YET ANOTHER CHARACTERIZATION OF THE PISOT SUBSTITUTION CONJECTURE

by

Paul MERCAT & Shigeki AKIYAMA

Abstract. — We give a sufficient geometric condition for a subshift to be measurably isomorphic to a domain exchange and to a translation on a torus. And for an irreducible unit Pisot substitution, we introduce a new topology on the discrete line and give a simple necessary and sufficient condition for the symbolic system to have pure discrete spectrum. This condition gives rise to an algorithm based on computation of automata. To see the power of this criterion, we provide families of substitutions that satisfies the Pisot substitution conjecture : 1) $a \mapsto a^k bc$, $b \mapsto c$, $c \mapsto a$, for $k \in \mathbb{N}$ and 2) $a \mapsto a^l b a^{k-l}$, $b \mapsto c$, $c \mapsto a$, for $k \in \mathbb{N}_{\geq 1}$, for $0 \leq l \leq k$ using different methods. And we also provide a example of S-adic system with pure discrete spectrum.

Contents

Key words and phrases. — Rauzy fractal, substitution, quasicrystal, cut-and-project, model set, Meyer set, tiling, Pisot number, Pisot substitution conjecture, pure discrete spectrum, algebraic coincidence, S-adic system.

1. Introduction

Sturmian systems are well-known examples of subshifts that are conjugate to translations on the torus $\mathbb{R}/\mathbb{Z}$. In 1982, Gérard Rauzy (see [**Rauzy 1982**]) gave a generalization to higher dimension for the subshift generated by the infinite fixed point of the Tribonnacci substitution:

$$\begin{cases} a & \mapsto & ab \\ b & \mapsto & ac \\ c & \mapsto & a \end{cases} .$$

He constructed a compact subset of $\mathbb{R}^2$, that we call now Rauzy fractal, and that has the property that it tiles the plane by translation. And we can define a domain exchange on this Rauzy fractal which is measurably conjugate to the subshift, and measurably conjugate to a translation on the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$.

In 2001, P. Arnoux and S. Ito (see [**Arnoux Ito 2001**]) generalized the work of Rauzy to any irreducible unit Pisot substitution. They introduced a combinatorial condition which is easy to check, called the strong coincidence, that permits to get a measurable conjugacy between the subshift and a domain exchange, which is also a finite extension of a translation on a torus.

To obtain a measurable conjugacy between the subshift of an irreducible unit Pisot substitution and a translation on a torus, several equivalent conditions (super coincidence, Geometric coincidence) has been studied ([**Ito Rao 2003, Barge Kwapisz 2006**]). This article gives another formulation of such coincidences and a short proof of its equivalence. The new criterion is checked by automata computation.

We introduce a topology on $\mathbb{Z}^d$ that permits to characterize easily when the subshift of a given irreducible unit Pisot substitution over d letters is measurably isomorphic to a translation on a $(d-1)$-dimensional torus: see theorem 3.3. And we show that this condition is equivalent to the non-emptiness of some computable regular language: see theorem 5.8.

In the last section, we use this condition to prove the pure discreteness for the family of substitution

$$s_k : \begin{cases} a \mapsto a^k bc \\ b \mapsto c \\ c \mapsto a \end{cases}$$

for $k \in \mathbb{N}$, where a^k means that the letter a is repeated k times. And we also prove the pure discreteness of the family of substitution

$$s_{l,k} : \begin{cases} a \mapsto a^l ba^{k-l} \\ b \mapsto c \\ c \mapsto a \end{cases}$$

for $k \in \mathbb{N}_{\geq 1}$, $0 \leq l \leq k$, by computing explicitly a automaton describing algebraic relations, and showing that the pure discreteness for the substitution

$$s_k : \begin{cases} a \mapsto a^k b \\ b \mapsto c \\ c \mapsto a \end{cases}$$

implies the pure discreteness for the other substitutions.

We also use the criterion to prove the pure discreteness for the S-adic system with the two substitutions

$$s_1 : \begin{cases} a & \mapsto & aab \\ b & \mapsto & c \\ c & \mapsto & a \end{cases} \quad \text{and} \quad s_2 : \begin{cases} a & \mapsto & aba \\ b & \mapsto & c \\ c & \mapsto & a \end{cases},$$

for all word in $\{s_1, s_2\}^{\mathbb{N}}$.

2. A criterion for a subshift to have purely discrete spectrum

In this section, we describe a general geometric criterion for a subshift to be measurably isomorphic to a translation on a torus. Let us start by introduce some notations.

2.1. Subshift. — We denote by $A^{\mathbb{N}}$ (respectively $A^{\mathbb{Z}}$) the set of infinite (respectively bi-infinite) words over an alphabet A. We denote by $|u|$ the length of a word u, and $|u|_a$ denotes the number of occurrences of the letter a in a word $u \in A^*$. And we denote by

$$\text{Ab}(u) = (|u|_a)_{a \in A} \in \mathbb{N}^A$$

the **abelianisation vector** of a word $u \in A^*$. The canonical basis of $\mathbb{R}^A$ will be denoted by $(e_a)_{a\in A} = (\mathrm{Ab}(a))_{a\in A}$.

The **shift** on infinite words is the application

$$S : \begin{array}{ccc} A^{\mathbb{N}} & \longrightarrow & A^{\mathbb{N}} \\ (u_i)_{i\in\mathbb{N}} & \longmapsto & (u_{i+1})_{i\in\mathbb{N}} \end{array}$$

We can also define the shift on bi-infinite words in an obvious way, and it becomes invertible.

We use the usual metric on $A^{\mathbb{N}}$:

$$d(u, v) = 2^{-n} \text{ where } n \text{ is the length of the maximal common prefix.}$$

The map S is continuous for this metric. Given an infinite word u, the compact set $\overline{S^{\mathbb{N}}u}$ is S-invariant. We call **subshift** generated by u, the dynamical system $(\overline{S^{\mathbb{N}}u}, S)$.

The same can be done for bi-infinite words.

2.2. Discrete line associated to a word. — Let $u \in A^{\mathbb{N}}$ be an infinite word over the alphabet A. Then, the associated **discrete line** is the following subset of $\mathbb{Z}^A$:

$$D_u := \left\{ Ab(v) \in \mathbb{Z}^A \;\middle|\; v \text{ finite prefix of } u \right\}.$$

If $u \in A^{\mathbb{Z}}$ is a bi-infinite word, then the corresponding discrete line is

$$D_u := -D_v \cup D_w,$$

where $v, w \in A^{\mathbb{N}}$ are infinite words such that $u = {}^t v w$, where ${}^t v = ...v_n...v_2v_0$ denotes the mirror of the word $v = v_0v_2...v_n...$.

For $u \in A^{\mathbb{N}}$, we can partition this discrete line into $d = |A|$ pieces. For every $a \in A$, let

$$D_{u,a} := \left\{ Ab(v) \in \mathbb{Z}^A \;\middle|\; va \text{ finite prefix of } u \right\}.$$

The sets $D_{u,a} + e_a$, $a \in A$, also gives almost a partition of D_u:

$$D_u = \{0\} \cup \bigcup_{a\in A} D_{u,a} + e_a.$$

For a bi-infinite word $u \in A^{\mathbb{Z}}$, we have the same, but we get a real partition, without the $\{0\}$. In both cases, these partitions permit to see the shift S on the word u as a domain exchange E:

$$E : \begin{array}{ccl} D_u & \longrightarrow & D_u \\ x & \longmapsto & x + e_a \text{ for } a \in A \text{ such that } x \in D_{u,a}. \end{array}$$

There is also a property of tiling for this discrete line: we have the following

Proposition 2.1. — *Let Γ_0 be the subgroup of $\mathbb{Z}^A$ generated by $(e_a - e_b)_{a,b\in A}$, and let u be any bi-infinite aperiodic word over the alphabet A. Then D_u is a fundamental domain for the action of Γ_0 on $\mathbb{Z}^A$. Moreover the translation T by e_a (for any $a \in A$) on $\mathbb{Z}^A/\Gamma_0$ is conjugate to the domain exchange E on D_u by the natural quotient map*

$\pi_0 : \mathbb{Z}^A \to \mathbb{Z}^A/\Gamma_0$, and the shift $(S^{\mathbb{Z}}u, S)$ is conjugate to the domain exchange (D_u, E) by the map

$$c : \begin{array}{ccc} S^{\mathbb{Z}}u & \to & D_u \\ S^n u & \mapsto & E^n 0 \end{array} .$$

Remark 2.2. — *We have the same for infinite non-eventually periodic words, but we get a fundamental domain for the action on the half-space*

$$\left\{ (x_a)_{a \in A} \in \mathbb{Z}^A \;\middle|\; \sum_{a \in A} x_a \geq 0 \right\},$$

and a conjugacy with the shift on $S^{\mathbb{N}}u$.

Proof. — The vectors $(e_a)_{a \in A}$ are equivalent modulo the group Γ_0. Hence, this discrete line is equivalent to $\mathbb{Z}e_a$ for any letter $a \in A$, and this is an obvious fundamental domain of $\mathbb{Z}^A$ for the action of Γ_0. The map c is well-defined and one-to-one because the word u is aperiodic. And it gives a conjugacy between the shift $(S^{\mathbb{Z}}u, S)$ and the domain exchange (D_u, E): $c \circ S = E \circ c$. The natural quotient map $\pi_0 : \mathbb{Z}^A \to \mathbb{Z}^A/\Gamma_0$ restricted to D_u is bijective, and it gives a conjugacy between the domain exchange (D_u, E) and the translation $(\mathbb{Z}^A/\Gamma_0, T)$: $\pi_0 \circ E = T \circ \pi_0$. □

If the discrete line D_u stays near a given line of $\mathbb{R}^A$ (this will be the case for example for periodic points of Pisot substitutions), then we can project onto a hyperplane $\mathcal{P}$ of $\mathbb{R}^A$ (for example the hyperplane of equation $\sum_{a \in A} x_a = 0$) along this line. The projection of $\mathbb{Z}^A$ is dense in the hyperplane for almost all lines, and the group Γ_0 becomes a lattice in the hyperplane. If the projection of the discrete line is not so bad, we can expect that the closure gives a tiling of the hyperplane, and that the closure of each piece of the partition of the discrete line doesn't intersect each other. And we can expect that the conjugacy given by the previous proposition becomes a conjugacy of the closures. Figure 1 shows the conjugacy given by the proposition 2.1, and what we get if everything goes well.

FIGURE 1. Commutative diagrams of the conjugacy between the shift S, the domain exchange E and the translation on the quotient T, before and after taking the closure

$$\begin{array}{ccc} S^{\mathbb{Z}}u & \xrightarrow{S} & S^{\mathbb{Z}}u \\ \downarrow c & & \downarrow c \\ D_u & \xrightarrow{E} & D_u \\ \downarrow \pi_0 & & \downarrow \pi_0 \\ \mathbb{Z}^A/\Gamma_0 & \xrightarrow{T} & \mathbb{Z}^A/\Gamma_0 \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \overline{S^{\mathbb{Z}}u} & \xrightarrow{S} & \overline{S^{\mathbb{Z}}u} \\ \downarrow \bar{c} & & \downarrow \bar{c} \\ \overline{\pi(D_u)} & \xrightarrow{E} & \overline{\pi(D_u)} \\ \downarrow \pi_0 & & \downarrow \pi_0 \\ \mathcal{P}/\pi(\Gamma_0) & \xrightarrow{T} & \mathcal{P}/\pi(\Gamma_0) \end{array}$$

Let us now give a general geometric criterion that permits to know that everything works well as in Figure 1.

2.3. Geometrical criterion for the pure discreteness of the spectrum. — Here is the main general geometric criterion for a subshift to have a pure discrete spectrum. We use the notations defined in subsection 2.2.

Theorem 2.3. — *Let $u \in A^{\mathbb{N}}$ be an infinite word over an alphabet A, and let π be a linear projection from $\mathbb{R}^A$ onto a hyperplane $\mathcal{P}$. We assume that we have the following:*
— *the restriction of π to $\mathbb{Z}^A$ is injective and has a dense image,*
— *the set $\pi(D_u)$ is bounded,*
— *the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is minimal.*

Then there exists a σ-algebra and a S-invariant measure μ such that the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is a finite extension of the translation of the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$, where T is the translation by $\pi(e_a)$ (for any $a \in A$) on the torus $\mathcal{P}/\pi(\Gamma_0)$, Γ_0 is the group generated by $\left\{e_a - e_b \;\middle|\; a, b \in A\right\}$, and λ is the Lebesgue measure. And it is also a topological semi-conjugacy.

If moreover the union

$$\bigcup_{t \in \pi(\Gamma_0)} \overline{\pi(D_u)} + t = \mathcal{P}$$

is disjoint in Lebesgue measure, then the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is uniquely ergodic and is isomorphic to the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$ and to a domain exchange on $\overline{\pi(D_u)}$.

In order to prove this theorem, we start by showing that we can extend by continuity the map $\pi \circ c : S^{\mathbb{N}}u \to \pi(D_u)$ that gives the conjugacy between the shift $(S^{\mathbb{N}}u, S)$ and the domain exchange $(\pi(D_u), E)$.

Lemma 2.4. — *Let $u \in A^{\mathbb{N}}$ be a non-eventually periodic infinite word over an alphabet A, and let π be a projection from $\mathbb{R}^A$ onto a hyperplane $\mathcal{P}$. We assume that $\pi(D_u)$ is bounded. Then the map*

$$\pi \circ c : \begin{array}{ccc} S^{\mathbb{N}}u & \to & \pi(D_u) \\ S^n u & \mapsto & E^n 0 \end{array}$$

can be extended by continuity at any point of the closure whose orbit is dense in $\overline{S^{\mathbb{N}}u}$.

To prove this lemma, we need the following geometric lemma, saying that we can always translate a bounded set of $\mathbb{R}^d$ in order to have a non empty but arbitrarily small intersection with the initial set.

Lemma 2.5. — *Let Ω be a bounded subset of $\mathbb{R}^d$. Then, we have*

$$\inf_{t \in \Omega - \Omega} \operatorname{diam}(\Omega \cap (\Omega - t)) = 0.$$

The proof is left as an exercise. It can be proven for example by considering a diameter and using the parallelogram law.

proof of lemma 2.4. — Let $w \in \overline{S^{\mathbb{N}}u}$ having dense orbit in $\overline{S^{\mathbb{N}}u}$ and let $\epsilon > 0$. By lemma 2.5, there exists $t \in D_u - D_u$ such that $\operatorname{diam}(\pi(D_u) \cap (\pi(D_u) - \pi(t))) \leq \epsilon$. Let n_1 and $n_2 \in \mathbb{N}$ such that $c(S^{n_2}u) - c(S^{n_1}u) = t$. We can assume that $n_1 \leq n_2$ up to replace t by $-t$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(S^{n_0}w, u) \leq 2^{-n_2}$. Now, for all $v \in S^{\mathbb{N}}u$ such that $d(w, v) \leq 2^{-(n_0+n_2)}$, we have that $c(S^{n_0+n_1}v) \in D_u \cap (D_u - t)$, because $c(S^{n_0+n_2}v) - c(S^{n_0+n_1}v) = t$. Hence, if we let $\eta = 2^{-(n_0+n_2)}$, we have

$$\forall v, v' \in D_u, \left\{ \begin{array}{c} d(v,w) \leq \eta \\ \text{and} \\ d(v',w) \leq \eta \end{array} \right\} \Longrightarrow d(c(v), c(v')) = d(c(S^{n_0+n_1}v), c(S^{n_0+n_1}v')) \leq \epsilon.$$

This proves that we can extend c by continuity at point w. □

Lemma 2.6. — *Let $u \in A^{\mathbb{N}}$ be an infinite non-eventually periodic word over an alphabet A, and let π be a projection from $\mathbb{R}^A$ onto a hyperplane $\mathcal{P}$. We assume that we have the following conditions:*

- *the restriction of the projection π to $\mathbb{Z}^A$ is injective and has a dense image,*
- *the set $\pi(D_u)$ is bounded,*
- *for every $a \in A$, the boundary of $\overline{\pi(D_{u,a})}$ has zero Lebesgue measure,*
- *the union $\bigcup_{a\in A} \overline{\pi(D_{u,a})} = \overline{\pi(D_u)}$, is disjoint in Lebesgue measure.*

Then the natural coding cod *of $(\pi(D_u), E)$ for the partition $D_u = \bigcup_{a\in A} D_{u,a}$, can be extended by continuity to a full measure part M of the closure. And we have*

$$\forall x \in M, \quad \lim_{\substack{y\to x \\ y\in\pi(D_u)}} (\pi \circ c)^{-1}(y) = \operatorname{cod}(x).$$

Proof. — Let $\Omega = \overline{\pi(D_u)}$ and $\forall a \in A,\ \Omega_a = \overline{\pi(D_{u,a})}$. We can extend the domain exchange E in an obvious way:

$$E' : \begin{array}{ccl} \bigcup_{a\in A} \mathring{\Omega}_a & \longrightarrow & \Omega \\ x & \longmapsto & x + \pi(e_a) \text{ for } a \in A \text{ such that } x \in \mathring{\Omega}_a. \end{array}$$

The part of full Lebesgue measure that we consider is the E'-invariant set

$$M := \bigcap_{n\in\mathbb{N}} E'^{-n}\Omega.$$

Let $\epsilon > 0$ and let $x \in M$. Let $n_0 \in \mathbb{N}_{\geq 1}$ such that $2^{-n_0} \leq \epsilon$. The set

$$M_{n_0} := \bigcap_{n=0}^{n_0} E'^{-n}\Omega$$

is an open set containing x, because E' is continuous and $E'^{-1}\Omega = \bigcup_{a\in A} \mathring{\Omega}_a + e_a$ is open. Hence there exists $\eta > 0$ such that $B(x, \eta) \subseteq M_{n_0}$. And for every $y \in B(x, \eta) \cap M$, the natural coding of (M, E') for the partition $M = \bigcup_{a\in A} M \cap \Omega_a + \pi(e_a)$

coincides with the coding of x for the n_0 first steps. Hence, cod is continuous on M. We get also the last part of the lemma by observing that if $y \in B(x,\eta) \cap \pi(D_u)$, then the coding of y (which is equal to $(\pi \circ c)^{-1}(y)$) also coincide with the coding of x for the n_0 first steps. □

proof of the theorem 2.3. — The hypothesis on the projection π show that u cannot be eventually periodic. Indeed, if u was eventually periodic with a period $v \in A^*$, then the hypothesis that $\pi(D_u)$ is bounded implies that $\pi(\mathrm{Ab}(v)) = 0$, but this contradict the hypothesis that the restriction of π to $\mathbb{Z}^A$ is injective.

The lemma 2.4 shows that we can extend the map $\pi \circ c$ by continuity to a map $\overline{c} : \overline{S^{\mathbb{N}}u} \to \overline{\pi(D_u)}$. If we compose $\overline{c}$ with the natural projection π_0 onto the torus $\mathcal{P}/\pi(\Gamma_0)$, we get a continuous function which is onto, because of the equality $\pi(\Gamma_0)+\overline{\pi(D_u)} = \mathcal{P}$ that comes from $\Gamma_0 + D_u = \mathbb{Z}^A$. And we have the equality

$$\pi_0 \circ \overline{c} \circ S = T \circ \pi_0 \circ \overline{c},$$

where T is the translation by $\pi(e_a)$ (for any $a \in A$) on the torus $\mathcal{P}/\pi(\Gamma_0)$. Indeed, this equality is true on the dense subset $S^{\mathbb{N}}u$ by the proposition 2.1, and the maps π_0, S and T are continuous. This proves that the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is topologically semi-conjugate to the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T)$.

Let's consider the σ-algebra that we get from the Borel σ-algebra with the continuous map $\pi_0 \circ \overline{c} : \overline{S^{\mathbb{N}}u} \to \mathcal{P}/\pi(\Gamma_0)$. A measure μ on this σ-algebra can be defined by $\mu((\pi_0 \circ \overline{c})^{-1}(A)) = \lambda(A)$ for any Borel set A of $\mathcal{P}/\pi(\Gamma_0)$, where λ is the Lebesgue measure. By continuity, this measure μ that we get on $\overline{S^{\mathbb{N}}u}$ is S-invariant, and for this measure the subshift $(\overline{S^{\mathbb{N}}u}, S, \mu)$ is semi-conjugate to $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.

To prove that the subshift is a finite extension of the translation on the torus, it remains to show that the number of preimages by $\pi_0 \circ \overline{c}$ is bounded and almost everywhere constant. The fact that it is bounded is a consequence of the hypothesis that $\pi(D_u)$ is bounded, and because $\pi(\Gamma_0)$ is a discrete subgroup of $\mathcal{P}$. But this number of preimages is also decreasing by the translation, so by ergodicity of the translation on the torus, it is almost everywhere constant. Hence we get that the subshift is a finite extension of the translation on the torus.

If we assume moreover that the union

$$\bigcup_{t \in \pi(\Gamma_0)} \overline{\pi(D_u)} + t = \mathcal{P}$$

is disjoint in measure, then we also have that the union

$$\bigcup_{a \in A} \overline{\pi(D_{u,a})} = \overline{\pi(D_u)}$$

is disjoint in measure. Indeed, if we have $\lambda(\overline{\pi(D_{u,a})} \cap \overline{\pi(D_{u,b})}) > 0$, then we have

$$\lambda(\overline{\pi(D_u)} + \pi(e_a - e_b) \cap \overline{\pi(D_u)}) \geq \lambda(\overline{\pi(D_{u,a})} \cap (\overline{\pi(D_{u,b})} + \pi(e_a))) > 0,$$

so we have $a = b$.

Then, we obtain that the boundary of each $\overline{\pi(D_{u,a})}$, $a \in A$, has zero Lebesgue measure, since

$$\partial\overline{\pi(D_{u,a})} \subseteq \overline{\pi(D_{u,a})} \cap \left(\bigcup_{b \in A \setminus \{a\}} \overline{\pi(D_{u,b})} \cup \bigcup_{t \in \pi(\Gamma_0) \setminus \{0\}} \overline{\pi(D_u)} + t \right).$$

Hence, we can use the lemma 2.6, and it gives

$$\forall x \in M,\ x = \lim_{\substack{y \to x \\ y \in \pi(D_u)}} \pi \circ c \circ (\pi \circ c)^{-1}(y) = \overline{c} \circ \operatorname{cod}(x).$$

In the same way, we also have $\forall x \in (\pi \circ c)^{-1}(M),\ \operatorname{cod} \circ \overline{c}(x) = x$. Therefore, cod is an almost everywhere defined reciprocal of $\overline{c}$. Hence, the map $\overline{c} : \overline{S^{\mathbb{N}} u} \to \overline{\pi(D_u)}$ is a measurable conjugacy between the subshift $(\overline{S^{\mathbb{N}} u}, S, \mu)$ and the domain exchange $(\overline{\pi(D_u)}, E, \lambda)$. The map $\pi_0 \circ \overline{c} : \overline{S^{\mathbb{N}} u} \to \mathcal{P}/\pi(\Gamma_0)$ is also invertible almost everywhere and is a measurable conjugacy between the subshift $(\overline{S^{\mathbb{N}} u}, S, \mu)$ and the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.

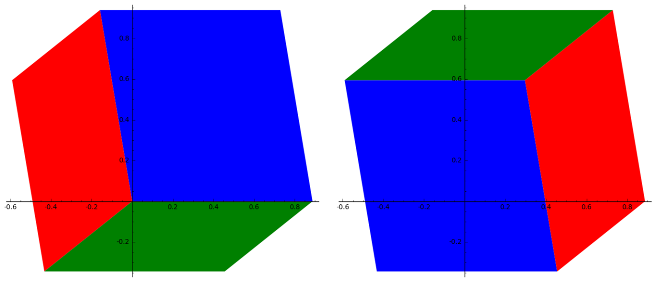
Then, the unique ergodicity of the translation on the torus implies that the subshift is also uniquely ergodic. Indeed, if μ' is an S-invariant measure of $\overline{S^{\mathbb{N}} u}$, then $\mu' \circ (\pi_0 \circ \overline{c})^{-1}$ is a T-invariant measure of the torus $\mathcal{P}/\pi(\Gamma_0)$, so it is the Lebesgue measure, and the μ'-measure of the set $(\pi_0 \circ \overline{c})^{-1}(M)$ is 1 (we assume that the measures are normalized to be probability measures). And the restriction of $\pi_0 \circ \overline{c}$ to $(\pi_0 \circ \overline{c})^{-1}(M)$ is injective and bi-continuous, thus we have $\mu' = \mu$. □

2.4. An easy example : generalization of Sturmian sequences. — An easy example where all works fine is obtained by taking a random line of $\mathbb{R}^d$ with a positive direction vector. We consider the natural $\mathbb{Z}^d$-tiling by hypercubes, and we take the sequence of hyperfaces that intersect the line. Almost surely, this gives a discrete line corresponding to some word u over the alphabet of the d type of hyperfaces. It is not difficult to see that the orthogonal projection π along the line onto a hyperplane $\mathcal{P}$ behave correctly for almost every choice of line. It gives a set whose closure tile the plane, on which a domain exchange acts. This dynamics is conjugate to the subshift generated by the word u. It is also conjugate to the translation by $\pi(e_1)$ on the torus $\mathcal{P}/\pi(\Gamma_0) \simeq \mathbb{T}^{d-1}$. Figure 2 shows the domain exchange for a line whose a direction vector is around (0.54973, 0.36490, 0.99501) in $\mathbb{R}^3$.

Remark 2.7. — *The word appearing in this last example is obtain by a simple algorithm: If the positive direction vector of the line is (v_1, v_2, v_3), and if the line goes through the point (c_1, c_2, c_3) then we have almost surely*

$$\exists (k_1, k_2, k_3) \in \mathbb{Z}^3,\ \forall j \in \{1,2,3\},\ k_j = \left\lfloor v_j \frac{k_{i_n} - c_{i_n}}{v_{i_n}} + c_j \right\rfloor$$
$$\Longrightarrow \quad \exists ! i_{n+1} \in \{1,2,3\},\ \forall j \in \{1,2,3\} \setminus \{i_{n+1}\},\ k_j = \left\lfloor v_j \frac{k_{i_{n+1}} + 1 - c_{i_{n+1}}}{v_{i_{n+1}}} + c_j \right\rfloor.$$

FIGURE 2. Domain exchange conjugate to a translation on the torus $\mathbb{T}^2$, and also conjugate to the subshift generated by a word corresponding to a discrete approximation of a line of $\mathbb{R}^3$.



The sequence $(i_n)_{n\in\mathbb{N}}$ define an infinite word over the alphabet $\{1, 2, 3\}$, and we get a bi-infinite word by invertibility of this algorithm.

3. Pure discreteness for irreducible unit Pisot substitution

In this section, we define a topology on $\mathbb{N}^A$ that permits to give a simple condition to get the pure discreteness of the spectrum of the subshift coming from an irreducible Pisot unit substitution, using the criterion of the previous section. And in the next section, we show that the reciprocal is true.

3.1. Substitutions. — Let s be a substitution (i.e. a word morphism) over a finite alphabet A of cardinality d. We denote by A^* the set of finite words over the alphabet A. Let M_s (or simply M when there is no ambiguity) be the **incidence matrix** of s. It is the $d \times d$ matrix whose coefficients are

$$m_{a,b} = |s(b)|_a \,, \ \forall (a, b) \in A^2,$$

where $|u|_a$ denotes the number of occurrences of the letter a in a word $u \in A^*$. A **periodic point** of s is a fixed point of some power of s. It is an infinite word $u \in A^{\mathbb{N}}$ such that there exists $k \in \mathbb{N}_{\geq 1}$ such that $s^k(u) = u$.

A substitution s is **primitive** if there exists a $n \in \mathbb{N}$ such that for all a and $b \in A$, the letter b appears in $s^n(a)$. A substitution s is **irreducible** if its incidence matrix is irreducible – i.e. if the degree of the Perron eigenvalue of the matrix equals the number of letters of the substitution.

If u is a periodic point of a primitive substitution, we can check that the subshift $(\overline{S^{\mathbb{N}}u}, S)$ depends only on the substitution and is minimal.

We say that a substitution s is **Pisot** if the maximal eigenvalue of its incidence matrix is a Pisot number – i.e. an algebraic integer greater than one, and whose conjugates have modulus less than one. If a substitution is Pisot irreducible, we can verify that the projection π onto a hyperplane, along the eigenspace for the

Pisot eigenvalue is bounded. We say that a Pisot number is an **unit** if its inverse is an algebraic integer. We say that a substitution is an **irreducible Pisot unit substitution** if the substitution is irreducible (i.e. the characteristic polynomial of the incidence matrix is irreducible), the highest eigenvalue of the incidence matrix is a Pisot number, and the determinant of the incidence matrix is ± 1. It is equivalent to say that the incidence matrix has only one eigenvalue of modulus greater or equal to one, and that this eigenvalue is a Pisot unit number.

3.2. Topology and main criterion. — Let A be a finite set, $\mathcal{P}$ be an hyperplane of $\mathbb{R}^A$ (for example the hyperplane of equation $\sum_{a \in A} x_a = 0$), and π be an irrational projection onto this hyperplane – that is a projection such that $\pi(\mathbb{Z}^A)$ is dense in $\mathcal{P}$. We define, for any subset S of $\mathcal{P}$, the **discrete line** of points that project to S:

$$Q_S = \left\{ x \in \mathbb{N}^A \;\middle|\; \pi(x) \in S \right\}.$$

This permits to define a topology on $\mathbb{N}^A$ by taking the following set of open sets

$$\left\{ Q_U \;\middle|\; U \text{ open subset of } \mathcal{P} \right\}.$$

And we can extends this topology to $\mathbb{Z}^A$ by considering the open sets of $\mathbb{Z}^A$

$$\left\{ S \subseteq \mathbb{Z}^A \;\middle|\; S \cap \mathbb{N}^A \text{ is an open subset of } \mathbb{N}^A \right\}.$$

Remark 3.1. — *We could define the topology directly on the whole space $\mathbb{Z}^A$ and work with bi-infinite words, but this topology permits to work with infinite words.*

Properties 3.2. — *The topology that we just defined has the following properties:*

- *If the projection π is such that $\pi(\mathbb{Z}^A)$ is dense in $\mathcal{P}$, then for any open subset U of $\mathcal{P}$, we have that $\pi(Q_U)$ is dense in U, and we have*
$$Q_U = \emptyset \Longleftrightarrow U = \emptyset.$$
- *For any subset $S \subseteq \mathcal{P}$ and any $t \in \mathbb{Z}^A$, the symmetric difference*
$$(Q_S + t) \Delta Q_{S+t}$$
is finite. In particular, we have $\overset{\circ}{Q_S} = \emptyset \Longleftrightarrow \overset{\circ}{Q}_{S+t} = \emptyset \Longleftrightarrow \overbrace{Q_S + t}^{\circ} = \emptyset$.
- *If $\det(M) \in \{-1, 1\}$, then for any subset S of $\mathcal{P}$, the symmetric difference $(MQ_S) \Delta Q_{MS}$ is finite. In particular, we have $\overbrace{MQ_S}^{\circ} = \emptyset \Longleftrightarrow \overset{\circ}{Q}_{MS} = \emptyset$.*
- *The space $\mathbb{N}^A$ is a Baire space for this topology.*

The fact that $\mathbb{N}^A$ is a Baire space follows from the fact that $\mathcal{P}$ is a Baire space, by the Baire category theorem. Indeed, if Q is a dense open set of $\mathbb{Z}^d$, then there exists a dense open set U of $\mathcal{P}$ such that $Q = Q_U$. Hence, a countable intersection of dense open subsets of $\mathbb{Z}^d$ is a dense subset of $\mathbb{Z}^d$.

This topology gives a necessary and sufficient condition for the subshift of a Pisot irreducible substitution, to have a pure discrete spectrum:

Theorem 3.3. — *Let s be an irreducible Pisot substitution over an alphabet A, and let $u \in A^{\mathbb{N}}$ be a periodic point of s. Then the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is a finite extension of the translation on a torus $\mathbb{T}^{d-1}$, where $d = |A|$. Moreover, if s is an irreducible unit Pisot substitution, then the subshift $(\overline{S^{\mathbb{N}}u}, S)$ has pure discrete spectrum if and only if $\exists a \in A,\ \overset{\circ}{D}_{u,a} \neq \emptyset$.*

3.3. Proof that an inner point implies the pure discreteness of the spectrum. — In this subsection, we prove the first statement and the sufficiency of the second statement. The necessity is proven in the next section.

First part of the proof of the theorem 3.3. — Up to replace the substitution s by a power, we can assume that the periodic point u is a fixed point. Let us show that the hypothesis of the theorem 2.3 are satisfied.

- The restriction of the projection π to $\mathbb{Z}^A$ is injective, and $\pi(\mathbb{Z}^A)$ is dense in $\mathcal{P}$: this is known for every irreducible Pisot substitution.
- The set $\pi(D_u)$ is bounded : it is well known that for any Pisot irreducible substitution, the Rauzy fractal $\overline{\pi(D_u)}$ is compact.
- The subshift $(\overline{S^{\mathbb{N}}u}, S)$ is minimal: this is true for every primitive substitution.

And we know that the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is uniquely ergodic. Hence, the theorem 2.3 gives us that the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is a finite extension of the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$.

Now if we assume that $\exists a \in A,\ \overset{\circ}{D}_{u,a} \neq \emptyset$, then we have the following

Lemma 3.4. — *We have for all $a \in A$, $\overset{\circ}{D}_{u,a} \neq \emptyset$ and $\overline{D_{u,a}} = \overline{\overset{\circ}{D}_{u,a}}$.*

Proof. — We have the equality

$$D_{u,a} = \bigcup_{b \xrightarrow{t} a} M D_{u,b} + t,$$

where $b \xrightarrow{t} a$ means that it is a transition in the automaton $\mathcal{A}^s$ (i.e. there exists words $u, v \in A^*$ such that $s(b) = uav$ and $\mathrm{Ab}(u) = t$). By primitivity, up to iterate enough this equality, every set $D_{u,b}$ appears in the union, so every set $D_{u,a}$ has non-empty interior as soon as one of them has non-empty interior.

Let c be the first letter of the fixed point u. If $x \in D_{u,a} = Q_{{}^tL^s_{c,a}}$, there exists arbitrarily large $n \in \mathbb{N}$ such that $x = Q_v$ for a word v of length n in the language ${}^tL^s_{c,a}$. And for such word v, we have $x + M^n D_{u,c} \subseteq D_{u,a}$. The sets $x + M^n D_{u,c}$ have non-empty interior, because $\det(M) \in \{-1, 1\}$, and converge to x when n tend to infinity. Hence $D_{u,a} \subseteq \overline{\overset{\circ}{D}_{u,a}}$ and this ends the proof. □

Hence, $\overset{\circ}{D}_u$ is a dense open subset of $\overline{\pi(D_u)}$. By Baire's theorem, for all $t \in \Gamma_0 \backslash \{0\}$, the empty set $\overset{\circ}{D}_u \cap (\overset{\circ}{D}_u + t)$ is a dense subset of $\overline{\overset{\circ}{D}_u} \cap (\overline{\overset{\circ}{D}_u} + t)$, therefore the sets

$\overset{\circ}{\overline{D_u}}$ and $(\overset{\circ}{\overline{D_u}} + t)$ are disjoint. This gives the wanted disjointness in measure since the boundary has zero Lebesgue measure.

The hypothesis of the theorem 2.3 are satisfied, thus the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is uniquely ergodic and measurably conjugate to the rotation by $\pi(e_a)$ on the torus $\mathcal{P}/\pi(\Gamma_0)$ with respect to the Lebesgue measure. In particular, it has pure discrete spectrum.

□

4. Algebraic coincidence ensures an inner point

In this section, we prove that pure discreteness of the subshift $(\overline{S^{\mathbb{N}}u}, S)$ ensures the non emptyness of $\overset{\circ}{D}_{u,a}$ for some a.

4.1. Algebraic coincidence of substitutive Delone set. — A Delone set is a relatively dense and uniformly discrete subset of $\mathbb{R}^d$. We say that $\mathbf{\Lambda} = (\Lambda_a)_{a\in A}$ is a *Delone multi-color set* in $\mathbb{R}^d$ if each Λ_a is a Delone set and $\cup_{a\in A}\Lambda_a \subset \mathbb{R}^d$ is Delone. Here a 'multi-set' $\mathbf{\Lambda}$ is simply a vector whose entries are Delone sets. We introduce this concept instead of taking their union, only because $\Lambda_a \cap \Lambda_b$ may not be empty for $a \neq b$. We think that each element of Λ_a has color a. A set $\Lambda \subset \mathbb{R}^d$ is a *Meyer set* if it is a Delone set and there exists a finite set F that $\Lambda - \Lambda \subset \Lambda + F$. A Delone set is a Meyer set if and only if $\Lambda - \Lambda$ is uniformly discrete in $\mathbb{R}^d$ [**Lagarias 1996**]. Note that a Meyer set has finite local complexity (FLC), i.e., for any $r > 0$ there are only finitely many transitionally inequivalent clusters (configurations of points) in a ball of radius r. $\mathbf{\Lambda} = (\Lambda_a)_{a\in A}$ is called a *substitution Delone multi-color set* if $\mathbf{\Lambda}$ is a Delone multi-color set and there exist an expansive matrix B and finite sets $\mathcal{D}_{ab}$ for $a, b \in A$ such that

$$\Lambda_a = \bigcup_{b\in A} (B\Lambda_b + \mathcal{D}_{ab}), \quad a \in A, \tag{1}$$

where the union on the right side is disjoint. The translation closure of the multi-color Delone set gives a topological dynamical system, which is minimal and uniquely ergodic if the substitution matrix $({}^{\#}\mathcal{D}_{ab})$ is primitive. Lagarias and Wang [**Lagarias Wang 2003**] proved that $|\det B|$ must be equal to the Perron Frobenius root of the substitution matrix.

Self-affine tiling dynamical system is the minimal and uniquely ergodic topological dynamical system given by a self-affine tiling with translation action (Solomyak [**Solomyak 1997**]). One can restates its translation dynamics by the translation action on the corresponding multi-colored Delone set $\mathbf{\Lambda}$ (see [**Lee Moody Solomyak 2003**]). Here a point in Λ_a represents a tile colored by a. The points are located in relatively the same position in the same colored tile. Lee [**Lee 2007**] introduced *algebraic coincidence* of substitutive multi-color Meyer set in $\mathbb{R}^d$ which is equivalent to pure discreteness of the corresponding dynamical system.

In this section, we prove that if 1-dimensional substitutive Meyer set associated to an irreducible Pisot unit substitution satisfy the algebraic coincidence, then there exists $a \in A$ such that $\overset{\circ}{D}_{u,a} \neq \emptyset$, which completes the proof of the main theorem.

4.2. Substitutive Meyer set from D_u. — Let s be a primitive substitution over an alphabet A whose substitution matrix is M. Assume that $v = \dots v_n \dots v_1$ and $u = u_1 \dots u_n \dots$ are one-sided infinite words that $vu \in A^{\mathbb{Z}}$ is a 2-sided fixed point of s. The u (resp. v) is a right (resp. left) infinite fixed of s and $v_1 u_1$ is the subword of $\sigma^n(a)$ for some $n \in \mathbb{N}$ and $a \in A$. The left abelianisation $D_{v,a}$ is defined by

$$\left\{ -\sum_{i=1}^{n} e_{v_i} \,\middle|\, v_n = a \right\}.$$

and $D_{vu,a} = D_{v,a} \cup D_{u,a}$. As s is a substitution, $D_{u,a}$ (resp. $D_{v,a}$) has one to one correspondence to the words $\{u_1 \dots u_n |\ n \in \mathbb{N}\}$ (resp. $\{v_n \dots v_1 |\ n \in \mathbb{N}\}$). We also define $D_{vu} = \bigcup_{a \in A} D_{vu,a}$. Then D_{vu} is a geometric realization of the fixed point $vu \in A^{\mathbb{Z}}$, that is, the set of vertices of a broken line naturally generated by corresponding fundamental unit vectors e_a ($a \in A$).

We project this broken line to make a self-similar tiling of the real line by tiles (intervals) corresponding to each letter. This is done by associating intervals whose lengths are given by the entry of a left eigenvector $\ell = (\ell_a)_{a \in A}$. The corresponding expanding matrix is of size 1 and equal to the Perron Frobenius root of M. Define $\psi : D_{u,v} \to \mathbb{R}$ by $\psi(\sum_i^n e_{u_i}) = \sum_{i=1}^n \ell_{u_i}$ and $\psi(-\sum_i^n e_{v_i}) = -\sum_{i=1}^n \ell_{v_i}$ according to the domain D_u or D_v. Put $\Lambda_a = \{\psi(v) |\ v \in D_{u,a}\}$ for $a \in A$. We normalize the eigenvector ℓ so that ψ becomes the orthogonal projection to the 1-dimensional subspace $\pi^{-1}(0)$ generated by the expanding vector of M. Then this is exactly the set of left end points of intervals which consists the tiling. It is clear that $\psi : D_{u,a} \to \Lambda_a$ is bijective and preserves addition structure, i.e., if $x \pm y \in D_{u,a}$ for $x, y \in D_{u,a}$ then $\psi(x \pm y) = \psi(x) \pm \psi(y)$ holds in Λ_a and vice versa. By this choice of the length, $\mathbf{\Lambda} = (\Lambda_a)_{a \in A}$ forms a substitution multi-colored Delone set. When s is a Pisot substitution, $\mathbf{\Lambda}$ is a substitution multi-colored Meyer set. The closure of the set of translations $\{\mathbf{\Lambda} - t |\ t \in \mathbb{R}\}$ by local topology forms a compact set X and $(X, \mathbb{R})$ is a topological dynamical system. By primitivity of s, this system is minimal and uniquely ergodic. Moreover the system $(X, \mathbb{R})$ is not weakly mixing if and only if s is a Pisot substitution ([**Solomyak 1997**]). Clark and Sadun [**Clark Sadun 2003**] showed that if s is an irreducible Pisot substitution, then $(X, \mathbb{R})$ shows pure discrete spectrum if and only if $(\overline{S^{\mathbb{N}} u}, S)$ does. Therefore we can use techniques developed in the tiling dynamical system to our problem.

4.3. Algebraic coincidence for D_u. — In this setting the algebraic coincidence in [**Lee 2007**] reads

$$\exists a\ \exists n \in N\ \exists \eta' \in \mathbb{R} \quad \beta^n \bigcup_{a\in A}(\Lambda_a - \Lambda_a) \in \Lambda_a - \eta' \tag{2}$$

and the projection ψ is bijective, (2) is equivalent to

$$\exists a\ \exists n \in N\ \exists \eta \in \mathbb{R}^d \quad M^n(\cup_{a\in A}(D_{u,a} - D_{u,a})) \in D_{u,a} - \eta. \tag{3}$$

Clearly we see $\eta \in D_{u,a}$. By primitivity of s, we easily see that for any $a, b \in A$, there exists $k \in \mathbb{N}$ that

$$\beta^k(\Lambda_a - \Lambda_a) \subset (\Lambda_b - \Lambda_b) \tag{4}$$

Yet we need another result depending heavily on irreducibility of substitution:

Lemma 4.1. — [**Sing 2006, Barge Kwapisz 2006**] *Let s be a primitive irreducible substitution and $\mathbf{\Lambda}$ be an associated substitution Delone multi-color set in $\mathbb{R}$. Then we have*

$$\left\langle \bigcup_{a\in A}(\Lambda_a - \Lambda_a) \right\rangle = \left\langle (\bigcup_{a\in A}\Lambda_a) - (\bigcup_{a\in A}\Lambda_a) \right\rangle. \tag{5}$$

Here $\langle X \rangle$ stands for the additive subgroup of $\mathbb{R}^d$ generated by the set described in X. Since

$$\psi^{-1}\left(\left\langle (\bigcup_{a\in A}\Lambda_a) - (\bigcup_{a\in A}\Lambda_a) \right\rangle\right)$$

contains all fundamental unit vector e_a ($a \in A$), it clearly coincides with $\mathbb{Z}^d$. Therefore Lemma 4.1 implies

$$\left\langle \bigcup_{a\in A}(D_{u,a} - D_{u,a}) \right\rangle = \mathbb{Z}^d \tag{6}$$

4.4. Proof of the existence of an inner point. — Note that the substitution matrix M is contained in $GL(d, \mathbb{Z})$, because s is a Pisot unit substitution.

Without loss of generality, we assume that u begins with $a \in A$, which implies $0 \in D_{u.a}$. We will prove that that there exists $N \in \mathbb{N}$ that $\pi(\eta)$ is an inner point of $D_{u,a}$ where $\eta \in D_{u,a}$ appeared in (3).

Let

$$\varphi : \begin{array}{ccc} \mathcal{P}(\mathbb{Z}^A) & \to & \mathcal{P}(\mathbb{Z}^A) \\ S & \mapsto & M^n(S - S) \end{array} \qquad \mathcal{D} : \begin{array}{ccc} \mathcal{P}(\mathbb{Z}^A) & \to & \mathcal{P}(\mathbb{Z}^A) \\ S & \mapsto & S - S \end{array},$$

where $n \in \mathbb{N}$ is such that

$$\forall b \in A,\ M^n(D_{u,b} - D_{u,b}) \subseteq D_{u,a} - \eta.$$

Lemma 4.2. — *For all $k \in \mathbb{N}_{\geq 1}$, we have*

$$\varphi^k \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right) \subseteq D_{u,a} - \eta.$$

Proof. — Easy, by induction. □

Lemma 4.3. — *Let $P \subseteq \mathbb{Z}^A$ such that $\psi(P)$ is relatively dense in $\mathbb{R}_+$ and $\pi(P)$ is bounded. Then, there exists $R > 0$ such that*

$$Q_{B(0,1)} \subseteq \bigcup_{x \in P} B(x, R),$$

where $B(x, R)$ is the ball of $\mathbb{Z}^A$ of center x and radius R.

Proof. — Let $M > 0$ such that $\forall x \in \mathbb{R}_+,\ d(x, \psi(P)) \leq M$. There exists $C_1 > 0$ and $C_2 > 0$ such that

$$\forall (x, y) \in (\mathbb{R}^A)^2,\ d(x, y) \leq C_1 d(\psi(x), \psi(y)) + C_2 d(\pi(x), \pi(y))$$

depending on the choice of the left eigenvector for the map ψ, and the choice of the linear projection π. We choose $R = C_1 M + C_2(\mathrm{diam}(\pi(P \cup \{0\})) + 1)$. Let $x \in Q_{B(0,1)}$, then we have $\psi(x) \in \mathbb{R}_+$, and for $y \in P$ such that $d(x, P) = d(x, y)$ we have

$$d(x, y) \leq C_1 d(\psi(x), \psi(P)) + C_2 d(\pi(x), \pi(y)) \leq C_1 M + C_2(\mathrm{diam}(\pi(P \cup \{0\})) + 1) \leq R.$$

Therefore, we have $x \in \bigcup_{y \in P} B(y, R)$. □

Lemma 4.4. — *There exists $N \in \mathbb{N}$ such that*

$$B(0, 1) \subseteq \mathcal{D}^N \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right),$$

where $B(0, 1)$ is the unit ball of $\mathcal{P}$.

Proof. — The set $\psi(D_{u,a})$ is relatively dense in $\mathbb{R}_+$ and $\pi(D_{u,a})$ is bounded. So, by lemma 4.3, there exists a $R > 0$ such that

$$Q_{B(0,1)} \subseteq \bigcup_{x \in D_{u,a}} B(x, R).$$

And we have, by (6),

$$\bigcup_{N \in \mathbb{N}} \mathcal{D}^N \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right) = \left\langle \bigcup_{b \in A} (D_{u,b} - D_{u,b}) \right\rangle = \mathbb{Z}^A.$$

Thus there exists $N \in \mathbb{N}_{\geq 3}$ large enough such that

$$B(0, R) \subseteq \mathcal{D}^{N-1} \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right),$$

where $B(0,R)$ is the ball of $\mathbb{Z}^A$ of center 0 and radius R. Then, we have

$$Q_{B(0,1)} \subseteq \bigcup_{x \in D_{u,a}} B(x,R) \subseteq B(0,R) - \mathcal{D}^2(D_{u,a}) \subseteq \mathcal{D}^N \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right).$$

□

Using these lemmas, we have the inclusion

$$M^{nN} Q_{B(0,1)} \subseteq M^{nN} \mathcal{D}^N \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right) = \varphi^N \left(\bigcup_{b \in A} D_{u,b} - D_{u,b} \right) \subseteq D_{u,a} - \eta.$$

And this implies that $D_{u,a}$ contains $M^{nN} Q_{B(0,1)} + \eta$, therefore it has non-empty interior.

5. Computation of the interior

In this section, we show that the interior of some subsets of $\mathbb{Z}^d$, for the topology defined in the subsection 3.2, can be described by a computable regular language. This gives a way to decide the Pisot substitution conjecture for any given irreducible Pisot unit substitution.

5.1. Regular languages. — Let Σ be a finite set, and let $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$ be the set of finite words over the alphabet Σ. A subset of $\mathcal{P}(\Sigma^*)$, is called a **language** over the alphabet Σ. We say that a language L over an alphabet Σ is **regular** if the set

$$\left\{ u^{-1} L \;\middle|\; u \in \Sigma^* \right\}$$

is finite, where $u^{-1}L := \left\{ v \in \Sigma^* \;\middle|\; uv \in L \right\}$.

An **automaton** is a quintuplet $\mathcal{A} = (\Sigma, Q, I, F, T)$, where Σ is a finite set called **alphabet**, Q is a finite set called **states**, $I \subseteq Q$ is the set of **initial states**, $F \subseteq Q$ is the set of **final states**, and $T \subseteq Q \times \Sigma \times Q$ is the set of **transitions**. We denote by $p \xrightarrow{t} q$ a transition $(p,t,q) \in T$, and we will write

$$q_0 \xrightarrow{t_1} q_1 \xrightarrow{t_2} \ldots \xrightarrow{t_n} q_n \in T$$

when for all $i = 1, 2, \ldots, n$ we have $(q_{i-1}, t_i, q_i) \in T$. We call **language recognized** by $\mathcal{A}$ the language $L_{\mathcal{A}}$ over the alphabet Σ defined by

$$L_{\mathcal{A}} = \left\{ u \in \Sigma^* \;\middle|\; \exists (q_i)_i \in Q^{|u|+1},\ q_0 \in I, q_{|u|} \in F, \text{ and } q_0 \xrightarrow{u_1} q_1 \xrightarrow{u_2} \ldots \xrightarrow{u_{|u|}} q_{|u|} \in T \right\}.$$

The following proposition is a classical result about regular languages
(see [**KN, HU, Sa, Carton**]).

Proposition 5.1. — *A language is regular if and only if it is the language recognized by some automaton.*

We say that an automaton is deterministic if I has cardinality one, and if for every state $q \in Q$ and every letter $t \in \Sigma$, there exists at most one state $q' \in Q$ such that $(q, t, q') \in T$ is a transition.

The **minimal automaton** of a regular language L, is the unique deterministic automaton recognizing L and having the minimal number of states. Such automaton exists, is unique, and the number of states is equal to the cardinal of the set $\left\{u^{-1}L \;\middle|\; u \in \Sigma^*\right\} \setminus \{\emptyset\}$. To an automaton, we can associate the **adjacency matrix** in $M_Q(\mathbb{Z})$ whose (s', s) coefficient is the number of transitions from state s to state s'. We denote by ${}^tL$ the mirror of a language L.

$${}^tL := \left\{u_n u_{n-1} ... u_1 u_0 \;\middle|\; u_0 u_1 ... u_{n-1} u_n \in L\right\}.$$

5.2. Discrete line associated to a regular language. — Given a word u over an alphabet $\Sigma \subseteq \mathbb{Z}^d$, and a matrix $M \in M_d(\mathbb{Z})$, we define

$$Q_{u,M} = \sum_{k=0}^{|u|-1} M^i u_i.$$

Given a language L over an alphabet $\Sigma \subseteq \mathbb{Z}^d$, and a matrix $M \in M_d(\mathbb{Z})$, we define the following subset of $\mathbb{Z}^d$.

$$Q_{L,M} = \left\{Q_{u,M} \;\middle|\; u \in L\right\} = \left\{\sum_{k=0}^{|u|-1} M^i u_i \;\middle|\; u \in L\right\}.$$

We will also call this set a discrete line, because when M has a Pisot number as eigenvalue and no other eigenvalue of modulus greater than one, then this set stays at bounded distance of a line of $\mathbb{R}^d$ – line which is the eigenspace of the matrix M for the Pisot eigenvalue. And we show now that every discrete line coming from a substitution is also the discrete line of some regular language. When it will be clear from the context what is the matrix, we will simply write Q_u and Q_L.

Remark 5.2. — *The notation Q_S was also defined for a part $S \subseteq \mathcal{P}$, but there is no ambiguity, because parts of $\mathcal{P}$ and languages are always different objects, and we use the same notation because in both cases it represents a discrete line.*

To a substitution s over the alphabet A, and $a, b \in A$, we associated the following deterministic automaton $\mathcal{A}^s_{a,b}$ with

- — set of states A,
- — initial state a,
- — set of final states $\{b\}$,
- — alphabet $\Sigma = \left\{t \in \mathbb{Z}^d \;\middle|\; \exists (c, u, v) \in A \times A^* \times A^*,\ s(c) = uv \text{ with } \mathrm{Ab}(u) = t \text{ and } |v| > 0\right\}$,
- — set of transitions $T = \left\{(c, t, d) \in A \times \Sigma \times A \;\middle|\; \exists u, v \in A^*,\ s(c) = udv \text{ and } \mathrm{Ab}(u) = t\right\}$.

We denote by $L^s_{a,b}$ the language of this automaton. We denotes by $\mathcal{A}^s$ the automaton $\mathcal{A}^s_{a,b}$ where we forget the data of the initial state and the set of final states.

Remark 5.3. — *This automaton is the abelianisation of what we usually call the prefix automaton.*

Proposition 5.4. — *If u is a fixed point of a substitution s whose first letter is a, then we have*

$$D_{u,b} = Q_{{}^tL^s_{a,b}, M_s}.$$

Remark 5.5. — *This proposition corresponds to write elements of the discrete line $D_{u,b}$ using the Dumont-Thomas numeration.*

Remark 5.6. — *If we want to describe the left infinite part of the discrete line associated to a bi-infinite fixed point of the substitution s, we have to consider the automata $\mathcal{A}^{{}^ts}_{a,b}$ where ${}^ts$ is the reverse substitution of s – that is $\forall a \in A$, ${}^ts(a) = {}^t(s(a))$. We can also describe a bi-infinite discrete line with only one automaton over the bigger alphabet $\Sigma_s \cup -\Sigma_{{}^ts}$.*

Remark 5.7. — *The automaton $\mathcal{A}^s$ permits to compute easily the map $E_1(s)$ defined in* [**Arnoux Ito 2001**]*:*

$$E_1(s)(x, e_a) = \sum_{a \xrightarrow{t} b \in T} (Mx + t, e_b),$$

where T is the set of transitions of $\mathcal{A}^s$.

We can also compute easily the map $E_1^*(s)$ when $\det(M) \in \{-1, 1\}$*:*

$$E_1^*(s)(x, e_b^*) = \sum_{a \xrightarrow{t} b \in T} (M^{-1}(x - t), e_a^*).$$

And we have

$$(y, e_b) \in E_1(s)(x, e_a) \iff a \xrightarrow{y-Mx} b \in T \iff (x, e_a^*) \in E_1^*(s)(y, e_b^*).$$

5.3. Computation of the interior. — We have seen in the previous section that the subshift associated to an irreducible Pisot substitution has pure discrete spectrum as soon as the interior of a piece of the discrete line is non-empty (see theorem 3.3), for the topology defined in subsection 3.2. In this section, we give a way to compute the interior (and hence to test the Pisot substitution conjecture) with the following

Theorem 5.8. — *Let L be a regular language over an alphabet $\Sigma \subseteq \mathbb{Z}^A$, M be an irreducible Pisot unimodular matrix, and π be the projection on a hyperplane $\mathcal{P}$ along the eigenspace of M for its maximal eigenvalue β. Then, there exists a regular language $\overset{\circ}{L} \subseteq L$ such that $Q_{\overset{\circ}{L}} = \overset{\circ}{Q}_L$. Moreover, this language $\overset{\circ}{L}$ is computable from L.*

Remark 5.9. — *The language $\overset{\circ}{L}$ doesn't depend on the choice of the hyperplane $\mathcal{P}$.*

With this theorem, the criterion given by the theorem 3.3 gives the following result:

Corollary 5.10. — *Let s be an irreducible Pisot unit substitution over an alphabet A. If there exist letters $a, b \in A$ such that a is the first letter of a fixed point of s, and the regular language ${}^t\overset{\circ}{L^s}_{a,b}$ is non-empty, then the subshift $(\overline{S^{\mathbb{N}}u}, S)$ has pure discrete spectrum.*

And, the Pisot substitution conjecture is equivalent to

Conjecture 5.11. — *For any irreducible Pisot unit substitution s over an alphabet A and for any letters $a, b \in A$, the regular language ${}^t\overset{\circ}{L^s}_{a,b}$ is non-empty.*

5.4. Proof of the theorem 5.8. — In order to compute the interior, we need a big enough alphabet.

Lemma 5.12. — *For any Pisot unit primitive matrix $M \in M_d(\mathbb{N})$, there exists $\Sigma' \subseteq \mathbb{Z}^A$ such that $0 \in \overset{\circ}{Q}_{\Sigma'^*, M}$.*

Proof. — Let's consider any substitution s whose incidence matrix is the irreducible unit Pisot matrix M. Let u be a periodic point for this substitution. We know that $\pi(D_u)$ is bounded and is a fundamental domain for the action of the lattice $\pi(\Gamma_0)$ on $\pi(\mathbb{Z}^A)$, where Γ_0 is the subgroup of $\mathbb{Z}^A$ spanned by $e_a - e_b$, $a, b \in A$. Hence, there exists a finite subset $S \subseteq \Gamma_0$ such that $D_u + S = \{x + y \mid (x, y) \in D_u \times S\}$ contains zero in its interior. Then, the alphabet $\Sigma' = \Sigma_s + S$ satisfy that $0 \in \overset{\circ}{Q}_{\Sigma' \Sigma_s^*} \subseteq \overset{\circ}{Q}_{\Sigma'^*}$. $\square$

The alphabet given by this lemma is not optimal. Here are two conjectures that gives natural choices of alphabet. The first one gives an alphabet of minimal size, and the second one gives the alphabet Σ that naturally comes from the substitution.

Conjecture 5.13. — *For all irreducible unit Pisot matrix M with spectral radius β, we have $0 \in \overset{\circ}{Q}_{\Sigma'^*}$, for $\Sigma' = \{-1, 0, 1, 2, ..., \lceil \beta \rceil - 2\}$.*

Conjecture 5.14. — *For all irreducible unit Pisot substitution s, we have $\overset{\circ}{Q}_{\Sigma_s{}^*} \neq \emptyset$.*

Remark 5.15. — *This last conjecture is a consequence of the Pisot substitution conjecture. But it should be easier to solve.*

Remark 5.16. — *We cannot assume in this last conjectures that the interior always contains 0, since we can only get the positive part of the hyperplane $\mathcal{P}$ with Pisot numbers whose conjugates are positive reals numbers. Nevertheless, if we have only $\overset{\circ}{Q}_{\Sigma'^*} \neq \emptyset$, then the set L_{int} computed in the proof of the theorem 5.8 satisfy*

$$\overset{\circ}{Q}_L \subseteq Q_{L_{int}} \subseteq \overline{\overset{\circ}{Q}_L},$$

so we have $\overset{\circ}{Q}_L = \emptyset \iff L_{int} = \emptyset$. Hence we can decide if Q_L has empty interior or not by computing L_{int} with this alphabet Σ'.

The following theorem is also useful to compute the interior. It is a variant of the main theorem of [**Mercat 2013**].

Theorem 5.17. — *Consider two alphabets Σ and Σ' in $\mathbb{Z}^A$, and a matrix $M \in M_A(\mathbb{Z})$ without eigenvalue of modulus one. Then the language*

$$L^{\mathrm{rel}} := \left\{(u,v) \in (\Sigma' \times \Sigma)^* \;\middle|\; Q_u = Q_v\right\}$$

is regular.

Remark 5.18. — *This language L^{rel} is related to what is usually called the zero automaton. See* [**Frou. Pel. 2017**] *and* [**Frou. Sak. 2010**] *for more details.*

proof of the theorem 5.8. — Consider the language

$$L_{int} := Z(S(Z(p_1(\Sigma'^* \times L0^* \cap L^{\mathrm{rel}})))),$$

where

— Σ' is an alphabet given by the lemma 5.12 and containing 0,
— $p_1 : (\Sigma' \times \Sigma)^* \to \Sigma'^*$ is the word morphism such that $\forall (x,y) \in \Sigma' \times \Sigma,\ p_1((x,y)) = x$,
— L^{rel} is the language defined in theorem 5.17,
— for any language L over the alphabet Σ', $S(L) := \left\{u \in \Sigma'^* \;\middle|\; u\Sigma'^* \subseteq L\right\}$,
— for any language L over the alphabet Σ', $Z(L) := \left\{u \in \Sigma'^* \;\middle|\; \exists n \in \mathbb{N},\ u0^n \in L\right\}$.

Then, we have

$$Q_{L_{int}} = \overset{\circ}{Q}_L,$$

and we have that the language L_{int} is "complete", that is:

$$L_{int} = \left\{u \in \Sigma'^* \;\middle|\; Q_u \in Q_{L_{int}}\right\}.$$

Indeed, for all $u \in \Sigma'^*$ we have

$$\begin{aligned}
u \in L_{int} &\iff \exists n \in \mathbb{N},\ u0^n \in S(Z(p_1(\Sigma'^* \times L0^* \cap L^{\mathrm{rel}}))),\\
&\iff \exists n \in \mathbb{N},\ u0^n\Sigma'^* \subseteq Z(p_1(\Sigma'^* \times L0^* \cap L^{\mathrm{rel}})),\\
&\iff \exists n \in \mathbb{N},\ \forall v \in \Sigma'^*,\ \exists k \in \mathbb{N},\ u0^n v0^k \in p_1(\Sigma'^* \times L0^* \cap L^{\mathrm{rel}}),\\
&\iff \exists n \in \mathbb{N},\ \forall v \in \Sigma'^*,\ \exists k \in \mathbb{N},\ \exists w \in L0^*,\ (u0^n v0^k, w) \in L^{\mathrm{rel}},\\
&\iff \exists n \in \mathbb{N},\ \forall v \in \Sigma'^*,\ Q_{u0^n v} \in Q_L,\\
&\iff \exists n \in \mathbb{N},\ Q_u + M^{n+|u|} Q_{\Sigma'^*} \subseteq Q_L,\\
&\iff Q_u \in \overset{\circ}{Q}_L.
\end{aligned}$$

But we can assume that $\Sigma \subseteq \Sigma'$ up to replace Σ' by $\Sigma \cup \Sigma'$. Then, we get the language $\overset{\circ}{L}$ by taking the intersection with L:

$$\overset{\circ}{L} = L \cap L_{int}.$$

This language verify what we want because $Q_{L_{int}} \subseteq Q_L$ and because L_{int} is complete. □

Remark 5.19. — *If we just want to test the non-emptiness of the language $\overset{\circ}{L}$, it is not necessary to compute all what is done in this proof. For example, the computation of the language L_{int} is enough (and we do not need that $\Sigma \subseteq \Sigma'$). And we don't even need to compute completely L_{int} if we only want to test if it is non-empty. And it is enough to have Σ' such that $Q_{\Sigma'^*}$ has non-empty interior.*

5.5. Examples. —

Example 5.20. — *For the Fibonnacci and for the Tribonnacci substitutions, we get ${}^t\overset{\circ}{L^s}_{a,b} = {}^tL^s_{a,b}$, for a the first letter of the fixed point u, and any letter b. Therefore the sets $D_{u,b}$ are open: $\overset{\circ}{D}_{u,b} = D_{u,b}$ (and we can check that they are also closed).*

Example 5.21. — *For the "flipped" Tribonnacci substitution:*

$$a \mapsto ab$$
$$b \mapsto ca$$
$$c \mapsto a$$

the minimal automaton of the language ${}^t\overset{\circ}{L^s_{a,a}}$ has 79 states (80 states for ${}^t\overset{\circ}{L^s_{a,b}}$, 81 for ${}^t\overset{\circ}{L^s_{a,c}}$). This automaton is plotted in figure 3, and the sets $\pi(D_{u,a})$ and $\pi(\overset{\circ}{D}_{u,a})$ for the fixed point u are drawn in figure 4.

FIGURE 3. Minimal automaton of the language ${}^t\overset{\circ}{L^s_{a,a}}$ of the example 5.21. The labels 0 correspond to the null vector, the labels 1 correspond to the vector e_a, and the labels $b^2 - b - 1$ correspond to the vector e_c. Final states are the double circles, and the initial state is the bold circle.

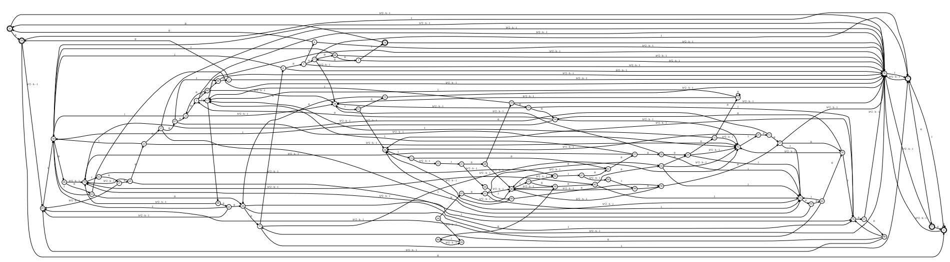
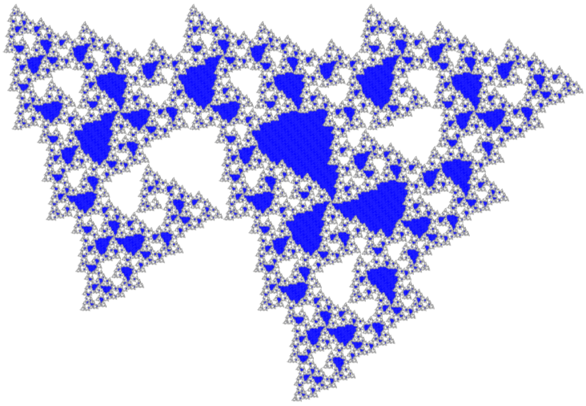


FIGURE 4. The sets $\pi(D_{u,a})$ (in gray and blue) and $\pi(\mathring{D}_{u,a})$ (in blue) for the example 5.21



Example 5.22. — *For the following substitution associated to the smallest Pisot number:*

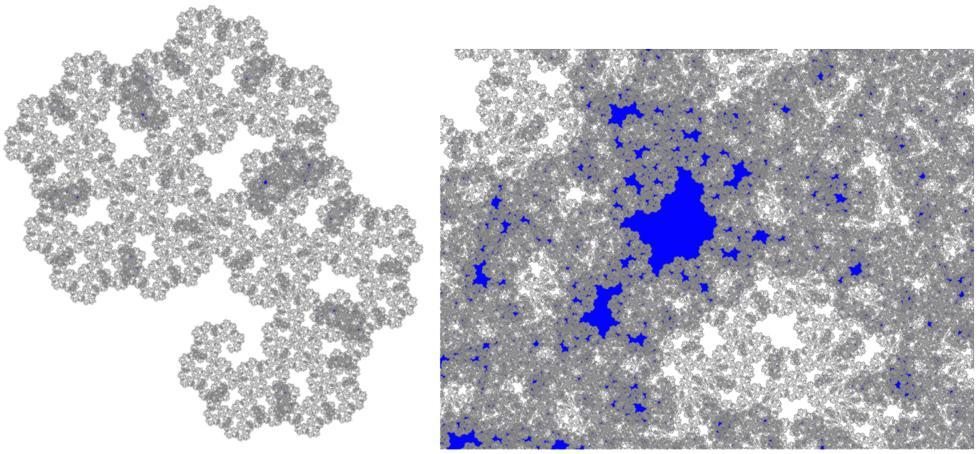
$$\begin{aligned} a &\mapsto b \\ b &\mapsto c \\ c &\mapsto ab \end{aligned}$$

the minimal automaton of the language ${}^t\mathring{L}^s_{a,a}$ *has* 1578 *states (*1576 *states for* ${}^t\mathring{L}^s_{a,b}$*,* 1577 *for* ${}^t\mathring{L}^s_{a,c}$*). The sets* $\pi(D_{u,a})$ *and* $\pi(\mathring{D}_{u,a})$ *are plotted on figure 5, where* u *is the periodic point starting by letter* a*.*

Remark 5.23. — *The first author have implemented the computing of the interior in the Sage mathematical software. The above examples has been computed using this implementation which is partially available here :* `https://trac.sagemath.org/ticket/21072`*. Unfortunately these tools are not easy to install and not well documented for the moment.*

Remark 5.24. — *To prove the Pisot substitution conjecture, it is enough for each irreducible Pisot substitution* s *and for any letter* a*, to find one particular "canonical" word in the language* ${}^t\mathring{L}^s_{a,a}$ *in order to prove it is non-empty.*

FIGURE 5. The sets $\pi(D_{u,a})$ (in gray and blue) and $\pi(\overset{\circ}{D}_{u,a})$ (in blue) for the example 5.22. Whole set at the left, and a zoom on it at the right.



6. Pure discreteness for various infinite family of substitutions

6.1. Proof of pure discreteness using a geometrical argument. — Using the theorem 3.3, we can prove the Pisot substitution conjecture for a new infinite family of substitutions:

Theorem 6.1. — *Let $k \in \mathbb{N}$, and let*

$$s_k : \begin{cases} a \mapsto a^k bc \\ b \mapsto c \\ c \mapsto a \end{cases}$$

where a^k means that the letter a is repeated k times. The subshift generated by the substitution s_k is measurably conjugate to a translation on the torus $\mathbb{T}^2$.

Proof. — The strategy of the proof is to use the theorem 3.3. For $k \geq 1$, let u be the fixed point of s_k starting with letter a. We show that $\pi(\overset{\circ}{D}_{u,a}) \neq \emptyset$ by showing that the point

$$t_k := \frac{k}{2} - \frac{\sqrt{k}}{2} I$$

is not in the closure of $\pi(\mathbb{Z}^A \backslash D_a)$:

$$t_k \notin \bigcup_{l \in A \backslash \{a\}} \overline{\pi(D_{u,l})} \cup \bigcup_{t \in \Gamma_0 \backslash \{0\}} \overline{\pi(D_u + t)},$$

FIGURE 6. Rauzy fractal of s_{20}

where $A = \{a, b, c\}$ is the alphabet of the substitution s_k, Γ_0 is the group generated by $(e_i - e_j)_{i,j \in A}$, where $(e_i)_{i \in A}$ is the canonical basis of $\mathbb{R}^A$, I denotes a complex number such that $I^2 = -1$, and π is the projection along the eigenspace for the maximal eigenvalue of the incidence matrix

$$M = \begin{pmatrix} k & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

of the substitution s_k, such that $\pi(e_a) = 1$, $\pi(e_b) = -\beta^2 + (k+1)\beta - (k-1)$ and $\pi(e_c) = \beta^2 - k\beta - 1$, where β is the complex eigenvalue of M such that $\mathrm{Im}(\beta) < 0$.

In order to do that, we approximate the sets $D_{u,l}$ by union of balls :

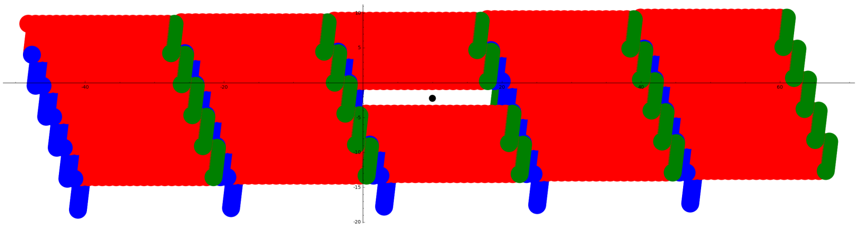


FIGURE 7. Strategy to prove that $t_k \notin \overline{\pi(\mathbb{Z}^d \setminus D_{u,a})}$
Approximation of the sets $\pi(D_{u,l})$ and their translated copies, by disks, for $k = 20$

Lemma 6.2. — *For all $k \geq 3$ and for every $l \in A$, we have the inclusion*

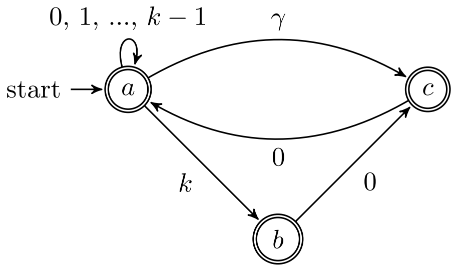
$$\overline{\pi(D_{u,l})} \subseteq \bigcup_{t \in S_l} B(t, \frac{1}{1 - \frac{1}{\sqrt{k}}}),$$

where

$$
\begin{aligned}
S_a &= \{\gamma\beta\} \cup \left\{ i+\beta j \;\middle|\; (i,j) \in \{0,1,...,k-1\}^2 \right\}, \\
S_b &= \left\{ k+\beta i \;\middle|\; i \in \{0,1,...,k-1\} \right\}, \\
S_c &= \{k\beta\} \cup \left\{ \gamma+\beta i \;\middle|\; i \in \{0,1,...,k-1\} \right\},
\end{aligned}
$$

where $\gamma = -\beta^2 + (k+1)\beta + 1 = \beta - \frac{1}{\beta}$.

FIGURE 8. Automaton describing $\pi(D_u)$



Proof. — For every $l \in A$, we have the equality

$$\pi(D_{u,l}) = \left\{ \sum_{k=0}^{|u|-1} u_i \beta^i \;\middle|\; u \in {}^tL_l \right\}$$

where L_l is the language of the automaton of Figure 8 where we replace the set of final states by $\{l\}$.

We get the proof of the lemma by considering words of length two, and by the inequality

$$\left| \sum_{k=2}^{|u|-1} u_i \beta^i \right| \leq \sum_{k=2}^{|u|-1} \max\left\{ |t| \;\middle|\; t \in \Sigma \right\} |\beta|^i \leq \frac{1}{1-\frac{1}{\sqrt{k}}}$$

for any word u over the alphabet Σ, where $\Sigma = \{0,1,...,k-1,k,\gamma\}$ is the alphabet of the languages L_l. Indeed, we have $\max\left\{ |t| \;\middle|\; t \in \Sigma \right\} = k$ and $|\beta| \leq \frac{1}{\sqrt{k}}$ for $k \geq 3$. $\square$

Lemma 6.3. — *For every $k \geq 1$, we have the inequalities*

$$\frac{1}{\sqrt{k+\frac{2}{k}}} < |\beta| < \frac{1}{\sqrt{k}}$$

$$\sqrt{k} - \frac{1}{\sqrt{k}} < |\gamma| < \sqrt{k+\frac{2}{k}} + \frac{1}{\sqrt{k}}$$

$$-\frac{1}{k} < \mathrm{Re}(\beta) < -\frac{1}{k+\frac{2}{k}}$$

$$-\frac{1}{\sqrt{k}} < \mathrm{Im}(\beta) < -\frac{1}{\sqrt{k+\frac{2}{k}}} + \frac{1}{k}$$

where $\mathrm{Re}(\beta)$ *is the real part of* β*, and* $\mathrm{Im}(\beta)$ *is the imaginary part.*

Proof. — Let β_+ be the real conjugate of β. We have $\beta_+ = k + \frac{1}{\beta_+} + \frac{1}{\beta_+^2} > 0$, so

$$k \leq \beta_+ \leq k + \frac{2}{k}.$$

And we have $|\beta|^2 = \frac{1}{\beta_+}$, hence we get the wanted inequalities for $|\beta|$. The inequalities for $\gamma = \beta - \frac{1}{\beta}$ follow. To get the real part, remarks that we have $k = \beta_+ + \beta + \overline{\beta} = \beta_+ + 2\,\mathrm{Re}(\beta)$, and this gives $\mathrm{Re}(\beta) = -\frac{1}{2\beta_+} - \frac{1}{2\beta_+^2}$.The inequalities for the imaginary part follow. □

Lemma 6.4. — *For all* $k \geq 14$*, we have* $t_k \notin \overline{\pi(D_{u,b})}$

Proof. — For all $i \in \{0, 1, ..., k-1\}$, we have $|k + i\beta - t_k| = \left|\frac{k}{2} + i\beta + \frac{\sqrt{k}}{2} I\right| \geq \frac{k}{2} - |i\beta| - \frac{\sqrt{k}}{2} \geq \frac{k}{2} - \frac{3\sqrt{k}}{2}$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 14$. □

Lemma 6.5. — *For all* $k \geq 31$*, we have* $t_k \notin \overline{\pi(D_{u,c})}$

Proof. — We have $|\gamma + i\beta - t_k| = \left|-\frac{k}{2} + \gamma + i\beta + \frac{\sqrt{k}}{2} I\right| \geq \frac{k}{2} - |i\beta| - |\gamma| - \frac{\sqrt{k}}{2} \geq \frac{k}{2} - \frac{3\sqrt{k}}{2} - \sqrt{k + \frac{2}{k}} - \frac{1}{\sqrt{k}}$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 31$.

We have $|k\beta - t_k| \geq \frac{k}{2} - \frac{3\sqrt{k}}{2}$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 14$. □

Let us show now that the point t_k is not in the translated copies of $\overline{\pi(D_u)}$ by the group $\pi(\Gamma_0)$. The group $\pi(\Gamma_0)$ is

$$\pi(\Gamma_0) = \left\{ c(\beta - k - 2) + d(\beta^2 - k\beta - 2) \;\middle|\; (c, d) \in \mathbb{Z}^2 \right\}.$$

Let $t_{c,d} := c(\beta - k - 2) + d(\beta^2 - k\beta - 2)$.

Lemma 6.6. — *For all* $k \geq 8$ *and for all* $(c, d) \in \mathbb{Z}^2$ *such that* $|c| \geq 1$ *and* $2|c| \geq |d|$*, we have*

$$|t_{c,d}| \geq k - 3\sqrt{k}.$$

Proof. — We have

$$|t_{c,d}| \geq |c|\,(k+2) - \frac{|c|}{\sqrt{k}} - |d|\,(\frac{1}{k} + \sqrt{k} + 2) \geq |c|\,(k - \frac{1}{\sqrt{k}} - \frac{2}{k} - 2\sqrt{k} - 2).$$

This is greater than $k - 3\sqrt{k}$ for $k \geq 8$. □

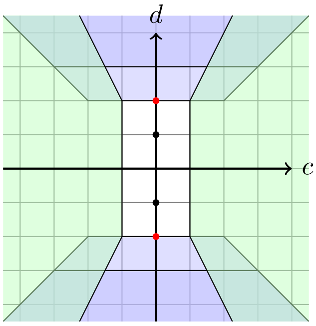
Lemma 6.7. — *For all* $k \geq 9$ *and for all* $(c, d) \in \mathbb{Z}^2$ *such that* $|c| \leq |d|$ *and* $2 \leq |d|$*, we have*

$$|\mathrm{Im}(t_{c,d})| \geq 2\sqrt{k} - 3.$$

And if moreover $|d| \geq 3$*, then we have* $|\mathrm{Im}(t_{c,d})| \geq 3\sqrt{k} - 5$.

Proof. — We have $\left|\mathrm{Im}(\beta^2 - k\beta - 2)\right| \geq k\,|\mathrm{Im}(\beta)| - |\beta|^2 \geq \frac{k}{\sqrt{k+\frac{2}{k}}} - 1 - \frac{1}{k}$. And we have $|\mathrm{Im}(\beta - k - 2)| = |\mathrm{Im}(\beta)| \leq \frac{1}{\sqrt{k}}$. Hence, $|\mathrm{Im}(t_{c,d})| \geq |d| \left(\frac{k}{\sqrt{k+\frac{2}{k}}} - 1 - \frac{1}{k} \right) - |c| \frac{1}{\sqrt{k}} \geq 2 \left(\frac{k}{\sqrt{k+\frac{2}{k}}} - 1 - \frac{1}{k} - \frac{1}{\sqrt{k}} \right)$. This is greater than $2\sqrt{k} - 3$ for $k \geq 9$. If moreover $|d| \geq 3$, then we have $|\mathrm{Im}(t_{c,d})| \geq 3 \left(\frac{k}{\sqrt{k+\frac{2}{k}}} - 1 - \frac{1}{k} - \frac{1}{\sqrt{k}} \right)$, and this is greater than $3\sqrt{k} - 5$ for $k \geq 6$. □

Figure 9. Zone covered by the lemma 6.6 (in green), and by the lemma 6.7 (in blue), and points remaining (the red points are remaining only for $l = c$)



Lemma 6.8. — *For all $k \geq 8$, $l \in A$ and $t \in S_l$, we have*

$$|t - t_k| \leq \frac{k}{2} + 2\sqrt{k} \qquad \textit{and}$$

$$|\mathrm{Im}(t - t_k)| \leq \begin{cases} \frac{3}{2}\sqrt{k} & \textit{if } t \notin \left\{ \gamma + \beta i \;\middle|\; i \in \{0, 1, ..., k-1\} \right\} \\ 2\sqrt{k} & \textit{otherwise.} \end{cases}$$

Proof. — For every $(i,j) \in \{0,1,...,k-1\}^2$, we have $|i + \beta j - t_k| \leq \frac{k}{2} + \frac{3}{2}\sqrt{k}$, $|k + \beta i - t_k| \leq \frac{k}{2} + \frac{3}{2}\sqrt{k}$, $|\gamma + \beta i - t_k| \leq \sqrt{k + \frac{2}{k}} + \frac{1}{\sqrt{k}} + 1 + \frac{k}{2} + \frac{\sqrt{k}}{2}$ (because the imaginary part of β is negative), $|k\beta - t_k| \leq \frac{k}{2} + \frac{3}{2}\sqrt{k}$ and $|\gamma\beta - t_k| = \left|\beta^2 - 1 - \frac{k}{2} + \frac{\sqrt{k}}{2} I\right| \leq \frac{1}{k} + 1 + \frac{k}{2} + \frac{\sqrt{k}}{2}$. Hence, the first inequality is true for $k \geq 8$.

We have $|\mathrm{Im}(i + \beta j - t_k)| = |\mathrm{Im}(k + \beta j - t_k)| = \left| j\,\mathrm{Im}(\beta) - \frac{\sqrt{k}}{2} \right| \leq \frac{3}{2}\sqrt{k}$, $|\mathrm{Im}(\gamma + \beta i - t_k)| \leq \left| \mathrm{Im}(\gamma) + \frac{\sqrt{k}}{2} \right| + i\,|\beta| \leq \sqrt{k + \frac{2}{k}} + \frac{1}{\sqrt{k}} - \frac{\sqrt{k}}{2} + \sqrt{k}$, $|\mathrm{Im}(k\beta - t_k)| \leq k\,|\beta| + \frac{\sqrt{k}}{2} \leq \frac{3}{2}\sqrt{k}$, $|\mathrm{Im}(\gamma\beta - t_k)| \leq |\gamma\beta| + \frac{\sqrt{k}}{2} \leq \frac{\sqrt{k}}{2} + 1 + \frac{1}{k}$. Hence, we get the wanted inequality for $k \geq 3$. □

Lemma 6.9. — *For all $k \geq 69$, we have $t_k \notin (\overline{\pi(D_u)} + t_{0,1}) \cup (\overline{\pi(D_u)} + t_{0,-1})$, and we have $t_k \notin \bigcup_{d \in \{-2,-1,0,1,2\}} (\overline{\pi(D_{u,c})} + t_{0,d})$.*

Proof. — For all $(i, j) \in \{0, 1, ..., k\}$, we have $|i + \beta j + t_{0,\pm1} - t_k| \geq |\mathrm{Im}(\beta j + t_{0,\pm1} - t_k)| = \left|\mathrm{Im}(\beta)(j \mp k) \pm \mathrm{Im}(\beta^2) + \frac{\sqrt{k}}{2}\right|$. If $\pm = +$, we have $|i + \beta j + t_{0,1} - t_k| \geq \frac{\sqrt{k}}{2} - \frac{1}{k}$ because $\mathrm{Im}(\beta) < 0$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 10$.

If $\pm = -$, we have $|i + \beta j + t_{0,-1} - t_k| \geq \frac{k}{\sqrt{k+\frac{2}{k}}} - \frac{\sqrt{k}}{2} - 1 - \frac{1}{k}$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 22$.

For $|d| \leq 1$, we have $|\gamma\beta + t_{0,\pm1} - t_k| \geq \frac{k}{2} - \frac{\sqrt{k}}{2} - \left|\beta^2 - 1\right| - (\frac{1}{k} + \sqrt{k} + 2) \geq \frac{k}{2} - 3\frac{\sqrt{k}}{2} - 3 - \frac{2}{k}$. This is greater than $\frac{1}{1-\frac{1}{\sqrt{k}}}$ for $k \geq 24$.

For all $i \in \{0, 1, ..., k-1\}$, and $|d| \leq 2$, we have $|\gamma + \beta i + t_{0,d} - t_k| \geq \frac{k}{2} - \sqrt{k + \frac{2}{k}} - \frac{1}{\sqrt{k}} - \sqrt{k} - |t_{0,d}| - \frac{\sqrt{k}}{2} \geq \frac{k}{2} - \frac{1}{\sqrt{k}} - \frac{3}{2}\sqrt{k} - |d|\,\frac{1}{k} - |d|\,\sqrt{k} - 2\,|d| = \frac{k}{2} - \frac{1}{\sqrt{k}} - \frac{7}{2}\sqrt{k} - \dfrac{2}{k} - 4$. This is greater than $\dfrac{1}{1 - \frac{1}{\sqrt{k}}}$ for $k \geq 69$.

For $|d| \leq 2$, we have $|k\beta + t_{0,d} - t_k| \geq \frac{k}{2} - \frac{3}{2}\sqrt{k} - |d|\,(\frac{1}{k} + \sqrt{k} + 2) \geq \frac{k}{2} - \frac{7}{2}\sqrt{k} - \frac{2}{k} - 4$. This is greater than $\dfrac{1}{1 - \frac{1}{\sqrt{k}}}$ for $k \geq 69$. □

Using the lemma 6.6 and 6.7, we have that for all the cases not covered by the lemma 6.9

$$|t_{c,d}| \geq k - 3\sqrt{k} \quad \text{or} \quad |\mathrm{Im}(t_{c,d})| \geq 3\sqrt{k} - 5 \quad \text{or} \quad |\mathrm{Im}(t_{c,d})| \geq 2\sqrt{k} - 3.$$

Hence, for all $l \in A$ and all $t \in S_l$, we have

$$|t + t_{c,d} - t_k| \geq k - 3\sqrt{k} - \frac{\sqrt{k}}{2} - \left|t - \frac{k}{2}\right| \geq \frac{k}{2} - \frac{9}{2}\sqrt{k} - \sqrt{k + \frac{2}{k}} - \frac{1}{k} \quad \text{or}$$

$$|t + t_{c,d} - t_k| \geq 3\sqrt{k} - 5 - \frac{\sqrt{k}}{2} - |\mathrm{Im}(t)| \geq \frac{3}{2}\sqrt{k} - 5 - \sqrt{k + \frac{2}{k}} - \frac{1}{k} \quad \text{or}$$

$$|t + t_{c,d} - t_k| \geq 2\sqrt{k} - 3 - \frac{\sqrt{k}}{2} - |\mathrm{Im}(t)| \geq \frac{1}{2}\sqrt{k} - 3 \quad \text{if } t \notin \left\{\gamma + i\beta \;\middle|\; i \in \{0, 1, ..., k-1\}\right\}.$$

This is greater than $\dfrac{1}{1 - \frac{1}{\sqrt{k}}}$ for $k \geq 126$ in the first case, for $k \geq 149$ in the second case and for $k \geq 69$ in the third case.

Consequently, we have proven that for every $k \geq 149$, we have $\pi(\overset{\circ}{D}_{u,a}) \neq \emptyset$ because t_k is not in the closure of of $\pi(\mathbb{Z}^A \backslash D_{u,a})$. By the theorem 3.3, we obtain the conclusion.

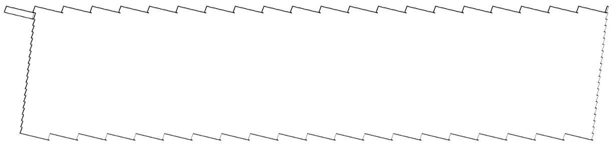
For $0 \leq k < 149$, we can check by computer, using what is done in the section 5, that the interior of $D_{u,a}$ is non-empty, by computing explicitly a regular language describing this interior and checking that this language is non-empty.

□

When we compute the interior of $D_{u,a}$ for these substitutions s_k, it appears that we get automata of the same shape for k large enough.

Conjecture 6.10. — *For all $k \geq 4$, the minimal automaton of the regular language $\overset{\circ}{L} := \left\{ u \in {}^t L_{a,a}^{s_k} \;\middle|\; Q_u \in \overset{\circ}{D}_{u,a} \right\}$ has 45 states.*

Figure 10. $\pi(D_{u,a} \backslash \overset{\circ}{D}_{u,a})$ for s_{20}



6.2. Proof of pure discreteness using automata. — In this subsection, we prove the pure discreteness using completely different technics but still as a corollary of theorem 3.3, for an another infinite familly of substitutions:

$$s_{l,k} : \begin{cases} a & \mapsto & a^l b a^{k-l} \\ b & \mapsto & c \\ c & \mapsto & a \end{cases} \qquad \forall\ 0 \leq l \leq k,$$

using the pure discreteness for the substitution

$$s_k : \begin{cases} a & \mapsto & a^k b \\ b & \mapsto & c \\ c & \mapsto & a \end{cases} \qquad \forall\ k \in \mathbb{N}_{\geq 1}.$$

This last substitution is a β-substitution, and the associated symbolic system is pure discrete after [**Barge 2015**] (but a similar argument as the one of the previous subsection can also be used to prove the pure discreteness for this family). We use this fact to prove the pure discreteness for the other family of substitutions.

The idea is to show that the part corresponding to letter a of the discrete line associated to $s_{l,k}$ contains a homothetic copy of the one for s_k. More precisely, we prove that $M^2 D_{u,a} \subseteq D_{v,a}$, where M is the incidence matrix of $s_{l,k}$ (it doesn't depends on l and k), v is the infinite fixed point of $s_{l,k}$ and u is the infinite fixed point of s_k. Hence, we have $\overset{\circ}{D}_{u,a} \neq \emptyset \Longrightarrow \overset{\circ}{D}_{v,a} \neq \emptyset$, and we can use the theorem 3.3.

Remark 6.11. — *The symbolic system associated to $s_{l,k}$ is conjugate to the one associated to $s_{k-l,k}$ by word-reversal. Therefore, we can assume without loss of generality that $1 \leq l \leq \left\lceil \frac{k}{2} \right\rceil$.*

6.2.1. Description of $D_{u,a}$ and $D_{v,a}$. — In all the following, u is the infinite fixed point of s_k, and v is the infinite fixed point of $s_{l,k}$, for integers $l, k \in \mathbb{N}$, with $k \geq l \geq 1$. We consider the following map

$$\varphi : \left\{ \begin{array}{ccc} \mathbb{Z}^A & \to & \mathbb{Q}(\beta) \\ (v_l)_{l \in A} & \mapsto & v_a + (\beta - k) v_b + (\beta^2 - \beta k) v_c. \end{array} \right. ,$$

where $A = \{a, b, c\}$. This linear map is one-to-one and has the property that the multiplication by the incidence matrix M in $\mathbb{R}^A$ becomes a multiplication by β in $\mathbb{Q}(\beta)$, because $(1, \beta - k, \beta^2 - \beta k)$ is a left eigenvector of M for the eigenvalue β. For a language over an alphabet $\Sigma \subseteq \mathbb{Q}(\beta)$, we denote

$$Q_L := \varphi(Q_{\varphi^{-1}(L)}) = \left\{ \sum_{i=0}^{|u|} u_i \beta^i \;\middle|\; u \in L \right\}.$$

Using the proposition 5.4, we have $\varphi(D_{u,a}) = Q_{L_k}$ and $\varphi(D_{v,a}) = Q_{L_{l,k}}$ where L_k is defined in Figure 11 and $L_{l,k}$ is the regular language defined on Figure 12, for β root of the polynomial $X^3 - kX^2 - 1$.

FIGURE 11. Automaton defining a language L_k such that $\varphi(D_{u,a}) = Q_{L_k}$, where a transition labeled by e means that there are $k-1$ transitions labeled by $1, 2, ..., k-1$

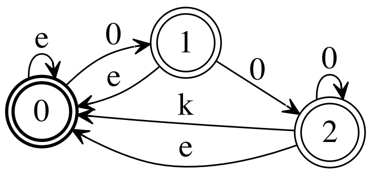
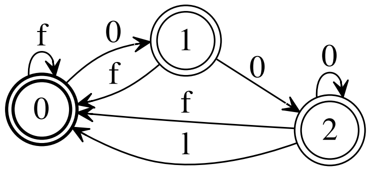


FIGURE 12. Automaton defining a language $L_{l,k}$ such that $\varphi(D_{v,a}) = Q_{L_{l,k}}$, where a transition labeled by f means that there are $l-1$ transitions labeled by $1, 2, ..., l-1$ and $k-l$ transitions labeled by $\beta - k + l, \beta - k + l + 1, ..., \beta - 2, \beta - 1$.



6.2.2. Zero-automaton. — In order to show that $M^2 D_{u,a} \subseteq D_{v,a}$, we need a way to go from the language $L_{l,k}$ to the language L_k. The following proposition permits to do it by describing algebraic relations between a word over the alphabet of L_k and a word over the alphabet of $L_{l,k}$. It works for $1 \leq l \leq k-2$, but by the remarks 6.11 we can assume it without loss of generality as soon as $k \geq 4$.

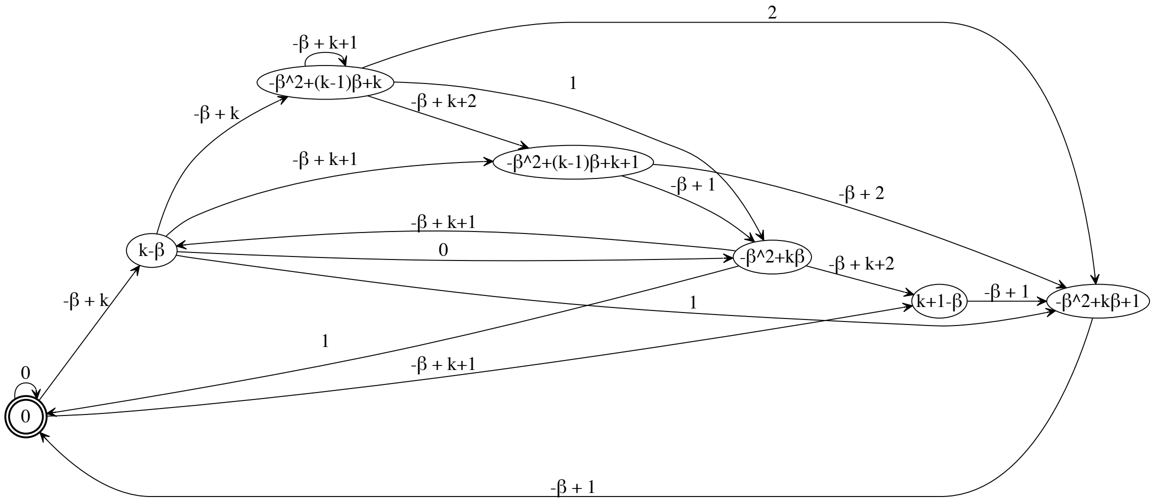
Proposition 6.12. — *Let L_0 be the language defined in Figure 14. We have*

$$L_0 \subseteq \left\{ u \in (\Sigma_k - \Sigma_{l,k})^* \;\middle|\; \sum_{i=0}^{|u|} u_i \beta^i = 0 \right\},$$

if $1 \leq l \leq k-2$, where

- *$\Sigma_k = \{0, 1, ..., k\}$ is the alphabet of the language L_k,*
- *$\Sigma_{l,k} = \{0, 1, ..., l, \beta - k + l, \beta - k + l + 1, ..., \beta - 1\}$ is the alphabet of the language $L_{l,k}$, and*

FIGURE 13. Automaton recognizing the language L_0'



Proof. — Let L_0' be the language of the automaton depicted in the picture 13. We verify easily that L_0' is the transposed (i.e. the word reversal) of the language L_0. And we easily check that the transitions of the automaton of Figure 13, satisfy the following.

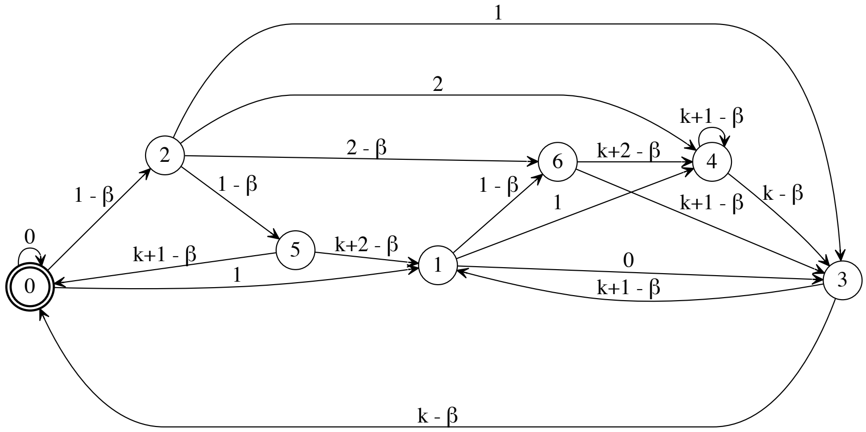
$$x \xrightarrow{t} y \qquad \Longrightarrow \qquad y = \beta x + t, \quad t \in \Sigma_k - \Sigma_{l,k}.$$

Hence, if we have a word $u_0 u_1 u_2 ... u_n \in L_0'$, it corresponds to a path from 0 to 0, so we have

$$0 \xrightarrow{u_0} u_0 \xrightarrow{u_1} \beta u_0 + u_1 \xrightarrow{u_2} ... \xrightarrow{u_{n-1}} \sum_{i=0}^{n-1} \beta^{n-1-i} u_i \xrightarrow{u_n} \sum_{i=0}^{n} \beta^{n-i} u_i = 0.$$

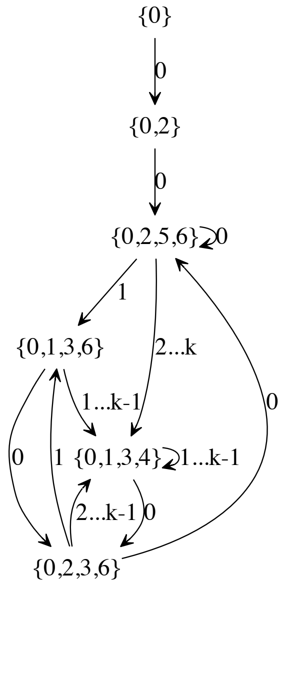
□

FIGURE 14. Automaton recognizing the language L_0



6.2.3. Proof that $M^2 D_{u,a} \subseteq D_{v,a}$. — We define a language L by the automaton $\mathcal{A}$ of Figure 15.

FIGURE 15. The automaton $\mathcal{A}$, recognizing a language L
$\{0\}$ is the initial state and every state is final



Lemma 6.13. — *The transitions of the automaton $\mathcal{A}$ of Figure 15 satisfy*

$$X \xrightarrow{b} Y \quad \Longrightarrow \quad Y \subseteq \left\{ y \in Q_0 \;\middle|\; \exists x \in X,\ \exists c \in \Sigma_{l,k},\ x \xrightarrow{b-c} y \in \mathcal{A}_0 \right\}$$

where $Q_0 = \{0, 1, 2, 3, 4, 5, 6\}$ is the set of states of the automaton $\mathcal{A}_0$ of Figure 14.

Proof. — We can check that using the following array: a star or a letter l means that the set $a \cap (b - \Sigma_{l,k})$ is non-empty for a given $(a, b) \in \Sigma_0 \times \Sigma_k$ (the converse is false, but we don't need it), where $\Sigma_0 = \{0, 1, 2, 1 - \beta, 2 - \beta, k - \beta, k + 1 - \beta, k + 2 - \beta\}$ is the alphabet of $\mathcal{A}_0$.

Σ_0 \ Σ_k	0	1	$2...l-1$	l	$l+1$	$l+2...k-1$	k
0	$*$	$*$	$*$	l			
1		$*$	$*$	$*$	l		
2			$*$	$*$	$*$		
$1-\beta$	$*$						
$2-\beta$	$*$	$*$					
$k-\beta$				$*$	$*$	$*$	
$k+1-\beta$					$*$	$*$	$*$
$k+2-\beta$						$*$	$*$

□

We remark that every state of the automaton $\mathcal{A}$ of Figure 15 contains 0. Hence, if we have a path $\{0\} \xrightarrow{b_1} X_1 \xrightarrow{b_2} ... \xrightarrow{b_n} X_n$ in this automaton, we have $0 \in X_n$, and by the lemma 6.13, we can find $(c_i)_{i=1}^n \in \Sigma_{l,k}^n$ such that we have the following path in $\mathcal{A}_0$:

$$0 \xrightarrow{b_1 - c_1} x_1 \xrightarrow{b_2 - c_2} ... \xrightarrow{b_n - c_n} 0.$$

Then, by definition of the automaton $\mathcal{A}_0$, we have

$$Q_b = \sum_{i=1}^{n} b_i \beta^i = \sum_{i=1}^{n} c_i \beta^i = Q_c.$$

And we have the following.

Lemma 6.14. — *The sequence $(c_i)_{i=1}^n \in \Sigma_{l,k}^*$ can be chosen such that the word $c_1 c_2 ... c_n$ is in $L_{l,k}$.*

Proof. — The language $L_{l,k}$ is the set of words over the alphabet $\Sigma_{l,k}^*$ such that every letter l is preceded by two letters 0. And we can check that in the proof of the lemma 6.13, the only place where we need to take $c_i = l$ is when we follow a transition of $\mathcal{A}$ labeled by l or by $l + 1$. And this occurs only when we follow an transition labeled by 0 or 1 in the automaton $\mathcal{A}_0$.

The only transitions of $\mathcal{A}$ that needs to take $c_i = l$ are $\{0,2,5,6\} \xrightarrow{l} \{0,1,3,4\}$, $\{0,2,5,6\} \xrightarrow{l+1} \{0,1,3,4\}$ and $\{0,1,3,6\} \xrightarrow{l+1} \{0,1,3,4\}$. But we can check that when we reach the state $\{0,2,5,6\}$, we have read at least two zeroes, so we can assume that the 0 of the state $\{0,2,5,6\}$ has been reached by following the path

$$0 \xrightarrow{0-0} 0 \xrightarrow{0-0} 0$$

in $\mathcal{A}_0$. This allows us to consider the transitions $0 \xrightarrow{l-l} 0$ and $0 \xrightarrow{(l+1)-l} 1$ of $\mathcal{A}_0$ and getting a word $c_1c_2...c_n$ that stays in the language $L_{l,k}$. In the same way, we reach the state $\{0,1,3,6\}$ after reading a 0 and then a 1, so we can assume that the 1 in the state $\{0,1,3,6\}$ has been reached by following the path

$$0 \xrightarrow{0-0} 0 \xrightarrow{1-0} 1$$

in $\mathcal{A}_0$. This allows us to consider the transition $1 \xrightarrow{(l+1)-l} 4$ of $\mathcal{A}_0$ and getting a word $c_1c_2...c_n$ that stays in the language $L_{l,k}$. □

We deduce from this lemma and from the equality $Q_b = Q_c$ that we have $Q_b \in Q_{L_{s,k}}$ for every word b in the language L. Hence, we have the inclusion $\beta^2 Q_{L_k} = Q_{0^2 L_k} \subseteq Q_L \subseteq Q_{L_{s,k}}$. So we have $M^2 D_{u,a} \subseteq D_{v,a}$.

By the theorem 3.3, we have for every $1 \le l \le k-2$,

$$\begin{aligned} s_k \text{ satisfy the Pisot substitution conjecture } &\Longrightarrow \overset{\circ}{D}_{u,a} \neq \emptyset \\ &\Longrightarrow \overset{\circ}{D}_{v,a} \neq \emptyset \quad (\text{because } M^2 D_{u,a} \subseteq D_{v,a}) \\ &\Longrightarrow s_{l,k} \text{ satisfy the Pisot substitution conjecture.} \end{aligned}$$

And it implies that $s_{l,k}$ satisfy the Pisot conjecture for every $0 \le l \le k$, $k \ge 4$, up to take the mirror. For $1 \le k < 4$, there is a finite number of possibilities, and we can check that it also works, for example by computing the interior of the discrete line.

7. Pure discreteness for a S-adic system

Let

$$s_1 : \begin{cases} a & \mapsto & aab \\ b & \mapsto & c \\ c & \mapsto & a \end{cases} \quad \text{and} \quad s_2 : \begin{cases} a & \mapsto & aba \\ b & \mapsto & c \\ c & \mapsto & a \end{cases}$$

be two substitutions over the alphabet $A = \{a,b,c\}$ having the same incidence matrix $M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Given an infinite word $i_0 i_1 ... \in \{1,2\}^{\mathbb{N}}$, we define a word $u \in A^{\mathbb{N}}$ by

$$u = \lim_{n\to\infty} s_{i_0} s_{i_1} ... s_{i_n}(a).$$

Remark that $s_{i_0} s_{i_1} ... s_{i_n}(a)$ is a prefix of $s_{i_0} s_{i_1} ... s_{i_n} s_{i_{n+1}}(a)$, so the limit exists.

We have the following

Theorem 7.1. — *For every word $i_0i_1... \in \{1,2\}^{\mathbb{N}}$, the subshift $(\overline{S^{\mathbb{N}}u}, S)$ is measurably isomorphic to a translation on a torus.*

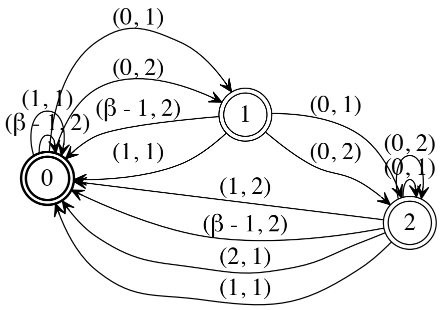
The idea of the proof is similar to the one of the previous subsection: we prove that for every sequence $i_0i_1...$ we have an inclusion of the form

$$t + M^k D_{u_{s_1},a} \subseteq D_{u,a},$$

for some $t \in \mathbb{Z}^3$ and $k \in \mathbb{N}$, where u_{s_1} is the infinite fixed point of s_1, and we use the theorem 2.3.

7.1. Representation of $D_{u,a}$ by an automaton. — Like for fixed points of substitutions, we can represent $D_{u,a}$ by an automaton. For simplicity, we will consider rather $\psi(D_{u,a}) \subseteq \mathbb{Q}(\beta)$, where β is the highest eigenvalue of M, and $\psi : \mathbb{R}^A \to \mathbb{R}$ is the linear map such that $\psi(e_a) = 1$, $\psi(e_b) = \beta - 2$ and $\psi(e_c) = \beta^2 - 2\beta$.

FIGURE 16. Automaton $\mathcal{A}$, recognizing a language L



Lemma 7.2. — *We have*

$$\psi(D_{u,a}) = \left\{ \sum_{i=0}^{n} u_i\beta^i \;\middle|\; n \in \mathbb{N}, (u_0, i_0)(u_1, i_1)...(u_n, i_n) \in L \right\},$$

$$\psi(D_{u_{s_1},a}) = \left\{ \sum_{i=0}^{n} u_i\beta^i \;\middle|\; n \in \mathbb{N}, (u_0, 1)(u_1, 1)...(u_n, 1) \in L \right\},$$

where L is the regular language recognized by the automaton of Figure 16.

Proof. — It is an easy generalization of the mirror of the abelianisation of the prefix automaton. It corresponds to the Dumont-Thomas numeration, but with an additional information that permits to take care of the fact that with can compose by one of the two substitutions of the S-adic system at each step. □

Now that we have a description of the discretes lines $D_{u,a}$ and $D_{u_{s_1},a}$, we use it to show that for every words $i_0i_1...$ we have an inclusion of the form $M^k D_{u_{s_1},a} + t \subseteq D_{u,a}$.

7.2. Proof of the inclusion. — Let $\Sigma_1 = \{0, 1, 2\}$ and $\Sigma_2 = \{0, 1, \beta - 1\}$. We define a regular language L_* over the alphabet $\Sigma_1 \times \{1, 2\}$ by

$$L_* = (\Sigma_1 \times \{1, 2\})^* \cap \sigma(L_0 \times L_1 \times L),$$

where

$$L_1 = \left\{ u_0 u_1 ... u_n \in \Sigma_1^* \;\middle|\; n \in \mathbb{N}, (u_0, 1)(u_1, 1)...(u_n, 1) \in L \right\}$$

$$L_0 = \left\{ u_0 u_1 ... u_n \in \Sigma^* \;\middle|\; n \in \mathbb{N}, \textstyle\sum_{i=0}^{n} u_i \beta^i = 0 \right\}$$

$$\Sigma = \Sigma_1 - \Sigma_2 = \{-1, 0, 1, 2, 1 - \beta, 2 - \beta, 3 - \beta\}$$

and σ is the word morphism

$$\sigma : \begin{array}{ccc} \Sigma \times \Sigma_1 \times \Sigma_L & \to & \Sigma_1 \times \{1, 2\} \cup \{*\} \\ (t, x, (y, i)) & \mapsto & \begin{cases} (x, i) & \text{if } x - y = t \\ * & \text{otherwise} \end{cases} \end{array}$$

where $\Sigma_L = \Sigma_2 \times \{1, 2\}$ is the alphabet of the language L.

Lemma 7.3. — *We have*

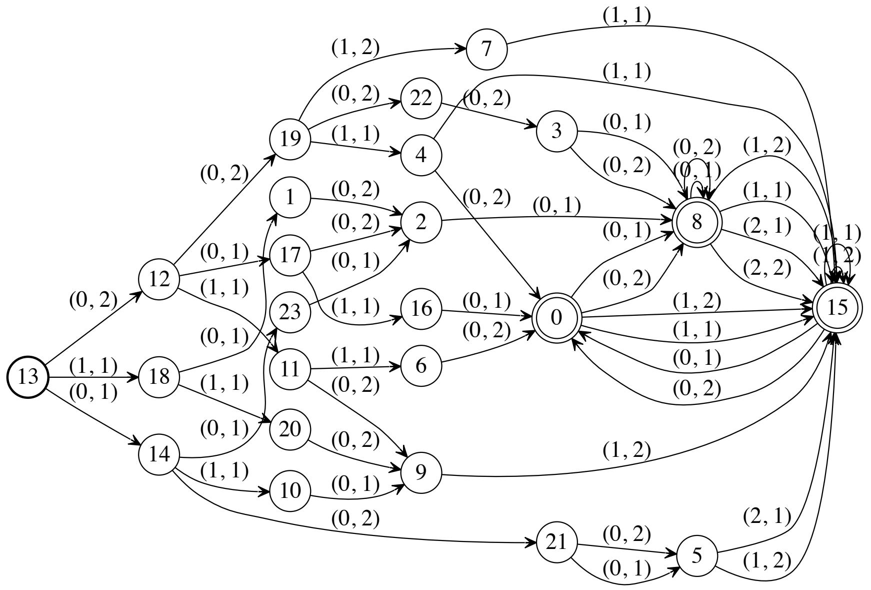
$$\psi(D_{u,a}) \supseteq \left\{ \textstyle\sum_{i=0}^{n} u_i \beta^i \;\middle|\; n \in \mathbb{N}, (u_0, i_0)(u_1, i_1)...(u_n, i_n) \in L_* \right\}.$$

Proof. — For all $n \in \mathbb{N}$, we have

$$\begin{aligned} & (x_0, i_0)(x_1, i_1)...(x_n, i_n) \in L_* \\ \Longrightarrow \quad & \exists (y_0, i_0)(y_1, i_1)...(y_n, i_n) \in L, \ \sum_{i=0}^{n} (x_i - y_i)\beta^i = 0 \\ \Longrightarrow \quad & \sum_{i=0}^{n} x_i \beta^i \in \psi(D_{u,a}). \end{aligned}$$

□

FIGURE 17. Automaton recognizing a language L_u



Lemma 7.4. — *We have*

$$L_* \supseteq L_u,$$

where L_u is the language defined in Figure 17.

Proof. — Computation done by computer. The language L_0 is regular thanks to [**Mercat 2013**], and its minimal automaton has 62 states. The minimal automaton of the language L_* has 210 states. □

We deduce from these two lemma that for every sequence $i_0 i_1 ...$, we have an inclusion of the form $\psi(D_{u,a}) \supseteq t + \beta^k \psi(D_{u_{s_1},a})$, for some $t \in \mathbb{Z}[\beta]$ and some $k \in \mathbb{N}$, thus we have

$$t' + M^k D_{u_{s_1},a} \subseteq D_{u,a}$$

for some $t' \in \mathbb{Z}^3$.

For example, if $i_0 = 1$, $i_1 = 1$, $i_2 = 2$ and $i_3 = 2$, then we have the inclusion

$$e_a + M e_a + M^3 e_a + M^4 D_{u_{s_1},a} \subseteq D_{u,a}.$$

7.3. Pure discreteness of the spectrum. — In order to use the theorem 2.3, we need the following property.

Lemma 7.5. — *The subshift $(\overline{S^{\mathbb{N}} u}, S)$ is minimal for every sequence $i_0 i_1 ... \in \{1, 2\}^{\mathbb{N}}$.*

Proof. — In the word u, there are at most 5 letters between two consecutive letters a. Indeed, $u = s_{i_0}s_{i_1}(v)$ for an infinite word $v \in A^{\mathbb{N}}$, and we have that $s_{i_0}s_{i_1}(a)$ is a word of length 7 with 4 letters a, $s_{i_0}s_{i_1}(b) = a$ and $s_{i_0}s_{i_1}(c)$ is a word of length 3 with 2 letters a.

Thus, there exists a constant C such that every factor of u of length $\geq C\beta^n$ contains $s_{i_0}s_{i_1}...s_{i_n}(a)$. Indeed, it suffices to see that $u = s_{i_0}s_{i_1}...s_{i_n}(w)$ with a word w that satisfies the above property.

If a word is in $\overline{S^{\mathbb{N}}u}$, then it contains arbitrarily large factors of u, so it contains $s_{i_0}s_{i_1}...s_{i_n}(a)$ for every $n \in \mathbb{N}$. Therefore this word is dense in $S^{\mathbb{N}}u$. □

A projection π along the eigenspace for the eigenvalue β of M onto some plane $\mathcal{P}$ is such that the restriction of π to $\mathbb{Z}^3$ is injective and has a dense image in $\mathcal{P}$. It is not difficult to see that the set $\pi(D_u)$ is bounded, by using for example the description of $\psi(D_u)$ given in the lemma 7.2.

In order to prove the disjointness in measure of the translated copies of $\overline{\pi(D_u)}$ by $\pi(\Gamma_0)$, we use the same strategy than in the proof of the theorem 3.3: we show that the interior of $D_{u,a}$ is non-empty, and we show that the interior of D_u is dense. It is know that we have $\overset{\circ}{D}_{u_{s_1},a} \neq \emptyset$ for the topology defined on the subsection 3.2 (we can prove it by computing this interior explicitly thanks to the theorem 5.8). And the matrix M is in $GL(3,\mathbb{Z})$, so if we consider an inclusion of the form $t + M^k D_{u_{s_1},a} \subseteq D_{u,a}$, $t \in \mathbb{Z}^3$, $k \in \mathbb{N}$ given by the previous subsection, then it implies that $\overset{\circ}{D}_{u,a} \neq \emptyset$, and this is true for every sequence $i_0i_1... \in \{1,2\}^{\mathbb{N}}$. Then, we have the following result.

Lemma 7.6. — *$\forall l \in \{a,b,c\}$, $\overset{\circ}{D}_{u,l}$ is dense in $D_{u,l}$.*

Proof. — Let $u_n = \lim_{k\to\infty} s_{i_n}s_{i_{n+1}}...s_{i_{n+k}}(a)$. We have just proven that for every $n \in \mathbb{N}$, $D_{u_n,a}$ has non-empty interior. And we have $u = s_{i_0}s_{i_1}...s_{i_{n-1}}(u_n)$, so we get the equality

$$\psi(D_{u,i}) = \bigcup_{i \xrightarrow{(t_0,i_0)} \dots \xrightarrow{(t_{n-1},i_{n-1})} j \in \mathcal{A}} \beta^n \psi(D_{u_n,j}) + \sum_{k=0}^{n-1} t_k \beta^k$$

for all $i,j \in \{a,b,c\}$. But the automaton $\mathcal{A}$ is such that we can reach any state from any state, even if we impose the right coefficients of labels read. Hence, we can approach (for our topology) any point of $D_{u,i}$ by subsets of $D_{u,i}$ of the form $M^k D_{u_k,a} + t$, $t \in \mathbb{Z}^3$, $k \in \mathbb{N}$. Such subsets have non-empty interior since $M \in GL(3,\mathbb{Z})$. This ends the proof. □

Lemma 7.7. — *The boundary of $\pi(D_u)$ has zero Lebesgue measure.*

In order to prove this lemma, let introduce some notations. For all $n \in \mathbb{N}$, let $u_n = \lim_{k\to\infty} s_{i_n}s_{i_{n+1}}...s_{i_{n+k}}(a)$, and for all $a \in A$, $R_a^n = \overline{\pi(D_{u_n,a})}$. We have the following

Lemma 7.8. — *For every $a \in A$, the sequence $(\lambda(R_a^n))_{n\in\mathbb{N}}$ is increasing and bounded.*

Proof. — By the lemma 7.2, we have the following equality

$$\psi(D_{u_n,a}) = \bigcup_{b \xrightarrow{(t,s_{n+1})} a \in \mathcal{A}} \beta\psi(D_{u_{n+1},b}) + t,$$

where $\mathcal{A}$ is the automaton of the figure 16. Without loss of generality, we can assume that $\pi = \sigma_- \circ \psi : \mathbb{Z}^A \to \mathbb{C}$, where σ_- is the Galois morphism

$$\sigma_- : \begin{array}{ccc} \mathbb{Q}(\beta) & \to & \mathbb{Q}(\gamma) \\ \beta & \mapsto & \gamma \end{array},$$

where γ is a complex conjugate of β. Thus, we have the equality

$$R_a^n = \bigcup_{b \xrightarrow{(t,s_{n+1})} a \in \mathcal{A}'} \gamma R_b^{n+1} + t,$$

where $\mathcal{A}'$ is the automaton $\mathcal{A}$ where we apply the Galois morphism σ_-.

Then, we have

$$\lambda(R_a^n) \leq \sum_{b \xrightarrow{(t,s_{n+1})} a \in \mathcal{A}'} \frac{1}{\beta}\lambda(R_b^{n+1}).$$

If we take the vector $X_n = (\lambda(R_a^n))_{a\in A} \in \mathbb{R}^A$, the previous inequality becomes

$$X_n \leq \frac{1}{\beta} M X_{n+1}.$$

But by the Perron-Frobenius theorem, we have the inequality $MX \leq \beta X$ for every $X \in \mathbb{R}_+^A$, so we get that X_n is increasing. The coefficient of X_n are also bounded by $\dfrac{\max_{t\in\Sigma'} |t|}{1 - |\gamma|}$, where Σ' is the alphabet of the automaton $\mathcal{A}'$. □

This lemma give the existence of the limit $\lambda_a^\infty = \lim_{n\to\infty} \lambda(R_a^n)$. We have the following lemma.

Lemma 7.9. — *There exists $\epsilon > 0$ and $\eta > 0$ such that for every $n \in \mathbb{N}$ and every $a \in A$, there exists $t \in \mathbb{C}$ and $r > 0$ such that the ball $B(t, r + \epsilon)$ is included in R_a, and such that $\lambda(B(t,r)) \geq \eta\lambda_a^\infty$.*

Proof. — It is an immediate consequence of the inclusions proven in the subsection 7.2. □

Lemma 7.10. — *There exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $a \in A$, we have $\lambda(\partial R_a^n) = 0$.*

Proof. — Let $n_0 \in \mathbb{N}$ such that

$$\forall a \in A,\ \lambda_a^\infty \leq (1+\eta)\lambda(R_a^{n_0}),$$

and let $k \in \mathbb{N}$ such that for every $a \in A$, every $(t,t') \in \mathbb{C}^2$, every $r > 0$, and every $n \in \mathbb{N}$,

$$\gamma^k R_a^n + t \cap B(t',r) \neq \emptyset \Longrightarrow \gamma^k R_a^n + t \subseteq B(t', r+\epsilon).$$

Let us show that for every $n \geq n_0$ we have

$$\forall a \in A,\ \lambda(\partial R_a^{n+k}) \leq c\lambda(R_a^{n+k}) \Longrightarrow \forall a \in A,\ \lambda(\partial R_a^n) \leq c(1-\eta^2)\lambda(R_a^n).$$

Let $a \in A$ and $n \geq n_0$. Let $t \in \mathbb{C}$ and $r > 0$ such that $B(t, r+\epsilon) \subseteq R_a^n$ and $\lambda(B(t,r)) \geq \eta\lambda_a^\infty$. Let

$$T_b = \left\{ \sum_{j=n}^{n+k-1} \gamma^{n+k-j-1} t_j \;\middle|\; b \xrightarrow{(t_n,i_n)} \dots \xrightarrow{(t_{n+k-1},i_{n+k-1})} a \in \mathcal{A}' \right\},$$

$$T_b' = \left\{ \sum_{j=n}^{n+k-1} \gamma^{n+k-j-1} t_j \;\middle|\; b \xrightarrow{(t_n,i_n)} \dots \xrightarrow{(t_{n+k-1},i_{n+k-1})} a \in \mathcal{A}' \text{ and } (\gamma^k \partial R_b^{n+k} + t) \cap B(t,r) = \emptyset \right\}.$$

Then we have

$$\begin{aligned}
\lambda(\partial R_a^n) &\leq \lambda\left(\bigcup_{b \in A} \bigcup_{t \in T_b'} (\gamma^k \partial R_b^{n+k} + t) \right) \\
&\leq \sum_{b \in A,\ t \in T_b'} \frac{1}{\beta^k} \lambda(\partial R_b^{n+k}) \\
&\leq \frac{c}{\beta^k} \sum_{b \in A,\ t \in T_b'} \lambda(R_b^{n+k}) \\
&\leq \frac{c}{\beta^k} \left[\left(\sum_{b \in A,\ t \in T_b} \lambda(R_b^{n+k}) \right) - \beta^k \lambda(B(t,r)) \right] \\
&\leq \frac{c}{\beta^k} \left(\beta^k \lambda(R_a^{n+k}) - \beta^k \eta \lambda_a^\infty \right) \\
&\leq c(1-\eta)\lambda_a^\infty \\
&\leq c(1-\eta^2)\lambda(R_a^n).
\end{aligned}$$

We deduce from these equalities that we have

$$\lambda(\partial R_a^n) \leq (1-\eta^2)^k \lambda(R_a^n) \xrightarrow[k \to \infty]{} 0.$$

□

Proof of the lemma 7.7. — We have the inclusion

$$\partial R_a \subseteq \bigcup_{b \xrightarrow{(t_0,i_0)} \dots \xrightarrow{(t_{n-1},i_{n-1})} a \in \mathcal{A}'} \gamma^n \partial R_b^n + \sum_{j=0}^{n-1} \gamma^{n-1-j} t_j.$$

And by the lemma 7.10, we have $\lambda(\partial R_b^n) = 0$ for $n \geq n_0$ and $b \in A$. Thus the boundary of R_a has zero Lebesgue measure. □

Thanks to the lemma 7.6, for every $t \in \Gamma_0 \backslash \{0\}$, the empty intersection $\overset{\circ}{D_u} \cap \overset{\circ}{D_u} + t$ is a dense open subset of $\overline{\overset{\circ}{D_u}} \cap \overline{\overset{\circ}{D_u}} + t$. Hence, the interior of $\overline{\pi(D_u)}$ and $\overline{\pi(D_u + t)}$ are disjoint. By the lemma 7.7, it proves that the Lebesgue measure of the intersection is zero.

Every hypothesis of the theorem 2.3 is satisfied, thus the subshift $(\overline{S^{\mathbb{N}} u}, S, \mu)$ is uniquely ergodic and measurably conjugate to the translation on the torus $(\mathcal{P}/\pi(\Gamma_0), T, \lambda)$. This ends the proof of the theorem 7.1.

References

[ABBLS 2015] S. Akiyama, M. Barge, V. Berthé, J. -Y. Lee, A. Siegel *On the Pisot substitution Conjecture*, Mathematics of Aperiodic Order, Springer Basel, 2015.
`https://www.irif.fr/~berthe/Articles/AkiyamaBargeBertheLeeSiegel.pdf`

[Akiyama Lee 2014] S. Akiyama, J.-Y. Lee *Overlap coincidence to strong coincidence in substitution tiling dynamics*, European Journal of Combinatorics, Elsevier, vol. 39, p. 233-243, 2014.
`https://arxiv.org/pdf/1403.0377.pdf`

[Arnoux Ito 2001] P. Arnoux, S. Ito *Pisot Substitutions and Rauzy fractals* , Bull. Belg. Math. Soc. 8, 181-207, 2001.
`http://iml.univ-mrs.fr/~arnoux/ArnouxIto.pdf`

[Barge Kwapisz 2006] M. Barge and J. Kwapisz, *Geometric theory of unimodular Pisot substitutions*, Amer. J. Math. **128** (2006), no. 5, 1219–1282.

[Barge 2015] M. Barge *The Pisot conjecture for β-substitutions*, arXiv:1505.04408v2, 2015.
`https://arxiv.org/pdf/1505.04408.pdf`

[Berthé Siegel 2005] V. Berthé, A. Siegel *Tilings associated to β-numeration and substitutions*, Electronic Journal of Combinatorial Number Theory, 2005.
`http://iml.univ-mrs.fr/~arnoux/integers.pdf`

[Brin Stuck 2002] M. Brin, G. Stuck *Introduction to Dynamical Systems*, ISBN 978-0521808415, Cambride University Press, 2002.
`https://imcs.dvfu.ru/lib.int/docs/Math/unsorted/Diff.Equations/Brin%20M.,%20Stuck%20G.,%20Introduction%20to%20Dynamical%20Systems,%202003.pdf`

[Carton] O. Carton *Langages formels, Calculabilité et Complexité* Vuibert, ISBN 978-2-311-01400-6, Paris, 2014. `https://gaati.org/bisson/tea/lfcc.pdf`

[Clark Sadun 2003] A. Clark and L. Sadun, *When size matters: subshifts and their related tiling spaces*, Ergodic Theory Dynam. Systems **23** (2003), no. 4, 1043–1057.

[Frou. Pel. 2017] Ch. Frougny, E. Pelantová *Beta-representations of 0 and Pisot numbers*, JTNB, 2017.
`https://arxiv.org/pdf/1512.04234.pdf`

[Frou. Sak. 2010] Ch. Frougny, J. Sakarovitch *Number representation and finite automata, Chapter 2* in Combinatorics, Automata and Number Theory, V. Berthé, M. Rigo (Eds), Encyclopedia of Mathematics and its Applications 135, Cambridge University Press, 2010.
`https://www.irif.fr/~cf/publications/cant-ch1.pdf`

[HU] J. E. Hopfcroft and J. D.Ullman, *Introduction to automata theory, languages and computation*, Addison-Wesley, 1979.

[Ito Rao 2003] Sh. Ito and H. Rao, *Atomic surfaces, tilings and coincidence. I. Irreducible case*, Israel J. Math. **153** (2006), 129–155.

[Kenyon 1997] R. Kenyon *Projecting the one dimensional Sierpinski gasket*, Israël Journal of Mathematics, 1997.
https://www.math.brown.edu/~rkenyon/papers/sier.ps

[KN] B. Khoussainov, A. Nerode *Automata Theory and its Applications* Springer Science & Business Media, ISBN 978-1-4612-0171-7, dec. 2012.

[Lagarias Wang 2003] J. C. Lagarias and Y. Wang, *Substitution Delone Sets*, Discrete Comput. Geom. **29** (2003), 175–209.

[Lagarias 1996] J.C. Lagarias, *Meyer's concept of quasicrystal and quasiregular sets*, Comm. Math. Phys. **179** (1996), no. 2, 365–376.

[Lee 2007] J.-Y. Lee, *Substitution Delone sets with pure point spectrum are inter-model sets*, J. Geom. Phys. **57** (2007), no. 11, 2263–2285.

[Lee Moody Solomyak 2003] J.-Y. Lee, R. V. Moody, and B. Solomyak, *Consequences of pure point diffraction spectra for multiset substitution systems*, Discrete Comput. Geom. **29** (2003), no. 4, 525–560.

[Mercat 2013] P. Mercat *Semi-groupes fortement automatiques*, Bull. SMF 141, fascicule 3, Paris, 2013.
http://www.i2m.univ-amu.fr/~mercat.p/Publis/Semi-groupes%20fortement%20automatiques.pdf

[Milt. Thus. 2014] M. Minervino, J. Thuswaldner *The geometry of non-unit Pisot substitutions*, preprint, 2014.
https://arxiv.org/pdf/1402.2002.pdf

[Rauzy 1982] G. Rauzy *Nombres algébriques et substitutions*, Bull. S.M.F., tome 10, p. 147-178, 1982.
http://www.numdam.org/article/BSMF_1982__110__147_0.pdf

[Sa] J. Sakarovitch, *Elements of automata theory*, Cambridge University Press, 2009.

[Sieg. Thusw. 2009] A. Siegel, J. Thuswaldner *Topological properties of Rauzy fractals*, Mémoires de la SMF, 118, pp.144, 2009. https://www.irisa.fr/symbiose/people/asiegel/Articles/Topological.pdf

[Sing 2006] B. Sing *Pisot substitutions and beyond*, PhD thesis, 2006.

[Solomyak 1997] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 695–738.

April 29, 2018

PAUL MERCAT, Aix-Marseille Université, 39 rue Frédéric Joliot-Curie 13453 Marseille cedex 13
E-mail : paul.mercat@univ-amu.fr • *Url :* https://www.i2m.univ-amu.fr/~mercat.p/

SHIGEKI AKIYAMA, Institute of Mathematics, University of Tsukuba 1-1-1 Tennodai, Tsukuba, Ibaraki, 305-8571 Japan • *E-mail :* akiyama@math.tsukuba.ac.jp
Url : http://math.tsukuba.ac.jp/~akiyama/