RAUZY FRACTALS CAN HAVE ANY SHAPE

by

Paul MERCAT

Abstract. — We present a convenient and powerfull tool to describe sets like Rauzy fractals and quasicrystals associated to substitutions. Using this tool, we give necessary and sufficient conditions for a quasicrystal to come from substitutions. The proof is constructive and allows us, among others, to construct substitutions having Rauzy fractals of any shape.

Contents

1. Introduction 1
2. Definitions and main results 3
3. Construction of a domain exchange 12
4. Construction of a substitution 14
5. Examples 18
6. Discussion 27
References 29

1. Introduction

Given a substitution, let's say for example the Fibonnacci substitution :

$$\begin{cases} a & \mapsto & ab \\ b & \mapsto & a \end{cases}$$

we can look at a fixed point. Up to replace the substitution by a power, such a fixed point always exists. Here, by iterating the letter a, we get the following fixed point :

$$abaababaabaababaababaabaababaabaababaaba...$$

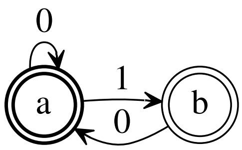
This infinite sequence has a lot of interesting properties in general. If we replace letters by intervals of convenient lengths, we get a self-similar tiling of $\mathbb{R}_+$. Convenient lengths are given by the non-negative coefficients of a Perron eigenvector of the incidence matrix M of the substitution. Here this matrix is $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \beta - 1 \end{pmatrix}$ is an eigenvector for the Perron eigenvalue β, which is the golden number. The set of points that appears at the boundaries of intervals in this self-similar tiling are elements of $\mathbb{Q}(\beta)$, and this set of points have very strong properties, because it is a non-periodic self-similar Meyer set. Here, we get the points

$$0,\ 1,\ \beta,\ \beta+1,\ \beta+2,\ 2\beta+1,\ 2\beta+2,\ 3\beta+1,\ 3\beta+2,\ 3\beta+3,\ 4\beta+2,\ 4\beta+3, ...$$

To get this set of points, we start from 0 and read the fixed point : we add 1 each time we read a letter a and we add $\beta - 1$ each time we read a letter b.

This set of points of $\mathbb{Q}(\beta)$ is described by the automaton of figure 1 below.

FIGURE 1. Automaton describing the quasicrystal associated to the Fibonnacci substitution



The initial state of this automaton is the one labelled by letter 'a', and the two states 'a' and 'b' are final states. The language recognized by this automaton, defined as the set of sequences of labels of paths from the initial state to a final state, is here exactly the set of words over the alphabet $\{0, 1\}$ that does not contain the subword 11.

The set of points of $\mathbb{Q}(\beta)$ described by such an automaton is the following :

$$\left\{ \sum_{k=0}^{n} a_k \beta^k \;\middle|\; n \in \mathbb{N},\ a_0 a_1 ... a_n \text{ labeling a path from state } a \right\}$$

Such a description of the quasicrystal will appear to be very usefull in the following.

The self-similar tiling of $\mathbb{R}_+$ is obtained by taking the embedding of $\mathbb{Q}(\beta)$ in $\mathbb{R}$ corresponding to the Perron eigenvalue of the incidence matrix M. If we look at the others embeddings, corresponding to eigenvalues less than 1, we get a bounded set whose adherence is called Rauzy fractal of the substitution. Here there is a unic such embedding, which is a real one, corresponding to the root of $x^2 - x - 1$ between -1 and 1. The Rauzy fractal is here the interval $[-1, \varphi]$ of $\mathbb{R}$ where φ is the golden number (i.e. greatest root of $x^2 - x - 1$).

Here is a result from **[MPP]** that gives exactly which intervalle is a Rauzy fractal.

Theorem 1.1 **(MPP)**. — *An intervalle* $[a, b]$ *is the Rauzy fractal of a substitution whose Perron number is a unit Pisot quadratic number if and only if we have*

— *it contains* 0,
— $a < b$,
— *and* $a, b \in \mathbb{Q}(\beta)$.

In this article we generalize this result by replacing intervals by g-β-sets (see below for a definition) which permits to describe conveniently every Rauzy fractal and quasicrystal arising from substitutions.

TODO : Tell history about what is known, cite the numerous people involved in such topic.

1.1. Organization of the paper. — This paper is organized as follow. The section 2 gives the main definitions and main results. We define Rauzy fractals, quasicrystals, cut-and-project sets, and we introduce g-β-sets which is the main tool of this paper. The section 3 explains how to construct a domain exchange conjugated to the shift on a given quasicrystal. This will be an important step to construct a substitution. The section 4 gives a proof of the main result of this paper, that is the construction of a substitution describing a given quasicrystal. In section 5, we give various examples obtained thanks to my implementation in Sage of the tools described in this article, and we give an explicit construction of Rauzy fractal approximating a shape. The last section is mainly devoted to two questions. One is about the complexity of substitutions coming from quasicrystals, and the other is about the definition of Rauzy fractals.

I thanks Pierre Arnoux, Arnaud Hilion and Pascal Hubert, for interesting discussions that helped me.

2. Definitions and main results

2.1. Rauzy fractals. — Let s be a substitution, or in others words, a word morphism over a finite alphabet $A = \{a_1, ..., a_n\}$. Up to replace s by a power, we can assume that s has a fixed point ω.

We defined the **broken line** associated to ω as the subset of $\mathbb{Z}^n$ defined by

$$\left\{ \begin{pmatrix} \text{number of occurences of } a_1 \text{ in } \omega_k \\ \text{number of occurences of } a_2 \text{ in } \omega_k \\ \vdots \\ \text{number of occurences of } a_n \text{ in } \omega_k \end{pmatrix} \in \mathbb{Z}^n \;\middle|\; k \in \mathbb{N} \right\}$$

where ω_k is the prefix of length k of the infinite word ω.

This broken line is very interesting since it is a geometrical object which completely encode the substitution and is stable by multiplication by the incidence matrix.

Definition 2.1. — First defined by Gérard Rauzy in 1982, *the* ***Rauzy fractal*** *(also called the* ***central tile****) is the closure of the projection of the broken line to the contracting space along the expanding line.*

With this definition, we can state one of our main result :

Theorem 2.2. — *Let $n \in \mathbb{N}_{\geq 1}$ and $P \subseteq \mathbb{R}^n$. The set P is arbitrarly approximated by Rauzy fractals, for the Hausdorff distance, if and only if P is bounded and $0 \in \overline{P}$. Moreover the proof is constructive.*

But to have a precise definition of what is a Rauzy fractal, we need to define what is the expanding line and what is the contracting space. This is particulary important for non-irreducible substitution (i.e. for substitutions on n letters whose Perron number has degree $d < n$), especially as substitutions given by this theorem are not irreducible in general. The "expanding line" has dimension 1 for Pisot numbers, but it can have greater dimension for other Perron numbers. Let's define it roughly for any Perron number.

Let M_s be the incidence matrix of the substitution s. By definition the coefficient (i, j) of this matrix is the number of occurrences of the letter a_j in the word $s(a_i)$ (maybe this is the transpose of the usual definition). By Perron-Frobenius theorem, there exists an eigenvector $v \in (\mathbb{R}_+)^n$, unic if the matrix is irreducible, for an eigenvalue λ which is the spectral radius of M_s, and moreover we can assume that $v \in (\mathbb{Q}(\lambda))^n$.

We can define a sort of broken line in $\mathbb{Q}(\lambda)$, by the following.

$$Q_\omega = \left\{ \sum_{k=1}^{N} v_{a_k} \;\middle|\; N \in \mathbb{N},\ a_1 a_2 ... a_N \text{ prefix of } \omega \text{ of length } N \right\}.$$

This is a projection of the broken line on $\mathbb{Q}(\lambda)$. This set is invariant by multiplication by the Perron eigenvalue λ and gives a self-similar tiling of $\mathbb{R}_+$. The definition of Q_ω depends of the choice of an eigenvector. We prefer to choose an eigenvector whose coefficients belongs to the integer ring $\mathcal{O}_\lambda$, in order to have $Q_\omega \subset \mathcal{O}_\lambda$.

For $\mathbb{Q}(\lambda)$, there are natural contracting and expanding spaces for the multiplication by λ. Indeed, consider the bigest sets P_+ and P_- of places (i.e. equivalence class of absolute values) of $\mathbb{Q}(\lambda)$ such that

$$\forall v \in P_+,\ |\lambda|_v > 1 \quad \text{and} \quad \forall v \in P_-,\ |\lambda|_v < 1.$$

If λ is an algebraic unit, the set P_+ corresponds to roots of the minimal polynomial of λ greater than 1 in absolute value, counting two conjugate complexes only once, and it is the same for P_- with the roots of modulus less than 1.

For each place v, we define a space E_v as the completion of $\mathbb{Q}(\lambda)$ for the absolute value v. If v is a real place (i.e. corresponding to a real root or the minimal polynomial of λ), then $E_v = \mathbb{R}$. If v is a complex place (i.e. corresponding to two conjugated complex roots or the minimal polynomial of λ), then $E_v = \mathbb{C}$. Otherwise, E_v is a

p-adic space, which is a finite extension of the p-adic field $\mathbb{Q}_p$ (which is the completion of $\mathbb{Q}$ for the p-adic absolute value).

Remark 2.3. — *When the substitution is irreducible and when the number λ is an algebraic unit, the spaces E_v can be seen as the eigenspaces of the matrix M_s.*

Now, we can define the expanding space

$$E_\lambda^+ := \prod_{v \in P_+} E_v,$$

and the contracting one

$$E_\lambda^- := \prod_{v \in P_-} E_v.$$

Let's take σ_+ and σ_- some embeddings of $\mathbb{Q}(\lambda)$ into the spaces E_+ and E_- respectively. We will also denote by σ_β the maximal real embedding when β is a Perron number.

Definition 2.4. — *We call **Rauzy fractal** of the substitution s the adherence of $\sigma_-(Q_\omega)$ in E_λ^-.*

Remark 2.5. — *This is the definition implemented in Sage by Timo Jolivet for unit Pisot numbers.*

Now that we have a precise and general definition of Rauzy fractal, we can give a more general version of the theorem 2.2.

Theorem 2.6. — *Let β be a Pisot number (not necessarly unit), and let $P \subseteq E_\beta^-$. The set P is arbitrarily approximated by Rauzy fractals, for the Hausdorff distance, if and only if P is bounded and $0 \in \overline{P}$.*

This theorem is proven in subsection 5.6.

Remark 2.7. — *I have developped tools in sage that permits to compute explicitly substitutions given by this theorem. See subsection 5.6 for more details.*

2.2. Quasicrystals. — In this subsection, we define what we call quasicrystal, and show that broken lines in $\mathbb{Q}(\beta)$ arising from substitutions whose Perron number is Pisot, are quasicrystals.

Definitions are compendious, but the reader will easily find more details in the literature.

Definition 2.8. — *A **Delone set** is a set uniformly discrete and relatively dense.*

Definition 2.9. — *A **quasicrystal** (or **Meyer set**) Q is a Delone set such that $Q - Q$ is also a Delone set.*

Remark 2.10. — *There are a lot of equivalent characterizations of Meyer sets.*

Proposition 2.11. — *If ω is a bi-infinite fixed point of a substitution whose Perron number is Pisot (eventually non-unit), then $\sigma_+(Q_\omega)$ is a quasicrystal of $\mathbb{R}$.*

Proof. — It is easy to see that $\sigma_+(Q_\omega)$ is a Delone set, because there are a finite number of intervalles between two consecutive points. Let β be the Pisot number associated to the substitution whose ω is a fixed point. Up to rescaling, we can assume that $Q_\omega \subseteq \mathcal{O}_\beta$. The difference $\sigma_+(Q_\omega) - \sigma_+(Q_\omega)$ is uniformly discrete, because $\sigma_-(Q_\omega) - \sigma_-(Q_\omega) = \sigma_-(Q_\omega - Q_\omega)$ is bounded (it is included in $R - R$ where R is the Rauzy fractal of the substitution), and because $\mathcal{O}_\beta$ is uniformly discrete in $E_\beta^- \times E_\beta^+$ and contains $Q_\omega - Q_\omega$. We get the relative denseness of $\sigma_+(Q_\omega - Q_\omega)$ using the inclusion $0 \in Q_\omega \subseteq Q_\omega - Q_\omega$. □

Remark 2.12. — *We have the same result for infinite fixed point. If ω is a infinite fixed point of a substitution whose Perron number is Pisot (eventually non-unit), then $\sigma_+(Q_\omega)$ is a quasicrystal of $\mathbb{R}_+$.*

2.3. g-β-sets. —

2.3.1. Definitions and main theorem. — In this paragraph we define g-β-sets and we state our main result.

Definition 2.13. — *A **g-β-set**, for an algebraic number β, is a subset of $\mathbb{Q}(\beta)$ of the form*

$$Q_{\beta,L} := \left\{ \sum_{i=0}^n a_i \beta^i \;\middle|\; n \in \mathbb{N},\ a_0 a_1 ... a_n \in L \right\}.$$

where L is a regular language over a finite alphabet $\Sigma \subset \mathbb{Q}(\beta)$.

In the following, we will work with a fixed algebraic integer β. Thus, we will denote the corresponding g-β-set by Q_L.

Remark 2.14. — *The name "g-β-sets" comes from the fact that it describes finite β-expansions of elements of $\mathbb{Q}(\beta)$, and the "g" stands for graph, as in gIFS (graph Iterated Function System).*

The following proposition is easy and shows that g-β-sets are usefull to describe quasicrystals associated to substitutions.

Proposition 2.15. — *Let ω be a fixed point of a substitution associated to a Perron number β, then the set Q_ω is a g-β-set.*

The following theorem is the main result of this paper. It gives a converse to the proposition 2.15.

Theorem 2.16. — *Let β be a Pisot number, eventually non-unit, and let Q be a g-β-set. We have the equality $Q = Q_\omega$, for a fixed point ω of a substitution whose Perron number is β, if and only the following three conditions are satisfied:*

— *Q is a quasicrystal (i.e. a Meyer set, see 2.2 for a definition),*

— $\beta Q \subseteq Q$,
— $0 \in Q$.
Moreover the proof is constructive.

Theorems 2.2 and 2.6 are easy corollaries of this theorem 2.16.

Remark 2.17. — *The theorem 2.16 can easily be extended to any Perron number without conjugate of modulus one, but in this case we have to replace the quasicrystal hypothesis by the finite condition (see subsection 3.1 for more details).*

2.3.2. Properties. — In this paragraph we will see that g-β-sets have a lot of nice and powerfull properties.

There are several languages that gives the same g-β-set, but there is a canonical one if we choose an alphabet, by the following proposition.

Definition 2.18. — *If Q is a g-β-set, and if $\Sigma \subset \mathbb{Q}(\beta)$ is a finite alphabet, then we define the language*

$$L_Q^\Sigma = \left\{ a_0 a_1 ... a_n \in \Sigma^* \;\middle|\; n \in \mathbb{N},\ \textstyle\sum_{i=0}^n a_i \beta^i \in Q \right\}.$$

We have obviously that $Q = Q_{L_Q^\Sigma}$ as soom as $Q \subseteq Q_{\Sigma^*}$. The following proposition tell us that moreover the language L_Q^Σ is regular.

Proposition 2.19. — *If Q is a g-β-set, for an algebraic number β without conjugate of modulus one, then L_Q^Σ is a regular language. Moreover, the language L_Q^Σ is computable from any language L such that $Q = Q_L$.*

Proof. — For a language $L \subseteq \Sigma^*$, let us denote

$$Z(L) := \left\{ a_0 ... a_n \in \Sigma'^* \;\middle|\; n \in \mathbb{N}, \exists k \in \mathbb{Z}, a_0 ... a_n 0^k \in L \right\}$$

where $\Sigma' = \Sigma \cup \{0\}$. When $k < 0$, the notation $a_0 ... a_n 0^k$ means that the word 0^k is a suffix of the word $a_0 ... a_n$ (otherwise it is not defined), and it means that it is the word such that $(a_0 ... a_n 0^k) 0^{-k} = a_0 ... a_n$.

It is not difficult to see that the language $Z(L)$ is regular if L is regular. Let L be a regular language over an alphabet Σ' such that $Q = Q_L$. Then we have

$$L_Q^\Sigma = Z(p_1(L^{rel}_{\Sigma' \cup \Sigma \cup \{0\}} \cap (\Sigma \cup \{0\})^* \times Z(L))) \cap \Sigma^*,$$

where p_1 is the projection on the first coordinate, and L_Σ^{rel} is defined by

$$L_\Sigma^{rel} := \left\{ (u_0 ... u_n, v_0 ... v_n) \in (\Sigma \times \Sigma)^* \;\middle|\; n \in \mathbb{N}, \textstyle\sum_{i=0}^n (u_i - v_i) \beta^i = 0 \right\}.$$

The language $L^{rel}_{\Sigma' \cup \Sigma \cup \{0\}}$ is regular thanks to theorem 1.1 in [**Me**], hence L_Q^Σ is regular. $\square$

Remark 2.20. — *This last proposition permits to change the alphabet used to represent a given g-β-set.*

In the following, we will denote by $\mathcal{O}_\beta$ the integer ring of the number field $\mathbb{Q}(\beta)$.

Definition 2.21. — *For a g-β-set Q, and for a lattice $\mathcal{O} \subset E^- \times E^+$, we call* ***adherence*** *in $\mathcal{O}$ the set*

$$\overline{Q}^{\mathcal{O}} := \left\{ x \in \mathcal{O} \;\middle|\; \sigma_-(x) \in \overline{\sigma_-(Q)} \right\}.$$

Remark 2.22. — $\sigma_+(\overline{Q_L}^{\mathcal{O}})$ *is the cut-and-project set obtained with the window* $\overline{\sigma_-(Q_L)}$ *for the lattice $\mathcal{O}$. See subsection 2.4 below.*

Remark 2.23. — *The adherence in $\mathcal{O}$ is the adherence for the topology on $\mathcal{O}$ induced by the one on E_β^-. In others words, the set of open sets for this topology is* $\left\{ Q_\Omega \;\middle|\; \Omega \text{ open set of } E_\beta^- \right\}$ *where* $Q_\Omega = \left\{ t \in \mathcal{O} \;\middle|\; \sigma_-(t) \in \Omega \right\}$.

Properties 2.24. — *For a fixed algebraic number β with no conjugate of modulus one, the set of g-β-sets is stable by*

1. *intersection,*
2. *union,*
3. *complementary (in another g-β-set),*
4. *Minkowski sum (i.e. the sum of two g-β-sets is a g-β-set),*
5. *multiplication by an element of $\mathbb{Q}(\beta)$,*
6. *translation by an element of $\mathbb{Q}(\beta)$,*
7. *adherence, interior, boundary, for the topology of $\mathcal{O}_\beta$ induced by E_-.*

And finite subsets of $\mathbb{Q}(\beta)$ are g-β-sets. Moreover, everything is computable, and non-emptiness, inclusion and equality are decidable.

Remark 2.25. — *If $\beta = \lambda^n$ for some integer $n \in \mathbb{N}_{\geq 1}$, then a g-$\beta$-set is also a g-$\lambda^k$-set for any $k \in \mathbb{N}_{k \geq 1}$.*

Question . — *Most of these properties are also true for numbers β with conjugates of modulus one. Are they all true ?*

Proof of properties 2.24. — Let Q and Q' be two g-β-sets, and let respectively L and L' be regular languages coming from proposition 2.19 for some alphabets. Up to take the union of the two alphabets, we can assume that we have $L = L_Q^\Sigma$ and $L' = L_{Q'}^\Sigma$ for the same alphabet Σ. Then we have

$$Q_L \cap Q_{L'} = Q_{L \cap L'}, \quad Q_L \cup Q_{L'} = Q_{L \cup L'}, \text{ and } \quad Q_L \backslash Q_{L'} = Q_{L \backslash L'},$$

hence properties 1, 2 and 3 follows from properties of regular languages.

If L is a regular language over the alphabet Σ, and if L' is a regular language over the alphabet Σ', then, we have

$$Q_L + Q_{L'} = Q_{S(L \times L')}$$

where S is the word morphism defined by $S(s, s') = s + s', \forall (s, s') \in \Sigma \times \Sigma'$. This proves property 4.

The property 5 is obvious since we have $\lambda Q_L = Q_{\lambda L}$: the language λL is the same as the language L but with the alphabet multiplied by λ.

Let $t \in \mathbb{Q}(\beta)$ and let L be a regular language over the alphabet Σ, then let

$$L' = \bigcup_{a \in \Sigma} (a+t)(a^{-1}L),$$

where $a^{-1}L = \left\{ w \in \Sigma^* \;\middle|\; aw \in L \right\}$. The language L' is a regular language over the alphabet $\Sigma \cup (\Sigma + t)$ and we have $Q_L + t = Q_{L'}$, hence this proves property 6.

The proof of property 7 is a little bit harder. By following the ideas of the proof of the main theorem of [**Me**], we can prove the following theorem :

Theorem 2.26. — *Let β be a algebraic number without conjugate of modulus one, and let $\Sigma \subset \mathbb{Q}(\beta)$ be a finite alphabet. The following language is regular :*

$$L_\Sigma^{rel\infty} := \left\{ (u,v) \in (\Sigma \times \Sigma)^* \;\middle|\; \exists (u',v') \in (\Sigma \times \Sigma)^{\mathbb{N}} \textit{ whose } (u,v) \textit{ is a prefix and } \sum_{i=0}^{+\infty} (u'_i - v'_i)\beta^i = 0 \textit{ in } E_\beta^- \right\}.$$

Let Q be a g-β-set for an alphabet Σ'. Using this theorem, we can define the following regular language

$$L := p_1(L_{\Sigma' \cup \Sigma}^{rel\infty} \cap \Sigma^* \times Z(L_Q^{\Sigma'})).$$

where $\Sigma \subset \mathcal{O}$ is an alphabet containing 0, and such that $\overline{Q}^{\mathcal{O}} \subseteq Q_{\Sigma^*}$ (such an alphabet Σ always exists). Then we define the language

$$L' := \left\{ u \in \Sigma^* \;\middle|\; \text{there exists an infinite number of } k \in \mathbb{N} \text{ such that } u0^k \in L \right\}.$$

We can check that L' is a regular language, and that we have $\overline{Q}^{Q_{\Sigma^*}} = \overline{Q}^{\mathcal{O}} = Q_{L''}$. Using properties 1 and 3 we also have the interior and boundary, so we have proven property 7.

It is easy to see that finite subsets of $\mathbb{Q}(\beta)$ are g-β-sets. All this proof is constructive, since proofs of theorem 2.26 and theorem 1.1 in [**Me**] are constructive.

$\square$

proof of remark 2.25. — Let $Q_{\beta,L}$ be a g-β-set for a regular language L over an alphabet Σ. Then we have $Q_{\beta,L} = Q_{\beta^n,L'}$ where L' is the regular language over the alphabet Σ^n defined by

$$L' := \left\{ a_0 ... a_k \in (\Sigma^n)^* \;\middle|\; a_0...a_k \in L \text{ seen as a word of length } n(k+1) \text{ over the alphabet } \Sigma \right\}.$$

Conversely, if $Q_{\beta^n,L}$ is a g-β-set for L a regular language over an alphabet Σ, then we have $Q_{\beta^n,L} = Q_{\beta,L'}$ where L' is the regular language over the alphabet $\Sigma \cup \{0\}$ defined by

$$L' := \left\{ 0^{n-1}a_0 0^{n-1}a_1 ... 0^{n-1}a_k \in (\Sigma \cup \{0\})^* \;\middle|\; a_0 a_1 ... a_n \in L \right\}.$$

This ends the proof of remarks 2.25. $\square$

The fact that g-β-sets come naturally to describe quasicrystals arising from substitutions and has a lot of nice properties show that it is an interesting fundamental object.

Remark 2.27. — *We see from theses properties that we can construct g-β-sets with any shape in the contracting space E^-. This allows us to construct Rauzy fractals of any shape. See subsection 5.6.*

Remark 2.28. — *Most of these operations on g-β-sets has been implemented efficiently in Sage, but there is still a lot of work to make this code more robust, more documented and more usable by others people than me.*

With these properties of g-β-sets and with the theorem 2.16, we easily get some stability for Rauzy fractals.

Corollary 2.29. — *Let R_1 and R_2 be two Rauzy fractals of substitutions whose Perron numbers are power of the same Pisot number β (eventually non-unit). Then we have*
- *If $0 \in \mathring{R}_1$ then $R_1 \cup \lambda R_2$ and $R_1 + \lambda R_2$ are Rauzy fractals for all $\lambda \in \mathbb{Q}(\beta)$,*
- *If $0 \in \mathring{R}_1$ and $0 \in \mathring{R}_2$ then $R_1 \cap R_2$ is a Rauzy fractal,*
- *If $t \in \mathring{R}_1 \cap \sigma_-(\mathbb{Q}(\beta))$ then $R_1 - t$ is a Rauzy fractal.*

More generally, the sets $\overline{\mathring{R}_1}$, $R_1 \cap R_2$, $R_1 \cup R_2$, $\overline{R_1 \backslash R_2}$, $R_1 + \lambda$ and $R_1 + \lambda R_2$, for $\lambda \in \sigma_-(\mathbb{Q}(\beta))$, are Rauzy fractals if they have non-empty interior, contain zero, and are invariant by multiplication by β^n, for some $n \in \mathbb{N}_{\geq 1}$, in E_β^- (β acting diagonally, by $\sigma_v(\beta)$ in E_v). Moreover, everything is computable.

2.4. Cut-and-project sequences. — Substitutions give examples of self-similar quasicrystals. Another way to construct sets with comparable properties is the cut-and-project method. This consist on taking a lattice and keeping only elements whose projection belongs to a choosen windows. Here the lattice we take is a ideal I of $\mathcal{O}_\beta$ for a unit Pisot number β.

Definition 2.30. — *Let $\mathcal{O}_\beta$ be the integer ring of a unit Pisot number β, and I be an ideal of $\mathcal{O}_\beta$. The* ***cut-and-project sequence*** *of window $\Omega \subset E^-$ for the ideal I is the set $\sigma_+(Q_{I,\Omega}) \subseteq E^+ = \mathbb{R}$ where*

$$Q_{I,\Omega} := \left\{ x \in I \;\middle|\; \sigma_-(x) \in \Omega \right\}.$$

When β and I are fixed, we will denote $Q_{I,\Omega} = Q_\Omega$.

Definition 2.31. — *A* ***model set*** *is a cut-and-project sequence for a window of non-empty interior.*

Remark 2.32. — *Cut-and-project sets can be defined in a much more general setting. We restrict to ideals of $\mathcal{O}_\beta$ because every model set arising from a substitution whose Perron number is a unit Pisot number, has this form.*

Remark 2.33. — *For β a unit Pisot number, a g-β-set is a model set if and only if it has non-empty interior in I, for the topology of I induced by E_β^-, where I is the group generated by the g-β-set.*

Here is a link between Meyer sets and model sets.

Proposition 2.34. — *A model set is a Meyer set.*

Here is a sort of converse (but the converse is not true, see example 5.7).

Proposition 2.35. — *Let $Q \subseteq \mathcal{O}_\beta$. If $\sigma_+(Q)$ is a Meyer set, it is included in a model set. And there exists a finite set $T \in \mathcal{O}_\beta$ such that $\sigma_+(Q+T)$ is a model set.*

Hence, quasicrystals are really closed to model sets.

This permits to show that Rauzy fractals as non-empty interior, thanks to the following general lemma.

Lemma 2.36. — *A finite union of closed sets of empty interior has empty interior.*

Corollary 2.37. — *A Rauzy fractal of a substitution whose Perron number is unit Pisot, always has non-empty interior.*

We can naturally ask the following conjecture. We say that a matrix is **irreducible** if there exists a positive power of the matrix whose coefficients are all positives.

Conjecture 2.38. — *Every quasicrystal coming from a substitution whose Perron number is a unit Pisot number and whose incidence matrix is irreducible, is a model set.*

See example 5.7 for a counter example if we remove the hypothesis about the incidence matrix.

It is decidable to test this conjecture :

Proposition 2.39. — *It is decidable to test if a β-invariant Meyer g-β-set is a model set for β unit Pisot number.*

Proof. — If Q is a Meyer set and is a g-β-set, we can compute the finite set of translations $t_1, t_2, ..., t_n$ of the shift. Then we can decide if the g-β-set has non-empty interior in the ideal $I = \sum_{i=1}^n t_i \mathcal{O}_\beta$ using properties 2.24, and this is equivalent to decide if it is a model set. □

With the stability by adherence of g-β-sets and with the theorem 2.16, we easily get the following corollary.

Corollary 2.40. — *Given a substitution whose Perron number is a Pisot number, there exists another substitution with the same Rauzy fractal and satisfying the conjecture 2.38 (i.e. it describes a quasicrystal which is a model set).*

The following result is easy and shows that the theorem 1.1 is a consequence of my theorem 2.16.

Proposition 2.41. — *Let $[a, b]$ be an interval of $\mathbb{R}$ and let β be a quadratic Pisot number. Then, $Q_{[a,b]}$ is a g-β-set if and only if $a, b \in \mathbb{Q}(\beta)$.*

Idem for $Q_{(a,b)}$, $Q_{(a,b]}$ and $Q_{[a,b)}$.

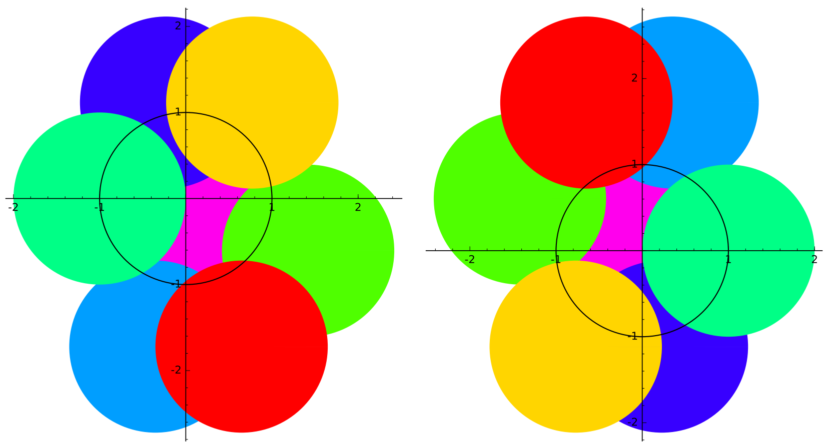
3. Construction of a domain exchange

The first step, to construct a substitution from a quasicrystal, is to construct a domain exchange which describe the shift on the quasicrystal. This is done by the following proposition. For the moment, we don't need to know that it is a g-β-sets.

This construction is an easy generalization to any quasicrystal of the fact that we can color the Rauzy fractal in order to have a domain exchange.

Proposition 3.1. — *Let β be a Pisot number (eventually non unit), and let $Q \subseteq \mathbb{Q}(\beta)$ such that $\sigma_+(Q)$ is a quasicrystal of $\mathbb{R}$ or $\mathbb{R}^+$. Then there exists a domain exchange with a finite number of pieces such that the union of the pieces is Q. Moreover, this domain exchange is conjugated to the shift on $\sigma_+(Q)$.*

Figure 2. Construction of a domain exchange in the unit disk, for the integer ring $\mathcal{O}_\beta$, where β is the Tribonnacci number.



$-2\beta^2 + 2\beta$, $\beta^2 - \beta - 1$, $\beta - 1$, 1, $-\beta^2 + 2\beta + 1$, $\beta^2 - \beta$, β

Proof. — The fact that $\sigma_+(Q)$ is a quasicrystal tells us that there exists a finite number of translations in the shift, i.e. to go from one point of $\sigma_+(Q) \subseteq E_+ = \mathbb{R}$ to the next one.

The shift on the quasicrystal can be described by a greedy algorithm that consist at taking the smallest translation possible (in $E^+ = \mathbb{R}$) to go from one point to the next one. If not possible test the second smallest translation, etc...

If we look what gives this algorithm in Q, we get the domain exchange. Indeed, the set of all points of Q that can be translated by some translation t and stay in Q is simply $Q \cap (Q - t)$. Thus, we get the pieces of the domain exchange by the following.

If $t_0, ...t_n \in \mathbb{Q}(\beta)$ are the possible translations of the shift, with

$$0 < \sigma_+(t0) < \sigma_+(t1) < ... < \sigma_+(t_n),$$

then the $(n+1)$ pieces of the domain exchanges are

$$Q \cap (Q - t_0), \quad Q \cap (Q - t_1) \backslash (Q - t_0), \quad Q \cap (Q - t_2) \backslash \left((Q - t_0) \cup (Q - t_1)\right), ...$$

□

The picture 2 shows the construction of a domain exchange on the model set $Q_{\mathcal{O}_\beta, \Omega}$ where Ω is the unit disk in $\mathbb{C}$, and β is the Tribonnacci number (i.e. root of $x^3 - x^2 - x - 1$).

Remark 3.2. — *The domain exchange described in the figure 2 for the open unit disk gives exactly the list of Pisot numbers (including non-unit ones) of degree* 3 *in* $\mathbb{Q}(\beta)$*, where* β *is the Tribonnacci number (i.e. greatest root of* $x^3 - x^2 - x - 1$*). Indeed if* x *is a Pisot number of degree three in* $\mathbb{Q}(\beta)$*, the next Pisot number is obtained by looking in which piece is the conjugate* $\overline{x}$*, and adding the corresponding translation to* x*.*

Remark 3.3. — *If moreover* Q *is a g-*β*-set, then the pieces of the domain exchange conjugated to the shift are also g-*β*-sets, and are computable. This is a consequence of the proof of proposition 3.1 and properties 2.24 of g-*β*-sets.*

3.1. Finite condition. — The following definition gives a necessary and sufficient condition for the existence of an finite domain exchange conjugated to the shift on a given quasicrystal. This permits to extend proposition 3.1 to any Perron number, and to extend theorem 2.16 to any Perron number without conjugate of modulus one.

Definition 3.4. — *For* β *a Perron number (eventually non-unit), we say that a set* $Q \subset \mathbb{Q}(\beta)$ *(respectively a set* $\Omega \subseteq E_\beta^-$*) satisfy the* ***finite condition*** *on* $\mathbb{R}$ *if* $\sigma_\beta(Q)$ *is discrete and if there exists a finite set of translations* $T \in \mathbb{Q}(\beta)$ *such that*

$$\sigma_\beta(T) \subset \mathbb{R}_+^* \text{ and } Q \subseteq Q + T \text{ (respectively } \Omega \subseteq \Omega + \sigma_-(T)\text{)}$$

In others words, the finite condition is equivalent to say that the set of differences of two consecutive points is finite.

The finite condition for $\mathbb{R}_+$ is the same, except that we replace the inclusion $Q \subseteq Q + T$ by $Q \subseteq \{0\} \cup (Q + T)$.

Proposition 3.5. — *Let β be a Perron number. There exists a domain exchange on $Q \subset \mathbb{Q}(\beta)$ (respectively on $\Omega \subset E_\beta^-$) conjugated to the shift on $\sigma_\beta(Q)$ (respectively on $\sigma_\beta(Q_\Omega)$) if and only if Q satisfy the finite condition.*

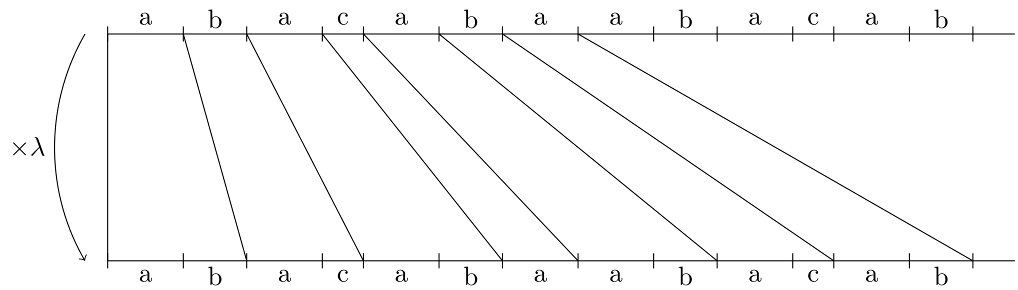
And here is the generalization of the main theorem.

Theorem 3.6. — *Let β be a Perron number, eventually non-unit, without conjugate of modulus one, and let Q be a g-β-set. We have the equality $Q = Q_\omega$ for a fixed point ω of a substitution whose Perron number is β if and only the following three conditions are satisfied:*
— *Q satisfy the finite condition,*
— *$\beta Q \subseteq Q$,*
— *$0 \in Q$.*
Moreover the proof is constructive.

4. Construction of a substitution

In this section, we give a proof of the theorems 2.16 and 3.6.

If we know that a quasicrystal $\sigma_+(Q)$ of $\mathbb{R}$ or $\mathbb{R}_+$ comes from the fixed point of a substitution for a Pisot number λ, it is not difficult to guess what is the substitution. Indeed, it is enough to take intervals between two consecutive points, multiply it by λ, and see how the result is covered by others intervals.



But we have to take care of the fact that one interval can have several substitutions rules, corresponding to the fact that several letters of a substitution can give intervals of same lengths.

If we look at what happens in the contracting space E^-, we have to do a sort of induction on λQ for the domain exchange on Q, and we have to iterate it up to stabilization. But it's not really an induction : we have to distinguish between different possible trajectories for points in λQ before they come back to λQ, otherwise the induction only give the same domain exchange on λQ than in Q.

Algorithm 1 Computing a substitution describing a given quasicrystal of $\mathbb{R}_+$

Require: Q (quasicrystal, or window in E_λ^- of the cut-and-project set, from which we want to compute a substitution)
Require: λ Pisot number such that $\lambda Q \subset Q$

$P \leftarrow \{$ set of pieces of the domain exchange $\}$ (see proof of proposition 3.1 to see how to compute them)
stop $\leftarrow$ false
while not stop **do**
  stop $\leftarrow$ true
  for $Q' \in P$ **do**
    $t \leftarrow 0$
    repeat
      if $\lambda Q' + t$ is not a subset of an element of P **then**
        replace in P the element Q' by the non-empty ones $Q' \cap \lambda^{-1}(Q'' - t)$ for $Q'' \in P$.
        stop $\leftarrow$ false
        leave the for loop
      else
        $t \leftarrow t + t_0$ where t_0 is the translation of the domain exchange for $\lambda Q' + t$.
      end if
    until not $\lambda Q' + t \subset \lambda Q$
  end for
end while
for $Q' \in P$ **do**
  The substitution rule for letter Q' is given by the orbit, under the domain exchange, of $\lambda Q'$ before coming back to λQ (i.e. the rule is given by the successive elements of P).
end for

The algorithm 1 compute the smallest substitution describing a given quasicrystal Q for a Pisot number λ such that $\lambda Q \subset Q$.

Remark 4.1. — *For λ Pisot such that $\lambda Q \subset Q$, if there exists a domain exchange with a finite number of pieces, this algorithm 1 terminates if and only if Q is g-λ-set.*

Remark 4.2. — *A more efficient version of this algorithm has been implemented in Sage.*

proof of theorems 2.16 and 3.6. — We will assume that Q is a quasicrystal of $\mathbb{R}_+$. It is not difficult to adapt the proof to a quasicrystal of $\mathbb{R}$.

Let β be a Perron number, eventually non-unit, without conjugate of modulus one. Let's define a ring $\mathcal{R}$, invariant by multiplication by β and by β^{-1}, by the following

$$\mathcal{R} = \left\{ t \in \mathbb{Q}(\beta) \;\middle|\; \forall \text{ place } v \notin P_+ \cup P_-,\ |t|_v \leq 1 \right\}.$$

Then, we have

Lemma 4.3. — *The ring $\mathcal{R}$ is discrete in $E_- \times E_+$.*

See [**Me**] for more details.

In the case where β is an unit number, the integer ring $\mathcal{O}_\beta$ is also stable by multiplication by β and by β^{-1} and is discrete in $E_- \times E_+$, hence we can take $\mathcal{R} = \mathcal{O}_\beta$ in the following.

Let L be a regular language over the alphabet Σ, such that $Q = Q_L$. Up to rescaling, we assume that the alphabet Σ is included in the ring $\mathcal{R}$. Therefore we have $Q \subset \mathcal{R}$.

The hypothesis that $\sigma_+(Q)$ is a quasicrystal (i.e. a Meyer set) or that it satisfy the finite condition, permits to obtain a domain exchange with a finite number of pieces by proposition 3.1 or proposition 3.5. Let's call it f_Q. In other words, f_Q is the shift on the quasicrystal Q. The following lemma tells that there is a finite returning time in βQ for this domain exchange f_Q.

Lemma 4.4. — *There exists an integer $N \in \mathbb{N}$, such that for all $x \in \beta Q$, there exists an integer $0 < n \leq N$ such that $(f_Q)^n(x) \in \beta Q$.*

Indeed, if T_0 is the set of translations of the domain exchange, we can take

$$N = \left\lfloor \frac{\max\left\{\sigma_\beta(\beta t) \;\middle|\; t \in T_0\right\}}{\min\left\{\sigma_\beta(t) \;\middle|\; t \in T_0\right\}} \right\rfloor.$$

Now let's define T be the smallest set containing

$$\left\{\textstyle\sum_{k=0}^n t_i \;\middle|\; (t_i)_{i=0}^n \in T_0^{n+1},\ 0 \leq n < N\right\} \cap (Q - Q)$$

and such that

$$(\beta^{-1}T - \Sigma) \cap (Q - Q) \subseteq T.$$

Lemma 4.5. — *T is finite.*

Indeed, $\sigma_-(T) \subset \sigma_-(Q) - \sigma_-(Q)$ is bounded, and it is easily seen that $\sigma_+(T)$ is also bounded, because the map $T \mapsto \beta^{-1}(T - \Sigma)$ is contracting in E_β^+. The discreteness of $T \subset \mathcal{R}$ in $E_- \times E_+$ gives the finiteness.

The fact that L is a regular language tells us that there exists a deterministic automaton $\mathcal{A}$, with a finite number of states S, recognizing the language L. For

$i \in S$, we define the language L_i of state i by

$$L_i := \{a_1 a_2 ... a_n \in \Sigma^* \text{ path in } \mathcal{A} \text{ from } i \text{ to a final state}\}.$$

We denote by $i \xrightarrow{t} j$ an edge in the automaton. We have the following relations

$$Q_{L_i} = \bigcup_{i \xrightarrow{t} j} t + \beta Q_{L_j}$$

for all $i \in S$.

To construct a substitution having a fixed point ω such that $Q_\omega = Q$, we start by constructing the set of letters of the substitution. We define A_0 as the smallest set stable by intersection, union, complementary in Q, and containing

$$(Q_{L_i} - t) \cap Q$$

for all $i \in S$ and $t \in T$.

Lemma 4.6. — *A_0 is finite.*

The sets $\left\{Q_{L_i} - t \;\middle|\; i \in S,\ t \in T\right\}$ and $\left\{Q \backslash (Q_{L_i} - t) \;\middle|\; i \in S,\ t \in T\right\}$ are finite. If we take all possible intersections of elements of these sets, there are in finite number. There are finitely many unions of such intersections. And any element of A is obtain that way.

Using the set A_0, we can define the alphabet A by

$$A := \left\{a \in A_0 \;\middle|\; \forall b \in A_0,\ b \subseteq a \Longrightarrow a = b\right\} \backslash \{\emptyset\}.$$

By construction, we have the following property.

Lemma 4.7. — *The set A is a finite partition of Q.*

Now that we have defined the alphabet of the substitution, let us show what is the substitution rule of a letter. The following lemma show that such a substitution rule is well defined.

Lemma 4.8. — *For all $a \in A$ and for all $0 \le n < N$, there exists a unic $b \in A$ such that $(f_Q)^n(\beta a) \subseteq b$.*

Proof. — For $a \in A$ and $0 < n < N$, there exists $t \in T$ such that $(f_Q)^n(\beta a) = \beta a + t$ and $\beta a + t \subseteq Q$. By lemma 4.7, there exists $b \in A$ such that $(\beta a + t) \cap b \neq \emptyset$. This is equivalent to $a \cap \beta^{-1}(b - t) \neq \emptyset$. But the set $Q \cap \beta^{-1}(b - t)$ is an element of A_0, because b is an union of intersections of sets of the form $(Q_{L_i} - t) \cap Q$ and $Q \backslash (Q_{L_i} - t)$, and we have

$$\beta^{-1}(Q_{L_i} - t) = \bigcup_{i \xrightarrow{t'} j} t' - \beta^{-1} t + Q_{L_j}.$$

The translation $\beta^{-1} t - t'$ is in T by definition of T, and we can see that $Q \cap \beta^{-1}(b - t)$ is an element of A_0. The element $Q \cap \beta^{-1}(b - t) \cap a$ is also an element of A_0 and is included in $a \in A$.We deduce that $\beta a + t \subseteq b$. $\square$

We define a substitution s over the alphabet A by $s(a) = a_1 a_2 ... a_{n_a}$ where a_i is the element of A such that $(f_Q)^{i-1}(\beta a) \subseteq a_i$, and

$$n_a := \min \left\{ k \in \mathbb{N}_{\geq 1} \;\middle|\; (f_Q)^k(\beta a) \in \beta Q \right\}.$$

Lemma 4.9. — *Let a_0 be the unic element of A containing 0. The substitution s admit a fixed point ω starting from letter a_0, and we have $Q_\omega = Q$.*

Proof. — TODO, easy. □

We have constructed a substitution from the quasicrystal. The converse is easy using proposition 2.11 (or remark 2.12) and proposition 2.15. This terminates the proof of theorems 2.16 and 3.6.

□

5. Examples

In this section, we give various examples of g-β-sets, and substitutions computed from them. More examples and details about these examples can be found here :
`http://www.i2m.univ-amu.fr/~mercat.p/RauzyFractals`.

5.1. Cantor sets.

— It is easy to describe sets like Cantor sets or Sierpiński carpets with g-β-sets. If we add a part of non-empty interior and make it such that it is invariant by multiplication by some power β^n of the Pisot number β (we can guarantee that by putting 0 in the interior for example), then the theorem 2.16 applies and we can get a substitution describing the quasicrystal and hence having a Rauzy fractal with a part that is a Cantor set or a Sierpiński carpet. Here are two such examples.

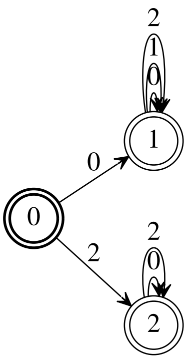
Exemple 5.1. — *Take the quadratic unit Pisot number $\beta = \sqrt{2} + 1$, and take the regular language L defined by the automaton of the figure 4. Then we can compute the smallest substitution whose a fixed point ω verifies $Q_\omega = Q_L$ and whose Perron number is β. We get the following substitution*

$$\begin{array}{rcl} 1 & \mapsto & 125 \\ 2 & \mapsto & 7 \\ 3 & \mapsto & 126 \\ 4 & \mapsto & 448 \\ 5 & \mapsto & 9348 \\ 6 & \mapsto & 93 \\ 7 & \mapsto & 12 \\ 8 & \mapsto & 4 \\ 9 & \mapsto & 5 \end{array}$$

whose Rauzy fractal is an interval union a Cantor set as described by the automaton.

FIGURE 3. Rauzy fractal of the example 5.1

FIGURE 4. Regular language describing an g-β-set which is an intervalle union a Cantor set for $\beta = \sqrt{2} + 1$



Exemple 5.2. — *Take the Tribonnacci Pisot number β, root of $x^3 - x^2 - x - 1$, and take L the regular language defined by the automaton of the figure 5. Then we can compute the domain exchange conjugated to the shift on $\sigma_+(Q_L)$. See figure 7. The smallest substitution whose a fixed point ω verify $Q_\omega = Q_L$, and whose Perron number is β^4, has 15 letters and is displayed in figure 6.*

FIGURE 5. Regular language describing a g-β-set which is a Sierpiński carpet union a set of non-empty interior for β the Tribonnacci number

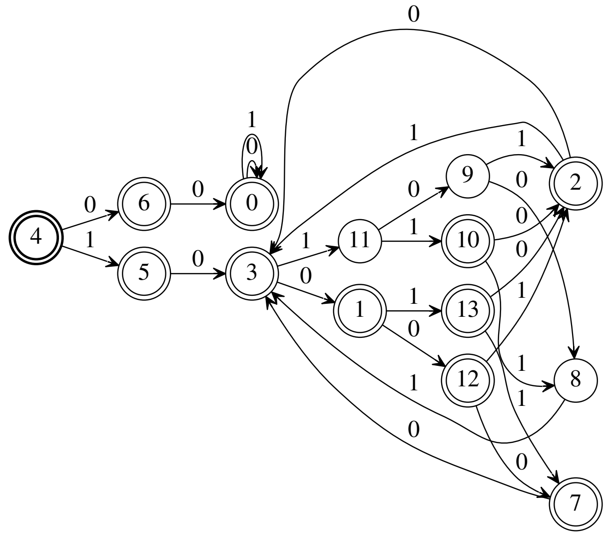
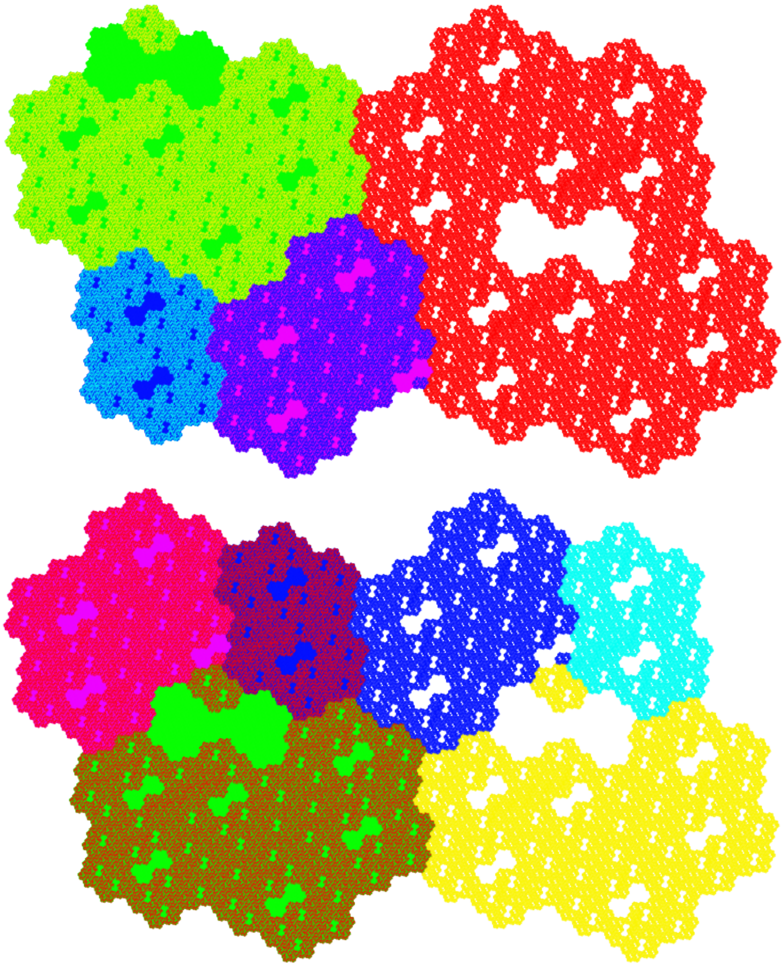


FIGURE 6. Smallest substitution whose Perron number is β^4 for the example 5.2

$$\begin{array}{rcl}
a & \mapsto & ndofmc, \\
b & \mapsto & bcibcoa, \\
c & \mapsto & ndofmclbfmcoandof, \\
d & \mapsto & ndofmcocofmcoandof, \\
e & \mapsto & ndofebakhjjhjgjh, \\
f & \mapsto & ndofmclbfmcoa, \\
g & \mapsto & jhjgjhj, \\
h & \mapsto & jhjgjhjjhjg, \\
i & \mapsto & ebfegjhjjhjg, \\
j & \mapsto & jhjgjhjjhjgjh, \\
k & \mapsto & lbfmcoandofebcijgjh, \\
l & \mapsto & ebfegjhjjhjgjh, \\
m & \mapsto & ndofmcoa, \\
n & \mapsto & ocofmcoa, \\
o & \mapsto & mcofmcoa.
\end{array}$$

FIGURE 7. Domain exchange with 6 pieces, describing the shift on $\sigma_+(Q_L)$ for the regular language L defined in figure 5



5.2. Quadratic Pisot numbers. — Quadratic Pisot numbers of a given quadratic field are always described by substitutions. This is a consequence of [**MPP**]. Indeed, the set of Pisot numbers of degree d of a given number field of degree d, form a model set, for the integer ring and for the windows $\Omega := \left\{x \in E^- \;\middle|\; \forall v \in P_-, |x|_v < 1\right\}$. When moreover the number field is real quadratic, this windows Ω is the intervalle $]-1,1[$, hence we can compute a substitution describing $Q_{]-1,1[}$ thanks to the main result of [**MPP**].

Proposition 5.3. — *Let β be a quadratic Pisot number. There exists a substitution whose a fixed point give the ordered list of quadratic Pisot numbers (including non-unit ones) of the field $\mathbb{Q}(\beta)$.*

Using my tools implemented in sage, I can compute the g-β-set $Q_{]-1,1[}$, for any quadratic Pisot number β, and then compute a substitution describing the quasicrystal.

The following substitution describes the list of quadratic Pisot numbers (including non-unit ones) of the field $\mathbb{Q}(\sqrt{5})$.

$$\begin{cases} 1 & \mapsto & 121 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 313 \end{cases}$$

The fixed point generated by 3 is

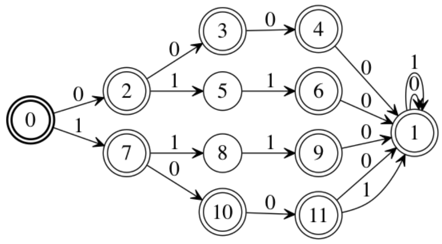
$$31312131312131213131213131213121313121 31...$$

We associocate to letter 1 the number 1, to letter 2 the number $\varphi - 1$ and to letter 3 the number φ, where φ is the golden number. These numbers are the entries of the Perron eigenvector of the incidence matrix. Then if we take the first n letters of this fixed point, the sum of the corresponding numbers is the n^{th} Pisot number of the field $\mathbb{Q}(\sqrt{5})$. We get the following list

$$\varphi,\ \varphi+1,\ 2\varphi+1,\ 2\varphi+2,\ 3\varphi+1,\ 3\varphi+2,\ 4\varphi+2,\ 4\varphi+3,\ 5\varphi+3,\ 5\varphi+4,\ 6\varphi+3, ...$$

5.3. Irreducible example. — I have implemented in sage a tool that permit to draw a Rauzy fractal of any shape with the mouse, like in a drawing software, and to compute the corresponding substitution. The following example has been obtain by drawing randomly using this tool.

FIGURE 8. Regular language describing a g-β-set for the example 5.4, for β the Tribonnacci number



Exemple 5.4. — *Let's take the following irreducible substitution*

$$\begin{cases} 1 & \mapsto & 131 \\ 2 & \mapsto & 1313112 \\ 3 & \mapsto & 131311213113 \end{cases}$$

Then the shift on the fixed point generated by 1 *is conjugated to the translation by* 1 *on the torus* $\mathbb{C}/(\beta^2+\beta-2,\beta)$*, where* β *is a root of* x^3-x^2-x-1*. The figure 9 shows the Rauzy fractal with its domain exchange, and the picture 10 shows the tiling corresponding to the torus.*

FIGURE 9. Domain exchange of the irreducible substitution of example 5.4

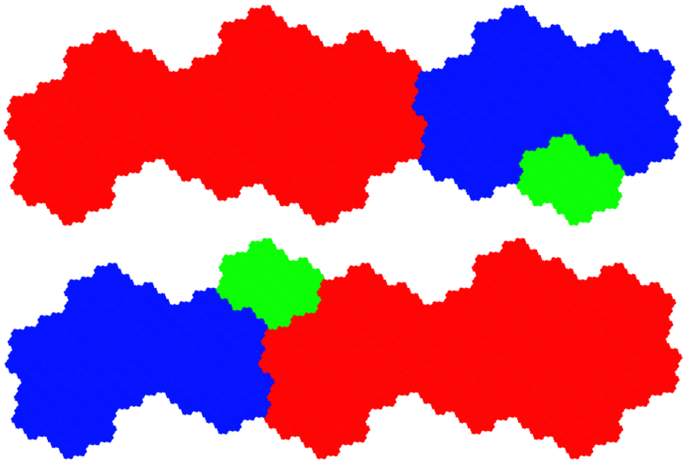
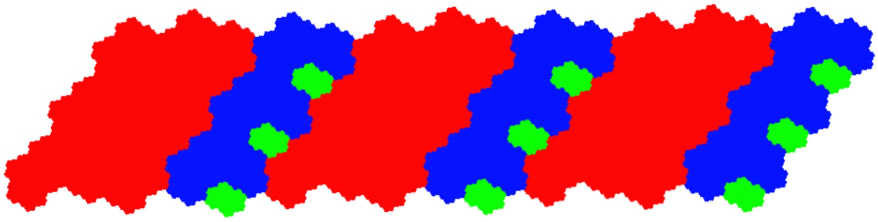


FIGURE 10. Tiling of $\mathbb{C}$ for the example 5.4



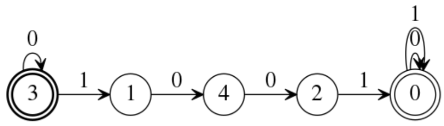
5.4. β-invariant example. — Given any g-β-set Q, we can define the smallest set Q' such that $\beta Q' \subset Q'$. The set obtained is still a g-β-set.

Lemma 5.5. — *Let Q be the smallest β-invariant set containing a given g-β-set. Then Q is a g-β-set.*

Hence, if we take any g-β-set of non-empty interior in $\mathcal{O}_\beta$, we can complete it into a g-β-set satisfying the hypothesis of theorem 2.16 and compute a substitution giving this quasicrystal.

The following example has been obtained thanks to this lemma.

FIGURE 11. Regular language describing a g-β-set given by lemma 5.5

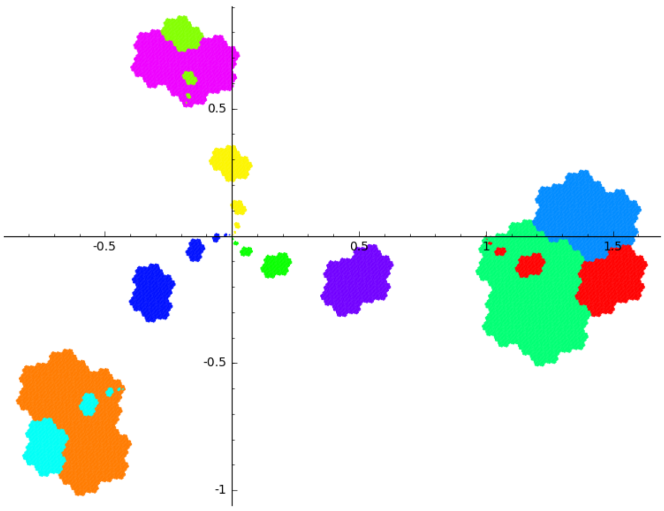


Exemple 5.6. — *For β the Tribonacci number and L the regular language defined in figure 11, the smallest substitution giving the quasicrystal Q_L is the following substitution over* 12 *letters.*

$$\begin{array}{rcl} a & \mapsto & g, \\ b & \mapsto & fk, \\ c & \mapsto & e, \\ d & \mapsto & aj, \\ e & \mapsto & hi, \\ f & \mapsto & hb, \\ g & \mapsto & d, \\ h & \mapsto & ab, \\ i & \mapsto & fc, \\ j & \mapsto & fi, \\ k & \mapsto & fj, \\ l & \mapsto & lb. \end{array}$$

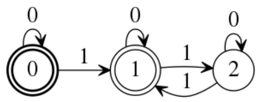
There cannot exists a substitution with less letters, since the domain exchange has 12 *pieces.*

FIGURE 12. Rauzy fractal of the example 5.6



5.5. Example of empty interior. — There exists examples of substitutions which does not satisfy the conjecture 2.38 if the incidence matrix of the substitution is not irreducible. Here is such a example.

FIGURE 13. Regular language describing a β-invariant g-β-set which is a quasicrystal but not a model set, for β the Tribonnacci number.



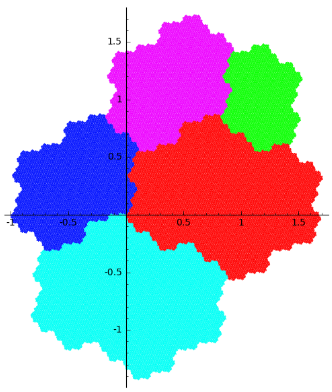
Exemple 5.7. — *Let β be the Tribonnacci number. The g-β-set defined in figure 13 is a quasicrystal and satisfy the hypothesis of the theorem 2.16. Hence, we can compute the following substitution that describe the quasicrystal.*

$$\begin{array}{rcl} 1 & \mapsto & 164, \\ 2 & \mapsto & 24, \\ 3 & \mapsto & 14, \\ 4 & \mapsto & 5, \\ 5 & \mapsto & 1, \\ 6 & \mapsto & 34. \end{array}$$

But this quasicrystal is not a model set. Indeed, the ideal of $\mathcal{O}_\beta$ generated by the quasicrystal is the whole $\mathcal{O}_\beta$ since it contains 1, but it is easy to check that this g-β-set has empty interior in $\mathcal{O}_\beta$, which mean that it is not a model set.

If we remove the letter 2 in this substitution, the incidence matrix becomes irreducible, and the quasicrystal described by this new substitution becomes a model set, as predicted by the conjecture 2.38.

FIGURE 14. Rauzy fractal of the example 5.7



5.6. Examples with various shapes. — In this subsection, we show examples of Rauzy fractals approximating various shapes. These examples has been obtained using my implementation of the g-β-sets in Sage.

For β a Perron number, it is easy to approximate any bounded open subset of E_β^- by a g-β-set with arbitrarly precision, using the following proposition.

Proposition 5.8. — *Let β be a Perron number and let $P \in E_\beta^-$ be an bounded open set. If $\Sigma \subseteq \mathbb{Q}(\beta)$ is such that $Q_{\mathcal{O}_\beta,P} \subseteq Q_{\Sigma^*}$ (such Σ always exists), then the Hausdorff distance between the g-β-set $Q_{L_n\Sigma^*}$ and the set P tends to 0 when $n \to \infty$, where*

$$L_n = \left\{u_0...u_{n-1} \in \Sigma^n \;\middle|\; \textstyle\sum_{k=0}^{n-1} u_k\beta^k \in P\right\}.$$

This permits to prove the theorem 2.6. Indeed, any bounded set $P \subset E_\beta^-$ is arbitrarly approximated by open sets, hence by g-β-sets. Moreover, if β is a Pisot number and if $\overline{P}$ contains 0, we can assume that such a g-β-set Q_L is a quasicrystal and that there exists $n \in \mathbb{N}_{\geq 1}$ such that $\beta^n Q \subset Q$, up to replace the language L by $L \cup 0^n\Sigma^*$. Hence, we can apply the theorem 2.16, and this proves the theorem 2.6.

I have implemented in Sage a tool that takes a Perron number β and a shape (given by its characteristic function), and gives the g-β-set described above, approximating the shape with an arbitrarily chosen precision. The shape can be given by an image or by inequalities for example.

FIGURE 15. Substitution over 328 letters whose Perron number is the 6^{th} power of the Tribonnacci number. The minimal automaton recognizing the language L_Q of the g-β-set Q has 714 states, for β the Tribonnacci number, and for the alphabet $\{0, 1\}$.

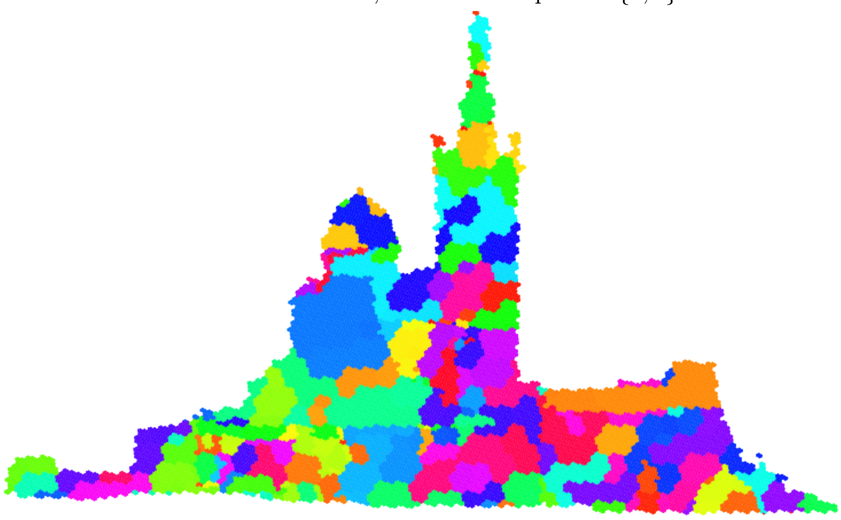


Figure 16. Domain exchange with 18 pieces for the substitution described in figure 15.

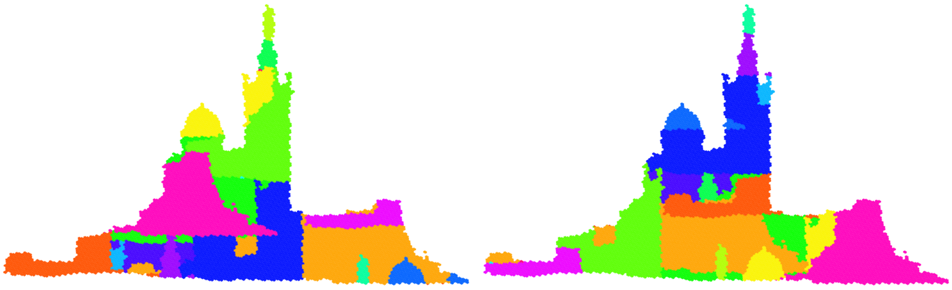


Figure 17. Rauzy fractals aproximating various shapes

See `http://www.i2m.univ-amu.fr/~mercat.p/RauzyFractals` for more details about these examples and for others examples.

6. Discussion

In this section, I discuss some points that I would think interesting to develop.

6.1. Complexity of substitutions. — Given a g-β-set Q for a Perron number β such that there exist a fixed point of a substitution for which $Q = Q_\omega$, what is the complexity of the substitution ? That is, what is the minimal number of letters of such substitution and what is its Pisot number ?

A first obvious remark is that the number of letters is at least the number of pieces of the domain exchange. An interesting fact is that it doesn't depends very much of the complexity of the g-β-set but rather of the shape (this is not completely true). A natural question is the following

Question . — *Let β be a Pisot number and be Q a g-β-set. Is it always possible to find a Pisot number $\lambda \in \mathbb{Q}(\beta)$ with $E_\beta^+ = E_\lambda^+$ and $E_\beta^- = E_\lambda^-$, and such that there exists a substitution whose Perron number is λ, with a fixed point ω satisfying $Q_\omega = Q$, and such that the number of letters is the same than the number of pieces of the domain exchange ?*

I think that this is false, but I have no clue.

Experimentally, the number of letters decrease with n when we search the smallest substitution for the Pisot number $\lambda = \beta^n$. This is true most of the time, and particularly with complicated g-β-sets, but not completely true (there are examples where it is not decreasing with n).

It would be interesting to understand what are the best Pisot numbers λ (i.e. the ones for which there exists a substitution with the minimum number of letters), and how to compute them.

6.2. Non-Pisot numbers. — For non-Pisot numbers, g-β-sets can still represent any subset Q_ω arising from the fixed point ω of a substitution, but this subset is no more a quasicrystal (except maybe for Salem numbers). If Q_L satisfy the finite condition we can still compute a domain exchange coding the shift on $\sigma_\lambda(Q_L)$, but this condition is less practical than for Pisot numbers. Maybe this case can be better understood using g-M-sets where M is a suitable matrix ?

For Perron numbers having at least a conjugate of modulus 1, we have to take care of the space $E_0 = \prod_{p \in P_0} k_p$ where P_0 is the set of archimedian places for which β has modulus one, and not only to the contracting space E_- and the expanding one E_+, otherwise, the integer ring $\mathcal{O}_\beta$ and the ring $\mathcal{R}$ are no more discrete. Lot of my tools doesn't work anymore in this case.

6.3. Definition of Rauzy fractal. — The Rauzy fractal is defined as the adherence of a countable subset of E_λ^-. I think that this is not a good definition for the following reasons :

— The quasicrystal $\sigma_+(Q_\omega)$ associated to a fixed point ω of a substitution can always be seen trivially as a cut-and-project set, because up to rescaling we have

$$\sigma_+(Q_\omega) = \left\{ \sigma_+(x) \in E_\lambda^+ \;\middle|\; x \in \mathcal{O}_\lambda,\ \sigma_-(x) \in \sigma_-(Q_\omega) \right\}.$$

It would be nice if the Rauzy fractal $R \subset E_\lambda^-$ was always a windows :

$$\sigma_+(Q_\omega) = \left\{ \sigma_+(x) \in E_\lambda^+ \;\middle|\; x \in \mathcal{O}_\lambda,\ \sigma_-(x) \in R \right\}.$$

For any Rauzy fractal, I can construct example for which this is the case, but this is not the case in general.

— The domain exchange on the Rauzy fractal comes from a real bijective domain exchange on the quasicrystal without overlaps (it is bijective if we take a whole bi-infinite fixed point, otherwise 0 may not have inverse, but every other element has an inverse). We loose informations about the domain exchange by taking the one defined on the Rauzy fractal. Maybe this wouldn't be the case with a right definition of Rauzy fractal ?

— We have seen that Rauzy fractals always have non-empty interior, and this comes from the only fact that a Rauzy fractal is closed. The following question naturally arise : Is there a particular topological property for a set $\sigma_-(Q_\omega) \subseteq E^-$ comparable to the closeness ? This could be a solution to solve conjecture 2.38 by generalizing lemma 2.36. A good definition of Rauzy fractal could help to do that.

I think that a good definition of Rauzy fractal could be given using Buchi automata, because it is a natural generalization for g-β-sets and because it permits to represent subsets of E^- of non-empty interior and more precisely than by taking the adherence.

But maybe it is a better idea to consider directly the quasicrystal with the topology induced by E^- on the generated group.

References

[MPP] Zuzana Masáková, Jiří Patera, Edita Pelantová *Substitution rules for cut and project sequences*,
`http://www.crm.umontreal.ca/pub/Rapports/2600-2699/2640.pdf`

[Me] P. Mercat *Semi-groupes fortement automatiques*, Bull. SMF 141, fascicule 3 (2013)
`http://www.i2m.univ-amu.fr/~mercat.p/Publis/Semi-groupes%20fortement%20automatiques.pdf`

September 27, 2016

Paul MERCAT, Université Aix-Marseille • *E-mail :* paul.mercat@univ-amu.fr
Url : https://www.i2m.univ-amu.fr/~mercat.p