Rauzy fractals, one dimensional Meyer sets, β-numeration and automata

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Zero Entropy System

Example of substitution

Let's take the following substitution over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

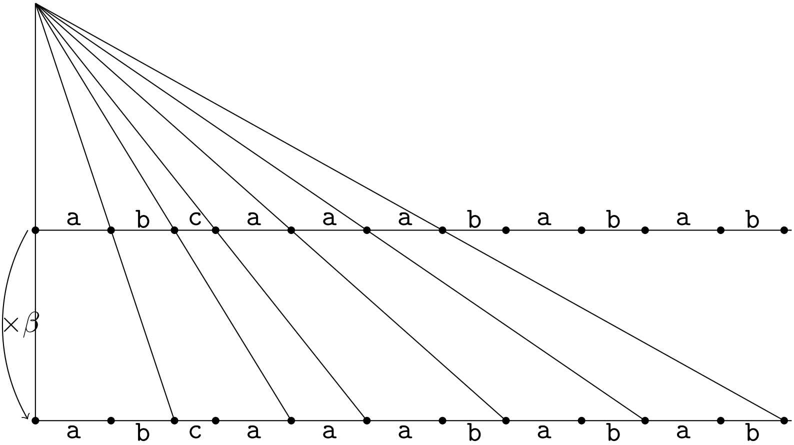
$$s : \begin{cases} a \mapsto ab \\ b \mapsto ca \\ c \mapsto a \end{cases}$$

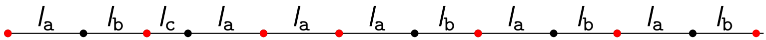
Then by iterating the letter $\mathtt{a}$ we get an infinite fixed point :

$$\begin{aligned} & \mathtt{a} \\ s(\mathtt{a}) = & \mathtt{ab} \\ s^2(\mathtt{a}) = & \mathtt{abca} \\ s^3(\mathtt{a}) = & \mathtt{abcaaab} \\ \ldots & \\ s^\infty(\mathtt{a}) = & \mathtt{abcaaababcaabcaabcaaababcaaababcaaabab}\ldots \end{aligned}$$

Self-similar tiling of $\mathbb{R}_+$

If we replace letters of this fixed point by intervalles of convenient lengths, we get a self-similar tiling of $\mathbb{R}_+$.





To get such a self-similar tiling of $\mathbb{R}_+$, the lengths of each intervalles must satisfy the equality

$$ {}^tM_s \cdot \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix} = \beta \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix}, $$

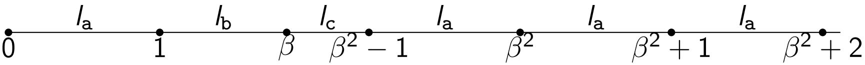
where $M_s = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is the incidence matrix of the substitution and β is the Perron eigenvalue of M_s. Hence we can assume that the lengths $l_i,\ i \in \{a, b, c\}$ live in $\mathbb{Q}(\beta)$.

Quasicrystal of $\mathbb{R}_+$

If we take for example

$$l_a = 1, \quad l_b = \beta - 1, \quad l_c = \beta^2 - \beta - 1,$$

we get the following subset Q of $\mathbb{Q}(\beta)$.



$$Q = \{0, 1, \beta, \beta^2 - 1, \beta^2, \beta^2 + 1, \beta^2 + 2, \beta^2 + \beta + 1, \beta^2 + \beta + 2, ...\}$$

This set have very strong properties since we have :

Proposition

Q is a β-invariant Meyer set of $\mathbb{R}_+$.

But what is a Meyer set ?

Meyer sets are a mathematical model for quasicrystals.

Definition

A ***Meyer set*** *of $\mathbb{R}_+$ is a set $Q \subset \mathbb{R}_+$ such that*

- *Q is a Delone set of $\mathbb{R}_+$,*
- *$Q - Q$ is a Delone set of $\mathbb{R}$.*

Definition

Q is a ***Delone set*** *of E if*

- *Q is* ***uniformly discrete***

$$\exists \epsilon > 0,\ \forall (x, y) \in Q^2, B(x, \epsilon) \cap B(y, \epsilon) = \emptyset,$$

- *Q is* ***relatively dense*** *in E*

$$\exists R > 0,\ E \subseteq \bigcup_{x \in Q} B(x, R).$$

Rauzy fractal

The quasicrystal Q is a part of $\mathbb{Q}(\beta)$, hence we can look at the action of the Galois group. Here, β has two complexes conjugated as conjugates, hence we have an embedding

$$\sigma : \mathbb{Q}(\beta) \hookrightarrow \mathbb{C} ,$$

by choosing one of the complex conjugates.

Proposition

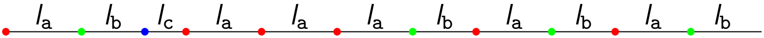
The set $\sigma(Q) \subseteq \mathbb{C}$ is bounded.

We call the closure $\overline{\sigma(Q)}$ a **Rauzy fractal**.

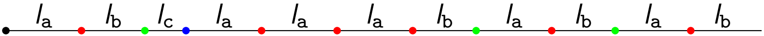
The Rauzy fractal $\overline{\sigma(Q)} \subset \mathbb{C}$

Coloring the Rauzy fractal

Moreover, we can color in red the points of $\sigma(Q)$ that are left bound of an interval of length 1 (i.e. coming from letter $\mathtt{a}$), in green the points that are left bound of an intervalle of length $\beta - 1$ (i.e. coming from letter $\mathtt{b}$), and the other ones, for $\beta^2 - \beta - 1$, in blue.

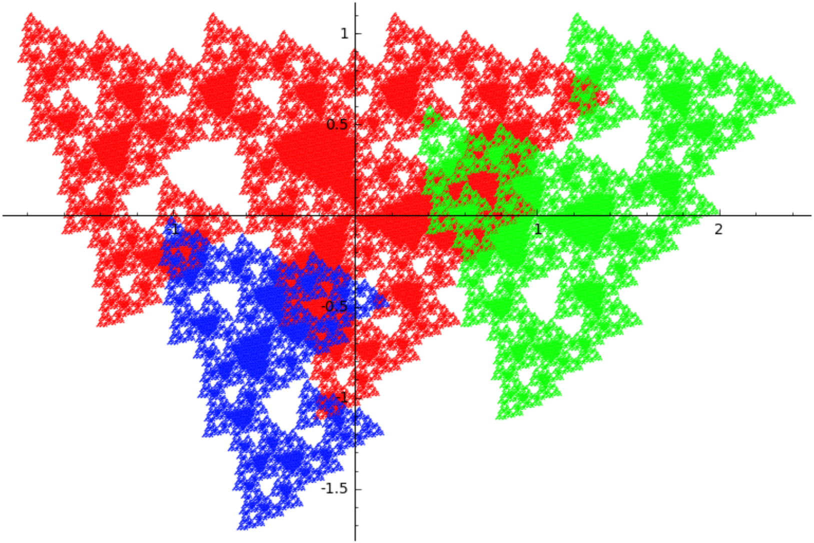


We can also color in the same way by considering the right bound rather than the left one.



Proposition

Let $u = s^\infty(a)$. Then, the subshift $(\overline{S^{\mathbb{Z}} u}, S)$ is measurably conjugated to a domain exchange on the Rauzy fractal $\overline{\sigma(Q)}$, for the Haar measure.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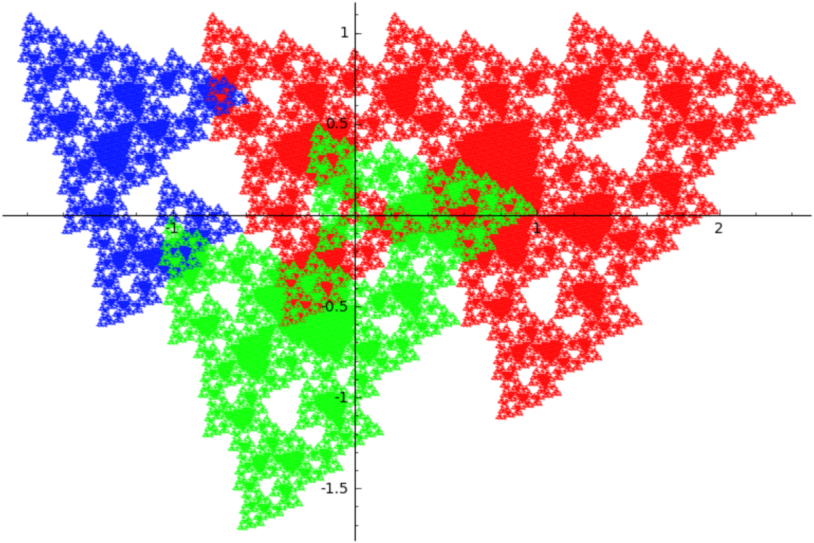
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Generalization to any substitution

If s is any substitution over a alphabet A, everything generalizes :

- **fixed point** : Up to replace s by a power, s has a fixed point ω.
- **self-similar tiling** : We get a self-similar tiling of $\mathbb{R}_+$ or $\mathbb{R}$ by replacing letters by intervals of lengths $l_a, a \in A$ given by a Perron left eigenvector of the incidence matrix.
- **quasicrystal** : We get a set $Q_\omega \subset \mathbb{R}$ by taking the bounds of intervals of this self-similar tiling, and up to rescaling we have $Q_\omega \subset \mathbb{Q}(\beta)$ where β is the Perron eigenvalue of the incidence matrix M_s. If β is a Pisot number, Q_ω is a Meyer set.
- **Rauzy fractal** : Q_ω is a subset of $\mathbb{Q}(\beta)$, therefore we can embed it into a natural contracting space E_β^c where it is a pre-compact subset.
- **Domain exchange** : If the substitution satisfies the strong coïncidence condition, then we can color the Rauzy fractal $\overline{\sigma_c(Q_\omega)}$ in order to define a domain exchange conjugated to the shift.

General definitions of contracting space and Rauzy fractal

There are natural contracting and expanding spaces for the multiplication by β on a number field $k = \mathbb{Q}(\beta)$. Call P the set of places of k (i.e. equivalence classes of absolute values), and let

$$P_e := \left\{ v \in P \;\middle|\; |\beta|_v > 1 \right\} \text{ and } P_c := \left\{ v \in P \;\middle|\; |\beta|_v < 1 \right\}.$$

The **contracting space** is $E_\beta^c := \prod_{v \in P_c} k_v$ and the expanding one is $E_\beta^e := \prod_{v \in P_e} k_v$, where k_v denotes the completion of k for the absolute value v. We denote by $\sigma_c = \prod_{v \in P_c} \sigma_v : \mathbb{Q}(\beta) \hookrightarrow E_\beta^c$ where $\sigma_v : \mathbb{Q}(\beta) \hookrightarrow k_v$ is a choice of one natural embedding.

Definition

We call ***Rauzy fractal*** *the adherence of $\sigma_c(Q_\omega)$ in E_β^c.*

For the previous example, where β is root of $x^3 - x^2 - x - 1$, we have $E_\beta^e = \mathbb{R}$ (there is one real place) and $E_\beta^c = \mathbb{C}$ (there is one complex place).

Rauzy fractals can approximate any shape

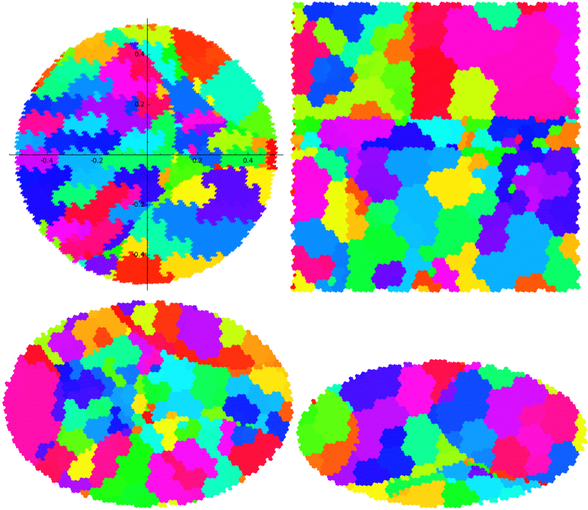
Theorem

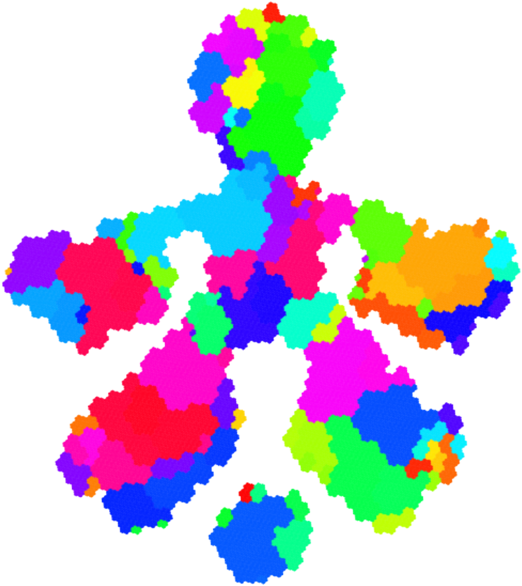
For any Pisot number β and for any $P \subset E_\beta^c$, bounded and containing 0, there exists substitutions whose Rauzy fractals approximate arbitrarily P, for the Hausdorff distance, and whose Perron numbers are powers of β. Moreover, the proof is constructive.

The Hausdorff distance between two subsets $A \subseteq E$ and $B \subseteq E$ of a metric space E is

$$d(A, B) = \max\left(\sup_{a\in A} \inf_{b\in B} d(a,b),\ \sup_{b\in B} \inf_{a\in A} d(a,b)\right).$$

Rauzy fractals aproximating various shapes





g-β-sets : a nice description of quasicrystals by automata

Definition (Main tool)

*A set $Q \subseteq \mathbb{Q}(\beta)$ is a **g-β-set** if we have*

$$Q = Q_{L,\beta} = \left\{ \sum_{k=0}^{n} a_k \beta^k \;\middle|\; n \in \mathbb{N},\ a_0 ... a_n \in L \right\},$$

where $\Sigma \subset \mathbb{Q}(\beta)$ is a finite alphabet and $L \subseteq \Sigma^*$ is a regular language.

Proposition

If ω is a fixed point of a substitution, then Q_ω is a g-β-set.

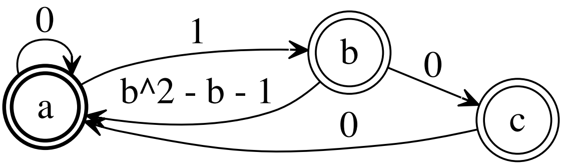
The aim of the following will be to give a reciprocal to this proposition.

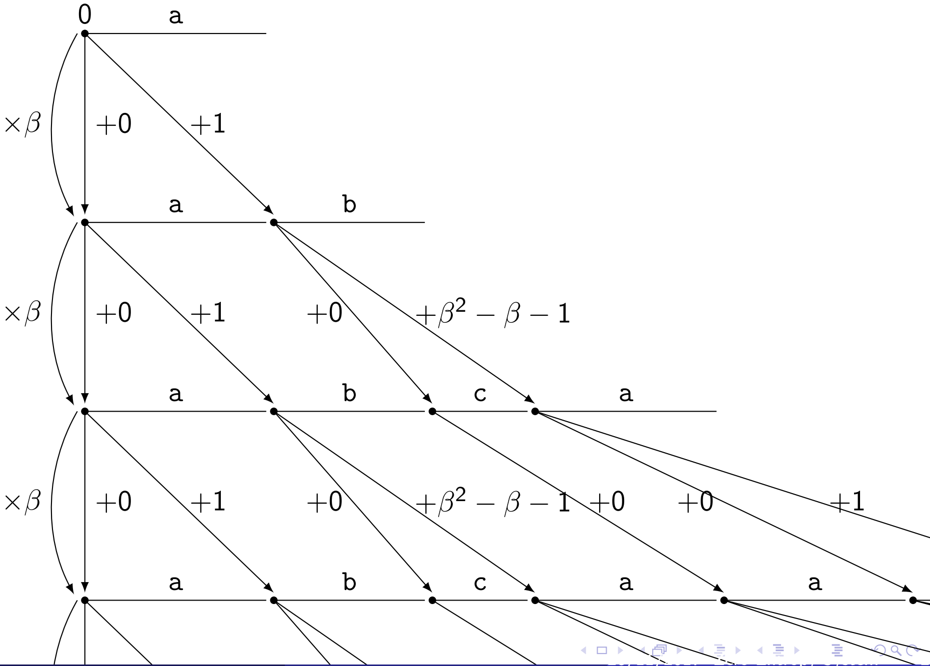
g-β-set coming from a substitution

For the example

$$s : \begin{cases} a \mapsto ab \\ b \mapsto ca \\ c \mapsto a \end{cases}$$

the mirror of the language L, recognized by the following automaton, define a g-β-set which is a quasicrystal coming from the substitution s, for β the Tribonnacci number.

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Stability of the set of g-β-sets

Properties (Properties of g-β-sets)

If β is an algebraic number without conjugate of modulus one, and if Q_1 and Q_2 are two g-β-sets, then

- $Q_1 \cup Q_2$, $Q_1 \cap Q_2$ *and* $Q_1 \backslash Q_2$ *are g-β-sets,*
- $Q_1 + Q_2$ *is a g-β-set,*
- $\forall t \in \mathbb{Q}(\beta)$, $Q_1 + t$ *is a g-β-set,*
- $\forall c \in \mathbb{Q}(\beta)$, cQ_1 *is a g-β-set,*
- $\forall k \geq 1, n \geq 1$, *a g-$\beta^k$-set is a g-$\beta^n$-set.*

Moreover, everything is computable, and emptyness and inclusion are decidable.

Hence, it is easy to approximate any shape by g-β-sets.

Main result : Characterization of Meyer sets coming from substitutions

It is easy to prove that Rauzy fractals can approximate any shape with the previous properties of g-β-sets and with the following theorem.

Theorem

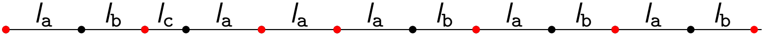
Let β be a Pisot number, and let $Q \subseteq \mathbb{Q}(\beta)$ a β-invariant Meyer set. Then, the Meyer set Q comes from a substitution if and only if it is a g-β-set that contains 0.

We have already seen that these conditions are necessary. Let us show that these are sufficient, and how to construct such substitution.

β-expansion algorithm in a β-invariant Meyer set

Let Q be a Meyer set and β be a Pisot number with $\beta Q \subset Q$ and $0 \in Q$. Then we can define the following algorithm that gives an unique finite β-expansion of any element of Q.

Data: $x \in Q$
Result: coefficients t_0 of a β-expansion of x
while $x \neq 0$ **do**
$\quad x \leftarrow x - t0$ for $t0 = \inf\{t \geq 0 | x - t \in \beta Q\}$;
$\quad x \leftarrow x/\beta$;
$\quad$print t_0 ;
end



The expansion of x is given by the successive elements t_0.

With the previous algorithm, we define the language

$$L_Q := \left\{ a_0 \dots a_n \in \Sigma_Q^* \;\middle|\; a_0 \dots a_n \text{ expansion of } x \text{ given by the algorithm } \right\} 0^*$$

over the finite alphabet

$$\Sigma_Q := \left\{ \inf\{t \geq 0 | x - t \in \beta Q\} \;\middle|\; x \in Q \right\}.$$

In others word, L_Q is the unique subset of Σ_Q^* containing the empty word ϵ, such that $Q = Q_{L_Q}$ and such that

$$a_0 \dots a_n \in L_Q \iff \begin{cases} a_0 = \min \left\{ t \in \Sigma_Q \;\middle|\; \sum_{k=0}^n a_k \beta^k \in \beta Q + t \right\} \\ a_1 \dots a_n \in L_Q \end{cases} .$$

Proposition

The following two sentences are equivalent.

- *Q comes from a substitution.*
- *L_Q is a regular language.*

Hence, to prove the main theorem, it is enough to prove the following

Lemma

We have the equivalence between :

- L_Q *is a regular language.*
- Q *is a g-*β*-set.*

The direct part is obvious. To prove the converse, we have to construct the language L_Q from any regular language L such that $Q = Q_L$.

Step 1/3 : get a regular language over the alphabet Σ_Q

Let L be a regular language over an alphabet $\Sigma \subset \mathbb{Q}(\beta)$ such that $Q = Q_L$.

Lemma (Change of the alphabet)

The following language is regular

$$L_{Q,\Sigma_Q} := \left\{a_0 \dots a_n \in \Sigma_Q^* \;\middle|\; n \in \mathbb{N},\ \textstyle\sum_{k=0}^n a_k \beta^k \in Q\right\},$$

and we have $Q_{L_{Q,\Sigma_Q}} = Q$.

Proof.

$$L_{Q,\Sigma_Q} = Z(p_1(L^{rel} \cap \Sigma_Q^* \times L0^*)) \text{ where } Z : L \mapsto \bigcup_{n \in \mathbb{N}} L0^{-n},$$

$$L^{rel} = \left\{(u, v) \in (\Sigma_Q \times \Sigma)^* \;\middle|\; \textstyle\sum_{k=0}^n (u_k - v_k)\beta^k = 0\right\}.$$

This last language is regular thanks to the main result of my paper « Semi-groupes fortement automatiques ». $\square$

Step 2/3 : stabilization by suffix

Lemma (Stabilization by suffix)

The greatest language $L' \subset L_{Q,\Sigma_Q}$ such that

$$u \in L' \implies \text{ every suffix of } u \text{ is in } L'$$

is a regular language, and we have $Q = Q_{L'}$.

Proof.

Take a deterministic automaton recognizing the mirror of L_{Q,Σ_Q}.
Remove every non final state.
Then this new automaton recognize the mirror of L'.
And we have $L_Q \subseteq L' \subseteq L_{Q,\Sigma_Q}$, hence $Q = Q_{L'}$. □

Step 3/3 : minimal words in lexicographic order

Lemma (Minimal words in lexicographic order describing Q)

We have the equality

$$L_Q = L' \backslash p_1(L' \times L' \cap L^{rel} \cap L^{>}),$$

where

$$L^{rel} := \left\{(u,v) \in (\Sigma_Q \times \Sigma_Q)^* \;\middle|\; \textstyle\sum_{k=0}^{n} (u_k - v_k)\beta^k = 0\right\}$$

and

$$L^{>} := \left\{(u,v) \in (\Sigma_Q \times \Sigma_Q)^* \;\middle|\; u > v \text{ for the lexicographic order}\right\},$$

where we choose the natural order on Σ_Q, given by the embedding into the expanding space $E^e_\beta = \mathbb{R}$.

Hence L_Q is regular, and this proves the theorem.

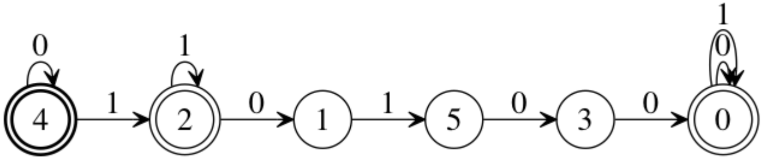
Proof of last lemma.

- $L' \times L' \cap L^{rel} \cap L^{>}$ is the couple of words of same length, giving the same element of Q, and with the left one strictly less than the right one for the lexicographic order.
- Hence $L' \backslash p_1(L' \times L' \cap L^{rel} \cap L^{>})$ is the set of elements of L' which are minimal in lexicographic order among the words of L' of same length describing the same point of Q.
- We deduce the equality with L_Q : the language is still stable by suffix and the first letter is the minimal one, as in the definition of L_Q.
- The language $L^{>}$ is easily seen to be regular : we can recognize it with an automaton having two states.
- The language L^{rel} is regular, thanks to my article « Semi-groupes fortement automatiques ».

$\square$

Example of g-β-set for β the Tribonnacci number

Let's take the g-β-set defined by



for $\beta^3 = \beta^2 + \beta + 1$. The regular language L described by this automaton is

$$L = 0^*1^* \cup 0^*1^+0100\{0,1\}^*.$$

This g-β-set satisfy every hypothesis of the theorem, hence we can compute a substitution from it.

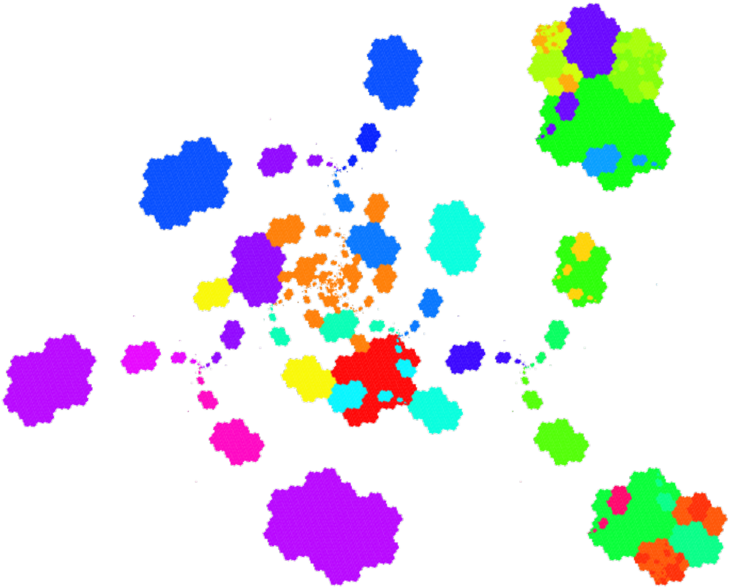
Corresponding substitution whose Perron number is β

$1 \mapsto 28, 12, 13$
$2 \mapsto 29, 1, 5$
$3 \mapsto 29, 1, 8, 13$
$4 \mapsto 4, 13$
$5 \mapsto 29, 3, 9$
$6 \mapsto 32, 11, 9$
$7 \mapsto 49$
$8 \mapsto 29, 3, 10, 13$
$9 \mapsto 29, 2$
$10 \mapsto 29, 3, 13$
$11 \mapsto 30, 26$
$12 \mapsto 32, 11, 10, 13$

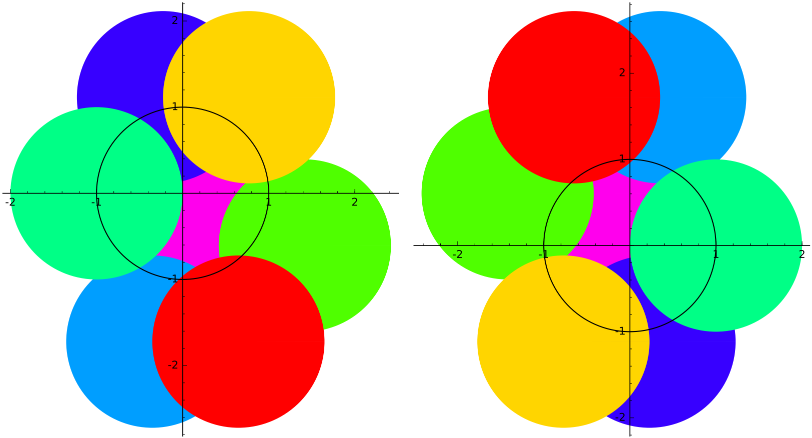
$13 \mapsto 29, 14$
$14 \mapsto 29, 1, 27$
$15 \mapsto 32, 11, 27$
$16 \mapsto 29, 19$
$17 \mapsto 4, 27$
$18 \mapsto 7$
$19 \mapsto 28, 6$
$20 \mapsto 33, 35$
$21 \mapsto 29, 34$
$22 \mapsto 17$
$23 \mapsto 18$
$24 \mapsto 19$

$25 \mapsto 25, 20$
$26 \mapsto 28, 15$
$27 \mapsto 29, 16$
$28 \mapsto 22$
$29 \mapsto 23$
$30 \mapsto 24$
$31 \mapsto 26$
$32 \mapsto 28, 21$
$33 \mapsto 31, 21$
$34 \mapsto 29$
$35 \mapsto 30$

Rauzy fractal



Construction of a domain exchange



$-\beta^2 + 2\beta, \ \beta^2 - \beta - 1, \ \beta - 1, \ 1, \ -\beta^2 + 2\beta + 1, \ \beta^2 - \beta, \ \beta$

Domain exchange on the model set defined by the unit disk window, and the integer ring $\mathcal{O}_\beta$ where β is the Tribonnacci number.

Another application of g-β-sets

Let s and h be the substitutions

$$s : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{cases} \qquad h : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases}$$

and let $R_s \subseteq \mathbb{C}$ and $R_h \subseteq \mathbb{C}$ be their Rauzy fractals.

Proposition

R_s is a countable union of homothetic transformations of R_h, union a set of dimension less than two.

Projecting a substitution on another with same β

We define the **projection** of a language on another by

$$\mathrm{Proj}(L, L') = \left\{ u \in L' \;\middle|\; \sum_{i=0}^{|u|-1} u_i \beta^i \in Q_L \right\} = Z(p_1(L' \times L0^* \cap \mathrm{L}^{\mathrm{rel}}))$$

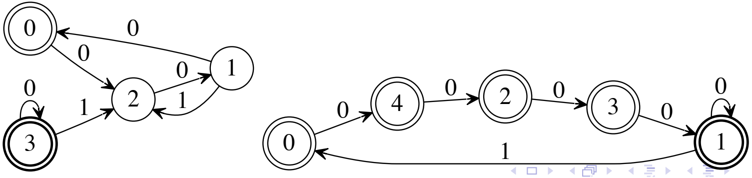
Proposition

There exists regular languages A and B such that

$$\mathrm{Proj}(0^3 L_s, L_h) = A L_h \cup B$$

with spectral radius of B less than β.

Figure – Minimal automata of L_s and L_h respectively



Computation of the dimension

The box dimension of the part of dimension less than two is

$$\dim_{MB}(\sigma_-(Q_{L_M})) = 2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$$

where $\gamma \approx 1.31477860592584...$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$ and β is the smallest Pisot number.

Theorem

Let $\overline{\beta}$ be a complex conjugate of the smallest Pisot number β, and let $L \subseteq \Sigma^*$ be a language over the alphabet $\Sigma = \{0, 1\}$ such that the elements of $\sigma_-(Q_L) = \left\{\sum_{i=0}^{|u|-1} u_i \overline{\beta}^i \;\middle|\; u \in L\right\} \subseteq \mathbb{C}$ are uniquely represented for a given length (i.e.
$\forall u, v \in L, \left(|u| = |v| \text{ and } \sum_{i=0}^{|u|-1} u_i \overline{\beta}^i = \sum_{i=0}^{|v|-1} v_i \overline{\beta}^i\right) \Longrightarrow u = v$).
Then we have $\dim_{MB}(\sigma_-(Q_L)) = \frac{\log(\gamma)}{\log(1/|\overline{\beta}|)} = 2\frac{\log(\gamma)}{\log(\beta)}$, where γ is the spectral radius of the minimal automaton of L.

Zoom in the Rauzy fractal of s