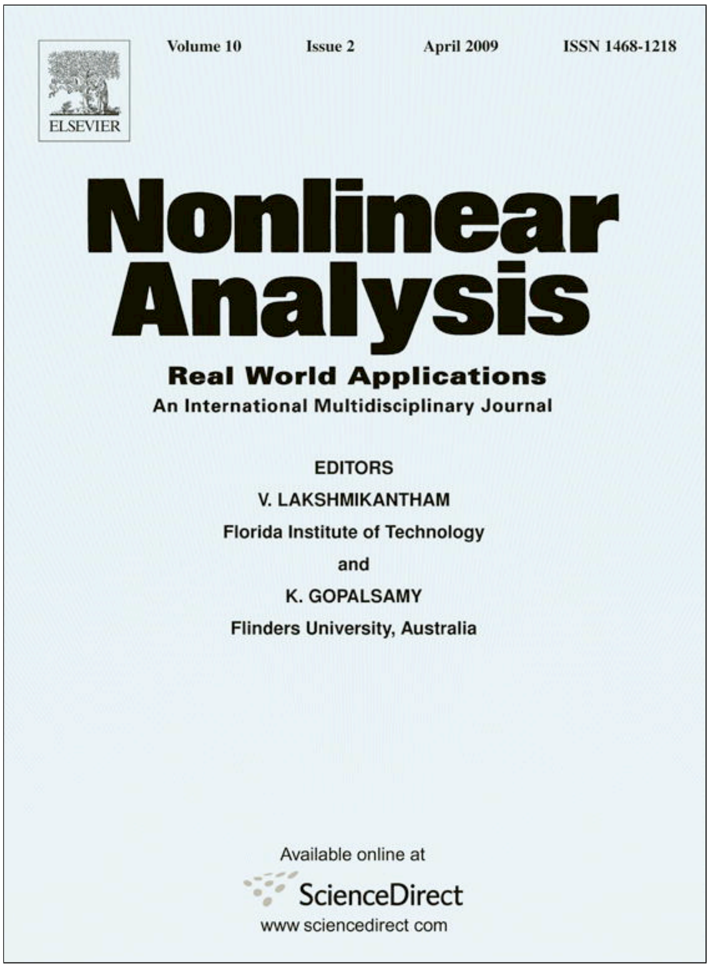
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A fast precipitation and dissolution reaction for a reaction–diffusion system arising in a porous medium[☆]

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Abstract

This paper is devoted to the study of a fast reaction–diffusion system arising in reactive transport. It extends the articles [R. Eymard, T. Gallouët, R. Herbin, D. Hilhorst, M. Mainguy, Instantaneous and noninstantaneous dissolution: Approximation by the finite volume method, ESAIM Proc. (1998); J. Pousin, Infinitely fast kinetics for dissolution and diffusion in open reactive systems, Nonlinear Anal. 39 (2000) 261–279] since a precipitation and dissolution reaction is considered so that the reaction term is not sign-definite and is moreover discontinuous. Energy type methods allow us to prove uniform estimates and then to study the limiting behavior of the solution as the kinetic rate tends to infinity in the special situation of one aqueous species and one solid species.

Keywords: Reaction–diffusion; Precipitation; Dissolution; Kinetics; Fast reaction

1. Introduction

In this paper we consider the reaction–diffusion system,

$$
(\mathrm{P}^\lambda)\quad \begin{cases} u_t = \Delta u - \lambda G(u,w) & \text{in } \Omega \times (0,T) \quad \text{(a)}\\ w_t = \lambda G(u,w) & \text{in } \Omega \times (0,T) \quad \text{(b)}\\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \quad \text{(c)}\\ u(x,0) = u_0(x), \qquad w(x,0) = w_0(x) & \text{for } x \in \Omega, \quad \text{(d)} \end{cases} \tag{1.1}
$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and T is a positive constant. We suppose that λ is a positive constant and that the function $G(\cdot,\cdot)$ is given by

$$
G(u,w) = (u - \overline{u})^+ - \operatorname{sign}^+(w)(u - \overline{u})^-, \tag{1.2}
$$

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where $\overline{u}$ is a given positive constant and

$$s^+ = \max(0, s), \qquad s^- = \max(0, -s), \quad \text{and} \quad \operatorname{sign}(s) = \begin{cases} 1 & \text{if } s > 0, \\ -1 & \text{if } s < 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The above system (P^λ) is a simplified adimensional model of reactive transport in a porous medium at the Darcy scale, where u stands for a concentration of an aqueous species, therefore mobile, and w stands for a concentration of a mineral species. The term $\lambda G(u, w)$ is a reaction rate that models either a precipitation if $u - \overline{u} \geq 0$, or a dissolution otherwise. The positive constant $\overline{u}$ is the thermodynamic constant of the dissolution reaction and λ is a constant rate. Reactive transport problems arise in the field of radioactive waste storage, oil industry or CO_2 storage. Indeed, water–rock interactions like precipitation and dissolution reactions have a strong impact both on flow and solute transport.

We focus on reactions which are very fast compared with the diffusion process so that λ is a large parameter. In this paper we extend a result of Eymard, Hilhorst, van der Hout and Peletier [6], which they obtained in the special case of a function $G(\cdot, \cdot)$ assumed to be nonnegative and nondecreasing in both arguments. This special assumption led to an easy way for estimating the time derivative of w in L^1 independently of λ. The Stefan problem obtained when $\lambda \to +\infty$ is the same as that of [7,11] but the problem (P^λ) considered in this paper has an additional precipitation term which prevents $G(\cdot, \cdot)$ from keeping a constant sign, and has led us to provide an original estimate. In [7], the main tool is a finite volume method used in any space dimension. In [11], a Legendre function (associated with the liquid concentration) is used in one-space dimension to deal with discontinuities. Note that in [3], the existence of a solution to the same problem with two aqueous species instead of one is proven; however, the study of the singular limit in this more complex case is still an open problem, since the techniques presented here do not seem to be easily adaptable. We suppose that the initial functions u_0 and w_0 satisfy the hypotheses:

$$(\mathrm{H}_0) \begin{cases} u_0, w_0 \in L^2(\Omega) \quad 0 \leq u_0 \leq M_1 \text{ and } 0 \leq w_0 \leq M_2 \text{ a.e in } \Omega, \\ \text{for some positive constants } M_1 \text{ and } M_2 \text{ such that } M_1 > \overline{u}. \end{cases}$$

We set $Q_T := \Omega \times (0, T)$ and denote by $W_2^{2,1}(Q_T) = \{u \in L^2(Q_T), \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial u}{\partial t} \in L^2(Q_T), i, j = 1, \ldots, N\}$ and by $C^{0,1}([0, T]; L^\infty(\Omega))$ the space of Lipschitz continuous functions with values in $L^\infty(\Omega)$. Next we define a notion of weak solution for Problem (P^λ).

Definition 1.1. (u^λ, w^λ) is a weak solution of Problem (P^λ) if for all $T > 0$

(i) $u^\lambda \in W_2^{2,1}(Q_T), \; w^\lambda \in C^{0,1}([0, T]; L^\infty(\Omega))$;

(ii) $\int_\Omega u^\lambda(T)\xi(T) - \int_\Omega u_0\xi(0) - \int_{Q_T} \left\{u^\lambda \xi_t - \nabla u^\lambda \nabla \xi - \lambda G(u^\lambda, w^\lambda)\xi\right\} = 0,$

$\int_\Omega w^\lambda(T)\xi(T) - \int_\Omega w_0\xi(0) - \int_{Q_T} \left\{w^\lambda \xi_t + \lambda G(u^\lambda, w^\lambda)\xi\right\} = 0,$

for all $\xi \in H^1(Q_T)$.

In view of its regularity, we remark that it satisfies the differential equations in Problem (P^λ) a.e. in Q_T. The purpose of this paper is to prove the following result.

Theorem 1. *Suppose that u_0 and w_0 satisfy the hypotheses* (H_0). *Then for every $\lambda > 0$, Problem* (P^λ) *has a unique nonnegative weak solution (u^λ, w^λ). Moreover there exist functions $U \in L^2(Q_T)$, $W \in L^2(Q_T)$ such that u^λ and w^λ converge strongly in $L^2(Q_T)$ to U and W respectively, as λ tends to ∞. The function $Z := -(U + W) + \overline{u}$ is the unique weak solution of the Stefan problem*

$$(\mathrm{SP}) \begin{cases} Z_t = \Delta(Z^+) & \text{in } \Omega \times (0, T) \quad \text{(a)} \\ \dfrac{\partial Z^+}{\partial n} = 0 & \text{on } \partial\Omega \times (0, T) \quad \text{(b)} \\ Z(x, 0) = -(u_0(x) + w_0(x) - \overline{u}) & \text{for } x \in \Omega. \quad \text{(c)} \end{cases} \tag{1.3}$$

Conversely the limit pair (U, W) is given by $(U, W) = (\overline{u} - Z^+, Z^-)$.

This paper is organized as follows : In Section 2, we present a physical derivation of (P^λ). In Section 3, we prove a comparison principle for Problem (P^λ), which implies the uniqueness of its weak solution. This result is quite natural since the monotonicity properties of G in u and in w make it a cooperative system [1]. In Section 4, we present some a priori estimates, which imply that as λ tends to ∞, the sum $-(u^\lambda + w^\lambda) + \overline{u}$ tends to the unique weak solution of the Stefan problem (SP).

Finally we refer to [2] for the study of the singular limit of a parabolic system where the nonlinear function has the same monotonicity properties as the function G.

2. The physical context

In this section, we first recall the physical derivation of a system of two parabolic partial differential equations coupled with an ordinary differential equation, which models reactive transport with one mineral species and two aqueous species which react according to a kinetic law [3,9]. We consider a chemical reaction of the form

$$W \Leftrightarrow \alpha U + \beta V, \tag{2.4}$$

where α, β are the algebraic stoichiometric coefficients, W the mineral, U and V the species in the liquid phase. Let u (resp. v and w) be the concentrations of U (resp. V and W). We assume that the aqueous species migrate into the saturated porous medium through a molecular diffusion process without any convection. It is shown in [3] that the concentrations (u, v, w) satisfy the system

$$\begin{cases} u_t - \Delta u = -\alpha\, w_t, \\ v_t - \Delta v = -\beta\, w_t, \\ w_t = \lambda(F(u,v)^+ - \mathrm{sign}^+(w)F(u,v)^-), \end{cases} \tag{2.5}$$

where λ is a constant rate coefficient and F represents the thermodynamical equilibrium gap; assuming that the reaction is elementary and using the transition state theory [10], one can deduce that a possible expression of the function F is given by [3]

$$F(u,v) = u^\alpha v^{\beta^+} - K v^{\beta^-}, \tag{2.6}$$

with K the thermodynamic constant of the reaction (2.4). We set $\alpha = 1$, $\beta = 0$, $\overline{u} = K$, and we define

$$G(u,w) := F(u,v)^+ - \mathrm{sign}^+(w)F(u,v)^- = (u-\overline{u})^+ - \mathrm{sign}^+(w)(u-\overline{u})^-.$$

Substituting the expressions for F and G in the system (2.5) yields the differential equations in Problem (P^λ).

3. Comparison principle and essential bounds

The following comparison principle holds.

Theorem 2. *Let (u,w) and (ϕ,ψ) be such that $u,\phi \in W_2^{2,1}(Q_T)$ and $w,\psi \in C^{0,1}([0,T]; L^\infty(\Omega))$ and suppose that they satisfy*

$$u_t \geq \Delta u - \lambda G(u,w) \quad \text{a.e. in } Q_T \tag{3.7}$$

$$w_t \geq \lambda G(u,w) \quad \text{a.e. in } Q_T \tag{3.8}$$

$$\phi_t \leq \Delta\phi - \lambda G(\phi,\psi) \quad \text{a.e. in } Q_T \tag{3.9}$$

$$\psi_t \leq \lambda G(\phi,\psi) \quad \text{a.e. in } Q_T \tag{3.10}$$

$$\frac{\partial u}{\partial n} = \frac{\partial \phi}{\partial n} = 0 \quad \text{a.e. on } \partial\Omega \times (0,T) \tag{3.11}$$

$$u(x,0) \geq \phi(x,0) \quad \text{for a.e. } x \in \Omega \tag{3.12}$$

$$w(x,0) \geq \psi(x,0) \quad \text{for a.e. } x \in \Omega. \tag{3.13}$$

Then

$$u(x,t) \geq \phi(x,t) \quad \text{a.e. in } Q_T \tag{3.14}$$

$$w(x,t) \geq \psi(x,t) \quad \text{a.e. in } Q_T. \tag{3.15}$$

The pair (u,w) is called a super-solution and the pair (ϕ,ψ) is called a subsolution of Problem (P^λ).

Problem (P^λ) is a cooperative system since the nonlinear function G is increasing in u and nonincreasing in w; nevertheless, the system we deal with in this paper is special in the sense that it consists of a PDE coupled with an ODE, and that the PDE's right-hand side is discontinuous; therefore, we shall give a sketch of the proof of Theorem 2, using the same ideas as in [12,1]. To begin with, we recall a technical result stated by Crandall and Pierre [5].

Lemma 3.1. *Let $p\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable and bounded and define q by $q(r) = \int_0^r p(s)\mathrm{d}s$. Let $\omega \in W^{1,1}(0,T,L^1(\Omega))$. Then $q(\omega) \in W^{1,1}(0,T,L^1(\Omega))$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}q(\omega) = p(\omega)\frac{\mathrm{d}}{\mathrm{d}t}\omega \quad \text{a.e.}$$

This lemma will be used several times in this article either with $p(s) = \operatorname{sign}(s)$ and thus $q(s) = |s|$ or with $p(s) = \operatorname{sign}^+(s)$ and thus $q(s) = s^+$.

Sketch of proof of Theorem 2. We subtract the inequality for u (3.7) from that for ϕ (3.9), and multiply the result by $\operatorname{sign}^{\delta,+}(\phi-u)$; similarly we subtract the inequality for ψ (3.10) from that for w (3.8), and multiply the result by $\operatorname{sign}^{\delta,+}(\psi-w)$, where $\operatorname{sign}^{\delta,+}$ is a smooth nondecreasing regularization of sign^+ which converges pointwise. Adding both inequalities and integrating the result on Ω we deduce that

$$\int_\Omega \frac{\partial}{\partial t}a_\delta(\phi-u) + \int_\Omega \frac{\partial}{\partial t}a_\delta(\psi-w) \le \int_\Omega \Delta(\phi-u)\operatorname{sign}^{\delta,+}(\phi-u) + \lambda\int_\Omega \tau_\delta(u,w,\phi,\psi)$$

with $a_\delta(s) = \int_0^s \operatorname{sign}^{\delta,+}(r)\mathrm{d}r$ which converges to s^+ and

$$\tau_\delta(u,w,\phi,\psi) = (G(u,w)-G(\phi,\psi))\operatorname{sign}^{\delta,+}(\phi-u) + (G(\phi,\psi)-G(u,w))\operatorname{sign}^{\delta,+}(\psi-w).$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \{a_\delta(\phi-u)+a_\delta(\psi-w)\} \le -\int_\Omega |\nabla(\phi-u)|^2\{\operatorname{sign}^{\delta,+}(\phi-u)\}' + \lambda\int_\Omega \tau_\delta(u,w,\phi,\psi). \quad \square$$

Because of the monotonicity properties of the function G one can show that $\lim_{\delta\downarrow 0}\tau_\delta \le 0$. Using Lebesgue's dominated convergence theorem and the hypotheses on the initial data, we deduce that for all $t \in [0,T]$

$$\iint_{Q_T} (\phi-u)^+(x,t)\mathrm{d}x + \iint_{Q_T} (\psi-w)^+(x,t)\mathrm{d}x \le 0,$$

so that $\phi \le u$ and $\psi \le w$ on Q_T.

Corollary 3.2. *Under hypotheses (H_0), let (u^λ, w^λ) be a weak solution of Problem (P^λ), then:*

$$u^\lambda(x,t) \le \tilde{u}(t) := \overline{u} + (M_1-\overline{u})\,\mathrm{e}^{\lambda t}, \quad \text{a.e. in } Q_T \tag{3.16}$$

$$w^\lambda(x,t) \le \tilde{w}(t) := M_2 + (M_1-\overline{u})(1+\mathrm{e}^{\lambda t}), \quad \text{a.e. in } Q_T. \tag{3.17}$$

Proof. We check that $(\tilde{u},\tilde{w})$ is a weak super-solution of Problem (P^λ) with the constant initial data $(\tilde{u}(x,0),\tilde{w}(x,0)) = (M_1, M_2+2(M_1-\overline{u}))$. The result then follows from the comparison principle given in Theorem 2. $\square$

4. A priori estimates

The purpose of this section is to prove the convergence Theorem 1. We first introduce some notations and give technical lemmas. We denote by $\operatorname{sign}^\delta$ a smooth nondecreasing approximation of the sign function which converges pointwise to its limit; we then define by $\mathcal{H}^\delta$ the regularization of the Heaviside function $\mathcal{H}^\delta(s) := \int_0^s \operatorname{sign}^\delta(\tau)\mathrm{d}\tau$. We prove below some a priori estimates.

Lemma 4.1. *Let* (u^λ, w^λ) *be the solution of* (P^λ). *Then there exists* $C_1 > 0$ *only depending on* T, Ω *and* $\overline{u}$, *such that*

$$\iint_{Q_T} |w_t^\lambda| \mathrm{d}x\mathrm{d}t = \lambda \iint_{Q_T} |G(u^\lambda, w^\lambda)| \mathrm{d}x\mathrm{d}t \le C_1, \tag{4.18}$$

and

$$\iint_{Q_T} |\nabla u^\lambda|^2 \mathrm{d}x\mathrm{d}t \le C_1. \tag{4.19}$$

Proof. We first prove (4.18). Multiplying (1.1)(a) by $\mathrm{sign}^\delta(u - \overline{u})$ and integrating the result on Ω we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left\{\mathcal{H}^\delta(u^\lambda - \overline{u})\right\} = -\int_\Omega \left\{\nabla(u^\lambda - \overline{u})\right\}^2 \left\{\mathrm{sign}^\delta\right\}'(u^\lambda - \overline{u}) - \lambda \int_\Omega G(u^\lambda, w^\lambda)\,\mathrm{sign}^\delta(u^\lambda - \overline{u}).$$

Using the nonnegativity of $s \mapsto \left\{\mathrm{sign}^\delta\right\}'$, we have the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left\{\mathcal{H}^\delta(u^\lambda - \overline{u})\right\} + \lambda \int_\Omega G(u^\lambda, w^\lambda)\,\mathrm{sign}^\delta(u^\lambda - \overline{u}) \le 0,$$

which we integrate on $(0, t)$, $0 < t \le T$ to obtain

$$\int_\Omega \left\{\mathcal{H}^\delta(u^\lambda - \overline{u})\right\}(t) + \lambda \int_{Q_t} G(u^\lambda, w^\lambda)\,\mathrm{sign}^\delta(u^\lambda - \overline{u}) \le \int_\Omega \left\{\mathcal{H}^\delta(u_0 - \overline{u})\right\}.$$

It then follows from Lebesgue's dominated convergence theorem that

$$\int_\Omega |u^\lambda - \overline{u}|(t) + \lambda \int_{Q_t} G(u^\lambda, w^\lambda)\,\mathrm{sign}(u^\lambda - \overline{u}) \le C.$$

In view of the special expression of $G(\cdot, \cdot)$, we remark that

$$G(u^\lambda, w^\lambda)\,\mathrm{sign}(u^\lambda - \overline{u}) = |G(u^\lambda, w^\lambda)|,$$

so that finally

$$\int_\Omega |u^\lambda - \overline{u}|(t) + \lambda \int_{Q_t} |G(u^\lambda, w^\lambda)| \le C$$

holds. Moreover in view of (1.1)(b) we deduce that

$$\iint_{Q_T} |w_t^\lambda| = \lambda \iint_{Q_T} |G(u^\lambda, w^\lambda)| \le C_1,$$

which coincides with (4.18). Next we prove (4.19). Multiplying (1.1)(a) by $u^\lambda - \overline{u}$ and integrating the result on Q_T we obtain

$$\iint_{Q_T} u_t^\lambda(u^\lambda - \overline{u}) = -\iint_{Q_T} |\nabla(u^\lambda - \overline{u})|^2 - \lambda \iint_{Q_T} G(u^\lambda, w^\lambda)(u^\lambda - \overline{u})$$

which implies that

$$\iint_{Q_T} |\nabla(u^\lambda - \overline{u})|^2 = -\frac{1}{2}\iint_{Q_T} \frac{\partial}{\partial t}|u^\lambda - \overline{u}|^2 - \lambda \iint_{Q_T} G(u^\lambda, w^\lambda)(u^\lambda - \overline{u}) \le \lambda \iint_{Q_T} |G(u^\lambda, w^\lambda)(u^\lambda - \overline{u})|.$$

Using (4.18) and the fact that u^λ is bounded we deduce (4.19). □

Next we prove estimates of differences between space and time translates of $\{u^\lambda\}$. We set for $r \in \mathbb{R}^+$:

$$\Omega_r = \{x \in \Omega, B(x, 2r) \subset \Omega\}.$$

Lemma 4.2. *There exists* $C_2 > 0$ *only depending on* T, Ω *and* $\overline{u}$ *such that*

$$\int_0^T \int_{\Omega_r} |u^\lambda(x+\xi,t) - u^\lambda(x,t)|^2 \mathrm{d}x\mathrm{d}t \leq C_2|\xi|^2, \tag{4.20}$$

and

$$\int_0^{T-\tau} \int_{\Omega} |u^\lambda(x,t+\tau) - u^\lambda(x,t)|^2 \mathrm{d}x\mathrm{d}t \leq C_2\tau \tag{4.21}$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \leq 2r$ *and* $\tau \in (0,T)$.

Proof. We first prove (4.20). We have that

$$\begin{aligned}
\int_0^T \int_{\Omega_r} |u^\lambda(x+\xi,t) - u^\lambda(x,t)|^2 \mathrm{d}x\mathrm{d}t &= \int_0^T \int_{\Omega_r} \left| \int_0^1 \nabla u^\lambda(x+\sigma\xi,t).\xi \mathrm{d}\sigma \right|^2 \mathrm{d}x\mathrm{d}t \\
&\leq \int_0^T \int_{\Omega_r} \left[\int_0^1 |\nabla u^\lambda(x+\sigma\xi,t)|^2 \mathrm{d}\sigma \int_0^1 |\xi|^2 \mathrm{d}\sigma \right] \mathrm{d}x\mathrm{d}t \\
&\leq |\xi|^2 \int_0^1 \left[\int_0^T \left(\int_{\Omega_r} |\nabla u^\lambda(x+\sigma\xi,t)|^2 \mathrm{d}x \right) \mathrm{d}t \right] \mathrm{d}\sigma \\
&\leq |\xi|^2 \int_0^1 \left[\int_0^T \left(\int_{\Omega} |\nabla u^\lambda(y,t)|^2 \mathrm{d}y \right) \mathrm{d}t \right] \mathrm{d}\sigma \\
&\leq |\xi|^2 \int_0^T \int_{\Omega} |\nabla u^\lambda(y,t)|^2 \mathrm{d}y\mathrm{d}t.
\end{aligned}$$

Using (4.19) we deduce (4.20). Next we prove (4.21).

$$\begin{aligned}
\int_0^{T-\tau} \int_{\Omega} |u^\lambda(x,t+\tau) - u^\lambda(x,t)|^2 \mathrm{d}x\mathrm{d}t &= \int_0^{T-\tau} \int_{\Omega} (u^\lambda(x,t+\tau) - u^\lambda(x,t)) \left(\int_0^\tau \partial_t u^\lambda(x,t+\sigma)\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^{T-\tau} \int_{\Omega} \left(u^\lambda(x,t+\tau) - u^\lambda(x,t)\right) \left(\int_0^\tau (\Delta u^\lambda - \lambda G(u^\lambda, w^\lambda))(x,t+\sigma)\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}t \\
&= \int_0^{T-\tau} \int_{\Omega} \left(u^\lambda(x,t+\tau) - u^\lambda(x,t)\right) \left(\int_0^\tau \Delta u^\lambda(x,t+\sigma)\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}t \\
&\quad - \lambda \int_0^{T-\tau} \int_{\Omega} \left(u^\lambda(x,t+\tau) - u^\lambda(x,t)\right) \left(\int_0^\tau G(u^\lambda, w^\lambda)(x,t+\sigma)\mathrm{d}\sigma \right) \mathrm{d}x\mathrm{d}t \\
&=: I + II.
\end{aligned} \tag{4.22}$$

Using (4.19) we have that

$$\begin{aligned}
|I| &\leq \int_0^\tau \left(\int_0^{T-\tau} \int_{\Omega} \left|\nabla u^\lambda(x,t+\tau) - \nabla u^\lambda(x,t)\right| |\nabla u^\lambda(x,t+\sigma)| \mathrm{d}x\mathrm{d}t \right) \mathrm{d}\sigma \\
&\leq 2\tau \int_0^T \int_{\Omega} |\nabla u^\lambda(x,t)|^2 \mathrm{d}x\mathrm{d}t \leq 2\tau C_1.
\end{aligned} \tag{4.23}$$

In view of (4.18) and Corollary 3.2, we obtain that

$$|II| \leq 2KC_1\tau. \tag{4.24}$$

Finally substituting (4.23) and (4.24) into (4.22) we deduce (4.21). □

Next we prove estimates of differences between space and time translates of $\{w^\lambda\}$.

Lemma 4.3. *There exists a positive function h such that $h(\xi) \to 0$ as $\xi \to 0$ and:*

$$\int_0^T \int_{\Omega_r} |w^\lambda(x+\xi,t) - w^\lambda(x,t)| \mathrm{d}x\mathrm{d}t \le h(\xi), \tag{4.25}$$

and

$$\int_0^{T-\tau} \int_{\Omega} |w^\lambda(x,t+\tau) - w^\lambda(x,t)| \mathrm{d}x\mathrm{d}t \le C_1\tau \tag{4.26}$$

for all $\xi \in \mathbb{R}^N$, $|\xi| \le 2r$ and $\tau \in (0,T)$.

Proof. We first remark that (4.26) immediately follows from (4.18). Next we prove (4.25). We set

$$\Omega_r' = \cup_{x\in\Omega_r} B(x,r)$$

and remark that by definition

$$\Omega_r \subset \Omega_r' \subset \Omega.$$

As it is done in [6], we introduce a function $\psi \in C_0^\infty(\Omega_r')$, such that

$$0 \le \psi \le 1 \quad \text{in } \Omega_r' \quad \text{and} \quad \psi = 1 \quad \text{in } \Omega_r.$$

Let $\xi \in \mathbb{R}^N$ with $|\xi| \le r$. For all $(x,t) \in \Omega_r' \times (0,T)$ we set

$$\tilde{u}^\lambda(x,t) = u^\lambda(x+\xi,t) \quad \text{and} \quad \tilde{w}^\lambda(x,t) = w^\lambda(x+\xi,t).$$

Next we show that

$$E := \left(G(u^\lambda,w^\lambda) - G(\tilde{u}^\lambda,\tilde{w}^\lambda)\right)\left(\mathrm{sign}(u^\lambda-\tilde{u}^\lambda) - \mathrm{sign}(w^\lambda-\tilde{w}^\lambda)\right) \ge 0, \quad \text{a.e. in } \Omega_r' \times (0,T). \tag{4.27}$$

We consider 4 different cases :

If $u^\lambda > \tilde{u}^\lambda$ and $w^\lambda > \tilde{w}^\lambda$ or if $u^\lambda < \tilde{u}^\lambda$ and $w^\lambda < \tilde{w}^\lambda$, then $(\mathrm{sign}(u^\lambda-\tilde{u}^\lambda) - \mathrm{sign}(w^\lambda-\tilde{w}^\lambda)) = 0$ and thus $E=0$.

If $u^\lambda \ge \tilde{u}^\lambda$ and $w^\lambda \le \tilde{w}^\lambda$ since $u \mapsto G(u,v)$ is increasing and $v \mapsto G(u,v)$ is nonincreasing we have that

$$G(\tilde{u}^\lambda,\tilde{w}^\lambda) \le G(u^\lambda,\tilde{w}^\lambda) \le G(u^\lambda,w^\lambda).$$

Thus $E \ge 0$.

Similarly if $u^\lambda \le \tilde{u}^\lambda$ and $w^\lambda \ge \tilde{w}^\lambda$, then $E \ge 0$. This concludes the proof of (4.27). In view of the ordinary differential equation for w^λ, (1.1)(b), we deduce from (4.27) that

$$(w^\lambda-\tilde{w}^\lambda)_t\left(\mathrm{sign}(u^\lambda-\tilde{u}^\lambda) - \mathrm{sign}(w^\lambda-\tilde{w}^\lambda)\right) \ge 0, \quad \text{a.e. in } \Omega_r' \times (0,T). \tag{4.28}$$

Multiplying the equality

$$(u^\lambda-\tilde{u}^\lambda)_t + (w^\lambda-\tilde{w}^\lambda)_t - \Delta(u^\lambda-\tilde{u}^\lambda) = 0,$$

by $[\mathrm{sign}^\delta(u^\lambda-\tilde{u}^\lambda)]\psi$ and integrating by part on $\Omega_r' \times (0,t)$ for $t \in (0,T)$ we obtain that

$$\int_0^t \int_{\Omega_r'} (u^\lambda-\tilde{u}^\lambda)_t \left[\mathrm{sign}^\delta(u^\lambda-\tilde{u}^\lambda)\right]\psi \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\Omega_r'} (w^\lambda-\tilde{w}^\lambda)_t \left[\mathrm{sign}^\delta(u^\lambda-\tilde{u}^\lambda)\right]\psi \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} \nabla(u^\lambda-\tilde{u}^\lambda)\nabla\left\{[\mathrm{sign}^\delta(u^\lambda-\tilde{u}^\lambda)]\psi\right\}\mathrm{d}x\mathrm{d}s = 0 \tag{4.29}$$

which by Lebesgue's dominated convergence theorem implies that

$$\int_0^t \int_{\Omega_r'} (u^\lambda-\tilde{u}^\lambda)_t \left[\mathrm{sign}(u^\lambda-\tilde{u}^\lambda)\right]\psi \mathrm{d}x\mathrm{d}s + \int_0^t \int_{\Omega_r'} (w^\lambda-\tilde{w}^\lambda)_t \left[\mathrm{sign}(u^\lambda-\tilde{u}^\lambda)\right]\psi \mathrm{d}x\mathrm{d}s$$
$$+ \int_0^t \int_{\Omega_r'} \nabla(u^\lambda-\tilde{u}^\lambda)\,\mathrm{sign}(u^\lambda-\tilde{u}^\lambda)\nabla\psi \mathrm{d}x\mathrm{d}s \le 0.$$

This with (4.28) gives that

$$\int_0^t\int_{\Omega_r'}|u^\lambda-\tilde{u}^\lambda|_t\psi\,\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\Omega_r'}|w^\lambda-\tilde{w}^\lambda|_t\psi\,\mathrm{d}x\mathrm{d}s+\int_0^t\int_{\Omega_r'}\nabla|u^\lambda-\tilde{u}^\lambda|\nabla\psi\,\mathrm{d}x\mathrm{d}s\le 0,$$

which implies after integration in time in the two first terms that

$$\int_{\Omega_r'}\left(|(u^\lambda-\tilde{u}^\lambda)(x,t)|+|(w^\lambda-\tilde{w}^\lambda)(x,t)|\right)\psi(x)\mathrm{d}x$$
$$\le\int_{\Omega_r'}(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|)\,\psi(x)\mathrm{d}x+\int_0^t\int_{\Omega_r'}|(u^\lambda-\tilde{u}^\lambda)(x,t)||\Delta\psi(x)|\mathrm{d}x\mathrm{d}s.$$

Integrating this inequality with respect to t on $(0,T)$ and using the fact that $0\le\psi\le 1$ in Ω_r' and $\psi=1$ in Ω_r we deduce that

$$\int_0^T\int_{\Omega_r}\left(|(u^\lambda-\tilde{u}^\lambda)(x,t)|+|(w^\lambda-\tilde{w}^\lambda)(x,t)|\right)\mathrm{d}x\mathrm{d}t$$
$$\le T\int_{\Omega_r'}(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|)\,\mathrm{d}x$$
$$+T\left(\int_0^T\int_{\Omega_r'}|u^\lambda-\tilde{u}^\lambda|^2\right)^{1/2}\left(\int_0^T\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}.$$

Also using (4.20) with Ω_r replaced by Ω_r' we deduce that

$$\int_0^T\int_{\Omega_r}|(w^\lambda-\tilde{w}^\lambda)(x,t)|\mathrm{d}x\mathrm{d}t\le T\int_{\Omega_r'}(|u_0(x)-u_0(x+\xi)|+|w_0(x)-w_0(x+\xi)|)\,\mathrm{d}x$$
$$+C(T)|\xi|\left(\int_{\Omega_r'}|\Delta\psi|^2\right)^{1/2}. \tag{4.30}$$

Therefore we have proved (4.25) with $h(\xi)$ being equal to the right-hand side of (4.30). □

Corollary 4.4. *Let (u^λ,w^λ) be the unique nonnegative solution of Problem (P^λ). There exist subsequences $\{u^{\lambda_m}\}$ and $\{w^{\lambda_m}\}$ and functions $U\in L^\infty(Q_T)$ and $W\in L^\infty(Q_T)$ such that*

$$u^{\lambda_m}\to U\quad\text{and}\quad w^{\lambda_m}\to W$$

strongly in $L^2(Q_T)$ as λ_m tends to ∞. Moreover as λ_m tends to ∞, $u^{\lambda_m}\to U$ weakly in $L^2(0,T;H^1(\Omega))$.

Proof. The first part of Corollary 4.4 follows from Lemmas 4.2 and 4.3 and the Riesz–Fréchet–Kolmogorov theorem ([4] Theorem IV.25 and Corollary IV.26). The last assertion follows from (4.19). □

Finally we characterize the limit pair (U,W), which amounts to proving Theorem 1.

Proof of Theorem 1. It is composed of three steps.
1. *Proof of the relation $G(U,W)=0$.* We set

$$G_\varepsilon(u,w):=(u-\overline{u})^+-\mathrm{sign}_\varepsilon^+(w)(u-\overline{u})^-,$$

where $\mathrm{sign}_\varepsilon^+$ is a smooth nonincreasing approximation of the sign^+ function such that

$$\mathrm{sign}_\varepsilon^+(x)=\begin{cases}1 & \text{if } x\ge\varepsilon\\ \dfrac{1}{\varepsilon}x & \text{if } 0\le x\le\varepsilon\\ 0 & \text{if } x\le 0.\end{cases}$$

Next we check that

$$0\le(u-\overline{u})G_\varepsilon(u,w)\le(u-\overline{u})G(u,w), \tag{4.31}$$

for all $(u,w)\in\mathbb{R}^2$. We consider three cases:

If $w \geq \varepsilon$, then $G(u, w) = G_\varepsilon(u, w) = u - \overline{u}$ so that

$$0 \leq (u - \overline{u})G_\varepsilon(u, w) \leq (u - \overline{u})G(u, w).$$

If $0 \leq w \leq \varepsilon$ and $u - \overline{u} \geq 0$, then $(u - \overline{u})G_\varepsilon(u, w) = (u - \overline{u})^2 = (u - \overline{u})G(u, w) \geq 0$.
If $0 \leq w \leq \varepsilon$ and $u - \overline{u} \leq 0$, then we have $(u - \overline{u})G_\varepsilon(u, w) = \frac{1}{\varepsilon}w(u - \overline{u})^2 \geq 0$ and moreover

$$(u - \overline{u})G_\varepsilon(u, w) \leq (u - \overline{u})G(u, w).$$

This concludes the proof of (4.31). Applying (4.31) at the point $(u^{\lambda_m}, w^{\lambda_m})$ and integrating the result on Q_T we deduce that

$$0 \leq \int_{Q_T} (u^{\lambda_m} - \overline{u})G_\varepsilon(u^{\lambda_m}, w^{\lambda_m}) \leq \int_{Q_T} (u^{\lambda_m} - \overline{u})G(u^{\lambda_m}, w^{\lambda_m}).$$

In view of Corollary 4.4 and (4.18) we deduce that $\lim_{\lambda_m \uparrow \infty} \int_{Q_T} (u^{\lambda_m} - \overline{u})G(u^{\lambda_m}, w^{\lambda_m}) = 0$ and thus since G_ε is continuous we have that

$$\int_{Q_T} (U - \overline{u})G_\varepsilon(U, W) = 0.$$

Therefore $G_\varepsilon(U, W) = 0$ or $U = \overline{u}$. Finally letting ε tend to 0 we obtain that $G(U, W) = 0$.
2. *We define* $Z := -(U + W) + \overline{u}$ and *search for relations between* U *and* W. We recall that

$$G(u, w) = (u - \overline{u})^+ - \text{sign}^+(w)(u - \overline{u})^-.$$

If $U \geq \overline{u}$, then $G(U, W) = U - \overline{u} = 0$ so that $Z = -W$. If $U < \overline{u}$, then $G(U, W) = \text{sign}^+(W)(U - \overline{u}) = 0$. Since $U \neq \overline{u}$ we deduce $\text{sign}^+(W) = 0$ and thus $W = 0$ and $Z = -U + \overline{u} \geq 0$.
Finally we obtain that

$$U = \overline{u} - Z^+ \quad \text{and} \quad W = Z^-. \tag{4.32}$$

3. *Characterization of* U *and* W
Let $(u^{\lambda_m}, w^{\lambda_m})$ be the unique solution of Problem (P^{λ_m}). Then

$$\int_{Q_T} (u^{\lambda_m} + w^{\lambda_m})\xi_t - \int_{Q_T} \nabla u^{\lambda_m} \nabla \xi = -\int_\Omega (u_0 + w_0)\xi(0),$$

for all $\xi \in C^{2,1}(\overline{Q_T})$ such that $\xi(T) = 0$. Letting λ_m tend to ∞ we deduce that

$$\int_{Q_T} (U + W)\xi_t - \int_{Q_T} \nabla U.\nabla \xi = -\int_\Omega (u_0 + w_0)\xi(0) \tag{4.33}$$

for all $\xi \in C^{2,1}(\overline{Q_T})$ such that $\xi(T) = 0$.
Using the relation $Z = -(U + W) + \overline{u}$ we deduce from the boundedness of U and W and from the equality (4.33) that Z is a weak solution of Problem (SP) in the following sense : (i) $Z \in L^\infty(Q_T)$; (ii) Z satisfies the integral equality

$$-\int_{Q_T} Z\xi_t + \int_{Q_T} \nabla Z^+.\nabla \xi = -\int_\Omega (u_0 + w_0 - \overline{u})\xi(0). \tag{4.34}$$

Since by [8] Proposition 5, Problem (SP) has a unique weak solution, it follows that Z is the unique weak solution of Problem (SP). This completes the proof of Theorem 1. □

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