RAUZY FRACTAL OF THE SMALLEST SUBSTITUTION ASSOCIATED TO THE SMALLEST PISOT NUMBER

by

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Abstract. — Up to words reversal and relabelling, there exists an unique substitution associated to the smallest Pisot number with a minimal number of letters. This is the substitution $s : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 12$. We study the Rauzy fractal of this substitution and show that it is the union of a countable number of Hokaiddo tiles and a fractal of dimension strictly less than 2 which is completely explicit. We complete the picture by showing that these Hokkaido tiles are arranged in three different manners to form tiles which are all pairwise disjoint. We also give an efficient algorithm to draw a zoom on a Rauzy fractal. And we show that the symbolic system of the substitution s is measurably isomorphic to a nice domain exchange with 4 pieces. The tools used in this article, using regular languages, are very general and can be easily adapted to study Rauzy fractals of any substitution associated to a Pisot number, and other fractals associated to algebraic numbers without conjugate of modulus one.

Contents

Key words and phrases. — Rauzy fractals, substitutions, Hokkaido, Pisot number.

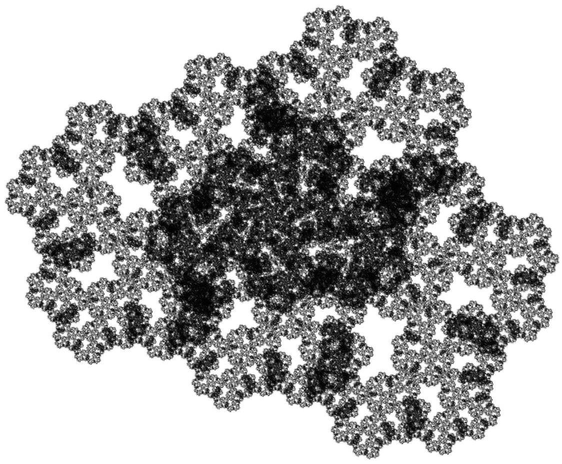
1. Introduction and main result

The smallest Pisot number β (also called the plastic number) is the greatest root of the polynomial $X^3 - X - 1$. This number is approximately 1.3247179572447460260... and has two conjugated complex conjugates of modulus strictly less than one. It is easy to check that, up to letters permutation, the substitutions

$$s: \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{array} \quad \text{and} \quad {}^t s: \begin{array}{l} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 21 \end{array}$$

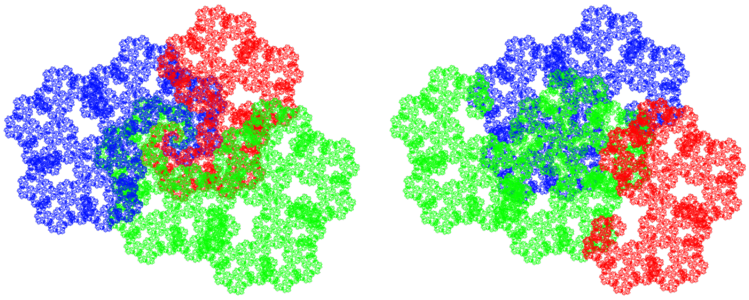
are the only ones on three letters to be associated to the smallest Pisot number β. And we get one of these two substitutions from the other one, by words reversal. Therefore, the study of one of these two substitutions is enough to understand completely both. In particular, they have the same Rauzy fractal, which looks like the following (see figure 1).

FIGURE 1. Rauzy fractal of the substitution $s : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 12$



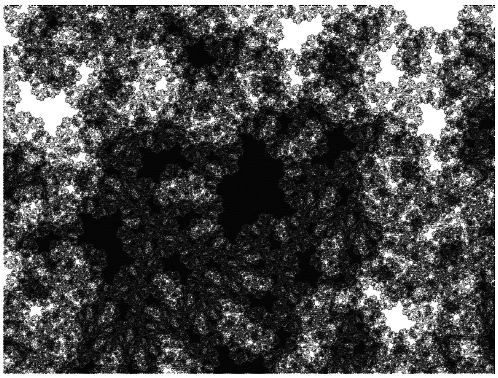
This Rauzy fractal is an interesting object that can be colored in order to define a domain exchange (see figure 2) which is measurably conjugated to a translation on the torus $\mathbb{T}^2$, and which is also measurably conjugated to the symbolic system generated by the substitution. A conjecture called the Pisot conjecture states that such conjugacies exist for every Pisot irreducible substitution.

FIGURE 2. Domain exchange of s



A well-known fact about Rauzy fractals is that it always has non-empty interior. But in this picture, we don't see very well this fact. Let us zoom in this Rauzy fractal in order to see what looks like the non-empty interior parts (see picture 3).

FIGURE 3. Zoom in the Rauzy fractal of the substitution $s : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 12$



The observation made by Timo Jolivet is that inside the Rauzy fractal of s, we recognize a well-known fractal called the Hokkaido fractal, and which is the Rauzy fractal of another substitution.

The **Hokkaido fractal** is the Rauzy fractal of the substitution

$$h : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases} .$$

FIGURE 4. The Hokkaido fractal

The name Hokkaido has been given by Shigeki Akiyama in reference to the Japanese island with the same name. This substitution h naturally occurs when we look at β-expansion for the smallest Pisot number β. It is studied in various papers like for example [**AL**], [**Ei Ito**], [**EIR**], and [**Sieg. Thusw.**]. There is a natural domain exchange on the Rauzy fractal of h which is measurably isomorphic to the symbolic system generated by h, and this domain exchange is exactly what we get if we induce the domain exchange for s on one of the small Hokkaido tile that occurs in the Rauzy fractal of s (see figure 10).

In this article, we will prove the observation of Timo Jolivet and we will give even a more precise description of the Rauzy fractal of s. The first step to do that will be to show that we can decompose the Rauzy fractal of the substitution s as the union of a fractal of dimension less than two and a countable union of homothetic transformations of the Hokkaido fractal (see pictures 5 and 6).

FIGURE 5. Part of dimension < 2

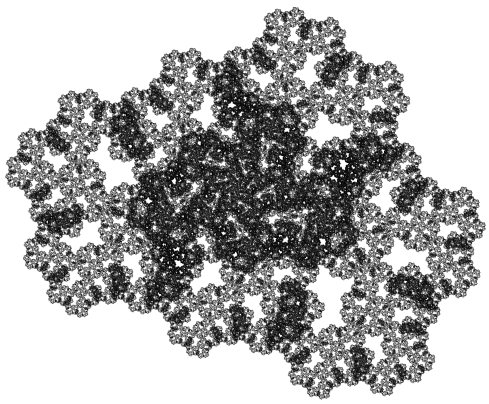
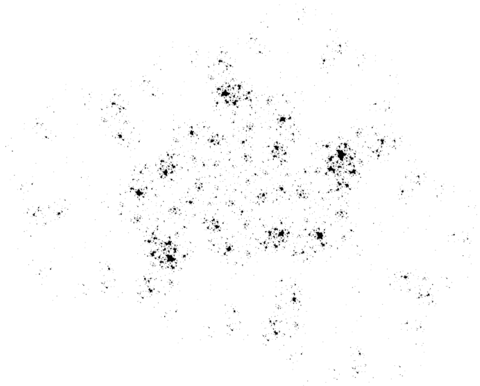
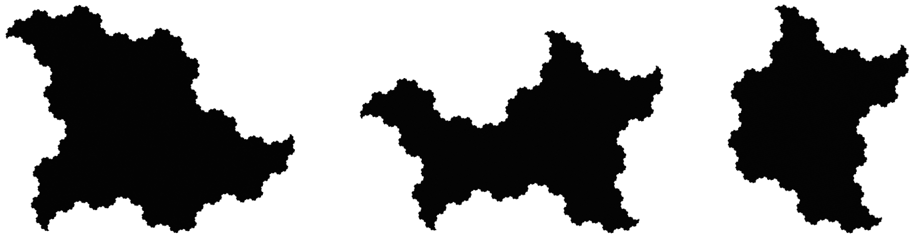


FIGURE 6. Countable union of Hokkaido tiles



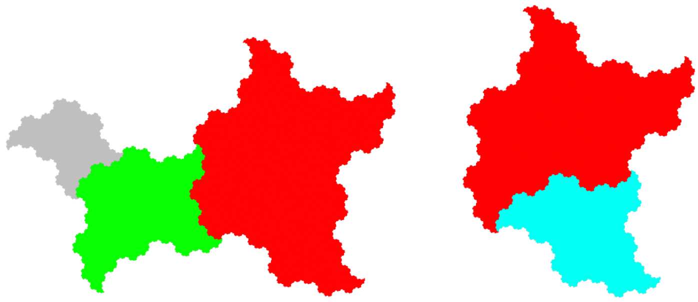
We show more precisely that in the countable union of Hokkaido fractals, there are three types of arrangements that are all pairwise disjoint.

FIGURE 7. Three types of arrangements of Hokkaido tiles



Moreover, these three arrangements are finite unions of homothetic transformations of the Hokkaido fractal (see figure 8), and one of them is exactly a single homothetic transformation of the Hokkaido fractal.

FIGURE 8. Links between the three different shapes
The second is a disjoint union of three time the third, and the third is a disjoint union of two times the first (up to a set of measure 0)



More precisely, what we show is the following.

Theorem 1.1. — *Let $R_s \subseteq \mathbb{C}$ and $H \subseteq \mathbb{C}$ be respectively the Rauzy fractals of the substitutions*

$$s : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{cases}, \qquad h : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases}.$$

Then we have

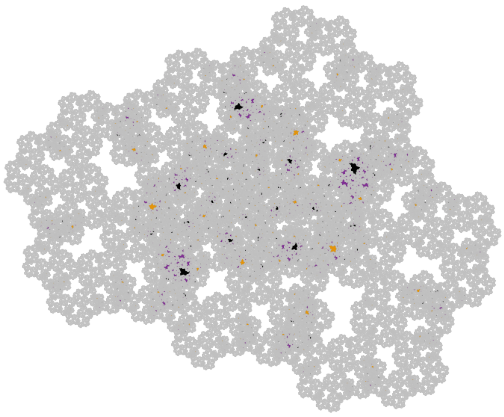
$$R_s = M \cup \bigcup_{i \in \mathbb{N}} (H_i \cup S_i \cup T_i),$$

where for every $i \in \mathbb{N}$

- *H_i is a homothetic transformation of H,*
- *S_i and T_i are respectively homothetic transformations of S and T, where S and T are finite unions of homothetic transformations of H,*
- *M is a fractal of dimension less than 2 (The exact Minkowski-Bouligand dimension is $2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$ where $\gamma \approx 1.31477860592584...$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$ and β is the smallest Pisot number.),*
- *$M \subseteq \overline{\mathbb{C} \backslash R_s}$,*

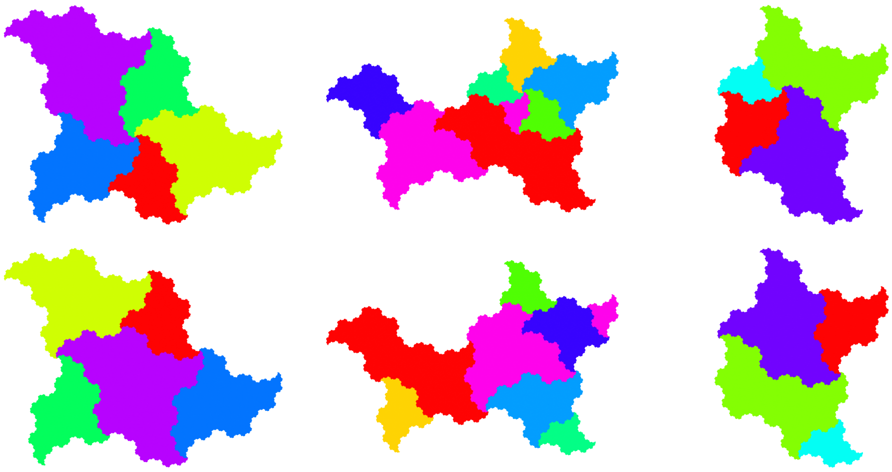
and $H_i, S_i, T_i, i \in \mathbb{N}$ are all pairwise disjoints.

FIGURE 9. Rauzy fractal with the occurrences of the first type of arrangement — $\sigma_-(Q_{B_1C_1L_h})$ — in black, the occurrences of the second type of arrangement — $\sigma_-(Q_{B_2C_2L_h})$ — in purple, the third type of arrangement — $\sigma_-(Q_{B_3C_3L_h})$ — in dark-yellow, and the part of dimension less than two in gray



The three types of shape appearing in the Rauzy fractal of s give three domain exchanges when we induce the symbolic system of s on one of the occurrence in the Rauzy fractal R_s:

FIGURE 10. Induction of the domain exchange of s on each type of arrangement



We will not prove this fact, but it can be achieved and computed using tools of [**Mercat2**].

The domain exchange that we get on the first type of shape is the same as the one we get from the Hokkaido substitution h. And it is interesting to remark that the domain exchange that we get on the third type of arrangement is naturally measurably isomorphic to a translation on the torus $\mathbb{T}^2$ (in particular, this third shape tile the plane). And this toral translation is the same as the one we get from s:

Proposition 1.2. — *There is a domain exchange with four domains on the third type of shape $\overline{\sigma_-(Q_{L_3})}$, where L_3 is defined in figure 18. This domain exchange is measurably isomorphic to a translation on the torus $\mathbb{T}^2$, and is also measurably isomorphic to the symbolic system of the substitution s.*

Hence this gives a much simpler geometrical representation than the natural one for the symbolic system of the substitution s.

And as for Hokkaido, we can get (using [**Mercat2**]) these domain exchange from substitutions, but with more letters:

$$s_2 : \begin{cases} a \mapsto ha \\ b \mapsto hfdejhab \\ c \mapsto hfc \\ d \mapsto hfchfd \\ e \mapsto ei \\ f \mapsto hab \\ g \mapsto eifdg \\ h \mapsto hfdej \\ i \mapsto fdej \\ j \mapsto fdg \end{cases} \qquad s_3 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 34 \\ 4 \mapsto 15 \\ 5 \mapsto 32 \end{cases} .$$

In particular, the natural coloring of the Rauzy fractal of this last substitution s_3 gives a decomposition of the third type of arrangement as an union of five Hokkaido tiles that are disjoint up to a set of Lebesgue measure zero. We will show that arrangements of the second type are also finite unions of Hokkaido tiles.

Remark 1.3. — *Computations and drawings have been done using the Sage mathematical software (see `http://www.sagemath.org` for more details), with additional tools developed by the author and available here: `https://trac.sagemath.org/ticket/21072`. Unfortunately, these tools are not easy to install and are not well documented yet. A worksheet with the computations of this article is available here: `https://old.i2m.univ-amu.fr/~mercat.p/SmallestPisotSubstitution.htm`*

The tools used in this article are very general and could be used to study Rauzy fractals of a large class of substitutions. Similar tools are developed in [**Sieg. Thusw.**], where they are able to decide a lot of topological properties of the Rauzy fractal of a given substitution. The computation of the dimension of the boundary of the Rauzy fractal is done using ideas of [**Lalley**]. In his article, Lalley gives an algorithm to compute the Hausdorff dimension of some sets by describing them by a finite graph. The same can be done to describe the boundary of a Rauzy fractal and to compute its dimension.

I thank Timo Jolivet to tell me about the observation that an Hokkaido tile appears inside the Rauzy fractal of the substitution s, and I also thank him to ask me if there is an efficient way to zoom in a Rauzy fractal. And I thank the referee [NAME ?] to ask me questions about the induction of the symbolic system of s: it made me discover the nice domain exchange with four pieces of the figure 10 which is measurably isomorphic to the symbolic system of s.

2. Definitions and notations

In this section, we present some tools and notations used to prove the main theorem 1.1. In particular, we recall what is a Rauzy fractal.

2.1. Substitutions. — Let A be an finite alphabet. We denote by $A^* := \bigcup_{n\in\mathbb{N}} A^n$ the set of finite words over the alphabet A. A **substitution** over the alphabet A is a word morphism from A^* to A^*, for the concatenation of words. A substitution is completely determined by images of letters of the alphabet.

2.2. Incidence matrix. — We call **incidence matrix** of a substitution s over n letters $\{a_1, a_2, ..., a_n\}$, the matrix $M_s \in M_n(\mathbb{Z})$ such that

$$Me_i = \left(|s(a_i)|_{a_j}\right)_{j=1}^{n}$$

where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$, and $|s(a_i)|_{a_j}$ is the number of occurrences of letter a_j in the word $s(a_i)$. For example, the incidence matrix of the substitution s defined above is

$$M_s = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

2.3. Periodic points. — If we have a substitution over an alphabet A, we can iterate the substitution starting from a letter of A. For example, for the alphabet $A = \{1, 2, 3, 4, 5\}$, for the substitution $h : 1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$, and for the letter 1, we get the words

1, 12, 123, 1234, 12345, 123451, 12345112, 12345112123, 123451121231234, ...

This gives a sequence of words where a word contains the previous one as prefix. Hence, it converges, for the usual topology on set of words, to an infinite word that we call a **fixed point** of the substitution h. A **periodic point** of a substitution is a fixed point of some power of the substitution. Such periodic point always exists. For the substitution s defined above, we have three infinite words that are periodic points:

```
1223233122331231212232331231212233121223...
2331231212233121223122323312312122312232...
3121223122323312122323312233123121223122...
```

If we consider bi-infinite words, there are 6 periodic points.

2.4. Broken line. — Take a periodic point $u = u_1u_2u_3...$ of a substitution over n letters $\{a_1, a_2, ..., a_n\}$. To a finite word v over the alphabet $\{a_1, a_2, ..., a_n\}$, we associated a vector of $\mathbb{Z}^n$ called **abelianisation**: $\mathrm{Ab}(v) := (|v|_{a_i})_{i=1}^n$ where $|v|_{a_i}$ is the number of occurrences of letter a_i in the word v. We call **discrete line** associated to the periodic point u, the set of points of $\mathbb{Z}^n$

$$D_u = \left\{\mathrm{Ab}(v) \;\middle|\; v \text{ finite prefix of } u\right\}.$$

Remark 2.1. — *The discrete line D_u associated to a fixed point u of a substitution s is M_s-invariant:*

$$M_s D_u \subseteq D_u.$$

Proof. — For all word v, we have $M_s \operatorname{Ab}(v) = \operatorname{Ab}(s(v))$. If v is a prefix of u, then $s(v)$ also. □

2.5. Rauzy fractal. — A Rauzy fractal is a geometric object giving informations about the substitution and its dynamical system. Let us give a precise definition. Let s be a substitution over n letters such that M_s have an unique eigenvalue β of modulus greater than one. We call **expanding space** the eigenspace (which is a line) associated to this greatest eigenvalue.

Proposition 2.2. — *Let u be a periodic point of s, then D_u is at bounded distance of the expanding space.*

The discrete line can be naturally mapped to $\mathbb{Q}(\beta)$, by taking a left eigenvector ${}^t w$ of the incidence matrix M_s for the greatest eigenvalue β, and applying the application $\varphi : \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{Q}(\beta) \\ X & \mapsto & {}^t wX \end{array}$. It is a natural map to consider since it gives a self-similar tiling in the expanding space $\mathbb{R}$: for any word $v \in A^*$, we have

$$\varphi(\operatorname{Ab}(s(v)) = \varphi(M_s \operatorname{Ab}(v)) = \beta\varphi(\operatorname{Ab}(v)),$$

so the set $\varphi(D_u)$ is invariant by multiplication by β if u is a fixed point of s. We denote by $\sigma_+ : \mathbb{Q}(\beta) \to \mathbb{R}$ the unique Galois embedding such that $\sigma_+(\beta) > 1$. We denote by σ_- the product of the other Galois embeddings, modulo the complex conjugation. The **contracting space** is the codomain of σ_-.

We call **Rauzy fractal** of s, and we denote by R_s, the adherence of a projection on the contracting space of the discrete line D_u for some periodic point u. More precisely

$$R_s = \overline{\sigma_-(\varphi(D_u))}.$$

For the substitutions s and h defined above, the contracting spaces are $\mathbb{C}$, because there is only one other embedding $\mathbb{Q}(\beta) \to \mathbb{C}$, corresponding to the two conjugated complex conjugates. Hence, we have $R_s \subseteq \mathbb{C}$ and $S_h \subseteq \mathbb{C}$.

For an irreducible substitution for an unit Pisot number, the Rauzy fractal can be seen as the adherence of a projection along the expanding space (i.e. the eigenspace for the Pisot eigenvalue) onto some hyperplane. When it is not irreducible, we project along a bigger space, in order to have something bounded.

2.6. Minkowski-Bouligand dimension. — We say that a set $S \subseteq \mathbb{C}$ has **Minkowski-Bouligand dimension** δ if we have

$$\delta = \lim_{\epsilon \to 0} \frac{\log(N_{\text{covering}}(\epsilon))}{\log(1/\epsilon)}$$

where $N_{\text{covering}}(\epsilon)$ is the minimal number of balls of radius ϵ necessary to cover S. We denote by $\dim_{MB}(S)$ the dimension of S if it exists. In this definition, we can replace $N_{\text{covering}}(\epsilon)$ by $N_{\text{packing}}(\epsilon)$ which is the maximal number of disjoint balls of radius ϵ centered on points of S. This gives an equivalent definition since we have, for all $\epsilon > 0$,

$$N_{\text{covering}}(2\epsilon) \leq N_{\text{packing}}(\epsilon) \leq N_{\text{covering}}(\epsilon/2).$$

If we denote by $\dim_H(S)$ the Hausdorff dimension of S, we have

$$\dim_H(S) \leq \liminf_{\epsilon \to 0} \frac{\log(N_{\text{covering}}(\epsilon))}{\log(1/\epsilon)}.$$

Hence, the Minkowski-Bouligand dimension is always greater than the Hausdorff dimension, when it exists. Contrarily to the Hausdorff dimension, the Minkowski-Bouligand dimension has the following property

$$\dim_{MB}(S) = \dim_{MB}(\overline{S}),$$

where $\overline{S}$ denotes the adherence of $S \subseteq \mathbb{C}$.

3. Regular languages and efficient zoom on a Rauzy fractal

An **alphabet** Σ is a finite set whose elements are called **letters**. A **language** is a set of finite words over an alphabet.

3.1. Regular languages and automata. — A language L over an alphabet Σ is **regular** if and only if the set of languages $\left\{u^{-1}L \;\middle|\; u \in \Sigma^*\right\}$ is finite, where

$$u^{-1}L = \left\{v \in \Sigma^* \;\middle|\; uv \in L\right\}.$$

Remark 3.1. — *This definition is not the usual one, but it is an useful characterization due to Myhill–Nerode.*

An **automaton** is a quintuple (Σ, Q, T, I, F), where
— Σ is a finite set called **alphabet**,
— Q is a finite set whose elements are called **states**,
— $T \subseteq Q \times \Sigma \times Q$ is the set of **transitions**,
— $I \subseteq Q$ is the set of **initial states**,
— $F \subseteq Q$ is the set of **final states**.

We say that a language $L \subseteq \Sigma^*$ is **recognized** by an automaton $\mathcal{A} = (\Sigma, Q, T, I, F)$, or that the language of $\mathcal{A}$ is L, if words of L are labels of paths from I to F following the set of transitions. More precisely,

$$L = \left\{u \in \Sigma^* \;\middle|\; \exists (q_i)_{i=0}^{|u|} \in Q^{|u|+1},\ q_0 \in I,\ q_{|u|} \in F, \text{ and } \forall i \in [\![1, |u|]\!],\ (q_{i-1}, u_i, q_i) \in T\right\}$$

Theorem 3.2. — *A language is regular if and only if it is recognized by an automaton.*

A proof of this result can be found in [**Carton**]. An automaton is **deterministic** if it has a single initial state and if for each state and each letter it has at most one transition from this state labeled by this letter. Given a regular language, there exists a canonical deterministic automaton recognizing this language. We call **minimal automaton** of a language $L \subseteq \Sigma^*$ the deterministic automaton recognizing the language L and having the minimal number of states. This automaton exists, is unique, and there is a natural bijection between its set of states and the set $\left\{u^{-1}L \;\middle|\; u \in \Sigma^*\right\} \setminus \{\emptyset\}$. The set of regular languages $\mathrm{Reg}(\Sigma)$ over an alphabet Σ has a lot of properties:

Properties 3.3. — *We have*
- $\emptyset \in \mathrm{Reg}(\Sigma)$,
- $\{\epsilon\} \in \mathrm{Reg}(\Sigma)$, *where ϵ is the empty word,*
- $\forall a \in \Sigma,\ \{a\} \in \mathrm{Reg}(\Sigma)$,
- $\forall A, B \in \mathrm{Reg}(\Sigma)$, *we have* $A \cup B \in \mathrm{Reg}(\Sigma)$, $A \cap B \in \mathrm{Reg}(\Sigma)$ *and* $A \backslash B \in \mathrm{Reg}(\Sigma)$,
- $\forall A, B \in \mathrm{Reg}(\Sigma)$, *we have* $AB \in \mathrm{Reg}(\Sigma)$, *where* $AB = \left\{uv \in \Sigma^* \;\middle|\; (u, v) \in A \times B\right\}$,
- $\forall A \in \mathrm{Reg}(\Sigma)$, *we have* $A^* \in \mathrm{Reg}(\Sigma)$, *where*
$$A^* = \left\{u_1u_2...u_n \in \Sigma^* \;\middle|\; (u_1, u_2, ..., u_n) \in A^n,\ n \in \mathbb{N}\right\},$$
- $\forall L \in \mathrm{Reg}(\Sigma),\ \sigma(L) \in \mathrm{Reg}(\Sigma')$, *where* $\sigma : \Sigma^* \to \Sigma'^*$ *is a word morphism,*
- $\forall L \in \mathrm{Reg}(\Sigma'),\ \sigma^{-1}(L) \in \mathrm{Reg}(\Sigma)$, *where* $\sigma : \Sigma^* \to \Sigma'^*$ *is a word morphism,*
- $\forall L \in \mathrm{Reg}(\Sigma),\ {}^tL = \left\{ {}^tu \;\middle|\; u \in L\right\} \in \mathrm{Reg}(\Sigma)$, *where* ${}^tu$ *is the word reversal of* u,
- *If* $0 \in \Sigma$, $\forall L \in \mathrm{Reg}(\Sigma),\ Z(L) = \left\{u \in \Sigma^* \;\middle|\; \exists n \in \mathbb{N},\ u0^n \in L\right\} 0^* \in \mathrm{Reg}(\Sigma)$,
- $\forall L \in \mathrm{Reg}(\Sigma),\ \mathrm{Pref}(L) = \left\{u \in \Sigma^* \;\middle|\; u \text{ prefix of a word of } L\right\} \in \mathrm{Reg}(\Sigma)$,
- $\forall L \in \mathrm{Reg}(\Sigma),\ \forall A \subseteq \Sigma,\ S^A(L) = \left\{u \in L \;\middle|\; \exists v \in A^{\mathbb{N}},\ \forall n \in \mathbb{N}, uv_n \in L\right\} \in \mathrm{Reg}(\Sigma)$, *where v_n is the prefix of length n of v.*

Moreover, every of these operations on regular languages are computable, and non-emptiness, inclusion and equality of regular languages are decidable.

Remark 3.4. — *The operation Z increase the language by adding words with more or less ending zeros,* $\mathrm{Pref}(L)$ *is the set of prefix of words of L, and $S^A(L)$ is the sub-language of L of words that can be prolongated infinitely many times by adding a letter of A and staying in L.*

Remark 3.5. — *These properties characterize the set of regular languages. Indeed, by the Kleene's theorem, the set of regular languages is also the smallest set of languages containing finite languages and invariant by union, complement, product and star operation.*

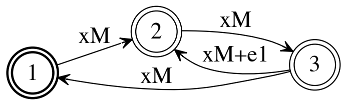
Proof. — Most of these properties of regular languages are very classical. See [**Carton**] for proof of these results. The two last properties can be shown using the characterization of regular languages by deterministic automata: we can construct automata for the new languages by keeping the same set of states, the same transitions and the same initial state, but changing the set of final states. In the last property, we have to keep a final state in the set of final states if and only if there is a path labeled in A from this state to a cycle labeled in A and composed only of final states. This gives an automaton recognizing the language $S^A(L)$. In the other property, we assume that the automaton has only accessible and co-accessible states (i.e. there exists a path from the state to a final state, and there exists a path from the initial state to the state). This can always been achieved up to removing non-accessible and non-co-accessible states. Then, we take the whole set of states as set of final states: the new automaton recognize the language $\mathrm{Pref}(L)$. To compute the language $Z(L)$ from a regular language L, take an automaton recognizing L, with final states F, and take as new set of final states $\left\{q \in F \;\middle|\; 0^* \in L_q\right\}$, where L_q is the language of the state q, that is the language of the automaton where we change the initial state to q. The concatenation of the language recognized by this automaton with 0^* is $Z(L)$. Hence $Z(L)$ is regular and computable from any regular language $L \subseteq \Sigma^*$. □

For more details about regular languages, see [**Carton**] and [**Khou. Nero.**].

Notation . — *In this article, initial states of automata are the bold circles. Final states are represented by double circles.*

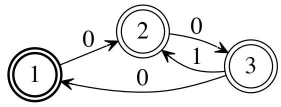
3.2. Representation of Rauzy fractals using regular languages. — Given a substitution s, we can naturally associate a graph, whose vertices are letters of the substitution, and with an edge from letter i to letter j for each j appearing in $s(i)$. The data of this graph is equivalent to the data of the incidence matrix. If moreover we add labels on these edges, we can completely encode a discrete line. For example, the following automaton represent the union of discrete lines for the three periodic points of s.

FIGURE 11. Automaton representing the union of discrete lines of s



If we start with vector $e1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and follow all the paths in this automaton, we get all the points of the union of discrete lines. In order to represent Rauzy fractals, we will project this to $\mathbb{Q}(\beta)$. The previous automaton becomes

FIGURE 12. Automaton representing the union of discrete lines of s projected on $\mathbb{Q}(\beta)$



Remark 3.6. — *This automaton is a variant of what we usually call the prefix automaton, with abelianized labels. It corresponds to the Dumont-Thomas numeration (see* **[BS]***).*

In general, a discrete line of a substitution σ is always represented in this way by a regular language over the alphabet $\left\{\mathrm{Ab}(u) \;\middle|\; u \text{ strict prefix of } \sigma(l) \text{ for a letter } l\right\}$, where $\mathrm{Ab} : u \mapsto (|u|_l)_{\text{letter } l}$ is the abelianisation. We consider the consider the image of this language under the natural mapping φ from $\mathbb{Z}^A$ to $\mathbb{Q}(\beta)$ given by an eigenvector of the incidence matrix for the Perron eigenvalue β. If L is such regular language, the mapping of the discrete line onto $\mathbb{Q}(\beta)$ is obtained by

$$\varphi(D_u) = Q_{{}^tL} = \left\{ \sum_{i=0}^{|u|-1} u_i\beta^i \;\middle|\; u \in {}^tL \right\}.$$

We obtain the Rauzy fractal with $\overline{\sigma_-(Q_{{}^tL})}$, where $\sigma_- : \mathbb{Q}(\beta) \to \mathbb{C}$ is a chosen Galois embedding. We also denote $Q_u = \sum_{i=0}^{|u|-1} u_i\beta^i$. And for an infinite word $u \in \Sigma^{\mathbb{N}}$, we will denote $\sigma_-(Q_u) = \sum_{i=0}^{+\infty} u_i\sigma_-(\beta^i) = \sum_{i=0}^{+\infty} u_i\sigma_-(\beta)^i$. There are several reasons to consider the mirror ${}^tL$ rather than directly the language L. One of them is that it permits to zoom efficiently on a Rauzy fractal.

3.3. Efficient zoom on Rauzy fractals. — Using an automaton recognizing the transposed of the language that we naturally get from a substitution, it is possible to compute efficiently the zoom on a Rauzy fractal. Indeed, when we browse paths in such automaton, we can know that this path u will not give a point in the chosen drawing area for most of paths. Because for a word uv we have

$$Q_{uv} = Q_u + \beta^{|u|}Q_v$$

and we have

$$\|\sigma_-\left(\beta^{|u|}Q_v\right)\| \leq \frac{\|\sigma_-(\beta)\|^{|u|}}{1 - \|\sigma_-(\beta)\|} \xrightarrow[|u|\to\infty]{} 0$$

for any word $v \in \Sigma^*$.

Hence, we can compute efficiently the intersection of the set

$$\left\{\sigma_-(Q_u) \;\middle|\; u \in L,\ |u| \leq n\right\},$$

with a given window, for any regular language L. And this set converges (for the Hausdorff metric), when n tends to infinity, to $\overline{\sigma_-(Q_L)}$, hence we can approximate any Rauzy fractal and compute efficiently a zoom on it.

4. Relations language and countable union of Hokkaido

In this section, we present a natural decomposition of the Rauzy fractal of the substitution $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 12$ as the union of a fractal of dimension less than two, and a countable union of Hokkaido tiles (i.e. tiles obtained from the Hokkaido fractal by homothetic transformations). In order to do this, we need a tool: the relations language.

4.1. Relations language. — Let β be the minimal Pisot number and $\Sigma = \{0, 1\}$. We call **relations language** the following language over the alphabet $\Sigma \times \Sigma$.

$$L^{\text{rel}} = \left\{ (u, v) \in (\Sigma \times \Sigma)^* \;\middle|\; \textstyle\sum_{i=0}^{|u|-1} u_i \beta^i = \sum_{i=0}^{|v|-1} v_i \beta^i \right\}.$$

Theorem 4.1. — *L^{rel} is a regular language.*

This result is a particular case of a result of Ch. Frougny (see [**Frou. Sak.**] and [**Frou. Pel.**]), and a more general version of this theorem is proven in [**Mercat**], but we give a proof here for completeness. This language permits to know what are the different β-expansions of one given algebraic integer. It will permits to know what are the common points of two discrete lines described by two different regular languages.

Proof. — The first observation is that we have

$$L^{\text{rel}} = \sigma^{-1}(L^0),$$

where $\sigma : (\Sigma \times \Sigma)^* \to \Sigma'^*$, with $\Sigma' = \{-1, 0, 1\}$, is the word morphism such that $\forall (a, b) \in \Sigma \times \Sigma,\ \sigma(a, b) = a - b$, and L^0 is the language

$$L^0 = \left\{ u \in \Sigma'^* \;\middle|\; \textstyle\sum_{i=0}^{|u|-1} u_i \beta^i = 0 \right\}.$$

Hence, we have L^{rel} is regular $\iff$ L^0 is regular $\iff$ $\left\{ u^{-1} L^0 \;\middle|\; u \in \Sigma'^* \right\}$ is finite. And we have for all $u \in \Sigma'^*$,

$$\begin{aligned} u^{-1} L^0 &= \left\{ v \in \Sigma'^* \;\middle|\; uv \in L^0 \right\} \\ &= \left\{ v \in \Sigma'^* \;\middle|\; \textstyle\sum_{i=0}^{|u|-1} u_i \beta^i + \beta^{|u|} \sum_{i=0}^{|v|-1} v_i \beta^i = 0 \right\} \\ &= \left\{ v \in \Sigma'^* \;\middle|\; \textstyle\sum_{i=0}^{|u|-1} u_i \beta^{i-|u|} + \sum_{i=0}^{|v|-1} v_i \beta^i = 0 \right\}. \end{aligned}$$

Hence $u^{-1} L^0$ is completely determined by $\sum_{i=0}^{|u|-1} u_i \beta^{i-|u|}$.

For β the smallest Pisot number, let $\sigma_+ : \mathbb{Q}(\beta) \to \mathbb{R}$ and $\sigma_- : \mathbb{Q}(\beta) \to \mathbb{C}$ be two Galois embedding of the number field $\mathbb{Q}(\beta)$, with $\sigma_+(\beta) = \beta$ and $\sigma_-(\beta) = \overline{\beta}$, where $\overline{\beta}$ is a complex conjugate of β. Then, we have the following

Theorem 4.2. — *$(\sigma_+ \times \sigma_-)(\mathbb{Z}[\beta])$ is a lattice of $\mathbb{R} \times \mathbb{C}$.*

See [**Lang**] for more details about this theorem. We have $S_u = \sum_{i=0}^{|u|-1} u_i \beta^{i-|u|} \in \mathbb{Z}[\beta]$ because $1/\beta = \beta^2 - 1$. Let us show now that $(\sigma_+ \times \sigma)(S_u)$ is bounded, for every relevant u. For all $u \in \Sigma'^*$, we have

$$|\sigma_+ (S_u)| = \left| \sum_{i=0}^{|u|-1} u_i \beta^{i-|u|} \right| \leq \sum_{i=0}^{|u|-1} \beta^{i-|u|} < \frac{1}{\beta - 1}.$$

If moreover we assume that $u^{-1}L^0 \neq \emptyset$, we have for some $v \in u^{-1}L^0$

$$|\sigma_-(S_u)| = \left| -\sigma_- \left(\sum_{i=0}^{|v|-1} v_i \beta^i \right) \right| \leq \sum_{i=0}^{|v|-1} \left|\overline{\beta}\right|^i < \frac{1}{1-\left|\overline{\beta}\right|}.$$

Therefore the set $(\sigma_+ \times \sigma_-)(S_u)$ is bounded in $\mathbb{R} \times \mathbb{C}$, uniformly in u, as soon as $u^{-1}L^0 \neq \emptyset$. Hence, by the theorem 4.2, the set $\left\{S_u \;\middle|\; u \in \Sigma'^* \text{ such that } u^{-1}L^0 \neq \emptyset\right\}$ is finite. This proves that the set $\left\{u^{-1}L^0 \;\middle|\; u \in \Sigma'^*\right\}$ is finite. Hence L^0 is regular, therefore L^{rel} also. □

Remark 4.3. — *The minimal automaton of the language L^{rel} has 179 states.*

We call **projection** of a regular language $L \subseteq \Sigma^*$ onto another regular language $L' \subseteq \Sigma^*$, the language

$$\text{Proj}(L, L') = Z(p_2(L^{\text{rel}} \cap Z(L) \times Z(L'))),$$

where Z is defined in properties 3.3, and $p_2 : (\Sigma \times \Sigma)^* \to \Sigma^*$ is the word morphism such that $\forall (a,b) \in \Sigma \times \Sigma,\ p_2(a,b) = b$.

Remark 4.4. — *We call the language* $\text{Proj}(L, L')$ *a projection onto the language L' because it corresponds to describe the elements of Q_L with words of the language $Z(L')$ (which is the language L' where we allow to add or to remove zeros at the end):*

$$\text{Proj}(L, L') = \left\{u \in Z(L') \;\middle|\; Q_u \in Q_L\right\}.$$

Of course, L' may not be large enough to describe all the elements of L, and we have

$$Q_{\text{Proj}(L,L')} = Q_L \cap Q_{L'}.$$

Proof of the remark. — We have

$$\begin{aligned} u \in \text{Proj}(L, L') &\iff \exists n \in \mathbb{N},\ u0^n \in p_2(L^{\text{rel}} \cap Z(L) \times Z(L')) \\ &\iff \exists n \in \mathbb{N},\ \exists v \in Z(L),\ (v, u0^n) \in L^{\text{rel}} \quad \text{and} \quad u \in Z(L') \\ &\iff Q_u \in Q_L \quad \text{and} \quad u \in Z(L'). \end{aligned}$$

□

Lemma 4.5. — *For all regular languages $L \subseteq \Sigma^*$ and $L' \subseteq \Sigma^*$, the languages $Z(L)$ and* $\text{Proj}(L, L')$ *are regular. We have the inclusion* $\text{Proj}(L, L') \subseteq Z(L')$, *and we have the equivalence*

$$\text{Proj}(L, L') = Z(L') \quad \iff \quad Q_{L'} \subseteq Q_L.$$

Moreover, $Z(L)$ and $\text{Proj}(L, L')$ *are computable from L and L'.*

Remark 4.6. — *Hence, it is decidable to check if we have $Q_A \subseteq Q_B$ for any regular languages A and B over the alphabet Σ.*

Proof. — The language $\mathrm{Proj}(L, L')$ is regular and computable from any regular languages L and L' since it is obtained by computable operations on regular languages. And we have $\mathrm{Proj}(L, L') \subseteq Z(L')$ by construction. By the remark 4.4, we have $\mathrm{Proj}(L, L') = Z(L') \iff Q_L \cap Q_{L'} = Q_{\mathrm{Proj}(L,L')} = Q_{Z(L')} = Q_{L'} \iff Q_{L'} \subseteq Q_L$. □

4.2. Countable union of Hokkaido. — The following theorem is a first step in order to prove the main theorem 1.1. It says that the Rauzy fractal that we want to study can be naturally decomposed into a fractal part of dimension less than two and another part which is a countable union of Hokkaido tiles.

Theorem 4.7. — *Let $R_s \subseteq \mathbb{C}$ and $H \subseteq \mathbb{C}$ be respectively the Rauzy fractals of the substitutions*

$$s : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 12 \end{cases}, \qquad h : \begin{cases} 1 \mapsto 12 \\ 2 \mapsto 3 \\ 3 \mapsto 4 \\ 4 \mapsto 5 \\ 5 \mapsto 1 \end{cases}.$$

Then we have

$$R_s = M \cup \bigcup_{i \in \mathbb{N}} H_i,$$

where for every $i \in \mathbb{N}$

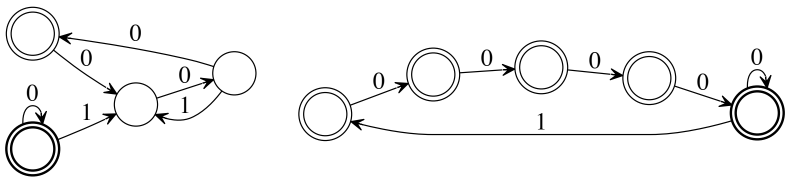
- *H_i is a homothetic transformation of H,*
- *M is a fractal of dimension less than 2 (The exact Minkowski-Bouligand dimension is $2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$ where $\gamma \approx 1.31477860592584...$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$ and β is the smallest Pisot number.),*
- *$M \subseteq \overline{\mathbb{C} \backslash R_s}$.*

This last assumption says that the adherence of M is the boundary of R_s. Hence, this decomposition of R_s is canonical in some sense.

Remark 4.8. — *It can be shown that this countable union of Hokkaido tiles is exactly the interior of the Rauzy fractal R_s, if we replace the Hokkaido tiles by their interior.*

Proof. — Let L_s be the language coming from the substitution s, and L_h be the language coming from the substitution h. We have $L_s \subset \Sigma^*$ and $L_h \subset \Sigma^*$ where $\Sigma = \{0, 1\}$.

FIGURE 13. Minimal automata of L_s and L_h respectively



We have $R_s = \overline{\sigma_-(Q_{L_s})}$ and $H = \overline{\sigma_-(Q_{L_h})}$, where $\sigma_- : \mathbb{Q}(\beta) \to \mathbb{C}$ is a Galois embedding of the number field $\mathbb{Q}(\beta)$ (i.e. σ_- is a field morphism with $\sigma_-(\beta) = \overline{\beta}$, where $\overline{\beta}$ is a complex conjugate of β).

There are a lot of languages $L \subset \Sigma^*$ satisfying $Q_{L_s} = Q_L$. Let us construct such a language L, with the property that $L \subseteq L_h$. But for that, we have to replace L_s by $0^3 L_s$ in order to have $Q_{0^3 L_s} = \beta^3 Q_{L_s} \subseteq Q_{L_h}$.

Let

$$L = \mathrm{Proj}(0^3 L_s, L_h),$$

where Proj is defined in the subsection 4.1. Then we have:

Lemma 4.9. — *The language L is regular and we have $Q_L = Q_{0^3 L_s}$ and $L \subseteq L_h$.*

Proof. — The fact that L is regular and included in L_h comes from lemma 4.5. The same lemma permits to show the equality $Q_L = Q_{0^3 L_s}$ by checking (by computer) that we have $\mathrm{Proj}(L, 0^3 L_s) = Z(0^3 L_s)$ and $\mathrm{Proj}(0^3 L_s, L) = Z(L)$. □

Remark 4.10. — *The minimal automaton of the language L has 197 states.*

Now that we have a language that describes R_s and which is included in L_h, the decomposition of R_s as a countable union of Hokkaido tiles and a fractal of dimension less than two will come from the following decomposition of the language L.

Lemma 4.11. — *We have*

$$L = L_M \cup Z(AL_h) \text{ (where the union is disjoint)},$$

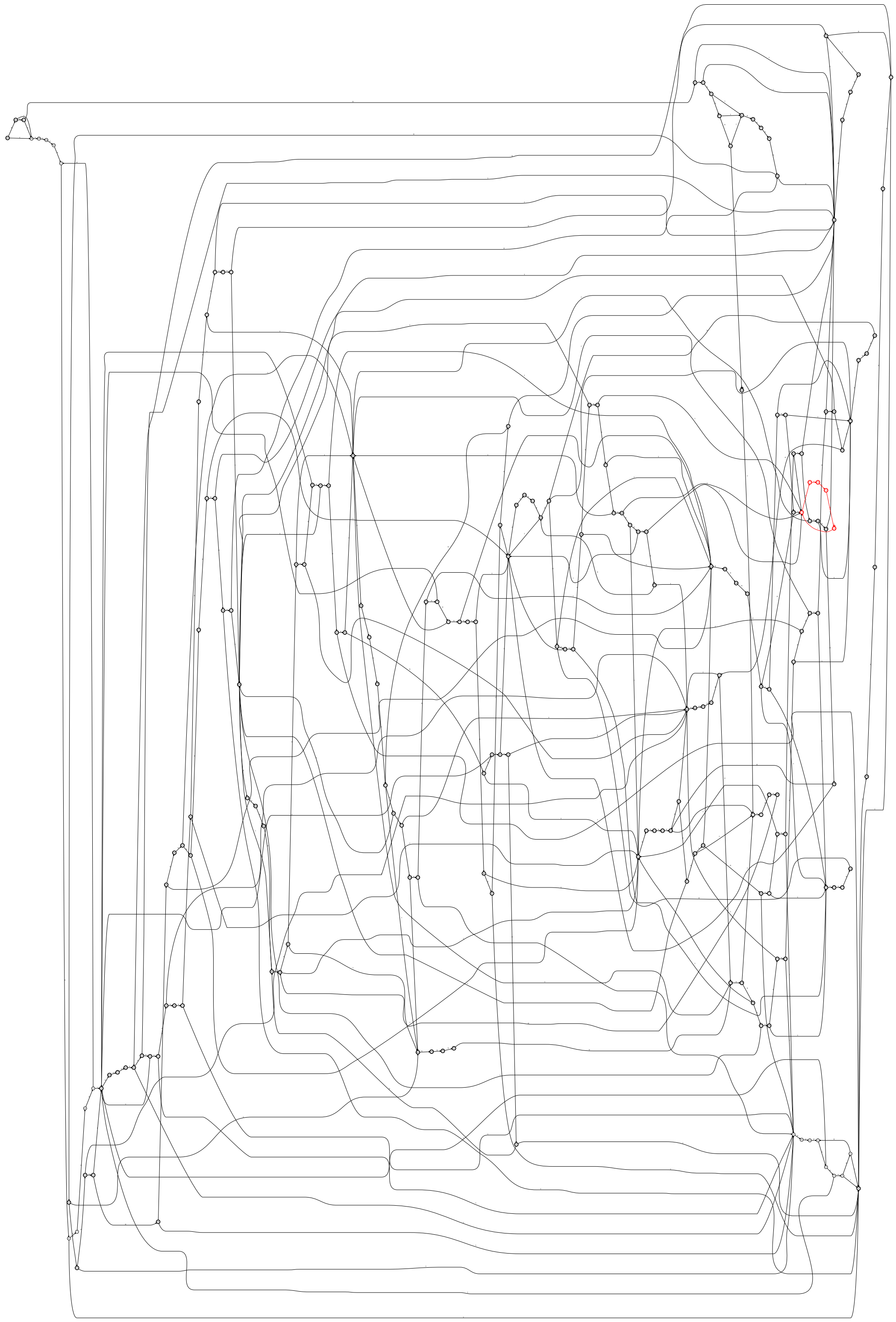
where L_M and A are computable regular languages over the alphabet Σ, with

$$\dim_{MB}(\sigma_-(Q_{L_M})) = 2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$$

where $\gamma \approx 1.31477860592584...$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$ and β is the smallest Pisot number.

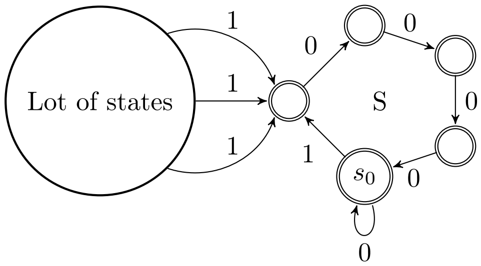
Remark 4.12. — *The regular language A describes exactly where are the Hokkaido tiles: Q_A is the set of points where a Hokkaido tile appear in $\overline{\beta}^3 R_s$. For the languages A and L_M constructed in the proof, the minimal automaton recognizing A has 197 states and the minimal automaton recognizing L_M has 191 states.*

Figure 14. Minimal automaton of the language L



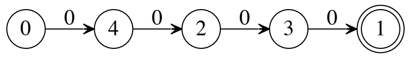
Remark 4.13. — *It could be shown that* $\sigma_-(Q_{AL_h}) = \sigma_-(Q_L) \cap \mathring{R}_s$ *and that* $\sigma_-(Q_{L_M}) = \sigma_-(Q_L) \cap \partial R_s$.

Proof. — This decomposition of L can be read on the minimal automaton A_L of L. Indeed, this automaton has this form:



More precisely, there is a sub-automaton S of A_L which is exactly the minimal automaton of L_h (see picture 13), except that it has no initial state. And there is no transition leaving from this sub-automaton, and the remaining of the automaton A_L. We get the language A by removing transitions from state s_0 which is the only state of S having two leaving transitions, and by replacing the set of final states by $\{s_0\}$. In other words, we get an automaton recognizing A by replacing the sub-automaton S by the one drawn on figure 15.

Figure 15



Then we get L_M as the complementary of $Z(AL_h)$ in L.

We obviously have that L_M and A are computable regular languages and that $L = L_M \cup Z(AL_h)$ with a disjoint union. To check the remaining of the lemma, we will use the following theorem.

Theorem 4.14. — *Let* $\overline{\beta}$ *be a complex conjugate of the smallest Pisot number* β, *and let* $L \subseteq \Sigma^*$ *be a language over the alphabet* $\Sigma = \{0,1\}$ *such that the elements of* $\sigma_-(Q_L) = \left\{\sum_{i=0}^{|u|-1} u_i \overline{\beta}^i \;\middle|\; u \in L\right\} \subseteq \mathbb{C}$ *are uniquely represented for a given length (i.e.* $\forall u, v \in L, \left(|u| = |v| \text{ and } \sum_{i=0}^{|u|-1} u_i \overline{\beta}^i = \sum_{i=0}^{|v|-1} v_i \overline{\beta}^i\right) \Longrightarrow u = v$*).*

Then we have

$$\dim_{MB}(\sigma_-(Q_L)) = \frac{\log(\gamma)}{\log(1/|\overline{\beta}|)} = 2\frac{\log(\gamma)}{\log(\beta)},$$

where γ *is the spectral radius of the minimal automaton of* L.

Proof. — Let $L_n = \left\{u \in L \;\middle|\; |u| = n\right\}$, and for all $u \in L$, let $x_u = \sum_{i=0}^{|u|-1} u_i \overline{\beta}^i$. Then we have $\sigma_-(Q_L) = \left\{x_u \;\middle|\; u \in L\right\}$, and we have the following

Lemma 4.15. — *There exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ and for all $u \neq v \in L_n$, $|x_u - x_v| \geq C \left|\overline{\beta}\right|^n$.*

Proof. — For all $n \in \mathbb{N}$, we have the inclusion

$$\left\{x_u - x_v \;\middle|\; (u,v) \in (L_n)^2\right\} \subseteq \left\{\sum_{i=0}^{n-1} a_i \overline{\beta}^i \;\middle|\; a \in \{-1,0,1\}^n\right\}.$$

Hence, it is enough to prove that the set $S = \left\{\sum_{i=0}^{n-1} a_i \overline{\beta}^{i-n} \;\middle|\; a \in \{-1,0,1\}^n\right\}$ is uniformly discrete to prove the lemma, thanks to the hypothesis that elements are uniquely represented for a given length. This follows from theorem 4.2, because $\sigma_+(S) \subseteq \mathbb{Z}[\beta]$, and the set $\sigma_+(S) = \left\{\sum_{i=0}^{n-1} a_i \beta^{i-n} \;\middle|\; a \in \{-1,0,1\}^n\right\}$ is bounded in $\mathbb{R}$ (by $\frac{1}{\beta-1}$), where σ_+ is the Galois embedding of $\mathbb{Q}(\overline{\beta})$ such that $\sigma_+(\overline{\beta}) = \beta$. □

Using this lemma, we have that the balls $B(x_u,\ \frac{1}{2}C\left|\overline{\beta}\right|^n),\ u \in L_n$, are all pairwise disjoints, hence we have

$$N_{\text{packing}}\left(\frac{1}{2}C\left|\overline{\beta}\right|^n\right) \geq \#L_n,$$

where N_{packing} is defined of subsection 2.6, applied here to the set $\sigma_-(Q_L)$. Therefore, we have

$$\liminf_{\epsilon \to 0} \frac{\log(N_{\text{packing}}(\epsilon))}{\log(1/\epsilon)} \geq \lim_{n\to\infty} \frac{\log(\#L_{n-1})}{\log\left(\frac{2}{C}\left|\overline{\beta}\right|^{-n}\right)} = \frac{\log(\gamma)}{\log(1/\left|\overline{\beta}\right|)}.$$

To prove the other inequality, let's consider for all $n \in \mathbb{N}$, and $u \in L_n$, the open ball

$$B_u = B\left(x_u,\ \frac{2\left|\overline{\beta}\right|^n}{1-\left|\overline{\beta}\right|}\right) \subseteq \mathbb{C}.$$

Up to replace L by the language $\mathrm{Pref}(L)$, which is a regular language with the same spectral radius, we have that for all $n \in \mathbb{N}$, the set of balls $\left\{B_u \;\middle|\; u \in L_n\right\}$ is a covering of $\sigma_-(Q_L)$, hence we have

$$N_{\text{covering}}\left(\frac{2\left|\overline{\beta}\right|^n}{1-\left|\overline{\beta}\right|}\right) \leq \#L_n,$$

where N_{covering} is defined on subsection 2.6. And we have $\#L_n \sim C\gamma^n$ for some constant $C > 0$. Therefore, we have

$$\limsup_{\epsilon\to 0} \frac{\log(N_{\text{covering}}(\epsilon))}{\log(1/\epsilon)} \leq \lim_{n\to\infty} \frac{\log(\#L_{n+1})}{\log\left(\frac{1-\left|\overline{\beta}\right|}{2}\left|\overline{\beta}\right|^{-n}\right)} = \frac{\log(\gamma)}{\log(1/\left|\overline{\beta}\right|)}.$$

Hence, we have $\dim_{MB}(\sigma_-(Q_L)) = \frac{\log(\gamma)}{\log(1/\left|\overline{\beta}\right|)}$. This ends the proof of the theorem 4.14. □

We can check (by computer) that the spectral radius of the minimal automaton of L_M is $\gamma \approx 1.31477860592584...$ which is the greatest root of the polynomial $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$. And the language L_M satisfy the hypothesis of the theorem 4.14, because it is included in L_h which comes from a substitution. Hence, we have

$$\dim_{MB}(\overline{\sigma_-(Q_{L_M})}) = \dim_{MB}(\sigma_-(Q_{L_M})) = 2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$$

This ends the proof of the lemma 4.11. □

The proof of the second property of the theorem 4.7 follows from this lemma 4.11. Indeed, let

$$M = \overline{\beta}^{-3}\overline{\sigma_-(Q_{L_M})}, \quad \forall a \in A,\ H_a = \overline{\beta}^{-3}\left(Q_a + \overline{\beta}^{|a|}R_h\right).$$

We have that $\overline{\sigma_-(Q_A)} \subseteq \overline{\sigma_-(Q_M)}$, because $\forall u \in A,\ \exists v \in L_M,\ u = v0$. Hence we have $R_s = M \cup \bigcup_{a\in A} H_a$, where H_a is homothetic to the Hokkaido tile, and A is countable. And we have that M is a fractal of dimension $2\frac{\log(\gamma)}{\log(\beta)} \approx 1.94643460326525...$ where $\gamma \approx 1.31477860592584...$ is the greatest root of $x^{13} - x^{12} - x^{10} + x^9 - 2\,x^4 + x^3 - 1$ and β is the smallest Pisot number. It remains to prove the last property. For this we use the lemma 6.6. We prove this lemma 6.6 later because it uses a new tool: the extended relations language $L^{\mathrm{rel}\,\infty}$. Using this lemma we can check that

$$\sigma_-(Q_M) \subseteq \overline{\sigma_-(Q_{L_h})\backslash\overline{\sigma_-(Q_{0^3L_s})}} \subseteq \overline{\mathbb{C}\backslash\overline{\beta}^3 R_s}$$

by computing

$$B = L_h\backslash p_1(S^{\{0\}\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap L_h 0^* \times \mathrm{Pref}(0^3 L_s 0^*)))$$

and checking (by computer) that

$$L_M \subseteq p_1(S^{\{0\}\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap L_M 0^* \times \mathrm{Pref}(B0^*))),$$

and this proves the wanted property. □

5. Extended relations language and three types of shapes

In this section, we do the second step of the proof of the main theorem 1.1. We have shown in the previous section that the Rauzy fractal R_s that we want to study is the union of a fractal of dimension less than two and a countable union of Hokkaido tiles. In this section, we show that these Hokkaido tiles are organized in three different manners that we will describe explicitly. In order to do that, we need a new tool: the extended relations language.

5.1. Extended relations language. — Let $\overline{\beta}$ be a complex conjugate of the minimal Pisot number β and $\Sigma = \{0,1\}$.

We call **extended relations language** the following language over the alphabet $\Sigma \times \Sigma$.

$$L^{\mathrm{rel}\,\infty} = \left\{ (u,v) \in (\Sigma\times\Sigma)^* \;\middle|\; \begin{array}{r} \exists (u',v') \in (\Sigma\times\Sigma)^{\mathbb{N}},\ \text{with } (u,v) \text{ prefix of } (u',v') \\ \text{and } \displaystyle\sum_{i=0}^{\infty} (u'_i - v'_i)\overline{\beta}^i = 0 \end{array} \right\}.$$

Theorem 5.1. — *$L^{\mathrm{rel}\,\infty}$ is a regular language.*

The proof is very similar to the proof of the theorem 4.1.

Proof. — The first observation is that we have

$$L^{\mathrm{rel}\,\infty} = \sigma^{-1}(L^\infty),$$

where $\sigma : (\Sigma\times\Sigma)^* \to \Sigma'^*$, with $\Sigma' = \{-1,0,1\}$, is the word morphism such that $\forall (a,b) \in \Sigma\times\Sigma,\ \sigma(a,b) = a-b$, and L^∞ is the language

$$L^\infty = \left\{ u \in \Sigma'^* \;\middle|\; \exists u' \in \Sigma'^{\mathbb{N}} \text{ extending } u,\ \textstyle\sum_{i=0}^\infty u'_i\overline{\beta}^i = 0 \right\}.$$

Hence, we have $L^{\mathrm{rel}\,\infty}$ is regular $\iff L^\infty$ is regular $\iff \left\{ u^{-1}L^\infty \;\middle|\; u \in \Sigma'^* \right\}$ is finite. And we have for all $u \in \Sigma'^*$,

$$\begin{aligned} u^{-1}L^\infty &= \left\{ v \in \Sigma'^* \;\middle|\; uv \in L^\infty \right\} \\ &= \left\{ v \in \Sigma'^* \;\middle|\; \begin{array}{r} \exists w \in \Sigma'^{\mathbb{N}},\ \sum_{i=0}^{|u|-1} u_i\overline{\beta}^i + \overline{\beta}^{|u|}\sum_{i=0}^{|v|-1} v_i\overline{\beta}^i + \\ \overline{\beta}^{|u|+|v|}\sum_{i=0}^\infty w_i\overline{\beta}^i = 0 \end{array} \right\} \\ &= \left\{ v \in \Sigma'^* \;\middle|\; \exists w \in \Sigma'^{\mathbb{N}},\ \sum_{i=0}^{|u|-1} u_i\overline{\beta}^{i-|u|} + \sum_{i=0}^{|v|-1} v_i\overline{\beta}^i + \overline{\beta}^{|v|}\sum_{i=0}^\infty w_i\overline{\beta}^i = 0 \right\}. \end{aligned}$$

Hence $u^{-1}L^\infty$ is completely determined by $S_u = \sum_{i=0}^{|u|-1} u_i\overline{\beta}^{i-|u|}$.

Let $\sigma_+ : \mathbb{Q}(\overline{\beta}) \to \mathbb{R}$ and $\sigma_- : \mathbb{Q}(\overline{\beta}) \to \mathbb{C}$ be the two Galois embeddings of the number field $\mathbb{Q}(\beta)$ such that $\sigma_-(\overline{\beta}) = \overline{\beta}$ and $\sigma_+(\overline{\beta}) = \beta$.

We have $S_u \in \mathbb{Z}[\overline{\beta}]$ because $1/\overline{\beta} = \overline{\beta}^2 - 1$. Let us show now that $(\sigma_+ \times \sigma_-)(S_u)$ is bounded, for every relevant u. For all $u \in \Sigma'^*$, we have

$$|\sigma_+(S_u)| = \left| \sum_{i=0}^{|u|-1} u_i\beta^{i-|u|} \right| \le \sum_{i=0}^{|u|-1} \beta^{i-|u|} \le \frac{1}{\beta-1}.$$

If moreover we assume that $u^{-1}L^\infty \neq \emptyset$, we have for some $v \in u^{-1}L^\infty$ and some $w \in \Sigma'^{\mathbb{N}}$,

$$|\sigma_-(S_u)| = \left| -\sigma_-\left(\sum_{i=0}^{|v|-1} v_i\overline{\beta}^i + \overline{\beta}^{|v|}\sum_{i=0}^\infty w_i\overline{\beta}^i \right) \right| \le \sum_{i=0}^\infty \left|\overline{\beta}\right|^i = \frac{1}{1-\left|\overline{\beta}\right|}.$$

Therefore the set $(\sigma_+ \times \sigma_-)(S_u)$ is bounded in $\mathbb{R} \times \mathbb{C}$, uniformly in u, as soon as $u^{-1}L^\infty \neq \emptyset$. Hence, by the theorem 4.2, the set $\{S_u \mid u \in \Sigma'^* \text{ such that } u^{-1}L^\infty \neq \emptyset\}$ is finite. This proves that the set $\{u^{-1}L^\infty \mid u \in \Sigma'^*\}$ is finite. Hence L^∞ is regular, therefore $L^{\mathrm{rel}\,\infty}$ also. $\square$

Remark 5.2. — *The minimal automaton of the language $L^{\mathrm{rel}\,\infty}$ has* 179 *states.*

5.2. Description of the three types of shapes. — In this subsection, we describe the three types of shapes appearing in the main theorem 1.1, and we show that these shapes are finite unions of Hokkaido tiles. These shapes are described by automata of the figures 16, 17 and 18. This means that the i^{th} shape is $\sigma_-(Q_{L_i})$ where L_i is the language of the i^{th} automaton. The figure 7 show the sets $\sigma_-(Q_{L_i})$.

Figure 16. Minimal automaton of the regular language L_1 describing the first type of shape

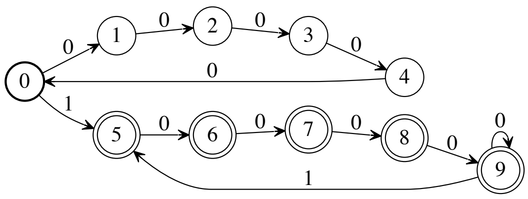


Figure 17. Minimal automaton of the regular language L_2 describing the second type of shape

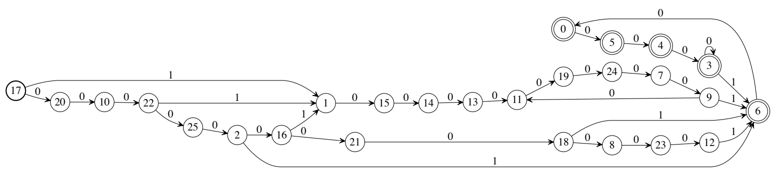
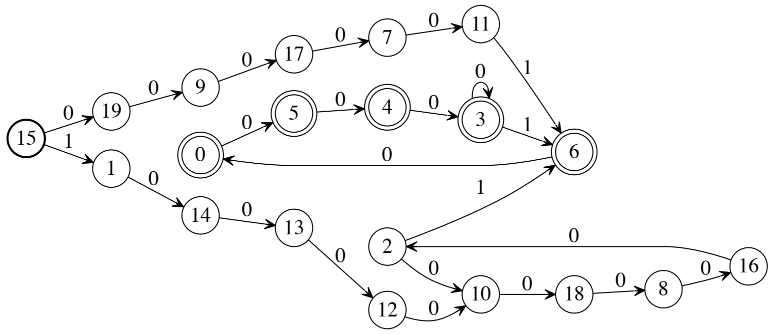


Figure 18. Minimal automaton of the regular language L_3 describing the third type of shape



As we can see in figures 16, 17 and 18, the three types of shapes are not trivially unions of finitely many Hokkaido tiles, but we prove it now.

5.2.1. First type of shape. — The first shape is a Hokkaido tile:

Proposition 5.3. — *Let L_1 be the regular language recognized by the automaton of the figure 16. We have*

$$Q_{L_1} = \beta^4 Q_{L_h} + 1.$$

Proof. — We have $\beta^4 Q_{L_h} + 1 = Q_{1000L_h}$. Hence, by lemma 4.5, the equality is obtained by checking that we have $\mathrm{Proj}(1000L_h, L_1) = Z(L_1)$ and $\mathrm{Proj}(L_1, 1000L_h) = Z(1000L_h)$. □

It gives us that $\overline{\sigma_-(Q_{L_1})} = \overline{\beta}^4 H + 1$, where $H = \overline{\sigma_-(Q_{L_h})}$ is the Hokkaido tile.

5.2.2. Second type of shape. — The second type of shape is a disjoint union of three homothetic transformations of the third type of shape, up to a set of Lebesgue measure zero (see figure 8). The following proposition permits to prove that we have the union:

Proposition 5.4. — *Let L_2 be the regular language recognized by the automaton of the figure 17. We have*

$$L_2 = \{\epsilon, 0^3, 0^6\} L_3,$$

where L_3 is the regular language recognized by the automaton of the figure 18.

Proof. — Easy verification. □

Hence, we get that $\overline{\sigma_-(Q_{L_2})} = \overline{\sigma_-(Q_{L_3})} \cup \overline{\beta}^3 \overline{\sigma_-(Q_{L_3})} \cup \overline{\beta}^6 \overline{\sigma_-(Q_{L_3})}$.

In order to prove the disjointness in measure of the union, we use the lemma 6.6. For every $A \neq B \in \{L_2, 0^3 L_2, 0^6 L_2\}$, we check (by computer) that we have

$$p_1(S^{\{0\}\times\{0,1\}}(L^{\mathrm{rel}\,\infty} \cap A0^* \times \mathrm{Pref}(B0^*))) = \emptyset,$$

and it gives us that $\sigma_-(Q_A) \cap \overline{\sigma_-(Q_B)} = \emptyset$. Hence, the tiles $\overline{\beta}^i \overline{\sigma_-(Q_{L_3})}$, for $i \in \{0, 3, 6\}$ intersect only in their boundary. But we will see that the boundary of some homothetic transformations of $\overline{\sigma_-(Q_{L_3})}$ is included in the boundary of R_s, hence it has zero Lebesgue measure. Indeed, it is known that boundaries of Rauzy fractals of primitive substitutions always has zero Lebesgue measure, see for example [**Milt. Thus.**].

5.2.3. Third type of shape. — The third type of shape is a disjoint union of two Hokkaido tiles, up to a set of measure zero (see figure 8). We start by proving that we have the union:

Proposition 5.5. — *Let L_3 be the regular language recognized by the automaton of the figure 18. We have*

$$L_3 = Z(0000010000L_h) \cup 100000000L_1.$$

Proof. — Easy verification. □

Hence, $\overline{\sigma_-(Q_{L_3})}$ is the union of the two Hokkaido tiles $\overline{\beta}^{10}H+\overline{\beta}^5$ and $\overline{\beta}^{13}H+\overline{\beta}^9+1$.

In order to prove the disjointness in measure of this union, we use the lemma 6.6, like for the second type of shape. It permits to prove that the two tiles intersect only in their boundary, but it is known that the boundary of the Hokkaido tile has zero Lebesgue measure.

5.3. Construction of the three types of shapes. — The aim of this subsection is to explain how the automata of the figures 16, 17 and 18 have been obtained. This is not clear from the construction that these shapes satisfy what we expect, but we can check it after having computed them explicitly. Hence, this subsection is not useful for the proof of the main theorem 1.1 since these automata have been given explicitly. In order to describe the three types of shapes formed by Hokkaido tiles glued together in the Rauzy fractal R_s, we need a way to know if the adherences of two given Hokkaido tiles have a non-empty intersection. This is done using the extended relations language. Let

$$\varphi(L) = AL_h \cap p_2(S^{\Sigma\times\Sigma}(Z(L) \times Z(AL_h) \cap L^{\mathrm{rel}\,\infty}))\Sigma^*,$$

This application gives the union of Hokkaido tiles whose adherences intersect the adherence of the set described by L. See lemma 6.10 for more details. Hence, the idea to construct the languages L_1, L_2 and L_3 is to choose an Hokkaido tile in the shape that we see when we zoom in the fractal, and then apply the function φ until it cover the whole shape. Unfortunately, this strategy doesn't work: the set doesn't stop to grow when applying φ. But we can get it work thanks to the following.

We define an equivalence relation on A by

$$u \sim' v \iff uR_h \cap vR_h \neq \emptyset,$$

and let $\sim$ be the transitive closure of $\sim'$.

Lemma 5.6. — *If $uvwR_h \cap uwR_h \neq \emptyset$ and if for all $n \in \mathbb{N}$, $uv^nw \in A$, then*

$$\forall n \in \mathbb{N},\ uv^nw \sim uw.$$

Proof. — For all $u, v, w \in \Sigma^*$, we have $uvR_h \cap uwR_h \neq \emptyset \iff vR_h \cap wR_h \neq \emptyset$. Hence, we have for all $n \in \mathbb{N}$,

$$uv^{n+1}wR_h \cap uv^nwR_h \neq \emptyset.$$

The result follows by transitivity of $\sim$. □

Using the function φ and the lemma 5.6, we have found three different shapes that are formed by Hokkaido tiles glued together, and that doesn't intersect any other Hokkaido tile. These shapes are drawn on the figure 7 and correspond to the automata of the figures 16, 17 and 18. We show in the following that no other shape appears.

6. Proof of the theorem 1.1

In this section, we show that the countable union of Hokkaido tiles given by the theorem 4.7 is organised as a pairwise disjoint countable union of homothetic transformations of the three types of tiles described in the previous section. This will terminates the proof of the main theorem 1.1. We want to show the following theorem.

Theorem 6.1. — *Let A be the regular language given by the theorem 4.7. Then, we have*

$$Q_{AL_h} = Q_{B_1C_1L_h \cup B_2C_2L_h \cup B_3C_3L_h},$$

where B_1, B_2, B_3, C_1, C_2, C_3 are regular languages such that

$$\forall i \in \{1, 2, 3\}, Q_{C_iL_h} = Q_{L_i},$$

where L_1, L_2 and L_3 are respectively the language recognized by the automaton of the figure 16, 17 and 18. And we have

— $\forall i \in \{1, 2, 3\},\ \forall u \neq v \in B_i,\ \overline{\sigma_-(Q_{uC_iL_h})} \cap \overline{\sigma_-(Q_{vC_iL_h})} = \emptyset$,

— $\forall i \neq j \in \{1, 2, 3\},\ \forall u \in B_iC_i, \forall v \in B_jC_j, \overline{\sigma_-(Q_{uL_h})} \cap \overline{\sigma_-(Q_{vL_h})} = \emptyset$.

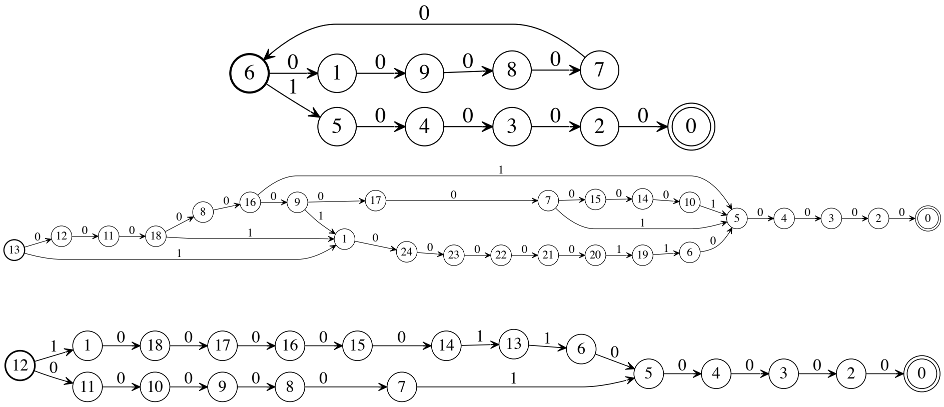
Moreover, everything is computable.

The language B_i describes where are the tiles of type i, and $\sigma_-(C_iL_h)$ is the shape of type i.

In order to prove this theorem, we start by constructing the languages B_i, C_i, $i \in \{1, 2, 3\}$, and then we show that these languages satisfy the required properties.

6.1. Construction of the languages B_1, B_2, B_3, C_1, C_2 and C_3. — In this part, we explain how to compute the languages given in the theorem 6.1. This is not clear from the construction that these languages satisfy what we expect, but we can check it after having computed them explicitly, and this is what we do in the next subsection 6.2. The first step is to consider the languages C_1, C_2 and C_3 recognized by the automata of the figure 19.

FIGURE 19. Minimal automata of regular languages C_1, C_2 and C_3 respectively



These languages comes from the three different shapes computed and described in the previous section. We can verify that we have

$$\forall i \in \{1,2,3\},\ Q_{C_i L_h} = Q_{L_i}.$$

In order to compute languages B_1, B_2 and B_3 describing the occurrence in the Rauzy fractal of tiles of each type, we do the following. Let Q_A be the set of states of the minimal automaton recognizing the language A, and for a state q, let L_q be the language of the state (that is the language of the automaton where we replace the initial state by q). Let's consider the sets

$$F_i = \left\{ q \in \mathrm{Q}_A \;\middle|\; \mathrm{Proj}(L_q, C_i) = Z(C_i) \right\} = \left\{ q \in \mathrm{Q}_A \;\middle|\; Q_{C_i} \subseteq Q_{L_q} \right\}$$

for $i \in \{1,2,3\}$. Then, we define D_i as the language of the minimal automaton of A where we replace the set of final states by F_i. The minimal automaton of D_1 has 192 states and the minimal automata of D_2 and D_3 have 191 states. These languages are not yet the right ones, because they describe tiles that are not disjoint: there are for example tiles of type 3 included in tiles of type 2. In order to get disjoint tiles, we start by projecting the concatenation of these languages with L_h on AL_h, and we consider that tiles of type 2 are not included in tiles of other types and that tiles of type 3 are not included in tiles of type 1 with this description. Let

$$\begin{aligned} E_2 &= \mathrm{Proj}(D_2 C_2 L_h, AL_h), \\ E_3 &= \mathrm{Proj}(D_3 C_3 L_h, AL_h) \backslash E_2, \\ E_1 &= \mathrm{Proj}(D_1 C_1 L_h, AL_h) \backslash (E_2 \cup E_3). \end{aligned}$$

Then E_i describes the union of tiles of type i occurring in the Rauzy fractal of s. We can convince ourself of it by drawing it, and we can check that $E_1 \cup E_2 \cup E_3 = Z(AL_h)$. The minimal automata of E_1, E_2 and E_3 have 205 states.

Then, we compute regular languages A_1, A_2 and A_3 such that for all $i \in \{1,2,3\}, Z(E_i) = Z(A_i L_h)$. This is done in the same way that for the construction of the language A from the language L: we recognize a sub-automaton of E_i corresponding to Hokkaido. Minimal automata of A_1, A_2 and A_3 have 205 states. The last step is done in the same way as for the construction of languages D_i. Let Q_i be the set of states of the minimal automaton of the language A_i, and let

$$F_i = \left\{q \in \mathrm{Q}_i \;\middle|\; \mathrm{Proj}(L_q L_h, L_i) = Z(L_i)\right\} = \left\{q \in \mathrm{Q}_i \;\middle|\; Q_{L_i} \subseteq Q_{L_q L_h}\right\},$$

for $i \in \{1,2\}$, and

$$F_3 = \left\{q \in \mathrm{Q}_i \;\middle|\; \mathrm{Proj}(L_q, C_3) = Z(C_3)\right\} = \left\{q \in \mathrm{Q}_i \;\middle|\; Q_{C_3} \subseteq Q_{L_q}\right\}.$$

We get an automaton recognizing the language B_i by replacing final states of the minimal automaton of A_i by F_i deprived of its final states. Minimal automata of B_1, B_2 and B_3 have 200, 191 and 191 states respectively.

Remark 6.2. — *This construction is weird, but we can check in the following subsection that it works!*

6.2. Proof of the theorem 6.1. — To prove the theorem 6.1, we take the languages B_1, B_2, B_3, C_1, C_2 and C_3 constructed above and we check that these languages satisfy every of the wanted properties.

Checking that we have

$$Q_{AL_h} = Q_{B_1 C_1 L_h \cup B_2 C_2 L_h \cup B_3 C_3 L_h},$$

is easy, using the lemma 4.5.

The following lemma permits to show the disjointness of adherences of tiles of the same type.

Lemma 6.3. — *We have for all $i \in \{1,2,3\}$, and $(u,v) \in B_i \times B_i$, $\overline{\sigma_-(Q_{uC_iL_h})} \cap \overline{\sigma_-(Q_{vC_iL_h})} \neq \emptyset$ if and only if*

$$\exists (u',v') \in C_i L_h \times C_i L_h,\ (uu', vv') \in S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap (B_i \,\mathrm{Pref}(Z(C_i L_h)) \times B_i \,\mathrm{Pref}(Z(C_i L_h))))$$

Hence, if we check that

$$S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap ((B_i \,\mathrm{Pref}(C_i L_h) \times (B_i \,\mathrm{Pref}(C_i L_h)) \backslash (((B_i \times B_i) \cap \Delta) \,\mathrm{Pref}(C_i L_h \times C_i L_h)))) = \emptyset,$$

it shows that any two different tiles of the same type that appear in Q_{AL_h} have disjoint adherences. This checking is done by computer. In order to prove the lemma 6.3, we will need the two following lemmas. The following lemma gives a characterization of the adherence of $\sigma_-(Q_L)$ for a regular language L. It will be useful to check algorithmically if tiles have disjoint adherences.

Lemma 6.4. — *For a regular language $L \subseteq \Sigma^*$, we have*

$$x \in \overline{\sigma_-(Q_L)} \iff \exists u \in \Sigma^{\mathbb{N}},\ \sigma_-(Q_u) = x \text{ and } \forall n \in \mathbb{N},\ u_n \in \mathrm{Pref}(Z(L)),$$

where u_n is the prefix of length n of u.

Proof. — The right-to-left implication is easy: Let $u \in \Sigma^{\mathbb{N}}$ such that $\sigma_-(Q_u) = x$ and $\forall n \in \mathbb{N}, u_n \in \operatorname{Pref}(Z(L))$. Then, by definition of $\operatorname{Pref}(L)$, $\forall n \in \mathbb{N}$, $\exists v_n \in \Sigma^*$, $u_n v_n \in Z(L)$. We have $Q_{u_n v_n} \in Q_L$ and $\sigma_-(Q_{u_n v_n}) = \sigma_-(Q_{u_n}) + \overline{\beta}^n \sigma_-(Q_{v_n}) \xrightarrow[n\to\infty]{} x$, therefore $x \in \overline{\sigma_-(Q_L)}$.

To prove the other implication, we use the fact that L is regular by considering an automaton recognizing $\operatorname{Pref}(Z(L))$. We can assume that all states of this automaton are co-reachable (that is there exists a path from the state, to a final state), up to remove the non-co-reachable ones. Then every state of the automaton is final since the language is stable by prefix. Then, using the fact that every word of $\operatorname{Pref}(Z(L))$ is extendable by $0 \in \Sigma$, we have the identity

$$\sigma_-(Q_{L_i}) = \bigcup_{i \xrightarrow{t} j} \overline{\beta}\sigma_-(Q_{L_j}) + t,$$

where L_i is the language of the state i of the automaton (that is the language of the automaton where we take i as initial state). If we take the adherence, we get

$$\overline{\sigma_-(Q_{L_i})} = \bigcup_{i \xrightarrow{t} j} \overline{\beta}\overline{\sigma_-(Q_{L_j})} + t,$$

for all state i. Hence, if $x \in \overline{\sigma_-(L_i)}$, we can consider a transition $i \xrightarrow{t} j$ in the automaton such that $\frac{x-t}{\overline{\beta}} \in \overline{\sigma_-(L_j)}$. By recurrence, there exists for all $n \in \mathbb{N}$, $(t_i)_{i=0}^{n-1} \in \Sigma^n$ such that $x = \sum_{i=0}^{n-1} t_i \overline{\beta}^i + \overline{\beta}^n y$ where $y \in \overline{\sigma_-(Q_{L_j})}$ for some state j. Hence, we get an infinite word $u \in \Sigma^{\mathbb{N}}$ such that $x = \sigma_-(Q_u)$. And for all $n \in \mathbb{N}$, we have $u_n \in \operatorname{Pref}(L)$ since every state of the automaton is final. □

Here is an useful corollary of this lemma.

Corollary 6.5. — *For two regular languages A and $B \subseteq \Sigma^*$, we have*

$$\begin{aligned} & x \in \bigcup_{u \in A} \overline{\sigma_-(Q_{uB})} \\ \iff\quad & \exists u \in A,\ \exists v \in \Sigma^{\mathbb{N}},\ \sigma_-(Q_{uv}) = x \text{ and } \forall n \in \mathbb{N},\ v_n \in \operatorname{Pref}(Z(B)), \end{aligned}$$

where v_n is the prefix of length n of u.

The following lemma permits to end the proof of the theorem 4.7.

Lemma 6.6. — *For A and B regular languages over an alphabet $\Sigma \subseteq \mathbb{Z}[\beta]$ containing 0, and for all $u \in A$, we have the equivalence*

$$\sigma_-(Q_u) \in \overline{\sigma_-(Q_B)} \quad \iff \quad u \in p_1(S^{\{0\}\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap A0^* \times \operatorname{Pref}(B0^*)))$$

Proof. — For $u \in A$, we have the equivalence

$$\begin{aligned}
& \sigma_-(Q_u) \in \overline{\sigma_-(Q_B)} \\
\iff\ & \exists v \in \Sigma^{\mathbb{N}},\ \sigma_-(Q_u) = \sigma_-(Q_v) \text{ and} \\
& \forall n \in \mathbb{N}, v_n \in \operatorname{Pref}(B0^*) \qquad\qquad \text{(by lemma 6.4)} \\
\iff\ & \exists v \in \Sigma^{\mathbb{N}},\ \forall n \in \mathbb{N},\ \exists k \in \mathbb{N},\ (u0^k, v_n) \in L^{\mathrm{rel}\,\infty} \cap A0^* \times \operatorname{Pref}(B0^*) \\
\iff\ & \exists v \in \Sigma^*,\ \exists v' \in \Sigma^{\mathbb{N}},\ \forall n \in \mathbb{N},\ (u0^n, vv'_n) \in L^{\mathrm{rel}\,\infty} \cap A0^* \times \operatorname{Pref}(B0^*) \\
\iff\ & \exists v \in \Sigma^*,\ (u, v) \in S^{\{0\}\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap A0^* \times \operatorname{Pref}(B0^*)) \\
\iff\ & u \in p_1(S^{\{0\}\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap A0^* \times \operatorname{Pref}(B0^*)))
\end{aligned}$$

where v_n is the prefix of length n of v. □

The following lemma reduce the problem of knowing if the adherences of $\sigma_-(Q_A)$ and $\sigma_-(Q_B)$ intersect, where A and B are regular languages, to a calculation with regular languages.

Lemma 6.7. — *Let A and B be two regular languages. Then we have*

$$\overline{\sigma_-(Q_A)} \cap \overline{\sigma_-(Q_B)} = \emptyset \quad \iff \quad S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap \operatorname{Pref}(Z(A)) \times \operatorname{Pref}(Z(B))) = \emptyset.$$

Proof. — We have the equivalences

$$\begin{aligned}
& (u, v) \in S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap \operatorname{Pref}(Z(A)) \times \operatorname{Pref}(Z(B))) \\
\iff\ & \exists (u', v') \in (\Sigma \times \Sigma)^{\mathbb{N}},\ \forall n \in \mathbb{N},\ (uu'_n, vv'_n) \in L^{\mathrm{rel}\,\infty} \cap \operatorname{Pref}(Z(A)) \times \operatorname{Pref}(Z(B)) \\
\iff\ & \exists (u', v') \in (\Sigma \times \Sigma)^{\mathbb{N}},\ \forall n \in \mathbb{N},\ \exists (u'', v'') \in (\Sigma \times \Sigma)^{\mathbb{N}}, Q_{uu'_n u''} = Q_{vv'_n v''} \\
& \text{and } \forall n \in \mathbb{N},\ (uu'_n, vv'_n) \in \operatorname{Pref}(Z(A)) \times \operatorname{Pref}(Z(B)) \\
\iff\ & \exists (u', v') \in (\Sigma \times \Sigma)^{\mathbb{N}},\ Q_{uu'} = Q_{vv'} \text{ and} \\
& \forall n \in \mathbb{N},\ (uu'_n, vv'_n) \in \operatorname{Pref}(Z(A)) \times \operatorname{Pref}(Z(B))
\end{aligned}$$

where u'_n and v'_n are respectively the prefix of length n of u' and v'. Hence, by lemma 6.4, we have the wanted equivalence. □

Remark 6.8. — *Hence, if A and B are regular languages, we can test algorithmically if we have $\overline{\sigma_-(Q_A)} \cap \overline{\sigma_-(Q_B)} = \emptyset$ or not.*

The following generalization is also useful.

Lemma 6.9. — *Let A, B, C and D be four regular languages. Then we have*

$$\left(\bigcup_{u\in A} \overline{\sigma_-(Q_{uB})} \right) \cap \left(\bigcup_{u\in C} \overline{\sigma_-(Q_{uD})} \right) = \emptyset$$

$$\iff \quad S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \ \cap\ A \operatorname{Pref}(Z(B)) \times C \operatorname{Pref}(Z(D))) = \emptyset.$$

We can now prove the lemma 6.3.

proof of lemma 6.3. — Let $(u, v) \in B_i \times B_i$. We have

$$\begin{aligned} & \exists (u', v') \in C_i L_h \times C_i L_h, \\ & (uu', vv') \in S^{\Sigma\times\Sigma}(L^{\mathrm{rel}\,\infty} \cap (\mathrm{Pref}(Z(B_i C_i L_h)) \times \mathrm{Pref}(Z(B_i C_i L_h)))) \\ \iff & \exists (u', v') \in (\Sigma \times \Sigma)^{\mathbb{N}},\ Q_{uu'} = Q_{vv'} \qquad \text{(by proof of lemma 6.7)} \\ & \text{ and every prefix of } (uu', vv') \text{ is in } \mathrm{Pref}(Z(B_i C_i L_h)) \times \mathrm{Pref}(Z(B_i C_i L_h)) \\ \iff & \overline{\sigma_-(Q_{uC_iL_h})} \cap \overline{\sigma_-(Q_{vC_iL_h})} \neq \emptyset. \qquad \text{(by lemma 6.4)} \end{aligned}$$

□

Now it only remains to prove that tiles of different types always have disjoint adherences. This is done by checking (by computer) that we have

$$\forall i \in \{1, 3\},\ Z(\varphi(B_i \,\mathrm{Pref}(Z(C_i L_h)))) = Z(B_i C_i L_h),$$

and

$$Z(\varphi(B_2 C_2 L_h)) = Z(B_2 C_2 L_h),$$

where φ is defined by

$$\varphi(L) = AL_h \cap p_2(S^{\Sigma\times\Sigma}(Z(L) \times Z(AL_h) \cap L^{\mathrm{rel}\,\infty}))\Sigma^*,$$

where $A = B_1C_1 \cup B_2C_2 \cup B_3C_3$. This gives the wanted disjointness thanks to the following lemma.

Lemma 6.10. — *For every regular languages L_1 and $L_2 \subseteq \Sigma^*$, we have*

$$\begin{aligned} & u \in \varphi(L_1 \,\mathrm{Pref}(Z(L_2))) \\ \iff & \exists a \in A,\ u \in aL_h \quad \text{and} \quad \overline{\sigma_-(Q_{aL_h})} \cap \bigcup_{u \in L_1} \overline{\sigma_-(uQ_{L_2})} \neq \emptyset. \end{aligned}$$

Hence, the equality $\varphi(B_i \,\mathrm{Pref}(Z(C_i L_h))) = B_i C_i L_h$ proves that a copy of Hokkaido in the set $\left\{aL_h \;\middle|\; a \in A\right\}$, whose adherence intersect the adherence of a tile of the set $\left\{bC_iL_h \;\middle|\; b \in B_i\right\}$, is necessarily in $\left\{aL_h \;\middle|\; a \in B_iC_i\right\}$. Therefore, the adherence of tiles of type i doesn't intersect any adherence of a copy of Hokkaido which is in a tile of an other type. And the equality $\varphi(B_2C_2 \,\mathrm{Pref}(Z(L_h))) = B_2C_2L_h$ proves that the adherence of a copy of Hokkaido which is in a tile of type 2 doesn't intersect any adherence of copy of Hokkaido which is in a tile of an other type. As tiles of type 2 and 3 are described as finite union of Hokkaido copies (i.e. the languages C_2 and C_3 are finite), this proves that the adherences of two tiles of different types doesn't intersect.

Proof of lemma 6.10. — We have

$$\begin{aligned}
& u \in \varphi(L) \\
\iff\ & \exists a \in A,\ u \in aL_h \text{ and } u \in p_2(S^{\Sigma\times\Sigma}(\mathrm{Pref}(Z(L)) \times \mathrm{Pref}(Z(AL_h)) \cap L^{\mathrm{rel}\,\infty}))\Sigma^* \\
\iff\ & \exists a \in A,\ u \in aL_h \text{ and } \exists v \in \Sigma^*, \exists w \in \Sigma^*,\ w \text{ prefix of } u, \\
& (v,w) \in S^{\Sigma\times\Sigma}(\mathrm{Pref}(Z(L)) \times \mathrm{Pref}(Z(AL_h)) \cap L^{\mathrm{rel}\,\infty}) \\
\iff\ & \exists a \in A,\ u \in aL_h \text{ and } \exists v \in \Sigma^*, \exists w \in \Sigma^*,\ w \text{ prefix of } u, \exists (v',w') \in (\Sigma\times\Sigma)^{\mathbb{N}}, \\
& \sigma_-(Q_{vv'}) = \sigma_-(Q_{ww'}) \text{ and } \forall n \in \mathbb{N}, (vv'_n, ww'_n) \in \mathrm{Pref}(Z(L)) \times \mathrm{Pref}(Z(aL_h)) \\
\iff\ & \exists a \in A,\ u \in aL_h \text{ and } \overline{\sigma_-(Q_{aL_h})} \cap \overline{\sigma_-(Q_L)} \neq \emptyset.
\end{aligned}$$

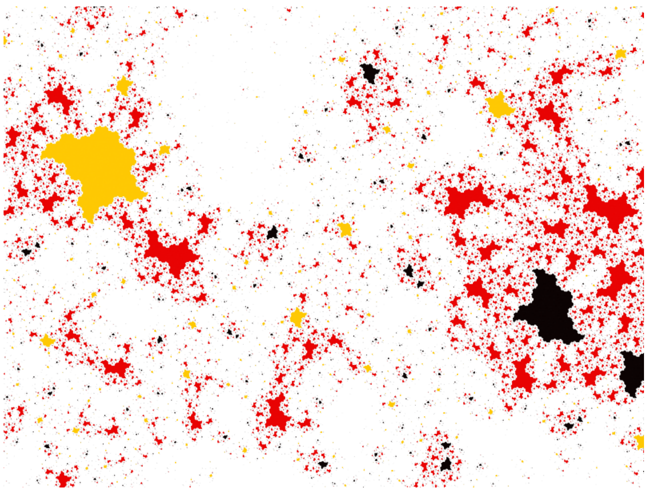
□

The theorem 6.1 almost finish the proof of the main theorem 1.1. In order to have the equality

$$R_s = \overline{\sigma_-(Q_{L_M})} \cup \bigcup_{u\in B_1} \overline{\sigma_-(Q_{uL_1})} \cup \bigcup_{u\in B_2} \overline{\sigma_-(Q_{uL_2})} \cup \bigcup_{u\in B_3} \overline{\sigma_-(Q_{uL_3})},$$

we need to check that $\overline{\sigma_-(Q_{B_1\cup B_2\cup B_3})} \subseteq \overline{\sigma_-(Q_{L_M})}$. This is true up to replace M by $M \cup B_1 \cup B_2 \cup B_3$. We can verify (thanks to theorem 4.14 applied to B_1, to B_2 and to B_3) that doing this doesn't change the dimension of $\overline{\sigma_-(Q_{L_M})}$. Hence, the theorem 1.1 is proven.

FIGURE 20. Zoom in the countable union of Hokkaido tiles, with arrangements of type 1 in black, arrangements of type 2 in red, and arrangements of type 3 in yellow



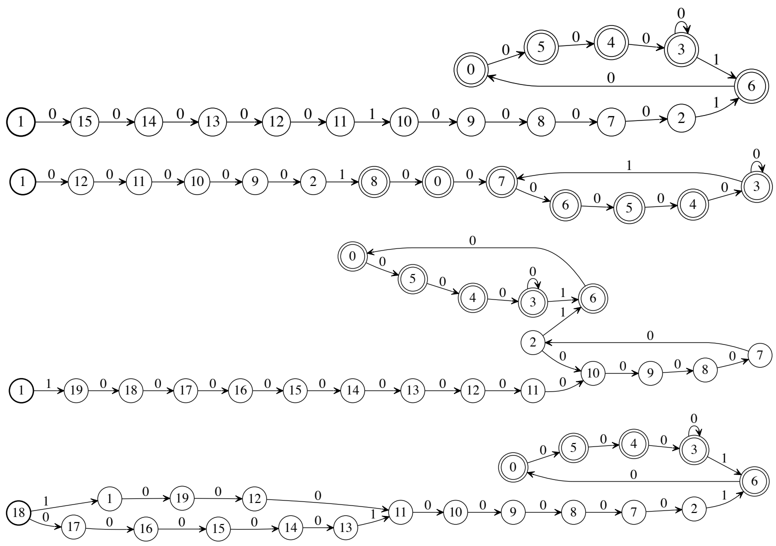
7. Proof of measurable conjugacies

The aim of this section is to prove the proposition 1.2. The proof is based on [**Aky. Me.**], following ideas of [**Arnoux Ito 2001**]. Let's start by proving that we have the domain exchange that we see in the figure 10 on the third type of shape.

Lemma 7.1. — *There is a domain exchange with four domains on $\overline{\sigma_-(Q_{L_3})}$, where L_3 is the language defined in figure 18, for the translations $\{\overline{\beta}^6, \overline{\beta}^7, \overline{\beta}^8, \overline{\beta}^6(1+\overline{\beta}-\overline{\beta}^2)\}$.*

Proof. — We consider domains $(\overline{\sigma_-(Q_{M_i})})_{i\in\{1,2,3,4\}}$ where the languages M_1, M_2, M_3 are defined in the figure 21.

FIGURE 21. Minimal automata of the languages M_1, M_2, M_3 and M_4



We can check that we have $M_1 \cup M_2 \cup M_3 \cup M_4 = L_3$, so the domains cover the third type of shape $\overline{\sigma_-(Q_{L_3})}$. And we can check that the domains are disjoint up to a set of measure zero using the lemma 6.6: we can check (by computer) that the language

$$p_1(S^{0\times\{0,1\}}(L^{\mathrm{rel}\,\infty} \cap A0^* \times \mathrm{Pref}(B0^*)))$$

is empty, so $\sigma_-(Q_A) \cap \overline{\sigma_-(Q_B)} = \emptyset$, for every $A \neq B \in \{M_1, M_2, M_3, M_4\}$.

The domain M_1 corresponds to the translation $\overline{\beta}^6 = \overline{\beta}^3 + \overline{\beta}^4$, the domain M_2 corresponds to the translation $\overline{\beta}^8$, the domain M_3 corresponds to the translation $\overline{\beta}^6(1+\overline{\beta}-\overline{\beta}^2) = \beta^4$ and the domain M_3 corresponds to the translation $\overline{\beta}^7 = \overline{\beta}^2(1+\overline{\beta}+\overline{\beta}^2)$. It is not difficult to see that the languages N_1, N_2, N_3 and N_4 defined in the figure 23 satisfy $Q_{M_1} + \beta^3 + \beta^4 = Q_{N_1}$, $Q_{M_2} + \beta^8 = Q_{N_2}$, $Q_{M_3} + \beta^4 = Q_{N_3}$ and $Q_{M_4} + \beta^7 = Q_{N_4}$.

FIGURE 22. Zoom in the Rauzy fractal, with arrangements of type 1 in black, arrangements of type 2 in purple, arrangements of type 3 in dark-yellow, and the part of dimension < 2 in gray

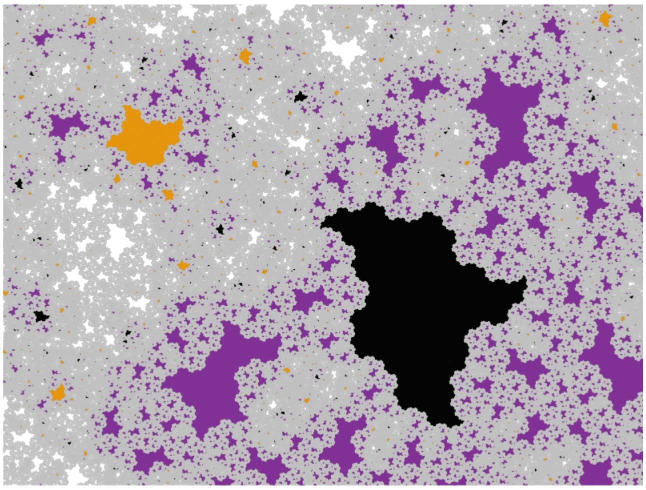
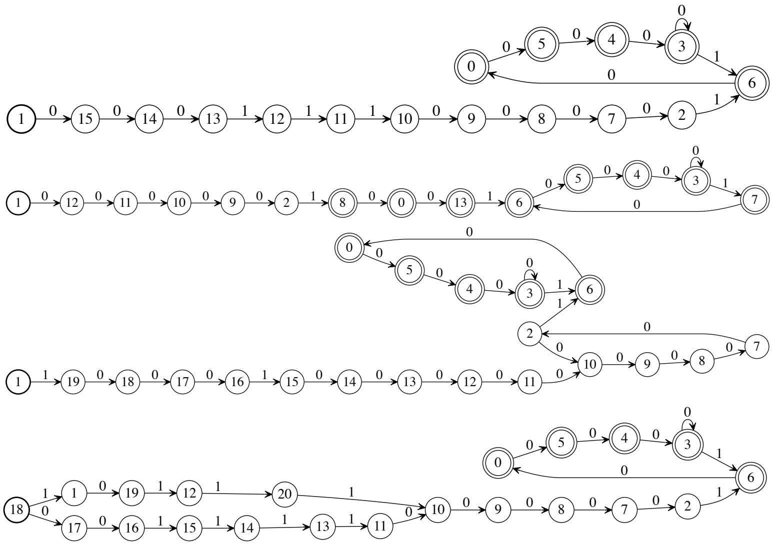


FIGURE 23. Minimal automata of the languages N_1, N_2, N_3 and N_4



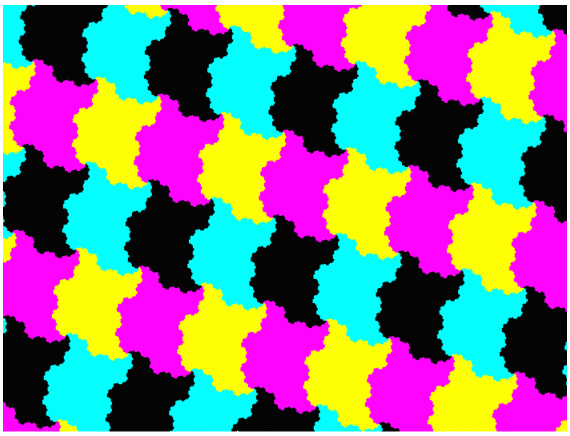
We can check (by computer, using lemma 4.5) that $Q_{N_1 \cup N_2 \cup N_3 \cup N_4} = Q_{L_3}$, so the domains still cover the shape after translations. And we check that the domains after translations are pairwise disjoint, like for the domains not translated. The only difference is that one of the language computed is not empty but contains a finite number of words (corresponding to the point β^5): this is because $\overline{\beta}^5 \in \sigma_-(Q_{N_1}) \cap \overline{\sigma_-(Q_{N_3})}$. $\square$

Lemma 7.2. — *Let Γ_0 be the discrete additive subgroup of $\mathbb{C}$ generated by $\{\overline{\beta}^6(1-\overline{\beta}), \overline{\beta}^6(\overline{\beta}-\overline{\beta}^2)\}$. The natural quotient map $\pi_0 : \mathbb{C} \to \mathbb{C}/\Gamma_0$ gives a measurable conjugacy between the domain exchange described in the previous lemma and the translation by β^6 on the torus $\mathbb{C}/\Gamma_0 \simeq \mathbb{T}^2$.*

Proof. — Let us show that Γ_0 is a fundamental domain for the action of the group Γ_0. The fact that $\Gamma_0 + \overline{\sigma_-(Q_{L_3})} = \mathbb{C}$ can be obtained by checking that $Q_{L_3} - \beta^5$ comes from the substitution s_3 defined in the introduction. And this also gives the fact that the boundary has zero Lebesgue measure. And the fact that the various translates of the tile are disjoint in measure can be checked as above, using the lemma 6.6. $\square$

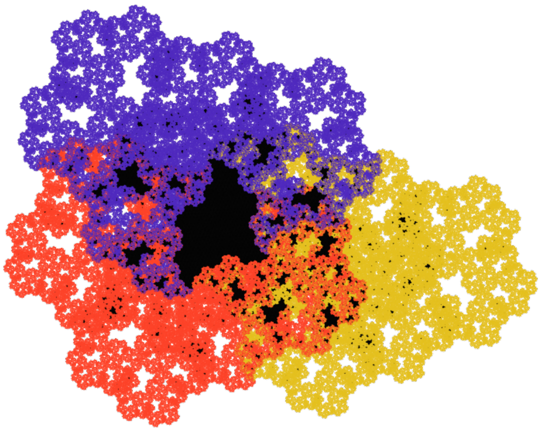
To end the proof of the proposition 1.2, it is known that the substitution s satisfy the Pisot conjecture, and it suffices to remark that the toral translation that we naturally get from the substitution s is the same as the one we consider here (see [**Arnoux Ito 2001**] for more details). Indeed, the translations of the domain exchange of s are 1, $\overline{\beta}$ and $\overline{\beta}^2$, so the group of differences is the group $\beta^{-6}\Gamma_0$, so we get the translation by 1 on the torus $\mathbb{C}/\gamma^{-6}\Gamma_0$, which is equivalent to the translation by β^6 on the torus $\mathbb{C}/\Gamma_0$. This ends the proof of the proposition 1.2.

FIGURE 24. Tiling of $\mathbb{C}$ with the third type of shape, for the action of Γ_0



Pierre Arnoux tolds me about an observation of Julien Bernat that a big Hokkaido tile appears inside the union of three translated copies of the substitution associated to the minimal Pisot number. The tools used in this article allow us to prove this observation, and we can decompose the union as set of dimension less than two union a countable union Hokkaido tiles, like in theorem 4.7.

FIGURE 25. Three translated (by 0, $\overline{\beta}^2 - 1$ and $\overline{\beta} - 1$) copies of the Rauzy fractal of s, with the countable union of Hokkaido tiles corresponding to the union in black.



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