Rauzy fractals can approximate any shape

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Example of substitution

Let's take the following substitution over the alphabet $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$:

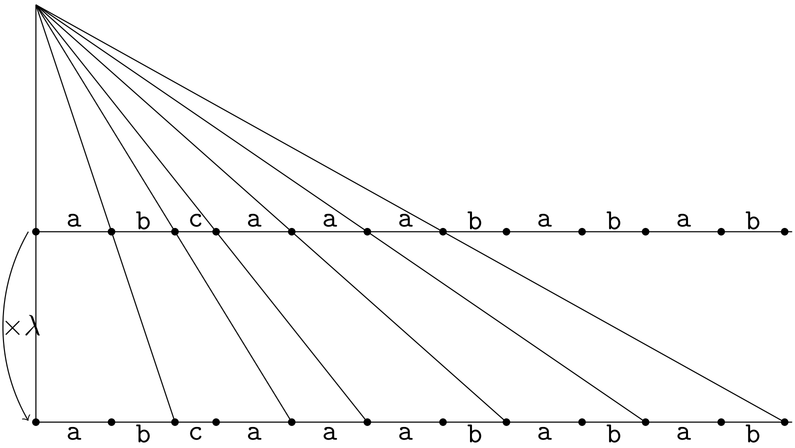
$$s : \begin{cases} a \mapsto ab \\ b \mapsto ca \\ c \mapsto a \end{cases}$$

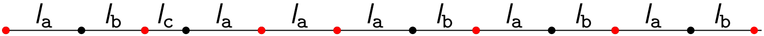
Then by iterating the letter a we get a fixed point :

$$\begin{aligned} & \mathtt{a} \\ s(\mathtt{a}) = & \ \mathtt{ab} \\ s^2(\mathtt{a}) = & \ \mathtt{abca} \\ s^3(\mathtt{a}) = & \ \mathtt{abcaaab} \\ \dots & \\ s^\infty(\mathtt{a}) = & \ \mathtt{abcaaababcaabcaabcaaababcaaababcaaabab}\dots \end{aligned}$$

Self-similar tiling of $\mathbb{R}_+$

If we replace letters of this fixed point by intervalles of convenient lengths, we get a self-similar tiling of $\mathbb{R}_+$.





To get such a self-similar tiling of $\mathbb{R}_+$, the lengths of each intervalles must satisfy the equality

$$M_s \cdot \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix} = \lambda \begin{pmatrix} l_a \\ l_b \\ l_c \end{pmatrix},$$

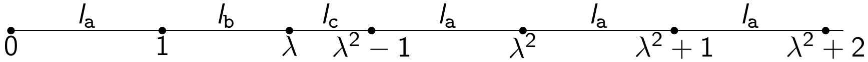
where $M_s = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ is the incidence matrix of the substitution and λ is the Perron eigenvalue of M_s. Hence we can assume that the lengths $l_i,\ i \in \{a, b, c\}$ live in $\mathbb{Q}(\lambda)$.

Quasicrystal of $\mathbb{R}_+$

If we take for example

$$l_a = 1, \quad l_b = \lambda - 1, \quad l_c = \lambda^2 - \lambda - 1,$$

we get the following subset Q of $\mathbb{Q}(\lambda)$.



$$Q = \{0, 1, \lambda, \lambda^2 - 1, \lambda^2, \lambda^2 + 1, \lambda^2 + 2, \lambda^2 + \lambda + 1, \lambda^2 + \lambda + 2, ...\}$$

This set have very strong properties since we have :

Proposition

Q is a λ-invariant Meyer set of $\mathbb{R}_+$.

But what is a Meyer set ?

Meyer sets are a mathematical model for quasicrystals.

Definition

*A **Meyer set** of $\mathbb{R}_+$ is a set $Q \subset \mathbb{R}_+$ such that*

- *Q is a Delone set of $\mathbb{R}_+$,*
- *$Q - Q$ is a Delone set of $\mathbb{R}$.*

Definition

*Q is a **Delone set** in E if*

- *Q is **uniformly discrete***

$$\exists \epsilon > 0,\ \forall (x, y) \in Q^2, B(x, \epsilon) \cap B(y, \epsilon) = \emptyset,$$

- *Q is **relatively dense** in E*

$$\exists R > 0,\ E \subseteq \bigcup_{x \in Q} B(x, R).$$

Rauzy fractal

The quasicrystal Q is a part of $Q(\lambda)$, hence we can look at the action of the Galois group. Here, λ has two complexes conjugated as conjugates, hence we have an embedding

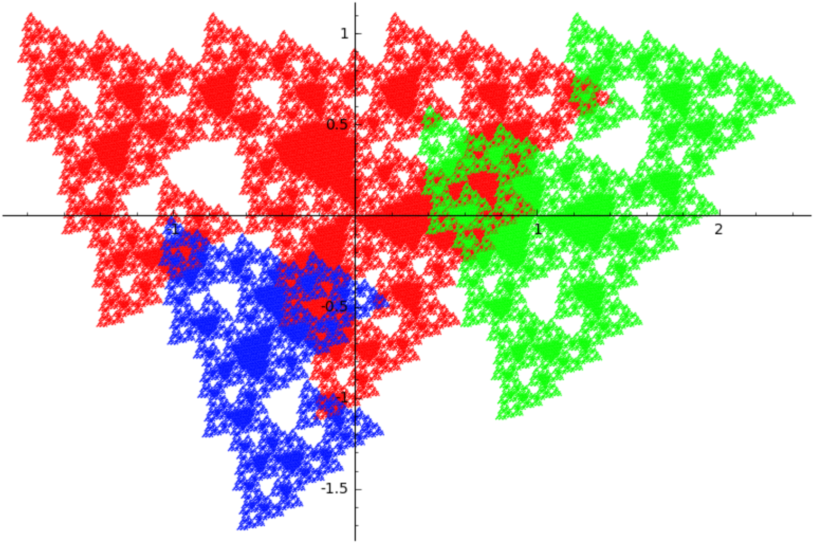
$$\sigma : \mathbb{Q}(\lambda) \quad \to \quad \mathbb{C} \ ,$$

by choosing one of the conjugates.

Proposition

The set $\sigma(Q) \subseteq \mathbb{C}$ is bounded. It's adherence is called a ***Rauzy fractal****.*

Moreover, we can color in red the points of $\sigma(Q)$ that are left bound of an interval of length 1 (i.e. coming from letter a), in green the points that are left bound of an intervalle of length $\lambda - 1$ (i.e. coming from letter b), and the other ones, for $\lambda^2 - \lambda - 1$, in blue.


1
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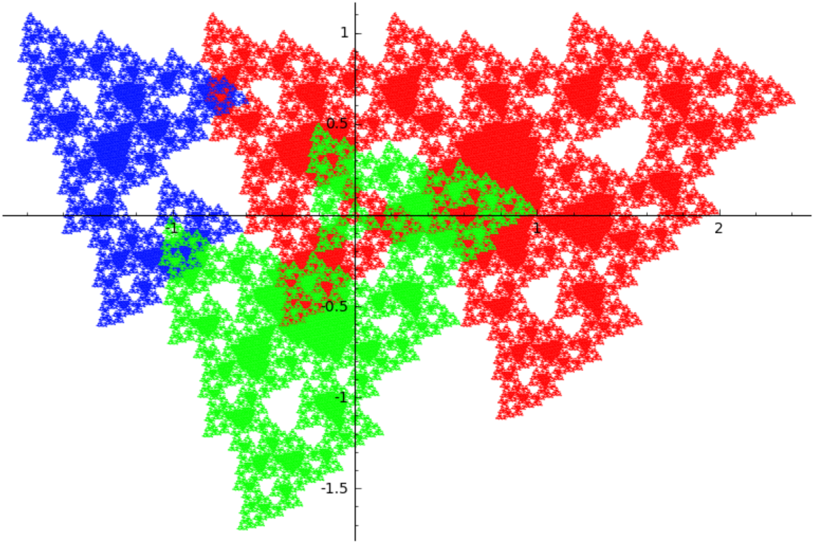
2
-0.5
-1
-1.5


1
0.5
-1
1
2
-0.5
-1
-1.5

If we color points in function of right bounds of intervals, and no more left bounds, we get the second coloration. We get a domain exchange which completely encode the substitution, and in particular the shift in the quasicrystal.

Generalization to any substitution

If s is any substitution over a alphabet A, everything generalizes :

- **fixed point** : Up to replace s by a power, s has a fixed point ω.
- **self-similar tiling** : We get a self-similar tiling of $\mathbb{R}_+$ or $\mathbb{R}$ by replacing letters by intervals of lengths l_a, $a \in A$ given by a Perron eigenvector of the incidence matrix.
- **quasicrystal** : We get a set $Q_\omega \subset \mathbb{R}$ by taking the bounds of intervals of this self-similar tiling, and up to rescaling we have $Q_\omega \subset \mathbb{Q}(\lambda)$ where λ is the Perron eigenvalue of the incidence matrix M_s. If λ is a Pisot number, Q_ω is a quasicrystal.
- **Rauzy fractal** : Q is a subset of $\mathbb{Q}(\lambda)$, therefore we can embed it into a natural contracting space E_λ^- where it is a pre-compact subset. And we can color it in order to define a domain exchange describing the shift on the fixed point.

General definitions of contracting spaces and Rauzy fractals

There are natural contracting and expanding spaces for the multiplication by β on a number field $k = \mathbb{Q}(\beta)$. Call P the set of places of k (i.e. equivalence classes of absolute values), and let

$$P_+ := \left\{ v \in P \;\middle|\; |\beta|_v > 1 \right\} \text{ and } P_- := \left\{ v \in P \;\middle|\; |\beta|_v < 1 \right\}.$$

The **contracting space** is $E_\beta^- := \prod_{v \in P_-} k_v$ and the expanding one is $E_\beta^+ := \prod_{v \in P_+} k_v$, where k_v denotes the completion of k for the absolute value v. We denote by σ_- some embedding into E_β^-.

Definition

*We call **Rauzy fractal** the adherence of $\sigma_-(Q_\omega)$ in E_β^-.*

For the previous example, where λ is root of $x^3 - x^2 - x - 1$, we have $E_\lambda^+ = \mathbb{R}$ (there is one real place) and $E_\lambda^- = \mathbb{C}$ (there is one complex place).

Rauzy fractals can approximate any shape

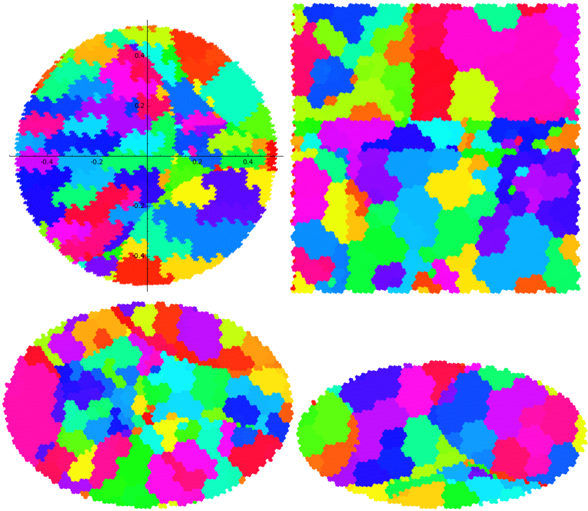
Theorem

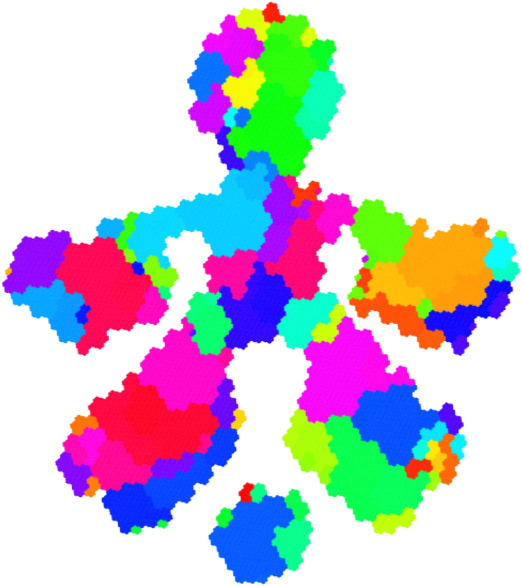
For any Pisot number β and for any $P \subset E_\beta^-$, bounded and containing 0, there exists substitutions whose Rauzy fractals approximate arbitrarily P, for the Hausdorff distance, and whose Perron numbers are powers of β. Moreover, the proof is constructive.

The Hausdorff distance between two subsets $A \subseteq E$ and $B \subseteq E$ of a metric space E is

$$d(A,B) = \max\left(\sup_{a\in A}\inf_{b\in B} d(a,b), \sup_{b\in B}\inf_{a\in A} d(a,b)\right).$$

Rauzy fractals aproximating various shapes





g-β-sets : a nice description of quasicrystals by automata

Definition (Main tool)

A set Q is a **g-β-set** *if we have $Q = Q_L$ for*

$$Q_L = \left\{ \sum_{k=0}^{n} a_k \beta^k \;\middle|\; n \in \mathbb{N},\; a_0 \ldots a_n \in L \right\},$$

where $\Sigma \subset \mathbb{Q}(\beta)$ is a finite alphabet and $L \subseteq \Sigma^*$ is a regular language.

Proposition

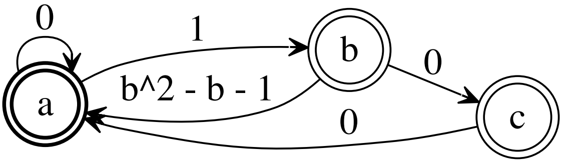
If ω is a fixed point of a substitution, then Q_ω is a g-β-set.

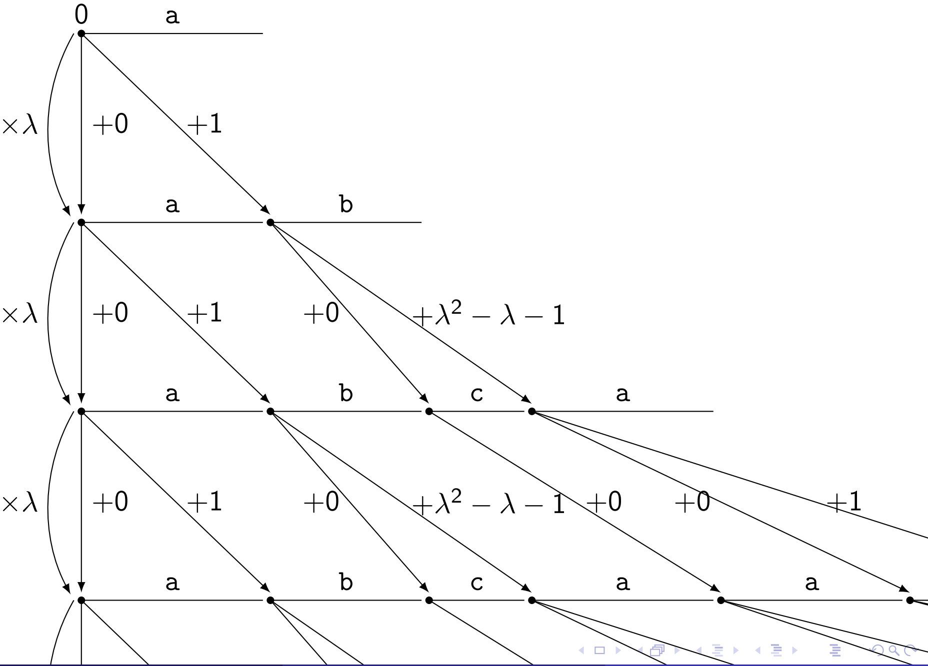
g-β-set coming from a substitution

For the example

$$s : \begin{cases} a \mapsto ab \\ b \mapsto ca \\ c \mapsto a \end{cases}$$

the mirror of the language L, recognized by the following automaton, define a g-β-set which is a quasicrystal coming from the substitution s, for β the Tribonnacci number.




0
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×λ
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+1
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×λ
+0
+1
+0
+λ² − λ − 1
+0
+0
+1
a
b
c
a
a

Stability of the set of g-β-sets

Properties (Properties of g-β-sets)

If Q_1 and Q_2 are two g-β-sets, then

- $Q_1 \cup Q_2$, $Q_1 \cap Q_2$ *and* $Q_1 \backslash Q_2$ *are g-β-sets,*
- $Q_1 + Q_2$ *is a g-β-set,*
- $\forall t \in \mathbb{Q}(\beta)$, $Q_1 + t$ *is a g-β-set,*
- $\forall c \in \mathbb{Q}(\beta)$, cQ_1 *is a g-β-set.*

Hence, it is easy to approximate any shape by g-β-sets.
-> See demonstration in Sage.
Can we get a substitution from a g-β-set ?

The main theorem

It is easy to prove that Rauzy fractals can approximate any shape with the properties of g-β-sets and with the following theorem.

Theorem

Let β be a Pisot number. A subset $Q \subset \mathbb{Q}(\beta)$ comes from a fixed point of a substitution whose Perron number is β, if and only if

- *Q is a quasicrystal (i.e. a Meyer set),*
- *$\beta Q \subseteq Q$,*
- *$0 \in Q$,*
- *Q is a g-β-set.*

We have already seen that these conditions are necessary. Let us show that these are sufficient.

β-expansion algorithm in a β-invariant quasicrystal

Let Q be a quasicrystal and β be a Pisot number with $\beta Q \subset Q$ and $0 \in Q$. Then we can define the following algorithm that gives an unic finite β-expansion of any elements of Q.

Data: $x \in Q$
Result: coefficients t_0 of a β-expansion of x
while $x \neq 0$ **do**
  $x \leftarrow x - t0$ for $t0 = \inf\{t \geq 0 | x - t \in \beta Q\}$;
  $x \leftarrow x/\beta$;
  print t_0 ;
end

The expansion of x is given by the successive elements t_0.

With the previous algorithm, we define the language

$$L_Q := \left\{a_0 \ldots a_n \in \Sigma_Q^* \;\middle|\; a_0 \ldots a_n \text{ expansion of } x \text{ given by the algorithm } \right\} 0^*$$

over the finite alphabet

$$\Sigma_Q := \left\{\inf\{t \geq 0 | x - t \in \beta Q\} \;\middle|\; x \in Q\right\}.$$

In others word, L_Q is the unic subset of Σ_Q^* containing the empty word ϵ, such that $Q = Q_{L_Q}$ and such that

$$a_0 \ldots a_n \in L_Q \iff \begin{cases} a_0 = \min\left\{t \in \Sigma_Q \;\middle|\; \sum_{k=0}^n a_k \beta^k \in \beta Q + t\right\} \\ a_1 \ldots a_n \in L_Q \end{cases}.$$

Proposition

The following two sentences are equivalent.

- *Q comes from a substitution.*
- *L_Q is a regular language.*

Hence, to prove the main theorem, it is enough to prove the following

Lemma

We have the equivalence between :

- L_Q *is a regular language.*
- Q *is a g-*β*-set.*

The direct part is obvious. To prove the converse, we construct the language L_Q from any regular language L such that $Q = Q_L$. Let's do it !

Step 1/3 : get a regular language over the alphabet Σ_Q

Let L be a regular language over an alphabet $\Sigma \subset \mathbb{Q}(\beta)$ such that $Q = Q_L$.

Lemma (Change of the alphabet)

The following language is regular

$$L_{Q,\Sigma_Q} := \left\{ a_0 ... a_n \in \Sigma_Q^* \;\middle|\; n \in \mathbb{N},\ \sum_{k=0}^{n} a_k \beta^k \in Q \right\},$$

and we have $Q_{L_{Q,\Sigma_Q}} = Q$.

Proof.

$$L_{Q,\Sigma_Q} = Z(p_1(L^{rel} \cap \Sigma_Q^* \times L0^*)) \text{ where } Z : L \mapsto \bigcup_{n \in \mathbb{N}} L0^{-n},$$

$$L^{rel} = \left\{ (u, v) \in (\Sigma_Q \times \Sigma)^* \;\middle|\; \sum_{k=0}^{n} (u_k - v_k) \beta^k = 0 \right\}.$$

This last language is regular thanks to the main result of my paper « Semi-groupes fortement automatiques ». □

Step 2/3 : stabilization by suffix

Lemma (Stabilization by suffix)

The greatest language $L' \subset L_{Q,\Sigma_Q}$ *such that*

$$u \in L' \implies \text{ every suffix of } u \text{ is in } L'$$

is a regular language, and we have $Q = Q_{L'}$.

Proof.

Take a deterministic automaton recognizing the mirror of L_{Q,Σ_Q}.
Remove every non final state.
Then this new automaton recognize the mirror of L'.
And we have $L_Q \subseteq L' \subseteq L_{Q,\Sigma_Q}$, hence $Q = Q_{L'}$. □

Step 3/3 : minimal words in lexicographic order

Lemma (Minimal words in lexicographic order describing Q)

We have the equality

$$L_Q = L' \backslash p_1(L' \times L' \cap L^{rel} \cap L^{>}),$$

where

$$L^{rel} := \left\{(u, v) \in (\Sigma_Q \times \Sigma_Q)^* \;\middle|\; \sum_{k=0}^{n}(u_k - v_k)\beta^k = 0\right\}$$

and

$$L^{>} := \left\{(u, v) \in (\Sigma_Q \times \Sigma_Q)^* \;\middle|\; u > v \textit{ for the lexicographic order}\right\},$$

where we choose the natural order on Σ_Q, given by the embedding into the expanding space $E_\beta^+ = \mathbb{R}$.

Hence L_Q is regular, and this proves the theorem.

Proof of last lemma.

- $L' \times L' \cap L^{rel} \cap L^{>}$ is the couple of words of same length, giving the same element of Q, and with the left one strictly less than the right one for the lexicographic order.
- Hence $L' \backslash p_1(L' \times L' \cap L^{rel} \cap L^{>})$ is the set of elements of L' which are minimal in lexicographic order among the words of L' of same length describing the same point of Q.
- We deduce the equality with L_Q : the language is still stable by suffix and the first letter is the minimal one, as in the definition of L_Q.
- The language $L^{>}$ is easily seen to be regular : we can recognize it with an automaton having two states.
- The language L^{rel} is regular, thanks to my article « Semi-groupes fortement automatiques ».

□

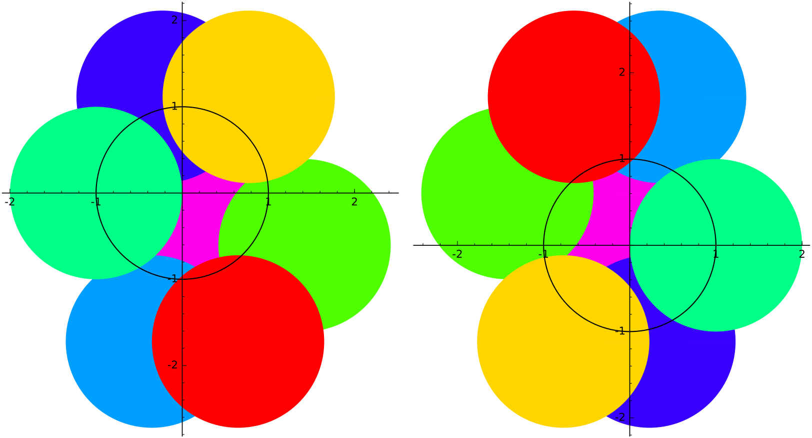
Others things I could speak about

There is still a lot to say about g-β-sets :

- details of my implementation in Sage and how make the computations efficiently,
- examples of computations with Sage,
- construction of a domain exchange on a g-β-set or on a model set,
- proof of stability properties of g-β-sets,
- other proof of the main theorem,
- a nice topology on g-β-sets which permits to decide if we have a model set or not,
- examples of substitutions that describe the list of Pisot numbers of a given quadratic field,
- questions, conjectures and things to develop,
- a simple proof that Rauzy fractals always have non-empty interior.

But maybe Nicolas wants to eat ?

Construction of a domain exchange

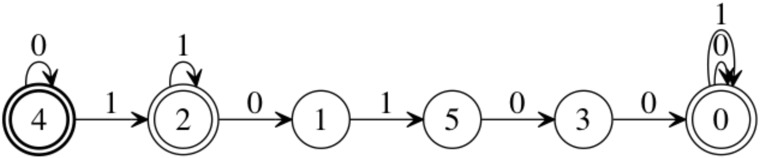


$-2\beta^2 + 2\beta, \ \beta^2 - \beta - 1, \ \beta - 1, \ 1, \ -\beta^2 + 2\beta + 1, \ \beta^2 - \beta, \ \beta$

Domain exchange on the model set defined by the unit disk window, and the integer ring $\mathcal{O}_\beta$ where β is the Tribonnacci number.

Example of g-β-set for β the Tribonnacci number

Let's take the g-β-set defined by



for $\beta^3 = \beta^2 + \beta + 1$. The regular language L described by this automaton is

$$L = 0^* 1^+ 0100\{0,1\}^*.$$

This g-β-set satisfy every hypothesis of the theorem, hence we can compute a substitution from it.

Corresponding substitution whose Perron number is β

$1 \mapsto 28, 12, 13$
$2 \mapsto 29, 1, 5$
$3 \mapsto 29, 1, 8, 13$
$4 \mapsto 4, 13$
$5 \mapsto 29, 3, 9$
$6 \mapsto 32, 11, 9$
$7 \mapsto 49$
$8 \mapsto 29, 3, 10, 13$
$9 \mapsto 29, 2$
$10 \mapsto 29, 3, 13$
$11 \mapsto 30, 26$
$12 \mapsto 32, 11, 10, 13$

$13 \mapsto 29, 14$
$14 \mapsto 29, 1, 27$
$15 \mapsto 32, 11, 27$
$16 \mapsto 29, 19$
$17 \mapsto 4, 27$
$18 \mapsto 7$
$19 \mapsto 28, 6$
$20 \mapsto 33, 35$
$21 \mapsto 29, 34$
$22 \mapsto 17$
$23 \mapsto 18$
$24 \mapsto 19$

$25 \mapsto 25, 20$
$26 \mapsto 28, 15$
$27 \mapsto 29, 16$
$28 \mapsto 22$
$29 \mapsto 23$
$30 \mapsto 24$
$31 \mapsto 26$
$32 \mapsto 28, 21$
$33 \mapsto 31, 21$
$34 \mapsto 29$
$35 \mapsto 30$

Rauzy fractal