show that a fixed point of a substitution of Pisot type satisfy most of the properties of the theorem 3.4, and also other properties:

Properties 5.22. — *Let s be an substitution of Pisot unit type over an alphabet A, let $u \in A^{\mathbb{N}}$ be a fixed point, and let $\pi : \mathbb{R}^A \to \mathcal{P}$ be a linear projection along the eigenspace for the maximal eigenvalue of the incidence matrix M_s. Then we have:*

1. *u is non-eventually periodic,*
2. *π is irrational,*
3. *$\pi(D_u)$ is bounded,*
4. *The Rauzy fractal $R = \overline{\pi(D_u)}$ cover the hyperplane $\mathcal{P}$ by translation by $\pi(\Gamma_0)$:*

$$\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathcal{P}.$$

5. *Each piece $R_a = \overline{\pi(D_{u,a})}$, of the Rauzy fractal $R = \overline{\pi(D_u)}$ has non-empty interior, $a \in A$,*

6. *The boundary of each piece* $\overline{\pi(D_{u,a})}$ *of the Rauzy fractal has zero Lebesgue measure.*

Proof. — The projection π is irrational: otherwise there exists an eigenvector of M_s with positive integer coefficients, and this would implies that there exists an rational eigenvalue greater than or equal to 1. But this is not possible with the hypothesis that s is of Pisot type. This proves the second property.

If the word u was eventually periodic, then the projection π would be rationnal and the abelianisation of a period would be an eigenvector with non-negative integer coefficients. This is not possible thank to the second property. Hence, the first property is proven.

The third property has been proven before.

The equality

$$\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathcal{P},$$

comes from the equality

$$\mathbb{Z}^A = \bigcup_{t \in \Gamma_0} D_u + t$$

which is given by the proposition 3.1. This proves the property 4.

Let us show that the Rauzy fractal R has non-empty interior. This is a consequence of the theorem of Baire, because the countable union of closed sets

$$\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathcal{P},$$

has non-empty interior, so one of the $R + t$ has non-empty interior, so R has non-empty interior. Let us show that for every $a \in A$, R_a has non-empty interior. We use a second time the theorem of Baire: the finite union of closed sets

$$R = \bigcup_{a \in A} R_a$$

has non-empty interior, so one of the closed sets R_a has non-empty interior. Then, we get that it is true for every set R_b, $b \in A$, using the gIFS equation:

$$R_b = \bigcup_{c \xrightarrow{t} b} N R_c + t.$$

Up to iterate this gIFS equation, the set R_a will appear in the union, because the graph of the automaton $\mathcal{A}^s$ is strongly connected since it corresponds to the irreducible matrix M_s. Hence, we get that R_b has non-empty interior, for every letter $b \in A$.

In order to prove that the Lebesgue measure of the boundary of each piece R_a has zero Lebesgue measure, we use the gIFS equation and the fact that one of the piece has non-empty interior. If we iterate the gIFS equation n times, we get

$$R_b = \bigcup_{a \xrightarrow{t_1} \dots \xrightarrow{t_n} b} N^n R_a + (t_n + N t_{n-1} + \dots + N^{n-1} t_1).$$

And the diameter of each set $N^n R_a$ tends to zero (it is bounded by $C\frac{1}{\beta^n}$). So there exists $n \in \mathbb{N}$ large enough such that there exists a term $N^n R_a + t$ in this union which is included in the interior of R_b. But we know that the union is disjoint in measure (we proved it before), and the intersection between this term and the rest of the union is exactly the boundary of $N^n R_a + t$. Hence, this boundary has zero Lebesgue measure, so the boundary of R_a also, because we have $N^n \partial R_a + t = \partial(N^n R_a + t)$. Using the irreducibility of s, we can iterate the gIFS equation enough such that this tile $N^n R_a + t$ is decomposed as a union where every R_c, $c \in A$, appears. So we can do the same reasoning for every piece R_a, $a \in A$, and it gives the wanted result.

□

These properties permits to use the theorem 3.4:

Corollary 5.23. — *The subshift* $(S^{\mathbb{N}}u, S)$ *is an extension of the translation* $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.

In order to have a real conjugacy, we need to check that the union

$$\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathcal{P}$$

is disjoint in measure. And if we have the disjonction in measure of the union

$$\bigcup_{a \in A} R_a = R,$$

then we have a mesurable conjugacy between $(\overline{S^{\mathbb{N}}u}, S)$ and $(\overline{\pi(D_u)}, E, \lambda)$.

Remark 5.24. — *The disjonction in measure of the first union gives the disjonction in measure for the second one.*

Proof. — By contrapositive, if the second union is not disjoint in measure, then there exists $a \neq b \in A$ such that $\lambda(R_a \cap R_b) > 0$. And the set $R_a \cap R_b$ has the property that if we translate it by $\pi(e_a)$ or by $\pi(e_b)$, then we stay in R, because $R_a + \pi(e_a) \subseteq R$ and $R_b + \pi(e_b) \subseteq R$. So we have $(R_a \cap R_b) + \pi(e_a) \in R$ and $(R_a \cap R_b) + \pi(e_a) + \pi(e_b - e_a) \in R$. Hence, the intersection $R \cap (R + \pi(e_b - e_a))$ contains $(R_a \cap R_b) + \pi(e_a)$ which have positive Lebesgue measure. And $e_b - e_a$ is a non-zero element of Γ_0. So the first union $\bigcup_{t \in \pi(\Gamma_0)} R + t = \mathcal{P}$ is not disjoint in measure. □

5.8. The strong coincidence condition. — In this subsection, we explicit a combinatorial condition that permits to get the disjonction in measure of the union

$$\bigcup_{a \in A} R_a = R,$$

which permits to get a measurable conjugacy between the subshift of a substitution and a domain exchange on the Rauzy fractal.

The idea is to use the disjointness in measure of the gIFS equation. We know that the union

$$R_c = \bigcup_{a \xrightarrow{t_1} \dots \xrightarrow{t_n} c} N^n R_a + (t_n + N t_{n-1} + \dots + N^{n-1} t_1).$$

is disjoint in measure. So if we find two terms of the form $N^n R_a + t$ and $N^n R_b + t$ in this union, with the same translation t but different letters a and $b \in A$, then it gives that the intersection $R_a \cap R_b = N^{-n}\left((N^n R_a + t) \cap (N^n R_a + t)\right) - N^{-n} t$ has zero Lebesgue measure.

For $(a, b) \in A^2$, if we denote by $C_{a,b}$ this condition, we have

$$\begin{aligned} C_{a,b} &\iff \exists c \in A,\ \exists n \in \mathbb{N} \text{ and } t \in \pi(\mathbb{Z}^A),\ N^n R_a + t \text{ and } N^n R_b + t \text{ appears in the union equal to } R_c. \\ &\iff \exists n \in \mathbb{N},\ \exists c \in A,\ \text{ and } \exists u, v, u', v' \in A^*,\ s(a) = ucv,\ s(b) = u'cv' \ \text{ and } \operatorname{Ab}(u) = \operatorname{Ab}(u'). \end{aligned}$$

Notice that the condition $\operatorname{Ab}(u) = \operatorname{Ab}(u')$ implies that $|u| = |u'|$.

Definition 5.25. — *We say that a substitution s satisfies the* ***strong coincidence condition*** *if we have*

$$\forall (a, b) \in A^2,\ \exists n \in \mathbb{N},\ \exists c \in A,\ \exists u, v, u', v' \in A^*,\ s(a) = ucv,\ s(b) = u'cv, \ \textit{ and } \operatorname{Ab}(u) = \operatorname{Ab}(u').$$

We have the following:

Proposition 5.26. — *If a substitution of Pisot unit type satisfies the strong coincidence condition, then the union*

$$\bigcup_{a \in A} R_a = R,$$

is disjoint in measure, and the subshift of the substitution is measurably conjugate to the domain exchange $(\overline{\pi(D_u)}, E, \lambda)$.

Example 5.27. — *The Tribonnacci substitution*

$$\begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{array}$$

satisfies the strong coincidence condition. Indeed, we have for all $l \in \{a, b, c\}$, $s(a) = \epsilon a v$ with $v \in A^*$.

Example 5.28. — *More generally, every substitution such that every word $s(a)$, $a \in A$, starts with the same letter, satisfies the strong coincidence condition.*

Example 5.29. — *The substitution*

$$s : \begin{array}{ccc} a & \mapsto & b \\ b & \mapsto & c \\ c & \mapsto & ab \end{array}$$

satisfies the strong coincidence condition. Indeed, we have $s^{13}(a) = bccabcababbccababb...$ *and* $s^{13}(b) = cababbcabbcbccabab...$ *and we have* $\mathrm{Ab}(bccabcababbccabab) = \mathrm{Ab}(cababbcabbcbccaba) = \begin{pmatrix}5\\7\\5\end{pmatrix}$, *and we check that there are also coincidences for the others couples of letters.*

5.9. Topology on $\mathbb{Z}^A$ and a criterion to have a tiling of $\mathcal{P}$. — In this subsection, we give a condition that permits to guarantee that the union

$$\bigcup_{t\in\pi(\Gamma_0)} R + t = \mathcal{P}$$

is disjoint in measure, where $R = \overline{\pi(D_u)}$ is the Rauzy fractal, and $\Gamma_0 =< e_a - e_b >_{a,b\in A}$. For this, we define a topology on $\mathbb{Z}^A$.

We define, for any subset S of $\mathcal{P}$, the **discrete line** of points that project to S:

$$Q_S = \left\{x \in \mathbb{N}^A \;\middle|\; \pi(x) \in S\right\}.$$

This permits to define a topology on $\mathbb{Z}^A$ by taking the following set of open sets

$$\left\{Q_U \;\middle|\; U \text{ open subset of } \mathcal{P}\right\}.$$

We check that this set is stable by unions, by finite intersections, and contains the empty set and the whole set $\mathbb{Z}^A$.

Remark 5.30. — *The notation Q_S is defined for a part $S \subseteq \mathcal{P}$, and is also defined for a regular language S. But there is no ambiguity, because parts of $\mathcal{P}$ and languages are different objects, and we use the same notation because in both cases it represents a discrete line.*

Remark 5.31. — *We could define the topology using the whole space $\mathbb{Z}^A$, but it will be more practical in the following to work only with the positive part of the space.*

Properties 5.32. — *The topology that we just defined has the following properties:*

- *If the projection π is such that $\pi(\mathbb{Z}^d)$ is dense in $\mathcal{P}$, then for any open subset U of $\mathcal{P}$, we have that $\pi(Q_U)$ is dense in U, and we have*

$$Q_U = \emptyset \Longleftrightarrow U = \emptyset.$$

- *For any subset $S \subseteq \mathcal{P}$ and any $t \in \mathbb{Z}^A$, the symmetric difference*

$$(Q_S + t)\Delta Q_{S+t}$$

is finite. In particular, we have $\overset{\circ}{Q_S} = \emptyset \Longleftrightarrow \overset{\circ}{Q}_{S+t} = \emptyset \Longleftrightarrow \overbrace{Q_S + t}^{\circ} = \emptyset$.

- *If* $\det(M) \in \{-1, 1\}$, *then for any subset S of $\mathcal{P}$, the symmetric difference* $(MQ_S)\Delta Q_{MS}$ *is finite. In particular, we have* $\overbrace{MQ_S}^{\circ} = \emptyset \Longleftrightarrow \overset{\circ}{Q}_{MS} = \emptyset$.
- *The space $\mathbb{Z}^d$ is a Baire space for this topology.*

The fact that $\mathbb{Z}^d$ is a Baire space follows from the fact that $\mathcal{P}$ is a Baire space, by the Baire category theorem. Indeed, if Q is a dense open set of $\mathbb{Z}^d$, then there exists a dense open set U of $\mathcal{P}$ such that $Q = Q_U$. Hence, a countable intersection of dense open subsets of $\mathbb{Z}^d$ is a dense subset of $\mathbb{Z}^d$.

This topology gives a necessary and sufficient condition for the subshift of a Pisot irreducible substitution, to have a pure discrete spectrum.

Theorem 5.33. — *Let s be an irreducible Pisot substitution over an alphabet A, and let u be a periodic point of s. Then the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is an extension of the translation on a torus $\mathbb{T}^{d-1}$, where $d = |A|$. Moreover, if s is an irreducible unit Pisot substitution such that there exists a letter $a \in A$ such that $\mathring{D}_{u,a} \neq \emptyset$, then, the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is mesurably isomorphic to the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.*

Remark 5.34. — *The reciprocal of this theorem is true, but it is difficult to prove, and we will not do it.*

Proof of the theorem 5.33. — Up to replace the substitution s by a power, we can assume that the periodic point u is a fixed point. We have already checked the first part of the theorem in the previous subsection: the subshift associated to the substitution is an extension of the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$, which is isomorphic to a translation on the torus $\mathbb{T}^{d-1}$, $d = |A|$.

Now if we assume that $\exists a \in A,\ \mathring{D}_{u,a} \neq \emptyset$, we have the following

Lemma 5.35. — *We have for all $a \in A$, $\mathring{D}_{u,a} \neq \emptyset$ and $\overline{D_{u,a}} = \overline{\mathring{D}_{u,a}}$.*

Proof. — We have the equality

$$D_{u,a} = \bigcup_{b \xrightarrow{t} a} M D_{u,b} + t,$$

where $b \xrightarrow{t} a$ means that it is a transition in the automaton $\mathcal{A}^s$ (i.e. there exists words $u, v \in A^*$ such that $s(b) = uav$ and $\mathrm{Ab}(u) = t$). By primitivity, up to iterate enough this equality, every set $D_{u,b}$ appears in the union, so every set $D_{u,a}$ has non-empty interior as soon as one of them has non-empty interior.

Let c be the first letter of the fixed point u. If $x \in D_{u,a} = Q_{{}^tL^s_{c,a}}$, there exists arbitrarily large $n \in \mathbb{N}$ such that $x = Q_v$ for a word v of length n in the language ${}^tL^s_{c,a}$. And for such word v, we have $x + M^n D_{u,c} \subseteq D_{u,a}$. The sets $x + M^n D_{u,c}$ have non-empty interior, because $\det(M) \in \{-1, 1\}$, and converge to x when n tend to infinity. Hence $D_{u,a} \subseteq \overline{\mathring{D}_{u,a}}$ and this ends the proof. □

Lemma 5.36. — *We have $\overline{D_u} = \overline{\mathring{D}_u}$.*

Proof. — We have $D_u = \bigcup_{a\in A} D_{u,a}$, so $\overset{\circ}{D_u} \supseteq \bigcup_{a\in A} \overset{\circ}{D}_{u,a}$. Hence, the lemma 5.35 gives the result. □

Hence, $\overset{\circ}{D_u}$ is a dense open subset of $\overline{\pi(D_u)}$. By Baire's theorem, for all $t \in \Gamma_0\backslash\{0\}$, the empty set $\overset{\circ}{D_u} \cap (\overset{\circ}{D_u} + t)$ is a dense subset of $\overline{\overset{\circ}{D_u}} \cap (\overline{\overset{\circ}{D_u}} + t)$, therefore the sets $\overline{\overset{\circ}{D_u}}$ and $(\overline{\overset{\circ}{D_u}} + t)$ are disjoint. This gives the wanted disjointness in measure since the boundary has zero Lebesgue measure.

The hypothesis of the theorem 3.4 are satisfied, hence we get that there exists a S-invariant measure on $\overline{S^{\mathbb{N}}u}$ such that the subshift is measurably conjugated to a domain exchange and to a translation on a torus. And it is know that the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is uniquely ergodic, so the measure that we get is the unique invariant measure. □

6. Un peu de théorie algébrique des nombres

La théorie algébrique des nombres est un vaste sujet sur lequel il y a beacoup à dire. Nous nous restreignons ici à quelques outils qui permettrons d'une part de généraliser le fait que l'on puisse voir la ligne discrète à l'aide d'un développement en base β, et d'autre part de travailler avec une classe de susbtitutions plus large.

Pour cela, on va s'intéresser d'un peu plus près à la notion de corps de nombres.

6.1. Corps de nombre. — On dit qu'un corps $\mathbb{L}$ est une **extension** d'un autre corps $\mathbb{K}$, si l'on a l'inclusion $\mathbb{K} \subseteq \mathbb{L}$. Quand $\mathbb{L}$ est une extension de $\mathbb{K}$, on peut voir $\mathbb{L}$ comme un $\mathbb{K}$-espace vectoriel. On appelle **degré de l'extension** la dimension de cette espace vectoriel, et on note

$$[\mathbb{L}|\mathbb{K}] := \dim_{\mathbb{K}}(\mathbb{L}).$$

Definition 6.1. — *Un* ***corps de nombres*** *est une extension finie de* $\mathbb{Q}$*.*

Example 6.2. — $\mathbb{Q}(\sqrt{2})$ *est un corps de nombre. En effet: c'est bien un corps, et c'est un* $\mathbb{Q}$*-espace vectoriel de dimension* 2 *dont* $(1, \sqrt{2})$ *est une base.*

Tous les corps de nombres sont de cette forme:

Theorem 6.3 **(Théorème de l'élément primitif)**. — *Si* $\mathbb{K}$ *est un corps de nombre de degré* d*, alors il existe un nombre algébrique* a *de degré* d *tel que* $\mathbb{K} = \mathbb{Q}(a)$*.*

6.2. Valeurs absolues et places. — Une valeur absolue sur un corps $\mathbb{K}$ est une application $|.| : \mathbb{K} \to \mathbb{R}_+$ vérifiant

— $\forall x \in \mathbb{K},\ |x| = 0 \iff x = 0$,
— $\forall x, y \in \mathbb{K},\ |x + y| \leq |x| + |y|$,
— $\forall x, y \in \mathbb{K},\ |xy| = |x|\,|y|$.

Si la valeur absolue vérifie de plus

$$\forall x, y \in \mathbb{K},\ |x+y| \le \max(|x|, |y|),$$

alors on dit que la valeur absolue est **ultramétrique**, sinon on dira qu'elle est **archimédienne**.

Une **valuation** à valeur réelle sur un anneau $\mathbb{K}$ est une application $v : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ telle que

— $\forall x \in \mathbb{K},\ v(x) = \infty \iff x = 0$,
— $\forall x, y \in \mathbb{K},\ v(x+y) \ge \min(v(x), v(y))$,
— $\forall x, y \in \mathbb{K},\ v(xy) = v(x) + v(y)$.

Si de plus la valuation est à valeur dans $\mathbb{Z} \cup \{\infty\}$, on dira que c'est une **valuation discrète**. On peut définir une valeur absolue ultramétrique à partir d'une valuation v par

$$|x| := e^{-v(x)},$$

et réciproquement, $v(x) := \ln(|x|)$ définit une valuation à partir d'une valeur absolue ultramétrique.

Example 6.4. — *L'opposé du degré d'un polynôme de $\mathbb{K}[X]$ est une valuation (avec la convention que 0 est de degré $-\infty$).*

Example 6.5. — *Sur le corps $\mathbb{Q}$, on peut définir une valuation v_p pour tout nombre premier p par*

$$\forall n \in \mathbb{Z},\ v_p(n) = \max\left\{k \in \mathbb{N} \;\middle|\; p^k \text{ divise } n\right\}.$$

Cette définition s'étend de façon unique au corps $\mathbb{Q}$ tout entier en posant

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b).$$

On appelle **valeur absolue p-adique** de $\mathbb{Q}$ la valeur absolue

$$|x| := p^{-v_p(x)}.$$

On choisit p et non pas e ou autre chose pour avoir la formule du produit (voir plus tard), mais cela n'a pas d'importance: si l'on change la base de l'exponentielle, on obtient des valeurs absolues équivalentes.

Une valeur absolue définie une distance: $d(x,y) = |x-y|$, et donc une topologie. On dit que deux valeurs absolues sont **équivalentes** si elles engendrent la même topologie. On appelle **place** de $\mathbb{K}$ une classe d'équivalence de valeurs absolues de $\mathbb{K}$.

On a le théorème suivant.

***Theorem 6.6* (Ostrowski)**. — *Les places non triviales de $\mathbb{Q}$ sont celles de la valeur absolue usuelle, et celles des valeurs absolues ultramétriques p-adiques.*

On pourra trouver une preuve de ce théorème ici: [**Bech.**].

6.3. Complétion d'un corps pour une valeur absolue et nombres p-adiques. — Étant donné un corps $\mathbb{K}$ muni d'une valeur absolue, on peut définir le complété de $\mathbb{K}$ pour cette valeur absolue, comme l'ensemble des suites de Cauchy de $\mathbb{K}$, quotienté par la relation d'équivalence "la différence des deux suites tends vers 0". Le complété est un corps.

C'est comme cela que l'on définit $\mathbb{R}$: c'est le complété de $\mathbb{Q}$ pour la valeur absolue usuelle. Si l'on considère le complété de $\mathbb{Q}$ pour une valeur absolue p-adique, on obtient le corps des **nombres p-adique** $\mathbb{Q}_p$. On peut alors faire de l'analyse p-adique, de la même façon que l'on fait de l'analyse avec l'ensemble des réels. Mais les choses se passent un peu différemment. Par exemple, en p-adique, une série converge si et seulement si son terme général tend vers 0.

On note $\mathbb{Z}_p$ et l'on appelle **entiers p-adiques** le sous-ensemble de $\mathbb{Q}_p$

$$\mathbb{Z}_p := \left\{ x \in \mathbb{Q}_p \;\middle|\; v_p(x) \geq 0 \right\}.$$

Pour la valeur absolue p-adique, un nombre est petit s'il est divisible par une grande puissance de p. Et contrairement à la valeur absolue usuelle, l'ensemble $\mathbb{Z}$ est borné pour les valeurs absolues p-adiques. On a $\mathbb{Z} \subset \mathbb{Z}_p$, et $\mathbb{Z}_p$ est borné puisque

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p \;\middle|\; |x|_p \leq 1 \right\}.$$

On peut voir l'ensemble des entiers p-adiques comme la limite projective des $\mathbb{Z}/p^n\mathbb{Z}$:

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}.$$

Cela signifie que pour tout $n \in \mathbb{N}$, il existe une application $\varphi_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ telle que $\forall x \in \mathbb{Z}_p$, $\varphi_{n+1}(x)$ est congru à $\varphi_n(x)$ modulo p^n.

Cela permet de représenter les nombres p-adiques à l'aide d'une sorte de développement en base p infini à gauche :

$$\mathbb{Z}_p = \left\{ \textstyle\sum_{k=0}^{\infty} a_i p^i \;\middle|\; \forall i \in \mathbb{N},\ a_i \in \mathbb{Z}/p\mathbb{Z} \right\}$$
$$\mathbb{Q}_p = \left\{ \textstyle\sum_{k=-n}^{\infty} a_i p^i \;\middle|\; n \in \mathbb{N},\ \forall -n \leq i,\ a_i \in \mathbb{Z}/p\mathbb{Z} \right\}$$

Le résultat suivant est utile pour savoir si un polynôme admet une racine dans $\mathbb{Q}_p$:

***Lemma 6.7* (Lemme de Hensel).** — *Soit $P \in \mathbb{Z}[X]$. S'il existe $x_0 \in \mathbb{Z}/p\mathbb{Z}$ tel que $P(x_0) = 0[p]$ et $P'(x_0) \neq 0[p]$, alors il existe $x \in \mathbb{Z}_p$ tel que $P(x) = 0$ et $x = x_0[p]$.*

Pour plus de détails sur les nombres p-adiques, voir par exemple [**Win.**] ou [**Col.**].

Le résultat suivant relie les différentes valeurs absolues de $\mathbb{Q}$:

Proposition 6.8. —

$$\forall x \in \mathbb{Q}\backslash\{0\}, \prod_{v \in \mathcal{P}_{\mathbb{Q}}} |x|_v = 1,$$

où $\mathcal{P}_{\mathbb{Q}}$ est l'ensemble des places de $\mathbb{Q}$, avec la normalisation suivante pour les places p-adiques:

$$|x|_p = p^{-v_p(x)}.$$

Example 6.9. — *Pour* $x = -\dfrac{2}{15}$, *on a*

$$|x| = \frac{2}{15},\ |x|_2 = \frac{1}{2},\ |x|_3 = 3,\ |x|_5 = 5,$$

et pour tout p *premier* $\notin \{2,3,5\}$ *on a* $|x|_p = 1$. *On a bien le produit qui vaut* 1.

Le résultat se généralise à tout corps de nombre:

Proposition 6.10 **(Formule du produit)**. — *Si* $\mathbb{K}$ *est un corps de nombres,*

$$\forall x \in \mathbb{K}\backslash\{0\},\ \prod_{v \in \mathcal{P}_{\mathbb{K}}} |x|_v = 1,$$

où $\mathcal{P}_{\mathbb{K}}$ *est l'ensemble des places de* $\mathbb{K}$*, avec la bonne normalisation.*

Un intérêt de la formule du produit est qu'il permet de construire des espaces dans lesquels certains anneaux qui nous intéressent sont discret:

Corollary 6.11. — *Soit* β *un nombre algébrique sur* $\mathbb{Q}$*. Alors l'anneau* $\mathbb{Z}[\beta]$ *est discret dans le produit* $\prod_{v\in\mathcal{P}} \mathbb{Q}(\beta)_v$*, où* $\mathcal{P}$ *est l'ensemble des places archimédiennes de* $\mathbb{Q}(\beta)$ *et des places ultramétriques* v *telles que* $|v| > 1$.

Proof. — Soit $\mathbb{K} = \mathbb{Q}(\beta)$ le corps de nombre engendré par β et $\mathcal{P}_{\mathbb{K}}$ les places de $\mathbb{K}$. Il suffit de montrer que 0 est un point isolé. Pour cela, montrons que la boule $B(0, \frac{1}{2}) \cap \mathbb{Z}[\beta]$ ne contient que 0.

Si β n'est pas un entier algébrique, il existe un entier $n \in \mathbb{N}_{\geq 1}$ tel que $n\beta$ est entier. (Il suffit de prendre pour n le coefficient dominant du polynôme minimal de β.) De plus, les diviseurs premiers de cet entier n sont exactement les p tels que $\mathcal{P}$ contient une place au dessus de $|.|_p$. On peut donc écrire les éléments de $\mathbb{Z}[\beta]$ sous la forme $\frac{1}{n^k}\sum_{i=0}^{d-1} a_i\beta^i$, avec $a_i \in \mathbb{Z}$.

Remarquons que pour tout $v \in \mathcal{P}_{\mathbb{K}}\backslash\mathcal{P}$, v est une place ultramétrique au dessus d'un nombre premier p premier à n, et on a

$$\left|\frac{1}{n^k}\sum_{i=0}^{d-1} a_i\beta^i\right|_v \leq \left|\frac{1}{n^k}\right|_v \max_{i=0}^{n}\{\left|a_i\beta^i\right|_v\} \leq 1,$$

puisque $\left|\frac{1}{n^k}\right|_v = 1$, n étant premier à p, $|a_i|_v \leq 1$, a_i étant entier, et $\left|\beta^i\right|_v = |\beta|_v^i \leq 1$, v n'étant pas dans $\mathcal{P}$.

Pour les autres places, par définition de la boule $B(0, \frac{1}{2})$ du produit $\prod_{v\in\mathcal{P}} \mathbb{Q}(\beta)_v$, on a $|x|_v \leq \frac{1}{2}$ pour tout $x \in B(0, \frac{1}{2}) \cap \mathbb{Z}[\beta]$.

D'après la formule du produit, le produit de toutes les valeurs absolue d'un tel $x \in B(0, \frac{1}{2}) \cap \mathbb{Z}[\beta]$ doit être égal à 1 si x est non nul. Mais ici ce produit est inférieur ou égal à $1/2$ donc $x = 0$. □

6.4. Description des places d'un corps de nombre. — Si v est une valeur absolue d'un corps de nombre $\mathbb{K}$, alors la restriction de v à $\mathbb{Q}$ est une valeur absolue. C'est donc la valeur absolue de $\mathbb{Q}$ usuelle ou bien la valeur absolue p-adique pour un certain nombre premier p. On dira que la valeur absolue v est "au dessus" de la place correspondante sur $\mathbb{Q}$.

Proposition 6.12. — *Si $\mathbb{K} = \mathbb{Q}(\beta)$ est un corps de nombres, alors les places au dessus d'une place v de $\mathbb{Q}$ sont en correspondance avec les facteurs irréductibles de P sur $\mathbb{Q}_v$, où P est le polynôme minimal de β et $\mathbb{Q}_v$ est le complété de $\mathbb{Q}$ pour la valeur absolue v. Plus précisément, si l'on a $P = P_1 P_2 ... P_k$, avec $P_i \in \mathbb{Q}_v[X]$ irréductibles sur $\mathbb{Q}_v[X]$, alors il existe des places v_1, v_2, ..., v_k de $\mathbb{K}$ telles que*

$$\mathbb{K}_{v_i} \simeq \mathbb{Q}_v[X]/(P_i).$$

Example 6.13. — *Si $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, le polynôme minimal de $\sqrt{2}$ est $P(X) = X^2 - 2$. Pour la valeur absolue usuelle de $\mathbb{Q}$, le complété de $\mathbb{Q}$ est $\mathbb{R}$, et le polynôme P se factorise sur $\mathbb{R}$ en deux polynômes de degrés 1, donc il existe deux valeurs absolues archimédiennes sur $\mathbb{K}$. Ce sont la valeur absolue usuelle de $\mathbb{R} \supset \mathbb{Q}(\sqrt{2})$ et celle du conjugué:*

$$\left|x + \sqrt{2}y\right|' = \left|x - \sqrt{2}y\right|.$$

Si l'on cherche les places au dessus de la valeur absolue ultramétrique $|.|_2$ de $\mathbb{Q}$, on vérifie que le polynôme $X^2 - 2$ n'a pas de racine dans $\mathbb{Q}_2$ et est donc irréductible. Il n'y a donc qu'une seule place au dessus de $|.|_2$, et le complété de $\mathbb{K}$ pour cette valeur absolue est une extension de $\mathbb{Q}_2$ de degré 2.

Au dessus de la place $|.|_7$ de $\mathbb{Q}$ il y a deux valeurs absolues sur $\mathbb{K}$: en effet, le polynôme $X^2 - 2$ admet une racine dans $\mathbb{Z}_7$ d'après le lemme de Hensel (puisque $P(3) = 3^2 - 2 = 0[7]$ et $P'(3) = 6 \neq 0[7]$), donc le polynôme P se factorise sur $\mathbb{Q}_7[X]$. Pour chacune de ces valeurs absolues, le complété du corps de nombres $\mathbb{Q}(\sqrt{2})$ est $\mathbb{Q}_7$.

7. Généralisation à une classe plus large de substitutions

En utilisant un peu de théorie algébrique des nombres, dont la section précédente donne un rapide aperçu, nous pouvons généraliser les fractales de Rauzy de façon à faire fonctionner ce que l'on a fait pour les substitutions de type Pisot unité pour une classe plus générale de substitutions.

7.1. Exemple non unité. — Considérons la substitution

$$s : \begin{cases} a & \mapsto & aaab \\ b & \mapsto & ab \end{cases}$$

Sa matrice d'incidence $M_s = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ n'est pas de déterminant 1. Néanmoins, on peut toujours faire la construction faite pour les substitutions de type Pisot unité. La ligne discrète D_u pour un point périodique u reste bien à distance bornée de la droite propre pour la plus grande valeur propre $\beta = 2 + \sqrt{2}$. Si l'on projette parallèlement à cette droite sur $\mathbb{R}$, on trouve bien une partie bornée dont on pourrait appeler l'adhérence fractale de Rauzy. Malheureusement, la substitution a beau vérifier la strong coincidence, les morceaux $\overline{\pi(D_{u,l})}$, $l \in \{a, b\}$, de l'échange de morceaux se chevauchent. Cela ne permet pas d'obtenir une représentation géométrique du sous-shift engendré par u comme on aimerait avoir.

7.1.1. Description du problème. — Le problème vient du fait que l'équation $gIFS$ n'est plus disjointe en mesure, à cause du déterminant de la matrice qui ne vaut pas ± 1. En considérant la projection

$$\pi : \begin{array}{ccc} \mathbb{Z}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & x + (\gamma - 3)y \end{array}$$

où $\gamma = 2 - \sqrt{2}$ est l'autre vp de M_s, on obtient l'équation gIFS:

$$R_a = \gamma R_a \cup (\gamma R_a + 1) \cup (\gamma R_a + 2) \cup \gamma R_b$$

$$R_b = (\gamma R_a + 3) \cup (\gamma R_b + 1)$$

où $R_a = \overline{\pi(D_{u,a})}$ et $R_b = \overline{\pi(D_{u,b})}$. La preuve de disjonction en mesure de ces unions utilisait de manière cruciale l'hypothèse que la matrice était unimodulaire. Ici, on obtient

$$\lambda(R_a) \leq 3\lambda(\gamma R_a) + \lambda(\gamma R_b),$$

mais on a $\lambda(\gamma R_a) = \frac{2}{\beta}\lambda(R_a)$ (le 2 qui apparaît est la valeur absolue du déterminant). Cela ne permet pas de déduire que le vecteur $(\lambda(R_a), \lambda(R_b))$ est un vecteur propre de M_s comme pour les substitutions de type Pisot unité. Et on peut vérifier que l'équation gIFS n'est en effet pas disjointe en mesure.

7.1.2. La solution : modifier la définition de la fractale de Rauzy. — La solution consiste à tenir compte de ce qui se passe dans le plongement dans l'espace 2-adique du corps de nombre $\mathbb{Q}(\beta)$, qui correspond à la place ultramétrique de $\mathbb{Q}(\beta)$ au dessus de la valeur absolue 2-adique de $\mathbb{Q}$.

Le polynôme minimal de β est $X^2 - 4X + 2$. Celui-ci est irréductible sur $\mathbb{Q}_2$, donc il existe une unique valeur absolue v_2 au dessus de $|.|_2$, et le complété E_2 de $\mathbb{Q}(\beta)$ pour cette valeur absolue est une extension de $\mathbb{Q}_2$ de degré 2.

L'idée pour faire tout fonctionner est de considérer la fractale de Rauzy comme une partie de $\mathbb{R} \times E_2$. On a deux plongement de $\mathbb{Q}(\beta)$ dans $\mathbb{R}$, correspondant aux deux valeurs absolues archimédiennes de $\mathbb{Q}(\beta)$, qui correspondent aux deux racines réelles du polynôme $X^2 - 4X + 2$.

Dans cet espace, tout va fonctionner correctement: l'équation gIFS devient disjointe en mesure (pour la mesure de Haar sur $\mathbb{R} \times E_2$), les morceaux de l'échange de domaine sont disjoints en mesure, les translatés de la fractale de Rauzy par le groupe $\pi(\Gamma_0)$ sont aussi disjoints en mesure.

7.2. Exemple pas de type Pisot. — Considérons la substitution

$$h : \begin{cases} a & \mapsto & ab \\ b & \mapsto & c \\ c & \mapsto & d \\ d & \mapsto & e \\ e & \mapsto & a \end{cases}$$

Sa matrice d'incidence est $M_h = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$. Le polynôme caractéristique est $\chi_{M_h}(X) = X^5 - X^4 - 1 = (X^2 - X + 1)(X^3 - X - 1)$. Le polynôme χ_{M_h} n'étant pas irréductible, la substitution h n'est pas de type Pisot: il y a en effet des valeurs propres de module 1 qui sont des racines de l'unité.

7.2.1. Problème. — Le problème avec la substitution h est que la ligne discrète D_u pour un point fixe $u \in A^{\mathbb{N}}$ n'est pas à distance bornée de l'espace propre E_β pour la plus grande valeur propre β. La fractale de Rauzy n'est donc même pas bornée. Cela vient du fait de l'existence de valeurs propres autres que β qui ne sont pas de module strictement inférieures à 1.

7.2.2. Solution. — La solution est de projeter sur un espace de dimension plus petite où l'action de la multiplication par la matrice M_h sera bien contractante. On définit la projection

$$\pi_\beta : \begin{array}{ccc} \mathbb{Z}^5 & \to & \mathbb{Q}(\beta) \\ X & \mapsto & {}^t V X \end{array}$$

où ${}^tV$ est un vecteur propre de M_h à gauche à coefficients dans $\mathbb{Q}(\beta)$.

La multiplication par M_h dans $\mathbb{Z}^5$ devient une multiplication par β dans $\mathbb{Q}(\beta)$:

$$\pi_\beta(M_h X) = \beta\pi(X),$$

pour tout $X \in \mathbb{Z}^5$.

Il y a ensuite un espace naturel dans lequel la multiplication par β devient contractante: il suffit de considérer le produit des complétés pour les valeurs absolues pour lequelles β est de module strictement inférieur à 1. Ici il n'y a qu'une seule telle valeur absolue, qui correspond aux conjugués complexes conjugués de β. On considère donc le plongement

$$\sigma : \begin{array}{ccc} \mathbb{Q}(\beta) & \to & \mathbb{C} \\ \beta & \mapsto & \gamma \end{array}$$

où γ est un des conjugués complexes de β. Ce plongement est un morphisme de corps, et donc l'image du générateur β suffit à complètement le caractériser.

Si l'on pose $\pi = \sigma \circ \pi_\beta$, on a $\pi : \mathbb{Z}^5 \to \mathbb{C}$, et on a $\pi(D_u) \subseteq \mathbb{C}$ borné. On peut donc définir la fractale de Rauzy par $\overline{\pi(D_u)}$. On obtient alors un échange de morceaux sur cette fractale de Rauzy, avec 5 morceaux $\overline{\pi(D_{u,i})}$, $i \in \{a, b, c, d, e\}$, disjoints deux à deux en mesure. Et cet échange de morceaux est mesurablement conjugué au sous-shift engendré par le mot u.

En revanche, cet échange de morceaux ne se quotiente pas en une translation du tore, parce-que le groupe $\pi(\Gamma_0)$ n'est pas discret dans $\mathbb{C}$.

7.3. Cadre général de la généralisation. — Nous supposerons que s est une substitution sur un alphabet A, qui satisfait les propriétés:

— s est primitive,

— la plus grande valeur propre β de la matrice d'adjacence M_s est un nombre de Pisot. On dira que s est Pisot.

On rappelle qu'un nombre de Pisot est un entier algébrique (i.e. une racine d'un polynôme à coefficients entiers unitaire) dont les conjugués sont tous de module strictement inférieur à 1.

Le nouveau cadre (substitution Pisot primitive) est nettement plus général, puisque l'on ne suppose plus

— que le déterminant de M_s vaut ± 1,

— que toutes les autres valeurs propres sont de modules strictement inférieurs à 1: on veut seulement que ce soit le cas pour les conjugués de la plus grande valeur propre.

Soit β la plus grande valeur propre de la matrice d'incidence M_s. Soit ${}^tV$ un vecteur propre à gauche de M_s à coefficients dans $\mathbb{Q}(\beta)$. On définit la projection

$$\pi_\beta : \begin{array}{ccc} \mathbb{Z}^A & \to & \mathbb{Q}(\beta) \\ X & \mapsto & {}^tVX \end{array}$$

La multiplication par M_s dans $\mathbb{Z}^A$ devient une multiplication par β dans $\mathbb{Q}(\beta)$: pour tout $X \in \mathbb{Z}^A$, on a

$$\pi_\beta(M_s X) = \beta \pi_\beta(X).$$

On va maintenant construire un plongement de $\mathbb{Q}(\beta)$ dans un espace où la multiplication par β devient strictement contractante.

Soit $\mathcal{P} = \left\{ v \in \mathcal{P}_{\mathbb{Q}(\beta)} \;\middle|\; |\beta|_v < 1 \right\}$ l'ensemble des places du corps de nombres $\mathbb{Q}(\beta)$ pour lesquelles β est de module strictement inférieur à 1. On pose alors

$$\sigma : \begin{array}{ccc} \mathbb{Q}(\beta) & \to & E \\ x & \mapsto & \prod_{v \in \mathcal{P}} \sigma_v(x) \end{array},$$

où $E = \prod_{v \in \mathcal{P}} \mathbb{Q}(\beta)_v$, avec $\sigma_v : \mathbb{Q}(\beta) \to \mathbb{Q}(\beta)_v$ est un plongement de $\mathbb{Q}(\beta)$ dans le complété $\mathbb{Q}(\beta)_v$.

Si l'on pose $\pi = \sigma \circ \pi_\beta$, on a alors

Proposition 7.1. — *Si la substitution s est Pisot, alors l'ensemble $\pi(D_u)$ est borné, pour un point périodique u.*

On appelera **fractale de Rauzy** l'adhérence de cet ensemble borné $\overline{\pi(D_u)}$.

Proof. — On a vu que l'on pouvait écrire la ligne discrète D_u pour un point périodique $u \in A^{\mathbb{N}}$, sous la forme

$$D_u = \left\{ \sum_{i=0}^n M_s^{n-i} u_i \;\middle|\; n \in \mathbb{N},\ u_0 u_1 ... u_n \in L \right\},$$

pour un langage rationnel L sur un alphabet fini Σ.

Si l'on projette par π_β, on obtient

$$\pi_\beta(D_u) = \left\{ \sum_{i=0}^n \beta^{n-i} u_i \;\middle|\; n \in \mathbb{N},\ u_0 u_1 ... u_n \in \pi_\beta(L) \right\}.$$

Si l'on applique alors l'application σ, on obtient alors un ensemble borné. Pour chaque place $v \in \mathcal{P}$, on a en effet pour tout $x \in \sigma_v(\pi_\beta(D_u))$,

$$|x| = \left| \sigma_v \left(\sum_{i=0}^n u_i \beta^i \right) \right| = \left| \sum_{i=0}^n u_i \beta^i \right|_v \leq \sum_{i=0}^n |u_i|_v \, |\beta|_v^i \leq \frac{\max_{t \in \Sigma} |t|_v}{1 - |\beta|_v}.$$

□

Pour $x = (x_v)_{v \in \mathcal{P}} \in E = \prod_{v \in \mathcal{P}} \mathbb{Q}(\beta)_v$, on notera

$$\beta.x := (\sigma_v(\beta) x_v)_{v \in \mathcal{P}}.$$

7.3.1. Propriétés. — Nous allons retrouver les propriétés de la fractale de Rauzy dans ce cadre plus général:

Properties 7.2. — *Soit s une substitution Pisot primitive, soit $u \in A^{\mathbb{N}}$ un point fixe, et soit $R = \overline{\pi(D_u)} \subseteq E = \prod_{v \in \mathcal{P}} \mathbb{Q}(\beta)_v$ la fractale de Rauzy, et $R_a = \overline{\pi(D_{u,a})}$, $a \in A$ les morceaux. On a les propriétés suivantes:*

— *R est compact,*

— *$R = \bigcup_{a \in A} R_a$,*

— *On a l'équation gIFS*

$$\forall a \in A,\ R_a = \bigcup_{b \xrightarrow{t} a \in \mathcal{A}^{PA}} \sigma(\beta) R_b,$$

où $\mathcal{A}^{PA}$ est l'automate des préfixes abélianisés. De plus, ces unions sont disjointes en mesure de Haar.

— *$\forall a \in A$, l'intérieur de R_a est dense dans R_a: $\overline{\mathring{R}_a} = R_a$,*

— *Si la substitution s vérifie la strong coincidence, c'est-à-dire*

$$\forall (a,b) \in A^2,\ \exists (u,v,u',v') \in A^*,\ \exists c \in A,\ s(a) = ucv,\ s(b) = u'cv',\ \text{ et } \operatorname{Ab}(u) = \operatorname{Ab}(u'),$$

alors l'union $R = \bigcup_{a \in A} R_a$ est disjointe en mesure, et l'échange de morceaux

$$E : \begin{array}{ccc} R & \to & R \\ x & \mapsto & x + \pi(e_a) \quad \text{si } x \in R_a \end{array}$$

est défini presque partout. Si de plus la projection π est injective (c'est le cas par exemple si le polynôme caractéristique est irréductible), alors cet échange de morceaux mesurablement conjugué au sous-shift engendré par u (qui est uniquement ergodique).

— *Si la substitution est unimodulaire, alors on a $\bigcup_{t\in\pi(\Gamma_0)} R+t=E$. Si de plus l'union est disjointe en mesure de Lebesgue, alors l'échange de morceaux est mesurablement conjugué à la translation par $\pi(e_a)$ (pour n'importe quelle lettre $a\in A$) sur le tore $E/\pi(\Gamma_0)$ pour la mesure de Lebesgue.*

Proof of the first properties. — $R=\overline{\pi(D_u)}$ est l'adhérence d'un ensemble bornée, donc il est borné. Or, E est localement compact (c'est un produit d'extensions finies de $\mathbb{R}$ et de $\mathbb{Q}_p$ qui sont localement compacts), donc la fractale de Rauzy R est compacte.

On a $D_u=\bigcup_{a\in A} D_{u,a}$ (avec une union disjointe). En projetant par π puis en prenant l'adhérence, on en déduit l'égalité

$$R=\overline{\pi(D_u)}=\overline{\bigcup_{a\in A}\pi(D_{u,a})}=\bigcup_{a\in A}\overline{\pi(D_{u,a})}=\bigcup_{a\in A}R_a,$$

(mais l'union n'est plus disjointe).

Pour un point fixe $u\in A^{\mathbb{N}}$ commençant par $c\in A$, on a l'équation gIFS

$$\forall a\in A,\ D_{u,a}=\Delta_{a,c}\cup\bigcup_{b\xrightarrow{t}a\in\mathcal{A}^{PA}} MD_{u,b}+t,$$

où $\Delta_{a,c}$ est l'ensemble vide si $a\neq c$ et $\Delta_{c,c}=\{0\}$. Si l'on projette et que l'on passe à l'adhérence, on en déduit l'équation gIFS

$$\forall a\in A,\ R_a=\bigcup_{b\xrightarrow{t}a\in\mathcal{A}^{PA}}\beta.R_b+\pi(t).$$

(Le point 0 n'a plus besoin d'être mis dans l'union, car on a $R_c\neq\emptyset$ et $R_c\supseteq\beta.R_c\implies 0\in R_c$.)

Montrons la disjonction en mesure de Haar μ de ces unions. On a

$$\mu(R_a)\le\sum_{b\xrightarrow{t}a\in\mathcal{A}^{PA}}\mu(\beta.R_b),$$

parce-que la mesure de Haar est invariant par translation. On en déduit

$$\mu(R_a)\le\sum_{b\in A}m_{a,b}\mu(\beta.R_b),$$

où $m_{a,b}$ est le coefficient (a,b) de la matrice d'incidence M, On a de plus

$$\mu(\beta.R_b)=\prod_{v\in\mathcal{P}}|\beta|_v\,\mu(R_b).$$

Or, d'après la formule du produit, on a $\beta\prod_{v\in\mathcal{P}}|\beta|_v=1$. (Il n'y a pas d'autre place pour laquelle β est de module différent de 1, puisque l'on a toutes les places archimédiennes, et β est un entier algébrique, donc le module de β est 1 pour toutes les places ultramétriques qui ne sont pas dans $\mathcal{P}$.) Si l'on pose $X = (\mu(R_a))_{a\in A}$, on a donc

$$\beta X \leq MX.$$

Mais d'après le théorème de Perron-Frobenius, la matrice M étant primitive (on suppose la substitution primitive), cette inégalité est une égalité. Ainsi, toutes les inégalités sont des égalités, et donc les unions dans l'équation gIFS sont disjointes en mesure de Haar.

La strong coincidence permet alors de déduire la disjonction en mesure de l'union $R = \bigcup_{a\in A} R_a$, puisque cela donne pour chaque couple $(a,b) \in A^2$, deux chemins de même longueur de a vers c et de b vers c dans l'automate $\mathcal{A}^{PA}$ qui correspondent à deux termes de l'union de l'équation gIFS itérée avec le même terme de translation t (ce qui équivaut à l'hypothèse sur les abélianisés qui sont égaux). Ainsi, l'union $(\gamma^n.R_a + t) \cup (\gamma^n.R_b + t)$ est disjointe par disjonction de l'équation gIFS, et on en déduit la disjonction en measure de R_a et R_b.

□

Exercise 7.3. — *Soit la substitution*

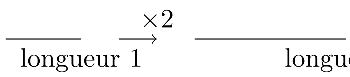
$$s: \begin{array}{ccc} 1 & \mapsto & 111112 \\ 2 & \mapsto & 111 \end{array}.$$

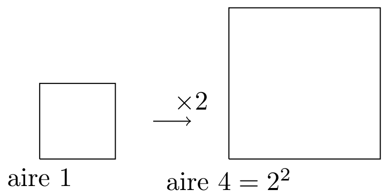
Dans quel espace vit la fractale de Rauzy de s ? Démontrer que le sous-shift engendré par s est mesurablement conjugué à un échange de morceaux.

8. Notion de dimension et bord d'une fractale de Rauzy

Les fractales de Rauzy sont des compacts d'intérieur non vides, mais leur bord peut être assez compliqué. Une façon de mesurer le bord, est de regarder sa dimension. Le but de ce chapitre est d'introduire cette notion de dimension et de voir sur un exemple comment calculer la dimension du bord d'une fractale de Rauzy.

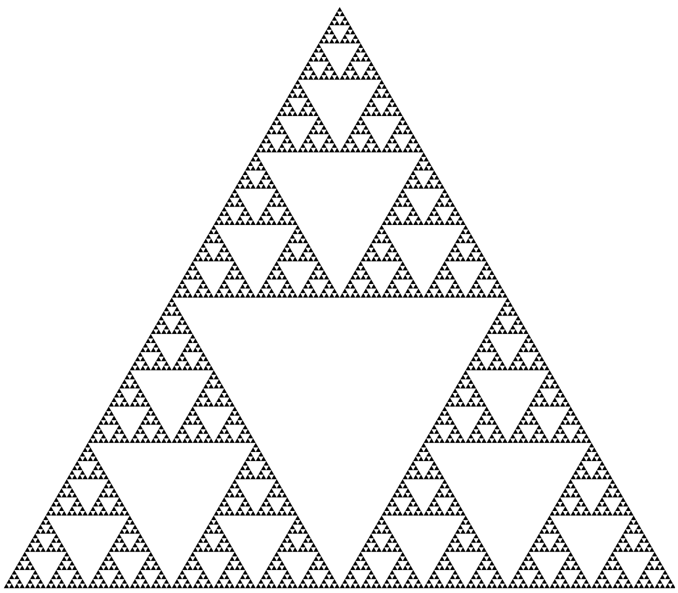
8.1. Introduction. — L'idée de la dimension est que c'est l'exposant δ tel que la "mesure"est multipliée par α^δ quand on fait une homothétie de rapport α.





Par exemple, pour l'ensemble triadique de Cantor, on obtient 2 copies de l'ensemble quand on fait une homothétie de rapport 3. La dimension est donc le nombre δ tel que $3^\delta = 2$, c'est-à-dire $\delta = \dfrac{\ln(2)}{\ln(3)}$.

Pour le triangle de Sierpinski, on obtient 3 copies du triangle quand on fait une homothetie de rapport 2, donc la dimension est le nombre δ tel que $2^\delta = 3$, c'est-à-dire $\delta = \dfrac{\ln(3)}{\ln(2)}$.



8.2. Plusieurs définitions de la dimension. — Nous donnons deux définitions formelles de la dimension: celle de Hausdorff, qui provient naturellement d'une mesure, et celle de Minkowski-Bouligand, qui est souvent plus facile à calculer.

8.2.1. Dimension de Hausdorff. — Soit E un espace métrique, et $\Lambda \subseteq E$ une partie. Pour $s > 0$ et $\epsilon > 0$, on pose

$H^s_\epsilon(\Lambda) := \inf\left\{\sum_{B\in\mathcal{B}}(\mathrm{diam}(B))^s \;\middle|\; \mathcal{B} \text{ ensemble de boules de rayon } \leq \epsilon \text{ qui recouvrent } \Lambda\right\}.$

La fonction $\epsilon \mapsto H^s_\epsilon(\Lambda)$ est décroissante, donc la limite

$$H^s(\Lambda) := \lim_{\epsilon\to 0} H^s_\epsilon(\Lambda)$$

existe dans $\mathbb{R} \cup \{\infty\}$.

Proposition 8.1. — *H^s est une mesure.*

On appelle H^s la mesure de Hausdorff de dimension s.

Properties 8.2. — *On a les propriétés*

$$H^s(\Lambda) > 0 \Longrightarrow \forall t < s,\ H^t(\Lambda) = \infty,$$

$$H^s(\Lambda) < \infty \Longrightarrow \forall t > s,\ H^t(\Lambda) = 0.$$

On définit la **dimension de Hausdorff** par

$$\dim_H(\Lambda) = \inf\left\{s > 0 \ \middle|\ H^s(\Lambda) = 0\right\} = \sup\left\{s > 0 \ \middle|\ H^s(\Lambda) = \infty\right\}.$$

Example 8.3. — *Pour l'ensemble triadique de Cantor C, on a un recouvrement naturel par 2^n boules de rayon $\frac{1}{3^n}$, donc on a*

$$H^s_{\frac{1}{3^n}}(C) \le 2^n \frac{1}{3^s n}.$$

Pour $s = \frac{\ln(2)}{\ln(3)}$, on a donc $H^s_{\frac{1}{3^n}}(C) \le 1$ pour tout $n \in \mathbb{N}$, donc $H^s(C) > 1$. On en déduit que l'on a

$$\dim_H(C) \le s = \frac{\ln(2)}{\ln(3)}.$$

Pour obtenir l'inégalité dans l'autre sens, on peut utiliser le lemme suivant.

Lemma 8.4. — *S'il existe une mesure de proba μ portée par Λ et s'il existe des constantes $C > 0$ et $s > 0$ telles que pour toute partie mesurable A on ait*

$$\mu(A) \le C(\mathrm{diam}(A))^s,$$

alors on a $\dim_H(\Lambda) \ge s$.

Proof. — Soit $\epsilon > 0$ et soit $\mathcal{B}$ un recouvrement de Λ par des boules de rayons inférieur ou égaux à ϵ. On a

$$\sum_{B \in \mathcal{B}} (\mathrm{diam}(B))^s \ge \frac{1}{C} \sum_{B \in \mathcal{B}} \mu(B) \ge \frac{1}{C}\mu(\Lambda) = \frac{1}{C}.$$

On a donc $H^s_\epsilon(\Lambda) \ge \frac{1}{C}$, et donc en passant à la limite quand ϵ tend vers 0, on obtient $H^s(\Lambda) \ge \frac{1}{C} > 0$. On obtient donc bien l'inégalité souhaitée $\dim_H(\Lambda) \ge s$. □

On peut utiliser ce lemme pour minorer la dimension de Hausdorff de l'ensemble de Cantor en construisant une mesure de la façon suivante. On considère μ_n la mesure de proba portée par l'union des 2^n boules de rayon $\frac{1}{3^n}$ qui recouvre Λ qui est uniforme par rapport à la mesure de Lebesgue. On a le lemme suivant

Lemma 8.5. — *L'ensemble des mesures de probabilité sur un compact est compact pour la convergence faible des mesures.*

On dit qu'une suite de mesures μ_n **converge faiblement** vers une mesure μ si l'on a

$$\lim_{n\to\infty} \int f d\mu_n = \int f d\mu,$$

pour toute fonction continue bornée f.

En utilisant ce lemme, on obtient qu'il existe une sous-suite convergente de la suite de mesure μ_n que l'on a définit. On peut alors démontrer que cette mesure vérifie les hypothèses du lemme 8.4, ce qui permet de démontrer la minoration $\dim_H(C) \geq \frac{\ln(2)}{\ln(3)}$.

8.2.2. Dimension de Minkowski-Bouligand. — On dit qu'une partie P d'une espace métrique E est **ϵ-séparée** si la distance entre deux points de P est toujours minorée par ϵ. On dit qu'une partie P **ϵ-couvre** une partie Λ si tout point de Λ est à distance au plus ϵ d'un point de P.

Etant donnée une partie Λ, on note

$$N_\epsilon^{sep}(\Lambda) = \max\left\{\#P \;\middle|\; P \text{ partie } \epsilon\text{-séparée de } \Lambda\right\},$$

$$N_\epsilon^{cov}(\Lambda) = \min\left\{\#P \;\middle|\; P \text{ partie de } \Lambda \text{ qui } \epsilon\text{-couvre } \Lambda\right\}.$$

On définit alors la **dimension de Minkowski-Bouligand** (ou box-dimension) par

$$\dim_{MB}(\Lambda) = \lim_{\epsilon\to 0} \frac{\log(N_\epsilon^{sep}(\Lambda))}{\log(1/\epsilon)} = \lim_{\epsilon\to 0} \frac{\log(N_\epsilon^{cov}(\Lambda))}{\log(1/\epsilon)}.$$

Cette limite peut ne pas exister, auquel cas on peut remplacer la limite par une limite inférieure (respectivement une limite supérieure), ce qui définit la dimension de Minkowski-Bouligand inférieure (respectivement supérieure).

Les deux types de recouvrement donnent bien la même définition de dimension, puisque l'on a

$$N_{2\epsilon}^{cov}(\Lambda) \leq N_\epsilon^{sep}(\Lambda) \leq N_{\epsilon/2}^{cov}(\Lambda).$$

Pour l'ensemble triadique de Cantor C, on peut vérifier que l'on a

$$N_{\frac{1}{3^n}}^{sep}(C) = 2^n = N_{\frac{1}{3^n}}^{cov}(C),$$

et on en déduit que l'on a

$$\dim_{MB}(C) = \lim_{n\to\infty} \frac{\log(2^n)}{\log(3^n)} = \frac{\ln(2)}{\ln(3)}.$$

8.3. Bord de Tribonnacci. — Dans cette sous-partie, nous montrons, sur l'exemple de Tribonnacci, comment décrire le bord d'une fractale de Rauzy et calculer sa dimension. La substitution de Tribonnacci est

$$s : \begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases}.$$

Nous avons vu précédemment que l'union $R = R_a \cup R_b \cup R_c$ est disjointe en mesure de Lebesgue. Donc l'union $R_a \cap R_b$ est une partie du bord de la fractale de Rauzy R.

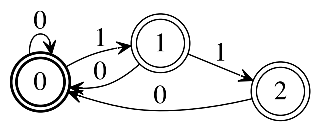
Pour décrire cette intersection, décrivons déjà R_a et R_b. Pour cela, on décrit $D_{u,a}$ et $D_{u,b}$ grâce à l'automate des préfixes abélianisés $\mathcal{A}^{PA}$.

$$D_{u,a} = \left\{ \sum_{i=0}^{n} M^{n-i} v_i \;\middle|\; v_0 v_1 ... v_n \text{ chemin de } a \text{ vers } a \text{ dans } \mathcal{A}^{PA} \right\},$$

$$D_{u,b} = \left\{ \sum_{i=0}^{n} M^{n-i} v_i \;\middle|\; v_0 v_1 ... v_n \text{ chemin de } a \text{ vers } b \text{ dans } \mathcal{A}^{PA} \right\}.$$

On peut ensuite projeter par π de façon à travailler dans le corps de nombre $\mathbb{Q}(\gamma)$, pour γ une valeur propre complexe de la matrice d'incidence M. Les ensembles $\pi(D_{u,a})$ et $\pi(D_{u,b})$ sont alors décrit de la même façon à l'aide de l'automate $\mathcal{A}^{PAP}$ de la figure 24.

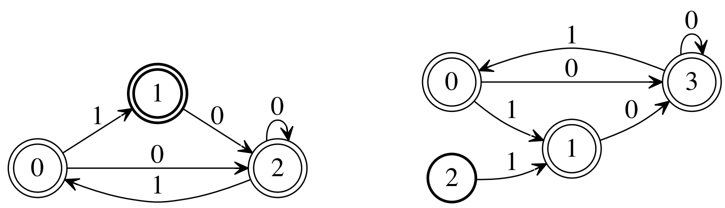
FIGURE 24. Automate des préfixes abélianisés projeté par π_β pour Tribonnacci



$$\pi(D_{u,j}) = \left\{ \sum_{i=0}^{n} \gamma^{n-i} v_i \;\middle|\; v_0 v_1 ... v_n \text{ chemin de } a \text{ vers } j \text{ dans } \mathcal{A}^{PAP} \right\}.$$

L'ensemble des mots $v_0 ... v_n$ de a vers i dans $\mathcal{A}^{PAP}$ est le language rationnel L_i de l'automate $\mathcal{A}^{PAP}$ ayant pour état initial a et pour état final i. Nous allons voir dans la suite qu'il est plus pratique de travailler avec le miroir ${}^tL_i$. Les languages ${}^tL_a$ et ${}^tL_b$ sont reconnu par les automates de la figure 25.

FIGURE 25. Automates minimaux $\mathcal{A}_a$ et $\mathcal{A}_b$ des langages ${}^tL_a$ et ${}^tL_b$ respectivement



On a alors
$$\pi(D_{u,j}) = \left\{ \sum_{i=0}^{n} \gamma^i v_i \;\middle|\; v_0v_1...v_n \in {}^tL_i \right\}.$$
L'avantage de cette réécriture est que si $v_0...v_n$ est le préfixe de longueur $n+1$ d'un mot infini, alors la somme $\sum_{i=0}^{n} \gamma^i v_i$ converge vers $\sum_{i=0}^{\infty} \gamma^i v_i$. Et cela permet de décrire l'adhérence grâce au lemme suivant.

Lemma 8.6. — *Pour tout $j \in A$, on a*
$$R_j = \left\{ \sum_{i=0}^{\infty} v_i\gamma^i \;\middle|\; \forall n \geq 2,\ v_0...v_n \in {}^tL_j \right\}.$$

Proof. — On a l'équation gIFS
$$\begin{cases} R_a &= \gamma R_a \cup \gamma R_b \cup \gamma R_c \\ R_b &= \gamma R_a + 1 \\ R_c &= \gamma R_b + 1 \end{cases}$$
Soit $x \in R_j$. Alors x est dans au moins un $\gamma R_{j'} + t'$ d'après l'équation gIFS, donc il existe $j_1 \in A$, $t_1 \in$ et $y \in R_{j_1}$ tel que $x = \gamma y + t_1$.

Par récurrence, on construit des suites $(t_k) \in \{0,1\}^{\mathbb{N}}$ et $(j_k) \in A^{\mathbb{N}}$ telles que
$$x \in \sum_{i=0}^{k} \gamma^i t_i + \gamma^{k+1} R_{j_k}$$
pour tout $k \in \mathbb{N}$. On a alors $x = \sum_{i=0}^{\infty} \gamma^i t_i$ puisque $\operatorname{diam}(\gamma^{k+1}R_{j_k}) \xrightarrow[k\to\infty]{} 0$. Et on vérifie que l'on a pour tout $n \geq 2$, $t_0t_1...t_n \in {}^tL_j$ (cela se voit sur les automates de la figure 25 qui ont tous les états de leur composante fortement connexe finaux).

Réciproquement, si $v \in A^{\mathbb{N}}$ est tel que pour tout $n \geq 2$, $v_0...v_n \in {}^tL_j$, alors on a $\sum_{i=0}^{n} v_i\gamma^i \in \pi(D_{u,j})$ pour tout $n \geq 2$, donc $\sum_{i=0}^{\infty} v_i\gamma^i \in \overline{\pi(D_{u,j})} = R_j$. □

Grâce à ce lemme, on peut décrire l'intersection $R_a \cap R_b$:
$$R_a \cap R_b = \left\{ \sum_{i=0}^{\infty} v_i\gamma^i \;\middle|\; \begin{array}{c} v_0v_1\text{.... chemin dans l'automate } \mathcal{A}_a,\ \exists w_0w_1\text{... chemin dans } \mathcal{A}_b \\ \text{tel que } \sum_{i=0}^{\infty} v_i\gamma^i = \sum_{i=0}^{\infty} w_i\gamma^i \end{array} \right\}$$
Mais pour expliciter ce qu'est cet ensemble, on aurait besoin de comprendre quel sont les couples de mots (v,w) tels que $\sum_{i=0}^{\infty} v_i\gamma^i = \sum_{i=0}^{\infty} w_i\gamma^i$, ce qui revient à décrire les mots $(a_i) \in \{-1,0,1\}^{\mathbb{N}}$ tels que
$$\sum_{i=0}^{\infty} a_i\gamma^i = 0.$$
Ces mots sont décrit par un automate fini d'après le lemme suivant.

Lemma 8.7. — *L'ensemble*
$$\left\{ \sum_{i=0}^{n-1} a_i\gamma^{i-n} \;\middle|\; n \in \mathbb{N},\ (a_i) \in \{-1,0,1\}^{\mathbb{N}},\ \sum_{i=0}^{\infty} a_i\gamma^i = 0 \right\}$$
est fini.

Proof. — Soit $(a_i) \in \{-1,0,1\}^{\mathbb{N}}$, telle que $\sum_{i=0}^{\infty} a_i\gamma^i = 0$. Pour tout $n \in \mathbb{N}$, on a

$$\left|\sum_{i=0}^{n-1} a_i\gamma^{i-n}\right| = \left|-\gamma^{-n}\sum_{i=n}^{\infty} a_i\gamma^i\right| =\leq \frac{1}{1-|\gamma|}.$$

Et pour l'autre valeur absolue $|.|_+$ archimédienne de $\mathbb{Q}(\beta)$, on a

$$\left|\sum_{i=0}^{n-1} a_i\gamma^{i-n}\right|_+ = \left|\sum_{i=0}^{n-1} a_i\beta^{i-n}\right| =\leq \frac{1/\beta}{1-1/\beta} = \frac{1}{\beta-1}.$$

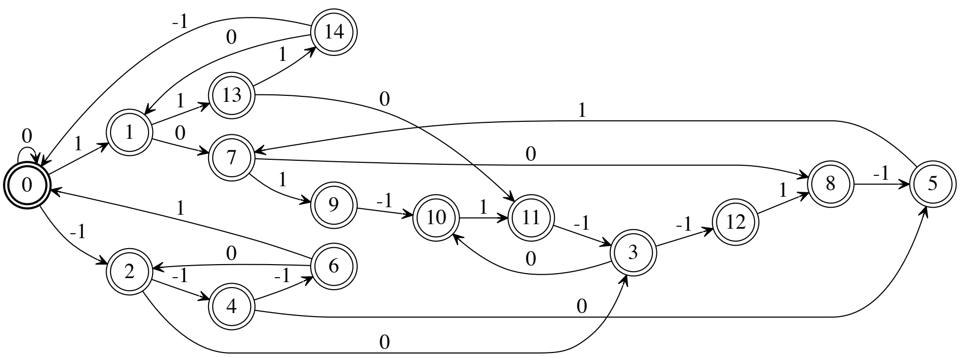
Ainsi, les sommes $\sum_{i=0}^{n-1} a_i\gamma^{i-n}$ sont uniformément bornées dans $\mathbb{R}\times\mathbb{C}$. Or, elles sont aussi dans $\mathbb{Z}[\gamma]$ qui est discrect dans $\mathbb{R}\times\mathbb{C}$ d'après le corollaire 6.11. Cela prouve bien qu'il n'y en a qu'un nombre fini. □

On peut donc construire un automate avec comme ensemble d'états $Q \subseteq \mathbb{Z}[\gamma]$ l'ensemble fini donné par ce lemme, comme alphabet $\{-1,0,1\}$, et avec pour transitions

$$x \xrightarrow{t} y \iff y = \gamma^{-1}(x+t).$$

On prend comme état initial 0 et comme état finaux tous les états. Cet automate est dessiné figure 26.

Figure 26. Automates décrivant les mots $(a_i) \in \{-1,0,1\}^{\mathbb{N}}$ tels que $\sum_{i=0}^{\infty} a_i\gamma^i = 0$.



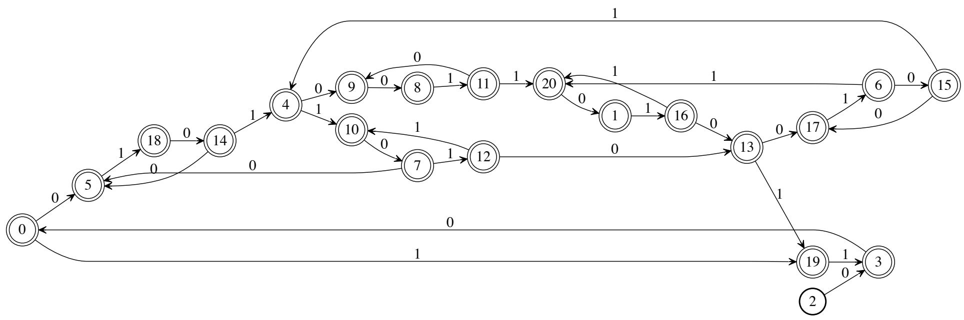
Un mot $(a_i) \in \{-1,0,1\}^{\mathbb{N}}$ est un parcours dans cet automate si et seulement si $\sum_{i=0}^{\infty} a_i\gamma^i = 0$. En effet, on a un chemin

$$0 \xrightarrow{t_0} -\beta^{-1}t_0 \xrightarrow{t_1} \ldots \xrightarrow{t_{n-1}} x$$

si et seulement si $x = \sum_{i=0}^{n-1} t_i \gamma^{i-n}$. Donc si (a_i) est un chemin infini, alors pour tout n, $\sum_{i=0}^{n-1} a_i \gamma^i = \gamma^n \sum_{i=0}^{n-1} t_i \gamma^{i-n} \xrightarrow[n \to 0]{} 0$. Et réciproquement, si l'on a $(a_i) \in \{-1,0,1\}^{\mathbb{N}}$ tel que $\sum_{i=0}^{\infty} a_i \gamma^i = 0$, alors (a_i) est un parcours dans l'automate.

En combinant cet automate et les automates $\mathcal{A}_a$ et $\mathcal{A}_b$, et après calcul, on peut alors décrire l'intersection $R_a \cap R_b$ par l'automate de la figure **??**.

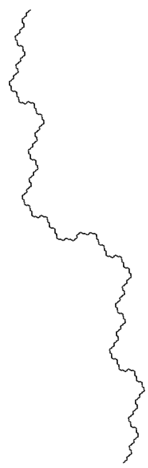
Figure 27. Automates $\mathcal{A}^{int}$ décrivant l'intersection $R_a \cap R_b$



On a le lemme suivant.

Lemma 8.8. —
$$R_a \cap R_b = \left\{ \sum_{i=0}^{\infty} a_i \gamma^i \;\middle|\; a_0 a_1 \ldots \text{ parcours dans l'automate } \mathcal{A}^{int} \text{ partant de l'état initial.} \right\}$$

Figure 28. L'intersection $R_a \cap R_b$



Maintenant que l'on a décrit l'intersection $R_a \cap R_b$ par un langage rationnel, pour calculer sa dimension on utilise le résultat suivant.

Theorem 8.9. — *Let $L \subseteq \Sigma^*$ be a language over the alphabet $\Sigma = \{0,1\}$ such that the elements of $Q_L = \left\{\sum_{i=0}^{|u|-1} u_i\gamma^i \;\middle|\; u \in L\right\} \subseteq \mathbb{C}$ are uniquely represented for a given length (i.e. $\forall u, v \in L$, $\left(|u| = |v| \text{ and } \sum_{i=0}^{|u|-1} u_i\gamma^i = \sum_{i=0}^{|v|-1} v_i\gamma^i\right) \implies u = v$).*

Then we have

$$\dim_{MB}(Q_L) = \frac{\log(\gamma)}{\log(1/|\gamma|)} = 2\frac{\log(\gamma)}{\log(\gamma)},$$

where γ is the spectral radius of the minimal automaton of L.

Proof. — Let $L_n = \left\{u \in L \;\middle|\; |u| = n\right\}$, and for all $u \in L$, let $x_u = \sum_{i=0}^{|u|-1} u_i\gamma^i$. Then we have $Q_L = \left\{x_u \;\middle|\; u \in L\right\}$, and we have the following

Lemma 8.10. — *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and for all $u \neq v \in L_n$, $|x_u - x_v| \geq C\,|\gamma|^n$.*

Proof. — For all $n \in \mathbb{N}$, we have the inclusion

$$\left\{x_u - x_v \;\middle|\; (u,v) \in (L_n)^2\right\} \subseteq \left\{\sum_{i=0}^{n-1} a_i\gamma^i \;\middle|\; a \in \{-1,0,1\}^n\right\}.$$

Hence, it is enough to prove that the set $S = \left\{\sum_{i=0}^{n-1} a_i\gamma^{i-n} \;\middle|\; a \in \{-1,0,1\}^n\right\}$ is uniformly discrete to prove the lemma, thanks to the hypothesis that elements are uniquely represented for a given length. This follows from corollary 6.11, because $\sigma_+(S) \subseteq \mathbb{Z}[\gamma]$, and the set $\sigma_+(S) = \left\{\sum_{i=0}^{n-1} a_i\beta^{i-n} \;\middle|\; a \in \{-1,0,1\}^n\right\}$ is bounded in $\mathbb{R}$ (by $\frac{1}{\beta-1}$), where σ_+ is the Galois embedding of $\mathbb{Q}(\gamma)$ such that $\sigma_+(\gamma) = \beta$. □

Using this lemma, we have that the balls $B(x_u,\ \frac{1}{2}C\,|\gamma|^n)$, $u \in L_n$, are all pairwise disjoints, hence we have

$$N^{\text{sep}}\left(\frac{1}{2}C\,|\gamma|^n\right) \geq \#L_n.$$

Therefore, we have

$$\liminf_{\epsilon \to 0} \frac{\log(N^{\text{sep}}(\epsilon))}{\log(1/\epsilon)} \geq \lim_{n\to\infty} \frac{\log(\#L_{n-1})}{\log\left(\frac{2}{C}|\gamma|^{-n}\right)} = \frac{\log(\gamma)}{\log(1/|\gamma|)}.$$

To prove the other inequality, let's consider for all $n \in \mathbb{N}$, and $u \in L_n$, the open ball

$$B_u = B\left(x_u,\ \frac{2\,|\gamma|^n}{1-|\gamma|}\right) \subseteq \mathbb{C}.$$

Up to replace L by the language $\text{Pref}(L)$, which is a regular language with the same spectral radius, we have that for all $n \in \mathbb{N}$, the set of balls $\left\{B_u \;\middle|\; u \in L_n\right\}$ is a covering of Q_L, hence we have

$$N_{\text{cov}}\left(\frac{2\,|\gamma|^n}{1-|\gamma|}\right) \leq \#L_n.$$

And we have $\#L_n \sim C\gamma^n$ for some constant $C > 0$. Therefore, we have

$$\limsup_{\epsilon\to 0} \frac{\log(N_{\mathrm{cov}}(\epsilon))}{\log(1/\epsilon)} \leq \lim_{n\to\infty} \frac{\log(\#L_{n+1})}{\log\left(\frac{1-|\gamma|}{2}|\gamma|^{-n}\right)} = \frac{\log(\gamma)}{\log(1/|\gamma|)}.$$

Hence, we have $\dim_{MB}(Q_L) = \frac{\log(\gamma)}{\log(1/|\gamma|)}$. This ends the proof of the theorem 8.9. □

La dimension de Minkowski-Bouligand de Q_L est la même que celle de l'ensemble $\overline{Q_L}$. Or, pour le langage L reconnu par l'automate $\mathcal{A}^{int}$ de la figure 27, on a

$$\overline{Q_L} = \left\{\sum_{i=0}^{\infty} a_i\gamma^i \;\middle|\; \forall n \geq 1,\ a_0...a_n \in L\right\} = R_a \cap R_b$$

puisque L est stable par préfixe de longueur au moins 1.

La dimension de Minkowski-Bouligand de $R_a \cap R_b$ est donc égale à $\dfrac{\ln(\alpha)}{\ln(1/|\gamma|)} = \dfrac{2\ln(\alpha)}{\ln(\beta)}$, où α est le rayon spectral du graphe de l'automate $\mathcal{A}^{int}$. On vérifie que $\alpha \simeq 1,39533699446707...$ est racine de $X^4 - 2X - 1$. Donc la dimension de $R_a \cap R_b$ est environ $1,09336416428231...$.

On pourrait démontrer que la dimension de Minkowski-Bouligand est égale à la dimension de Hausdorff (on conjecture même que c'est vrai pour tous les ensemble strictement auto-similaires).

9. Substitutions duales

Dans ce chapitre, nous présentons une construction dûe à P. Arnoux et S. Ito publiée en 2001 (voir [**Arnoux Ito 2001**]). Cette construction permet de donner une autre définition équivalente des fractales de Rauzy par approximations successives de la fractale de Rauzy par des unions de parallellogrammes. Ces approximation successives ont la propriété de paver le plan, et sont naturellement munies d'un échange de morceaux.

9.1. L'application linéaire $E_1(s)$. — Soit s une subtitution sur un alphabet A. On pose

$$\mathcal{F} = \left\{f \in \mathcal{F}(\mathbb{Z}^A \times A, \mathbb{R}) \;\middle|\; f \text{ à support fini}\right\},$$

où le support d'une fonction $f \in \mathcal{F}(\mathbb{Z}^A \times A, \mathbb{R})$ est

$$\mathrm{supp}(f) = \left\{x \in \mathbb{Z}^A \times A \;\middle|\; f(x) \neq 0\right\}.$$

$\mathcal{F}$ est un $\mathbb{R}$-espace vectoriel de dimension infinie. Une base est donnée par les fonctions

$$(x, e_i) : \begin{array}{ccc} \mathbb{Z}^A \times A & \to & \mathbb{R} \\ (y, j) & \mapsto & \begin{cases} 1 & \text{si } x = y \text{ et } i = j \\ 0 & \text{sinon.} \end{cases} \end{array}$$

On représente géométriquement ces fonctions (x, e_i) par un segment de longueur 1 basé en x et suivant le vecteur e_i. On peut généraliser cette représentation géométrique pour une partie de $\mathcal{F}$:

Definition 9.1. — *On dit qu'un élément $f \in \mathcal{F}$ est* **géométrique** *s'il est à valeurs dans* $\{0, 1\}$*:* $\mathrm{Im}(f) \subseteq \{0, 1\}$.

On identifie un élément de $\mathcal{F}$ qui est géométrique à l'union des segments de longueur 1 correspondant aux élements (x, i) qui sont dans le support.

On va définir une application linéaire sur l'espace vectoriel $\mathcal{F}$ qui correspond à l'action de la substitution sur la ligne brisée.

On définit l'application linéaire

$$E_1(s) : \begin{array}{ccl} \mathcal{F} & \to & \mathcal{F} \\ (x, e_i) & \mapsto & \sum_{i \xrightarrow{t} j \in \mathcal{A}^{PA}} (M_s x + t, e_j) \end{array}$$

Cette application linéaire correspond bien à ce que l'on attends de l'action de la substitution s sur la ligne brisée. En particulier, si a est la première lettre d'un point fixe de s, alors les supports de $E_1(s)^n(0, e_a)$ convergent vers la ligne brisée correspondant à ce point fixe.

Proposition 9.2. — *Pour deux substitutions s_1 et s_2, on a*

$$E_1(s_1) \circ E_1(s_2) = E_1(s_1 \circ s_2).$$

Pour démontrer cette proposition, nous utilisons les lemmes suivant.

Lemma 9.3. — *On a $M_{s_1 s_2} = M_{s_1} M_{s_2}$.*

Proof. —

$$\begin{aligned} M_{s_1 s_2} &= (|s_1 s_2(b)|_a)_{a,b \in A} \\ &= (\sum_{l \text{ lettre à chaque position du mot } s_2(b)} |s_1(l)|_a)_{a,b \in A} \\ &= \left(\sum_{c \in A} |s_2(b)|_c \, |s_1(c)|_a \right)_{a,b \in A} \\ &= M_{s_1} M_{s_2}. \end{aligned}$$

□

Lemma 9.4. — *On a l'équivalence*

$$i \xrightarrow{t} j \in \mathcal{A}^{PA}_{s_1 s_2} \iff i \xrightarrow{t'} k \in \mathcal{A}^{PA}_{s_2},\ k \xrightarrow{t''} j \in \mathcal{A}^{PA}_{s_1} \text{ et } t = M_{s_1} t' + t''.$$

Proof. —

$$
\begin{array}{rcl}
i \xrightarrow{t} j \in \mathcal{A}^{PA}_{s_1 s_2} & \iff & \exists u, v \in A^*,\ s_1 s_2(i) = ujv \text{ et } t = \mathrm{Ab}(u) \\
& \iff & \exists u', v', u'', v'' \in A^*,\ \exists k \in A,\ s_2(i) = u'kv',\ s_1(k) = u''jv'', \quad \text{et } t = \mathrm{Ab}(s_1(u')u'') = M_{s_1} A \\
& \iff & i \xrightarrow{t'} k \in \mathcal{A}^{PA}_{s_2},\ k \xrightarrow{t''} j \text{ et } t = M_{s_1} t' + t''.
\end{array}
$$

□

Preuve de la proposition 9.2. — Il sufit de vérifier l'égalité sur la base $\{(x, e_i) \in \mathcal{F} \mid x \in \mathbb{Z}^d, i \in A\}$. On a

$$
\begin{array}{rcl}
E_1(s_1 \circ s_2)(x, e_i) & = & \displaystyle\sum_{i \xrightarrow{t} j \in \mathcal{A}^{PA}_{s_1 s_2}} (M_{s_1 s_2} x + t, e_j) \\
& = & \displaystyle\sum_{i \xrightarrow{t_2} k \in \mathcal{A}^{PA}_{s_2} \text{ et } k \xrightarrow{t_1} j \in \mathcal{A}^{PA}_{s_1}} (M_{s_1 s_2} x + M_{s_1} t_2 + t_1, e_j) \\
& = & \displaystyle\sum_{i \xrightarrow{t_2} k} \sum_{k \xrightarrow{t_1} j} (M_{s_1} M_{s_2} x + M_{s_1} t_2 + t_1, e_j) \\
& = & E_1(s_1) \left(\displaystyle\sum_{i \xrightarrow{t_2} k} (M_{s_2} x + t_2, e_k) \right) \\
& = & E_1(s_1) E_1(s_2)(x, e_i).
\end{array}
$$

□

Example 9.5. — *Pour la substitution de Tribonnacci:*

$$
s : \begin{array}{rcl} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array}
$$

L'application linéaire $E_1(s)$ est définie par

$$
\begin{array}{rcl}
E_1(s)(x, e_1) & = & (Mx, e_1) + (Mx + e_1, e_2) \\
E_1(s)(x, e_2) & = & (Mx, e_1) + (Mx + e_1, e_3) \\
E_1(s)(x, e_3) & = & (Mx, e_1).
\end{array}
$$

En itérant $E_1(s)$ à partir de $(0, e_1)$, on trouve le début de la ligne discrète, avec les points connectés par des segments.

FIGURE 29. $E_1(s)^4(0, e_1)$ pour s la substitution de Tribonnacci

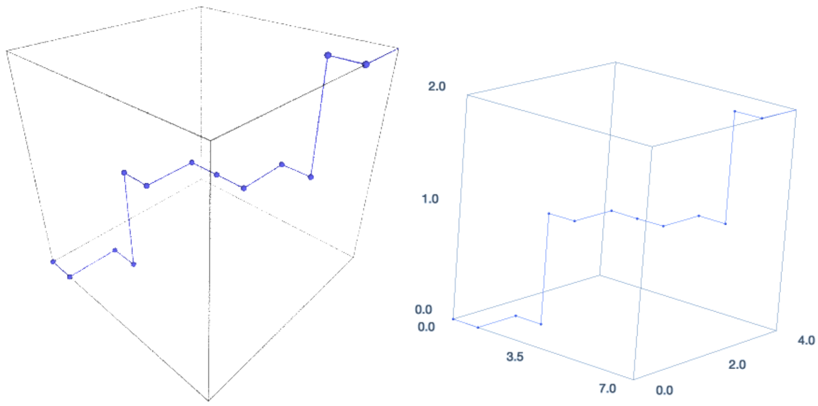
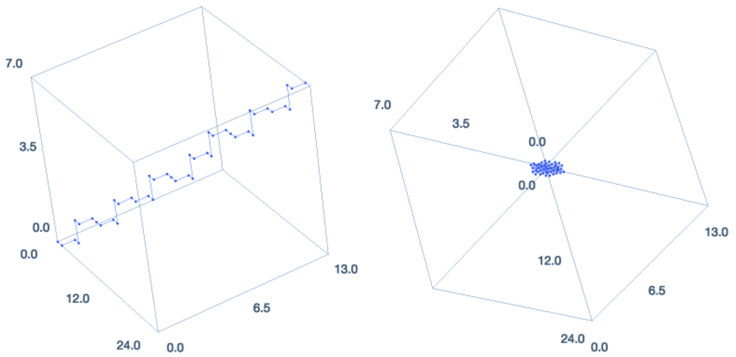


FIGURE 30. $E_1(s)^6(0, e_1)$ pour s la substitution de Tribonnacci



9.2. L'application linéaire duale $E_1^*(s)$. — Comme pour tout espace vectoriel, on peut considérer le dual du $\mathbb{R}$-espace vectoriel $\mathcal{F}$ en considérant les formes linéaires sur $\mathcal{F}$. Mais ici on ne garde pas toutes les formes linéaires, de façon à ne pas avoir un espace dual trop gros. On pose

$$\mathcal{F}^* = \left\{ \varphi \in \mathcal{L}(\mathcal{F}, \mathbb{R}) \;\middle|\; \exists K \subseteq \mathbb{Z}^A \times A \text{ fini}\ ,\ \forall f \in \mathcal{F},\ \varphi(f) \neq 0 \Longrightarrow \operatorname{supp}(f) \subseteq K \right\}.$$

Une base de $\mathcal{F}^*$ est donnée par les formes linéaires

$$(x, e_i^*) : \begin{array}{ccc} \mathcal{F} & \to & \mathbb{R} \\ (y, e_j) & \mapsto & \begin{cases} 1 & \text{si } x = y \text{ et } i = j \\ 0 & \text{sinon.} \end{cases} \end{array}$$

De la même façon que pour les élements de $\mathcal{F}$, on dira qu'un élement de $\mathcal{F}^*$ est géométrique si l'image est contenue dans $\{0, 1\}$. On représente géométriquement (x, e_i^*) par une petite hyperface orthogonale au vecteur e_i.

On considère l'application bilinéaire

$$< .|. >: \begin{array}{ccc} \mathcal{F}^* \times \mathcal{F} & \to & \mathbb{R} \\ (\varphi, f) & \mapsto & \varphi(f) \end{array}$$

On définit l'application linéaire $E_1^*(s) : \mathcal{F}^* \to \mathcal{F}^*$ duale (ou transposée) de l'application $E_1(s)$ par

$$\forall x \in \mathcal{F}, \forall y \in \mathcal{F}^*, < E_1^*(s)y|x > = < y|E_1(s)x > .$$

Proposition 9.6. — *Pour tout $(x,i) \in \mathbb{Z}^A \times A$, on a*

$$E_1^*(s)(x, e_i^*) = \sum_{j \xrightarrow{t} i \in \mathcal{A}^{PA}} (M_s^{-1}(x-t), e_j^*).$$

Proof. — Pour tout $(x,i) \in \mathbb{Z}^A \times A$ et tout $(y,j) \in \mathbb{Z}^A \times A$, on a

$$\begin{array}{rcl} < E_1^*(s)(x, e_i^*)|(y, e_j) > & = & < (x, e_i^*)|E_1(s)(y, e_j) > \\ & = & < (x, e_i^*)| \sum_{j \xrightarrow{t} k \in \mathcal{A}^{PA}} (M_s y + t, e_k) > \\ & = & \sum_{j \xrightarrow{t} k \in \mathcal{A}^{PA}} < (x, e_i^*)|(M_s y + t, e_k) > \\ & = & \sum_{j \xrightarrow{t} i \in \mathcal{A}^{PA}} < (M_s^{-1}(x-t), e_i^*)|(y, e_i) > \\ & = & < \sum_{j \xrightarrow{t} i \in \mathcal{A}^{PA}} (M_s^{-1}(x-t), e_j^*)|(y, e_j) > \end{array}$$

□

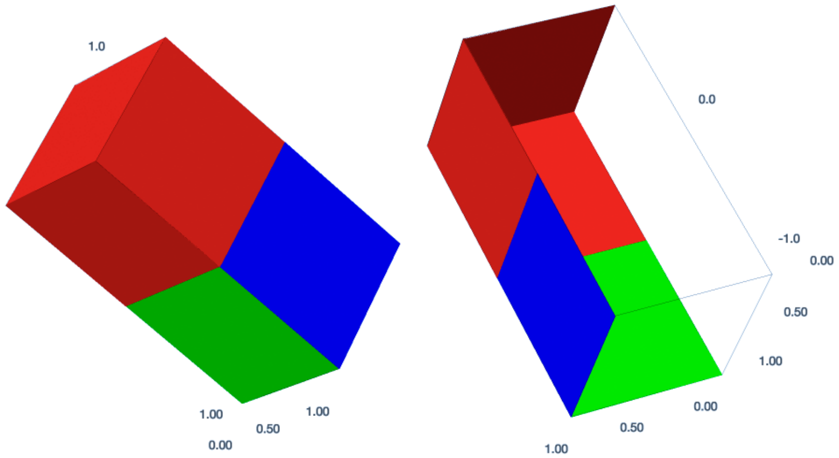
Example 9.7. — *Pour la substitution de Tribonnacci*

$$s: \begin{array}{ccc} 1 & \mapsto & 12 \\ 2 & \mapsto & 13 \\ 3 & \mapsto & 1 \end{array}$$

L'application linéaire $E_1^*(s)$ est l'application linéaire définie par

$$\begin{array}{rcl} E_1^*(s)(x, e_1^*) & = & (M^{-1}x, e_1^*) + (M^{-1}x, e_2^*) + (M^{-1}x, e_3^*) \\ E_1^*(s)(x, e_2^*) & = & (M^{-1}(x - e_1), e_1^*) \\ E_1^*(s)(x, e_3^*) & = & (M^{-1}(x - e_1), e_1^*). \end{array}$$

FIGURE 31. $E_1^*(s)(0, e_i)$ pour s la substitution de Tribonnacci avec $i = 1$ en rouge, $i = 2$ en vert, et $i = 3$ en bleu.



9.3. Propriétés. — Voyons quelles sont les propriétés de l'application linéaire $E_1^*(s)$, et en particulier ce que l'on obtient en itérant cette application.

Soit V_β un vecteur propre pour la plus grande valeur propre β de la matrice d'incidence M_s.

Soit

$$\mathcal{G} = \left\{ f \in \mathcal{F} \;\middle|\; \forall (x,i) \in \mathbb{Z}^A \times A,\ f((x,e_i)) \neq 0 \Longrightarrow V_\beta.x < 0 \text{ et } V_\beta.(x+e_i) \geq 0 \right\}$$

$$\mathcal{G}^* = \left\{ \varphi \in \mathcal{F}^* \;\middle|\; \forall (x,i) \in \mathbb{Z}^A \times A,\ \varphi((x,e_i^*)) \neq 0 \Longrightarrow V_\beta.x < 0 \text{ et } V_\beta.(x+e_i) \geq 0 \right\}$$

Un élément de $\mathcal{G}^*$ géométrique se représente géométriquement comme une union de petites hyperfaces qui restent proche du plan orthogonal au vecteur V_β. Cette sorte d'approximation discrète du plan est stable par la substitution duale $E_1^*(s)$:

Proposition 9.8. — *On a*

$$E_1^*(s)(\mathcal{G}^*) = \mathcal{G}^*.$$

Proof. — Par dualité, il suffit de démontrer l'équivalence, pour tout $f \in \mathcal{F}$,

$$f(\mathcal{G}) = \{0\} \Longleftrightarrow E_1(s)(f)(\mathcal{G}) = \{0\}. \tag{1}$$

En effet, on a

$$< E_1^*(s)(x,e_i^*)|(y,e_j) >=< (x,e_i^*)|E_1(s)(y,e_j) >, \tag{2}$$

et d'autre part, on a $(x,e_i^*) \in \mathcal{G} \Longleftrightarrow (x,e_i) \in \mathcal{G}$. Donc pour tout $(y,e_j) \notin \mathcal{G}$ et $(x,e_i^*) \in \mathcal{G}^*$, le produit 2 ci-dessus est nul, ce qui donne l'inclusion $E_1^*(s)(\mathcal{G}^*) \subseteq \mathcal{G}^*$. Et de même, pour tout $(y,e_j) \in \mathcal{G}$ et $(x,e_i^*) \notin \mathcal{G}^*$, le produit est nul, ce qui donne l'autre inclusion.

Pour démontrer l'équivalence 1, on remarque que l'image par $E_1(s)$ d'un élément (x, e_i) est un élément géométrique de $\mathcal{F}$ dont la représentation géométrique est une union de segments connectés les uns aux autres et reliant les points $M_s x$ et $M_s(x+e_i)$. De plus, cette union de segments intersecte le plan d'équation $V_\beta.X = 0$ en au plus un point puisque la suite de segment croit strictement pour la norme 1. Or, les inégalités $V_\beta.x < 0$ et $V_\beta.(x + e_i) \geq 0$ sont préservées par la matrice M_s puisque V_β est un vecteur propre, donc l'union de segments correspondant à $E_1(s)(x, e_i)$ intersecte le plan si et seulement si le segment correspondant à (x, e_i) intersecte le plan. Cela démontre bien l'équivalence 1. □

9.3.1. Itération de la substitution duale $E_1^*(s)$. — Soient

$$\mathcal{U} = \sum_{a \in A} (-e_a, e_a^*)$$

et

$$\mathcal{U}' = \sum_{a \in A} (0, e_a^*).$$

On a $\mathcal{U} \in \mathcal{G}^*$, et on peut se demander ce que devient l'itéré de $\mathcal{U}$ par $E_1^*(s)$.

Proposition 9.9. — *Pour tout $n \in \mathbb{N}$, on a que $E_1^*(s)^n(\mathcal{U})$ et $E_1^*(s)^n(\mathcal{U}')$ sont géométriques.*

Proof. — Il suffit de le démontrer pour $n = 1$, puisque l'on a $E_1^*(s)^n = E_1^*(s^n)$. On veut montrer que l'image par $E_1^*(s)$ d'une facette (x, e_i^*) donne des facettes toutes différentes. Par dualité, cela revient à démontrer que l'image par $E_1(s)$ d'un segment (y, e_j) est une union de segments tous distincts, ce qui est bien le cas. □

9.3.2. Lien avec la fractale de Rauzy. — On a l'égalité

$$\begin{aligned} E_1^*(s)^n(x, e_i^*) &= \sum_{j \xrightarrow{t_1} \dots \xrightarrow{t_n} i} (M^{-n}x + \sum_{k=1}^{n} M^{-k} t_k, e_j^*) \\ &= \sum_{j \xrightarrow{t_1} \dots \xrightarrow{t_n} i} (M^{-n}(x + \sum_{k=1}^{n} M^{n-k} t_k), e_j^*) \end{aligned}$$

Or, si $u \in A^{\mathbb{N}}$ est un point fixe commençant par la lettre a, on a pour tout $b \in A$,

$$D_{u,b} = \left\{ \sum_{k=1}^{n} M^{n-k} t_k \;\middle|\; n \in \mathbb{N},\ b \xrightarrow{t_1} \dots \xrightarrow{t_n} a \in \mathcal{A}^{PA} \right\}$$

On voit donc que la représentation géométrique de $E_1^*(s)^n(x, e_i^*)$ se rapproche d'une partie d'un plan discret dont le renormalisé par M_s^n converge vers la fractale de Rauzy dans le plan orthogonal au vecteur propre V_β.

On obtient un échange de morceaux en passant de $E_1^*(s)(\mathcal{U})$ à $E_1^*(s)(\mathcal{U}')$, et cet échange se quotiente en une translation du tore. La conjecture Pisot revient à dire que l'on a encore cet échange de morceaux et cette translation du tore quand on passe à la limite.

FIGURE 32. $E_1^*(s)^4(\mathcal{U}')$ pour s la substitution de Tribonnacci

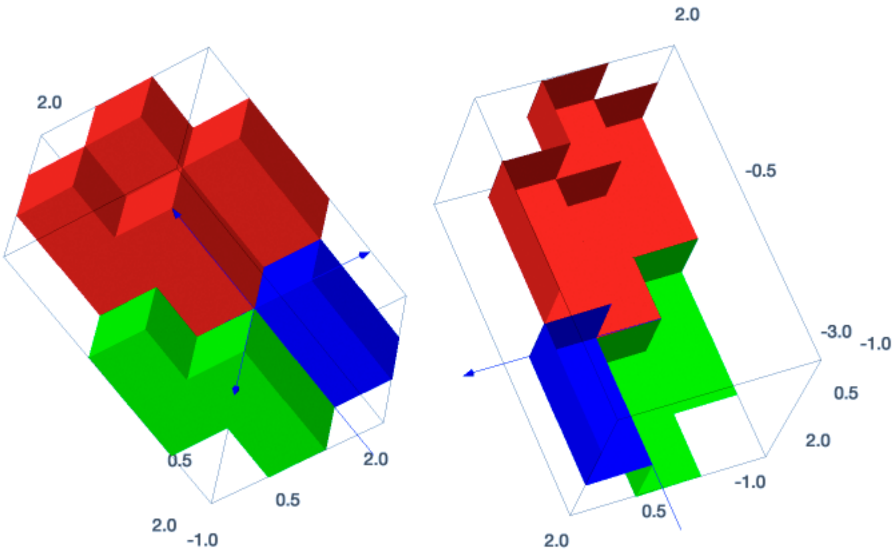


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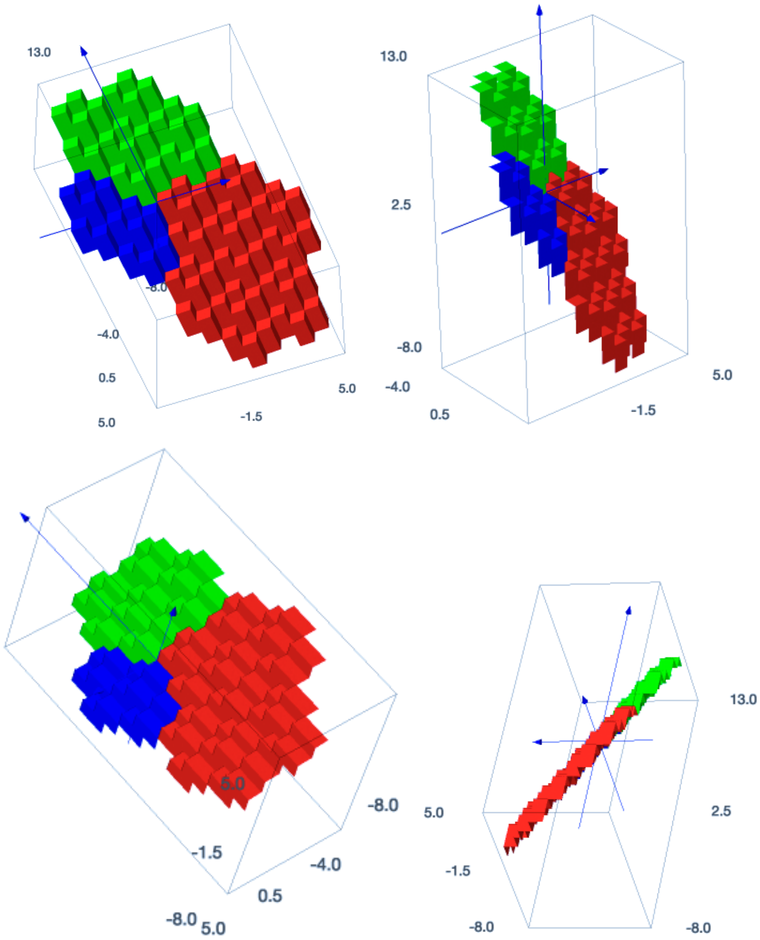
FIGURE 33. $E_1^*(s)^8(\mathcal{U}')$ pour s la substitution de Tribonnacci



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