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Dedicated to Professor W.-L. Wendland on the occasion of his 70th birthday

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Combined triangular FV-triangular FE method for nonlinear convection-diffusion problems

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Introduction

The finite volume method (FVM) represents an efficient and robust method for the solution of conservation laws and inviscid compressible flow. This technique is based on expressing the balance of fluxes of conserved quantities through boundaries of control volumes, combined with approximate Riemann solvers. On the other hand, the finite element method (FEM), based on the concept of a weak solution defined with the aid of suitable test functions is quite natural for the solution of elliptic and parabolic problems. However, it is not mandatory to adhere to these paths of discretization in their respective regimes of common use. The finite (control) volume method (cell–centred or vertex centred) may also be used for the discretization of elliptic problems (see [9,23]). Often the control volume approach is used in the framework of the FE methods in order to gain stability from an upwinding ([2,27,28]). For applications to compressible flow, see e. g. [17–19].

In the solution of convection–diffusion problems, including viscous compressible flow, it is quite natural to try to employ the advantages of both FV and FE methods in such a way that the FVM is used for the discretization of inviscid Euler fluxes, whereas the FEM is applied to the approximation of viscous terms. This idea leads us to the *combined finite volume–finite element method* (FV–FE method) proposed in [13]. (Sometimes it is also called the mixed FV–FE method.) The analysis and applications of this method were investigated in [1, 8, 14–16]. The numerical computations for the system of compressible viscous flow ([6–8, 12, 24]) demonstrate that the combined FV–FE method is feasible and produces good numerical results for technically relevant problems. The idea of using a combination of the FV and FE methods appears also in [3, 21, 22]. In [5] the combined FV–FE method was applied to the solution of a complex coupled problem.

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In [14–16] the convergence and error estimates were studied for the combination of piecewise linear finite elements and the dual finite volumes constructed over a triangular mesh. The papers [1] and [8] are concerned with the combination of nonconforming Crouzeix-Raviart piecewise linear finite elements and barycentric finite volumes. Numerical experiments presented in [6] or [12], using combined FV–FE techniques for the solution of compressible viscous flow however show that in some complicated problems the best results can be achieved with the aid of comforming triangular piecewise linear finite elements for the discretization of viscous terms, combined with triangular finite volumes for the discretization of convective terms. The theoretical analysis of this method has been still missing. Therefore, our goal is to fill in this gap and to derive error estimates for this combined FV–FE method.

1 Continuous problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain and $(0,T)$, where $T>0$, time interval. We consider the following initial-boundary value problem: Find the solution of the equation

$$\frac{\partial u}{\partial t} + \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} = \varepsilon\, \Delta u + g \quad \text{in } Q_T = \Omega \times (0,T)\,, \tag{1.1}$$

with the initial condition

$$u(x,0) = u^0(x),\ x \in \Omega\,, \tag{1.2}$$

and the boundary condition

$$u|_{\partial\Omega\times(0,T)} = 0\,. \tag{1.3}$$

We assume that the data have the following properties:

a) $f_s \in C^1(\mathbb{R})$, $f_s(0)=0$, $|f_s| \le C_{f'}$ $s=1,2$,
b) $\varepsilon > 0$,
c) $g \in C([0,T];\, L^2(\Omega))$,
d) $u^0 \in L^2(\Omega)$.

In virtue of assumption a), the functions f_s satisfy the Lipschitz condition with constant $C_{f'}$. The constant ε is the diffusion coefficient and the functions f_s are fluxes of the quantity u in the direction x_s.

In what follows we shall use the standard notation for function spaces: $L^p(\omega)$ – Lebesgue space, $W^{k,p}(\omega), H^k(\omega) = W^{k,2}(\omega)$ Sobolev spaces, $L^p(0,T;X)$ – Bochner space of functions defined in $(0,T)$ with values in a Banach space X, $C^k([0,T];X)$ – space of k-times continuously differentiable mappings of the interval $[0,T]$ with values in X (ω is a bounded domain, $k \ge 0$ integer, $p \in [1,\infty]$) – see, e. g. [25].

If $p \in [1,\infty)$ and $|\cdot|_X$ is a seminorm in X, then by $|\cdot|_{L^p(0,T;X)}$ we denote a seminorm in $L^p(0,T;X)$ defined by

$$|u|_{L^p(0,T;X)} = \left(\int_0^T |u(t)|_X^p \,\mathrm{d}t \right)^{1/p} \quad \text{for } u \in L^p(0,T;X)\,. \tag{1.4}$$

We shall use the following notation:

$$(u,v) = \int_\Omega u\, v \,\mathrm{d}x, \quad u,\, v \in L^2(\Omega)\,, \tag{1.5}$$

$$a(u,v) = \varepsilon \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \quad u,\, v \in H^1(\Omega)\,, \tag{1.6}$$

$$b(u,v) = \int_\Omega \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, v \,\mathrm{d}x, \quad u \in H^1(\Omega) \cap L^\infty(\Omega),\ v \in L^2(\Omega)\,, \tag{1.7}$$

$$|u|_{H^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{1/2}, \quad u \in H^1(\Omega) \tag{1.8}$$

$$\text{(seminorm in } H^1(\Omega))\,.$$

The analysis of the FV-FE method will be carried out under the assumption that the exact solution of problem (1.1) – (1.3) satisfies the regularity condition

$$\frac{\partial u}{\partial t} \in L^2(0,T;H^2(\Omega))\,.$$

This condition is, of course, stronger than can be established under the assumptions a)–d) on data. However, it is necessary for deriving error estimates.

2 Discrete problem

2.1 Triangulation

Let $\mathcal{T}_h$ be a partition of the closure $\overline{\Omega}$ of the domain Ω formed by a finite number of closed triangles called *finite elements*. We number all elements in such a way that we can write $\mathcal{T}_h = \{K_i\}_{i\in I}$, where $I \subset Z^+ = \{0,1,2,\ldots\}$ is a suitable index set. We assume that the triangulation $\mathcal{T}_h$ satisfies the following conditions:

$$\overline{\Omega} = \bigcup_{i\in I} K_i \tag{2.1}$$

and two different elements K_i, K_j are either disjoint or have a common vertex or a common side.

Further, we shall consider a mesh $\mathcal{D}_h = \{D_i\}_{i\in J}$ formed by closed triangles D_i, which will be called *finite volumes*. (It is possible to use closed convex polygons as finite volumes, but for simplicity of the treatment we consider a triangular finite volume mesh.) Symbol $J \subset Z^+$ denotes a suitable index set. We assume that the mesh $\mathcal{D}_h$ has the same properties as the triangulation $\mathcal{T}_h$. If two finite volumes $D_i, D_j \in \mathcal{D}_h$ have a common side, we call them neighbours. Then we use the notation

$$\Gamma_{ij} = \Gamma_{ji} = \partial D_i \cap \partial D_j \tag{2.2}$$

and

$$s(i) = \{j \in J; j \neq i, D_j \quad \text{is a neighbour of} \quad D_i\}\,. \tag{2.3}$$

The sides of finite volumes adjacent to the boundary $\partial\Omega$, which form this boundary, will be denoted by S_j and numbered by indices $j \in J_B \subset Z^- = \{-1,-2,\ldots\}$. Thus, $J \cap J_B = \emptyset$ and $\partial\Omega = \bigcup_{j\in J_B} S_j$. For a finite volume D_i adjacent to the boundary $\partial\Omega$ we write

$$\begin{aligned} &\gamma(i) = \left\{j \in J_B; S_j \subset \partial\Omega \cap \partial D_i\right\}\,, \\ &\Gamma_{ij} = S_j\,, \quad \text{for} \quad j \in \gamma(i)\,. \end{aligned} \tag{2.4}$$

If D_i is not adjacent to $\partial\,\Omega$, then we set $\gamma(i) = \emptyset$. We set

$$S(i) = s(i) \cup \gamma(i)\,. \tag{2.5}$$

Then

$$\begin{aligned} \partial D_i &= \bigcup_{j\in S(i)} \Gamma_{ij}\,, \\ \partial D_i \cap \partial\Omega &= \bigcup_{j\in\gamma(i)} \Gamma_{ij}\,, \\ |\partial D_i| &= \sum_{j\in S(i)} |\Gamma_{ij}|\,, \end{aligned} \tag{2.6}$$

where $|\partial D_i|$ is the length of ∂D_i and $|\Gamma_{ij}|$ is the length of the side Γ_{ij}. By $\boldsymbol{n}_{ij}$ we shall denote the unit outer normal to ∂K_i on the side Γ_{ij}.

For $k \in Z^+, K \in \mathcal{T}_h$ we denote by $P_k(K)$ the space of all polynomials on K of degree $\leq k$. In what follows the following finite element spaces

$$X_h = \{v_h \in C(\overline{\Omega});\ v_h|_K \in P_1(K)\ \forall K \in \mathcal{T}_h\}, \tag{2.7}$$

$$V_h = \{v_h \in X_h;\; v_h|_{\partial\Omega} = 0\}. \tag{2.8}$$

and the finite volume space

$$Y_h = \{v_h \in L^2(\Omega);\; v_h|_{D_i} \in P_0(D_i)\; \forall i \in J\} \tag{2.9}$$

will be used.

The relation between the FE and FV spaces is given by the so-called *lumping operator* $L_h :\; X_h \to Y_h$ or, more general, $L_h :\; C(\overline{\Omega}) \to Y_h$.

2.2 Derivation of the method

Let u be a classical solution of problem (1.1)–(1.3). We multiply equation (1.1) by a test function $v \in V_h$, integrate over Ω and apply Green's theorem. We obtain the identity

$$\left(\frac{\partial u}{\partial t}, v\right) + \sum_{i\in J} \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, v \,\mathrm{d}x + a(u,v) = (g,v)\,. \tag{2.10}$$

In order to approximate the terms with fluxes f_s, the test function v is replaced by $L_h v$:

$$\sum_{i\in J} \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\, v \,\mathrm{d}x \approx \sum_{i\in J} L_h v|_{D_i} \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\,\mathrm{d}x\,. \tag{2.11}$$

If we apply Green's theorem to the right-hand side and approximate fluxes with the aid of a so-called numerical flux H, we get

$$\begin{aligned}\int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s}\,\mathrm{d}x &= \int_{\partial D_i} \sum_{s=1}^{2} f_s(u)\, n_s \,\mathrm{d}S = \sum_{j\in S(i)} \int_{\Gamma_{ij}} \sum_{s=1}^{2} f_s(u)\, n_s \,\mathrm{d}S \\ &\approx \sum_{j\in S(i)} H(L_h u|_{D_i}, L_h u|_{D_j}, \boldsymbol{n}_{ij})\, |\Gamma_{ij}|\,. \end{aligned} \tag{2.12}$$

For the faces $\Gamma_{ij} \subset \partial\Omega$ (i.e. $j \in \gamma(i)$) we use the boundary condition (1.3), on the basis of which we set $H(L_h u|_{D_i}, L_h u|_{D_j}, \boldsymbol{n}_{ij}) = 0$. As a result we obtain the approximation of the convective terms represented by the form

$$b_h(u,v) = \sum_{i\in J} L_h v|_{D_i} \sum_{j\in s(i)} H(L_h u|_{D_i}, L_h u|_{D_j}, \boldsymbol{n}_{ij})\, |\Gamma_{ij}|. \tag{2.13}$$

Definition 2.1 We define an approximate solution of problem (1.1) – (1.3) as a function $u_h \in C^1([0,T];\, V_h)$ satisfying

$$a)\;\left(\frac{\partial u_h}{\partial t}, v_h\right) + b_h(u_h, v_h) + a(u_h, v_h) = (g, v_h)\;\forall v_h \in V_h, \tag{2.14}$$

$$b)\; u_h(0) = u_h^0 = \Pi_h u^0,$$

where Π_h is the operator of X_h-interpolation.

Condition (2.14) a) is equivalent to a system of ordinary differential equations, which can be solved, e.g. by the Runge-Kutta method.

3 Theoretical analysis

In what follows, in the domain Ω, we shall consider systems $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ of finite element meshes and $\{\mathcal{D}_h\}_{h\in(0,h_0)}$ of finite volume meshes, with $h_0 > 0$. (For simplicity, we shall not emphasize the dependence of index sets I and J on h by notation.) We shall use the notation $h_K = \operatorname{diam}(K)$, $h = \max_{K\in\mathcal{T}_h} h_K$ and by ρ_K we denote the radius of the largest circle inscribed into the element $K \in \mathcal{T}_h$.

3.1 Assumptions

Let us assume that the system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ is *shape regular*. This means that there exists a constant C_T independent of K and h such that

$$\frac{h_K}{\rho_K} \le C_T, \quad K \in \mathcal{T}_h,\ h \in (0,h_0)\,. \tag{3.1}$$

Further, let

$$\mathrm{diam}(D_i) \le C_D\, h,\ \forall i \in J, \tag{3.2}$$

with a constant $C_D > 0$ independent of i and h.

Let us define the set $\omega(D_i)$ by

$$\omega(D_i) = \cup\{K \in \mathcal{T}_h;\ K \cap D_i \ne \emptyset\}\,. \tag{3.3}$$

For a given element $K \in \mathcal{T}_h$, let R_K be the number of sets $\omega(D_i)$ containing the element K; we assume that there exists $R < +\infty$, independent of h, such that $R_K \le R$ for any $K \in \mathcal{T}_h$. This means that each element $K \in \mathcal{T}_h$ intersects at most R finite volumes D_i. Then

$$\sum_{i\in J} |v_h|^2_{H^1(\omega(D_i))} \le R\,|v_h|^2_{H^1(\Omega)}\,. \tag{3.4}$$

Moreover, let the *inverse assumption* be satisfied: There exists a constant $C_I > 0$ such that

$$h \le C_I h_K \quad \forall K \in \mathcal{T}_h,\ h \in (0,h_0)\,. \tag{3.5}$$

Now we shall specify properties of the numerical flux H:

Assumptions (H):

1) $H(u,v,\boldsymbol{n})$ is defined in $\mathbb{R}^2 \times B_1$, where $B_1 = \{\boldsymbol{n} \in \mathbb{R}^2; |\boldsymbol{n}| = 1\}$, and *Lipschitz-continuous* with respect to u, v:

$$\begin{aligned} &|H(u,v,\boldsymbol{n}) - H(u^*,v^*,\boldsymbol{n})| \le C_H(|u-u^*| + |v-v^*|),\\ &\qquad\qquad u,\ v,\ u^*,\ v^* \in \mathbb{R},\ \boldsymbol{n} \in B_1. \end{aligned} \tag{3.6}$$

2) $H(u,v,\boldsymbol{n})$ is *consistent*:

$$H(u,u,\boldsymbol{n}) = \sum_{s=1}^{2} f_s(u)\,n_s, \quad u \in \mathbb{R},\ \boldsymbol{n} = (n_1,n_2) \in B_1. \tag{3.7}$$

3) $H(u,v,\boldsymbol{n})$ is *conservative*:

$$H(u,v,\boldsymbol{n}) = -H(v,u,-\boldsymbol{n}), \quad u,\ v \in \mathbb{R},\ \boldsymbol{n} \in B_1. \tag{3.8}$$

In view of (3.6) and (3.7), the functions $f_s,\ s = 1,2,$ are Lipschitz-continuous with constant $2C_H$.

In the sequel we shall consider the lumping operator L_h defined by

$$L_h v|_{D_i} = \frac{1}{|D_i|}\int_{D_i} v\,\mathrm{d}x, \quad i \in J, \tag{3.9}$$

for functions v locally integrable in Ω.

Remark 3.1 Other choices of lumping operators are possible. For instance, if one assumes that the triangular finite element mesh $\mathcal{T}_h$ satisfies the Delaunay condition (see [9], Example 9.1), and one takes for $\mathcal{D}_h$ the dual Voronoï mesh, a possible choice for L_h is

$$L_h v|_{D_i} = v(P_i), \quad i \in J,$$

where $P_i,\ i \in J$, denote the vertices of triangular elements $K \in \mathcal{T}_h$. With this choice, the piecewise linear method on the mesh $\mathcal{T}_h$ yields a diffusion matrix which is identical to that of the finite volume method on the dual Voronoï mesh.

Hence, with the spatial discretization given in Definition 2.1, and an explicit or implicit Euler scheme together with a mass lumping technique for the time derivative term, using a monotone flux H, we get a scheme which is identical, up to the right–hand–side, to a finite volume scheme which was previously studied in [10, 26]. In fact in these papers, convergence is proven for a more general operator, namely a degenerate nonlinear parabolic equation, using a Kruzkov-like technique. This proof can easily be adpated to the right hand side which occurs when using the scheme of Definition 2.1. However, no error estimate is known yet.

In virtue of (1.6) and (2.10), the exact solution of problem (1.1) satisfies the identity

$$\left(\frac{\partial u}{\partial t}, v_h\right) + b(u, v_h) + a(u, v_h) = (g, v_h), \ \forall v_h \in V_h, \tag{3.10}$$

where

$$b(u, v) = \int_\Omega \sum_{s=1}^{2} \frac{\partial f_s(u)}{\partial x_s} v \, \mathrm{d}x \,. \tag{3.11}$$

Substituting here the approximate solution u_h instead of u, we obtain

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + b(u_h, v_h) + a(u_h, v_h) = (g, v_h) + \hat{\varepsilon}_h(u_h, v_h), \tag{3.12}$$

where the term $\hat{\varepsilon}_h = \hat{\varepsilon}_h(u_h, v_h)$ represents *the truncation error* for the convection term. By (2.14) a), it is possible to express the truncation error $\hat{\varepsilon}_h$ in the form

$$\hat{\varepsilon}_h(u_h, v_h) = b(u_h, v_h) - b_h(u_h, v_h) \,. \tag{3.13}$$

Subtracting (3.10) from (3.12), we find that

$$\left(\frac{\partial (u_h - u)}{\partial t}, v_h\right) + b(u_h, v_h) - b(u, v_h) + a(u_h - u, v_h) = \hat{\varepsilon}_h(u_h, v_h) \,. \tag{3.14}$$

Our main goal is to derive the estimate for the error $e_h = u_h - u$ of the method. By Π_h we shall denote the operator of the X_h-interpolation. One possibility is to use the Lagrange interpolation: For a function $\varphi \in C(\overline{\Omega})$ we define $\Pi_h \varphi$ as an element $\Pi_h\varphi \in X_h$ such that $(\Pi_h\varphi)(P) = \varphi(P)$ for all vertices P of the triangulation $\mathcal{T}_h$. The error e_h can be expressed as $e_h = \xi + \eta$, where

$$\xi = u_h - \Pi_h u, \quad \eta = \Pi_h u - u \,. \tag{3.15}$$

Since $\xi \in V_h$, we can use $v_h := \xi$ in (3.14). This yields the identity

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) + a(\xi, \xi) = \hat{\varepsilon}_h(u_h, \xi) + b(u, \xi) - b(u_h, \xi) - \left(\frac{\partial \eta}{\partial t}, \xi\right) - a(\eta, \xi), \tag{3.16}$$

which will serve as the basis for the derivation of the error estimate.

3.2 Auxiliary results

The symbols C and c will denote generic constants which can attain different values at different places.

Lemma 3.2 *There exists a constant $C > 0$ such that*

$$|b(u, v_h) - b(w, v_h)| \le C \, \|u - w\|_{L^2(\Omega)} \, |v_h|_{H^1(\Omega)}, \quad u, w \in H^1(\Omega), \ v_h \in V_h. \tag{3.17}$$

Proof. Using the definition of the form b, Green's theorem, Lipschitz continuity of $f_s, \ s = 1, 2$, and equality $v_h|_{\partial\Omega} = 0$, we find that

$$|b(u, v_h) - b(w, v_h)| = \left| \int_\Omega \sum_{s=1}^{2} \left(\frac{\partial f_s(u)}{\partial x_s} - \frac{\partial f_s(w)}{\partial x_s} \right) v_h \, \mathrm{d}x \right|$$

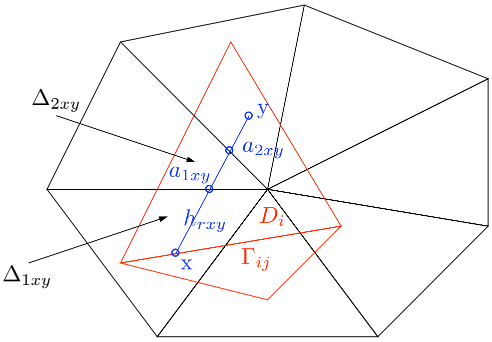

Fig. 1 (online colour at: www.zamm-journal.org) Estimation of $|v_h(x) - v_h(y)|$

$$
\begin{aligned}
&= \left| - \int_\Omega \sum_{s=1}^{2} (f_s(u) - f_s(w)) \frac{\partial v_h}{\partial x_s} \, dx \right| \\
&\le C_{f'} \int_\Omega |u - w| \, |\nabla v_h| \, dx \\
&\le C_{f'} \, \|u - w\|_{L^2(\Omega)} \, |v_h|_{H^1(\Omega)} \, .
\end{aligned}
$$

Thus, we have (3.17) with $C = C_{f'}$. □

In what follows, we shall derive several estimates important for the proof of the consistency of the method. Namely, we shall verify the validity of the estimate

$$
|b_h(u_h, v_h) - b(u_h, v_h)| \le C\, h \, |u_h|_{H^1(\Omega)} \, |v_h|_{H^1(\Omega)} \, , \quad u_h, \, v_h \in V_h \, . \tag{3.18}
$$

Lemma 3.3 *For $v_h \in V_h$, $x \in D_i$, $D_i \in \mathcal{D}_h$, we have*

$$
|v_h(x) - L_h v_h|_{D_i}| \le C \, |v_h|_{H^1(\omega(D_i))} \, , \tag{3.19}
$$

where $\omega(D_i)$ is defined by (3.3).

Proof. By the definition (3.9) of the lumping operator, we can write

$$
\begin{aligned}
v_h(x) - L_h v_h|_{D_i} &= v_h(x) - \frac{1}{|D_i|} \int_{D_i} v_h(y) \, dy \\
&= \frac{1}{|D_i|} \int_{D_i} (v_h(x) - v_h(y)) \, dy \, .
\end{aligned} \tag{3.20}
$$

Let $x, y \in D_i$. The straight segment connecting x with y intersects several elements from $\mathcal{T}_h$. We shall denote these intersections by $\Delta_{r,xy}$, $r = 1, \ldots, k_{xy}$. The length of $\Delta_{r,xy}$ will be denoted by $h_{r,xy}$. See Fig. 1. In view of (3.2),

$$
\sum_r h_{r,xy} \le \operatorname{diam}(D_i) \le c\, h. \tag{3.21}
$$

Using the linearity of v_h on each element $K \in \mathcal{T}_h$, we find that

$$
\begin{aligned}
|v_h(x) - v_h(y)| &= |v_h(x) - v_h(a_{1,xy}) + v_h(a_{1,xy}) - v_h(a_{2,xy}) + \ldots - v_h(y)| \\
&\le \underbrace{|v_h(x) - v_h(a_{1,xy})|}_{\nabla v_h|_{\Delta_{1,xy}} \cdot (x - a_{1,xy})} + \underbrace{|v_h(a_{1,xy}) - v_h(a_{2,xy})|}_{\nabla v_h|_{\Delta_{2,xy}} \cdot (a_{1,xy} - a_{2,xy})} + \ldots \\
&\le \sum_{r=1}^{k_{xy}} h_{r,xy} \underbrace{|\nabla v_h|_{\Delta_{r,xy}}}_{\text{const}} \le c\, h \max_{D_i} |\nabla v_h| \, .
\end{aligned} \tag{3.22}
$$

It follows from (3.20) and (3.22) that

$$
|v_h(x) - L_h v_h|_{D_i}| \le c \, \frac{h}{|D_i|} \int_{D_i} \max_{D_i} |\nabla v_h| \, dy
$$

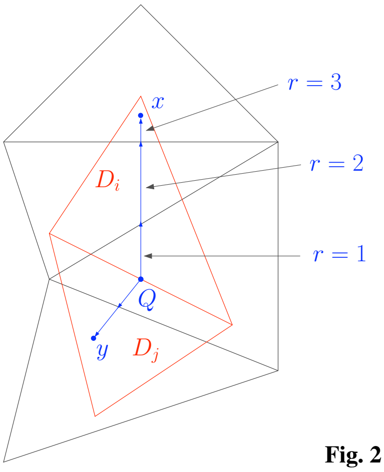

Fig. 2 (online colour at: www.zamm-journal.org) Estimation of $L_h v_h|_{D_i} - L_h v_h|_{D_j}$

$$
\begin{aligned}
&= c\, h \max_{D_i} |\nabla v_h| \\
&\leq c\, h \max_{\omega(D_i)} |\nabla v_h| \,. \qquad (3.23)
\end{aligned}
$$

Further, we use the inverse assumption (3.5), which implies the so-called *inverse estimate*: there exists a constant $C > 0$ independent of K, h, v_h such that

$$\|\nabla v_h\|_{L^\infty(K)} = |v_h|_{W^{1,\infty}(K)} \leq \frac{C}{h} |v_h|_{H^1(K)} \,, \qquad (3.24)$$

$$\forall v_h \in P_1(K), \quad \forall K \in \mathcal{T}_h, \quad \forall h \in (0, h_0) \,. \qquad (3.25)$$

(For proof, see, e. g. [4], Sect. 3.2.)

Now, it is possible to estimate $\max_{\omega(D_i)} |\nabla v_h|$. The inverse estimate and the inequality

$$\max_{n=1,\ldots,m} |a_n| \leq \left(\sum_{n=1}^{m} |a_n|^2 \right)^{1/2} \qquad (3.26)$$

imply that

$$
\begin{aligned}
\max_{\omega(D_i)} |\nabla v_h| = \max_{K \subset \omega(D_i)} |\nabla v_h|_K| &\leq c\, \frac{1}{h} \max_{K \subset \omega(D_i)} |v_h|_{H^1(K)} \\
&\leq c\, \frac{1}{h} \left(\sum_{K \subset \omega(D_i)} |v_h|^2_{H^1(K)} \right)^{1/2} \\
&= c\, \frac{1}{h} |v_h|_{H^1(\omega(D_i))} \,. \qquad (3.27)
\end{aligned}
$$

Substituting (3.27) in (3.23), we obtain estimate (3.19). □

Furthermore, we shall estimate the expression $L_h v_h|_{D_i} - L_h v_h|_{D_j}$, where the finite volumes D_i and D_j are neighbours. Let $x \in D_i$, $y \in D_j$ and let Q be the centre of the segment Γ_{ij}, see Fig. 2. The segment Qx connecting the point Q with x intersects several elements $K_{r,Qx} \in \mathcal{T}_h$, $r = 1, \ldots, n_{Qx}$. These parts of the segment can be represented by vectors $\vec{h}_{r,Qx}$, $r = 1, \ldots, n_{Qx}$. Then $v_h(x)$ can be expressed in the form

$$v_h(x) = v_h(Q) + \sum_{r=1}^{n_{Qx}} \nabla v_h|_{K_{r,Qx}} \cdot \vec{h}_{r,Qx} \,. \qquad (3.28)$$

Similarly we can write

$$v_h(y) = v_h(Q) + \sum_{r=1}^{n_{Qy}} \nabla v_h|_{K_{r,Qy}} \cdot \vec{h}_{r,Qy} . \qquad (3.29)$$

Lemma 3.4 *There exists a constant $C > 0$ such that*

$$\big|L_h v_h|_{D_i} - L_h v_h|_{D_j}\big| \le C\,(|v_h|_{H^1(\omega(D_i))} + |v_h|_{H^1(\omega(D_j))}). \tag{3.30}$$

P r o o f. Both expressions in the left-hand side of the estimate can be expressed with the aid of the integral average (3.9) and then (3.28) and (3.29) can be used. Further, using the obvious relation $\sum_r |\vec{h}_{r,Qx}| \le ch$ and the inverse estimate, we get similarly as in the proof of Lemma 3.3

$$\begin{aligned}
|L_h v_h|_{D_i} - L_h v_h|_{D_j}| &= \left| \frac{1}{|D_i|} \int\limits_{D_i} v_h(x)\,\mathrm{d}x - \frac{1}{|D_j|} \int\limits_{D_j} v_h(y)\,\mathrm{d}y \right| \\
&= \left| \frac{1}{|D_i|} \int\limits_{D_i} v_h(Q)\,\mathrm{d}x - \frac{1}{|D_j|} \int\limits_{D_j} v_h(Q)\,\mathrm{d}y \right. \\
&\quad + \frac{1}{|D_i|} \int\limits_{D_i} \sum_{r=1}^{n_{Qx}} \nabla v_h|_{K_{r,Qx}} \cdot \vec{h}_{r,Qx}\,\mathrm{d}x \\
&\quad \left. - \frac{1}{|D_j|} \int\limits_{D_j} \sum_{r=1}^{n_{Qy}} \nabla v_h|_{K_{r,Qy}} \cdot \vec{h}_{r,Qy}\,\mathrm{d}y \right| \\
&\le c\,h\,(\max_{D_i} |\nabla v_h| + \max_{D_j} |\nabla v_h|) \\
&\le c\,h\,(\max_{\omega(D_i)} |\nabla v_h| + \max_{\omega(D_j)} |\nabla v_h|) \\
&\le C\,(|v_h|_{H^1(\omega(D_i))} + |v_h|_{H^1(\omega(D_j))})\,.
\end{aligned}$$

This proves estimate (3.30). □

Lemma 3.5 *There exists a constant $C > 0$ such that*

$$|\hat{\varepsilon}_h(u_h, v_h)| = |b_h(u_h, v_h) - b(u_h, v_h)| \le C\,h\,|u_h|_{H^1(\Omega)}\,|v_h|_{H^1(\Omega)}. \tag{3.31}$$

P r o o f. We can write

$$\begin{aligned}
|\hat{\varepsilon}_h(u_h, v_h)| &= |b(u_h, v_h) - b_h(u_h, v_h)| \\
&\le \underbrace{|b(u_h, v_h) - b(u_h, L_h v_h)|}_{\sigma_1} + \underbrace{|b(u_h, L_h v_h) - b_h(u_h, v_h)|}_{\sigma_2}\,.\,.
\end{aligned} \tag{3.32}$$

Our goal is to estimate σ_1 and σ_2. We have

$$\begin{aligned}
|\sigma_1| &= \left| \int\limits_{\Omega} \sum_{s=1}^{2} \frac{\partial f_s(u_h)}{\partial x_s}\,(v_h - L_h v_h)\,\mathrm{d}x \right| \\
&= \left| \int\limits_{\Omega} \sum_{s=1}^{2} {f_s}'(u_h) \frac{\partial u_h}{\partial x_s}\,(v_h - L_h v_h)\,\mathrm{d}x \right| \\
&\le 2\,C_{f'} \int\limits_{\Omega} |\nabla u_h|\,|v_h - L_h v_h|\,\mathrm{d}x \\
&\le 2\,C_{f'}\,|u_h|_{H^1(\Omega)}\,\|v_h - L_h v_h\|_{L^2(\Omega)}\,.
\end{aligned} \tag{3.33}$$

Now it is necessary to estimate the norm $\|v_h - L_h v_h\|_{L^2(\Omega)}$. Due to (3.2), it holds that $|D_i| \le c\,h^2$. By Lemma 3.3 and (3.4),

$$\|v_h - L_h v_h\|^2_{L^2(\Omega)} = \sum_{i \in J} \int\limits_{D_i} |v_h(x) - L_h v_h|_{D_i}|^2\,\mathrm{d}x$$

$$\begin{aligned}
&\leq C \sum_{i\in J} |D_i|\, |v_h|^2_{H^1(\omega(D_i))} \\
&\leq C\, h^2 \sum_{i\in J} |v_h|^2_{H^1(\omega(D_i))} \\
&\leq R\, C\, h^2\, |v_h|^2_{H^1(\Omega)}\,. \qquad (3.34)
\end{aligned}$$

Hence,

$$\|v_h - L_h v_h\|_{L^2(\Omega)} \leq c\, h\, |v_h|_{H^1(\Omega)}\,. \qquad (3.35)$$

Now the substitution into (3.33) implies that

$$|\sigma_1| \leq c\, h\, |u_h|_{H^1(\Omega)}\, |v_h|_{H^1(\Omega)}\,. \qquad (3.36)$$

Further, we shall estimate the expression σ_2. By Green's theorem,

$$\begin{aligned}
|\sigma_2| &= |b(u_h, L_h v_h) - b_h(u_h, v_h)| \\
&= \left| \sum_{i\in J} \int_{D_i} \sum_{s=1}^{2} \frac{\partial f_s(u_h)}{\partial x_s}\, L_h v_h\, \mathrm{d}x - \sum_{i\in J} \sum_{j\in s(i)} L_h v_h|_{D_i}\, H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})\, |\Gamma_{ij}| \right| \\
&= \left| \sum_{i\in J} \int_{\partial D_i} \sum_{s=1}^{2} f_s(u_h) L_h v_h|_{D_i} n_s \mathrm{d}S - \sum_{i\in J} \sum_{j\in s(i)} L_h v_h|_{D_i} H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})\, |\Gamma_{ij}| \right| .
\end{aligned}$$

Since the numerical flux H is consistent, we can write

$$|\sigma_2| = \left| \sum_{i\in J} L_h v_h|_{D_i} \sum_{j\in s(i)} \left(\int_{\Gamma_{ij}} H(u_h, u_h, \boldsymbol{n}_{ij})\, \mathrm{d}S - H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})\, |\Gamma_{ij}| \right) \right| .$$

Now we shall use the Lipschitz continuity and conservativity of H. For given $i \in J$ and $j \in s(i)$, when we exchange i and j, using (3.8), we find that

$$\begin{aligned}
&L_h u_h|_{D_i} \left(\int_{\Gamma_{ij}} H(u_h, u_h, \boldsymbol{n}_{ij})\, \mathrm{d}S - H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij}) |\Gamma_{ij}| \right) \\
&\quad + L_h u_h|_{D_j} \left(\int_{\Gamma_{ij}} H(u_h, u_h, \boldsymbol{n}_{ji})\, \mathrm{d}S - H(L_h u_h|_{D_j}, L_h u_h|_{D_i}, \boldsymbol{n}_{ji}) |\Gamma_{ij}| \right) \\
&= (L_h u_h|_{D_i} - L_h u_h|_{D_j}) \left(\int_{\Gamma_{ij}} H(u_h, u_h, \boldsymbol{n}_{ij})\, \mathrm{d}S - H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij}) |\Gamma_{ij}| \right) .
\end{aligned}$$

Summing over $i \in J$ and $j \in s(i)$ and dividing by two yields

$$\begin{aligned}
|\sigma_2| &= \frac{1}{2} \left| \sum_{i\in J} \sum_{j\in s(i)} (L_h v_h|_{D_i} - L_h v_h|_{D_j}) \int_{\Gamma_{ij}} \big(H(u_h, u_h, \boldsymbol{n}_{ij}) - H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij}) \big)\, \mathrm{d}S \right| \\
&\leq \frac{1}{2} \sum_{i\in J} \sum_{j\in s(i)} \left| L_h v_h|_{D_i} - L_h v_h|_{D_j} \right| C_L \int_{\Gamma_{ij}} \big(|u_h(x) - L_h u_h|_{D_i}| + |u_h(x) - L_h u_h|_{D_j}| \big)\, \mathrm{d}S. .
\end{aligned}$$

In virtue of Lemma (3.3), the estimates $|\Gamma_{ij}| \le c\,h$ and (3.4) and the Cauchy inequality, we obtain

$$
\begin{aligned}
|\sigma_2| &\le C \sum_{i\in J}\sum_{j\in s(i)} \big(|v_h|_{H^1(\omega(D_i))} + |v_h|_{H^1(\omega(D_j))}\big) \int\limits_{\Gamma_{ij}} \big(|u_h|_{H^1(\omega(D_i))} + |u_h|_{H^1(\omega(D_j))}\big)\,\mathrm{d}S \\
&\le C\,h \sum_{i\in J}\sum_{j\in s(i)} \big(|v_h|_{H^1(\omega(D_i))} + |v_h|_{H^1(\omega(D_j))}\big)\big(|u_h|_{H^1(\omega(D_i))} + |u_h|_{H^1(\omega(D_j))}\big) \\
&\le C\,h \left(\sum_{i\in J}\sum_{j\in s(i)} \big(|v_h|_{H^1(\omega(D_i))} + |v_h|_{H^1(\omega(D_j))}\big)^2\right)^{1/2} \left(\sum_{i\in J}\sum_{j\in s(i)} \big(|u_h|_{H^1(\omega(D_i))} + |u_h|_{H^1(\omega(D_j))}\big)^2\right)^{1/2} \\
&\le C\,h \left(2\sum_{i\in J}\sum_{j\in s(i)} \Big(|v_h|^2_{H^1(\omega(D_i))} + |v_h|^2_{H^1(\omega(D_j))}\Big)\right)^{1/2} \left(2\sum_{i\in J}\sum_{j\in s(i)} \Big(|u_h|^2_{H^1(\omega(D_i))} + |u_h|^2_{H^1(\omega(D_j))}\Big)\right)^{1/2} \\
&\le 8\,C\,h \left(\sum_{i\in J} |v_h|^2_{H^1(\omega(D_i))}\right)^{1/2} \left(\sum_{i\in J} |u_h|^2_{H^1(\omega(D_i))}\right)^{1/2} \\
&\le 8\,R\,C\,h\,|v_h|_{H^1(\Omega)}\,|u_h|_{H^1(\Omega)}\,.
\end{aligned}
$$

Now, this and (3.36) already imply (3.31). □

Lemma 3.5 gives a consistency property of the method, and will lead to the error estimate.

3.3 Error estimate

In what follows, we shall assume that the exact solution u is sufficiently regular, namely, it satisfies the condition

$$\frac{\partial u}{\partial t} \in L^2(0,T;\, H^2(\Omega))\,. \tag{3.37}$$

This implies that

$$u \in C([0,T];\, H^2(\Omega)). \tag{3.38}$$

Lemma 3.6 *There exists a constant $C > 0$ such that*

$$\|\Pi_h v - v\|_{L^2(K)} \le C\,h^2\,|v|_{H^2(K)}, \tag{3.39}$$

$$|\Pi_h v - v|_{H^1(K)} \le C\,h\,|v|_{H^2(K)} \tag{3.40}$$

for all $v \in H^2(K)$, $K \in \mathcal{T}_h$ and $h \in (0, h_0)$.

Proof. See, e. g. [4]. □

From this lemma, the following estimates can be derived:

Lemma 3.7 *For all $h \in (0, h_0)$ we have*

$$\|\eta\|_{L^2(\Omega)} \le C\,h^2\,|u|_{H^2(\Omega)}, \tag{3.41}$$

$$|\eta|_{H^1(\Omega)} \le C\,h\,|u|_{H^2(\Omega)}, \tag{3.42}$$

$$\left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(\Omega)} \le C\,h^2 \left|\frac{\partial u}{\partial t}\right|_{H^2(\Omega)}, \tag{3.43}$$

where $\eta = \Pi_h u - u$.

Proof. Let us establish (3.41). Using (3.39), we have

$$\|\eta\|^2_{L^2(\Omega)} = \sum_{K\in\mathcal{T}_h} \|\eta\|^2_{L^2(K)} \le C\,h^4 \sum_{K\in\mathcal{T}_h} |u|^2_{H^2(K)} = C\,h^4\,|u|^2_{H^2(\Omega)}\,. \tag{3.44}$$

Other estimates are proven in a similar way. □

Now, starting from identity (3.16) and using the definitions of the forms $(\cdot,\cdot)$ and $a(\cdot,\cdot)$, we prove some additional estimates.

Lemma 3.8 *For a.e. $t \in (0,T)$, it holds:*

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\xi(t)\|^2_{L^2(\Omega)}, \tag{3.45}$$

$$a(\xi,\xi) = \varepsilon\,|\xi|^2_{H^1(\Omega)}, \tag{3.46}$$

$$\left|\left(\frac{\partial \eta}{\partial t}, \xi\right)\right| \le Ch^2\,\left|\frac{\partial u}{\partial t}\right|_{H^2(\Omega)}\,\|\xi\|_{L^2(\Omega)}, \tag{3.47}$$

$$|a(\eta,\xi)| \le \varepsilon\, C\, h\,|u|_{H^2(\Omega)}\,|\xi|_{H^1(\Omega)}, \tag{3.48}$$

where $\xi = u_h - \Pi_h u$, $\eta = \Pi_h u - u$.

Proof. Relation (3.45) is obtained by the differentiation of the integral $\int_\Omega |\xi(t)|^2\,\mathrm{d}x$ with respect to the parameter t. Relation (3.46) follows from (1.6). In the proof of (3.47) we use the Cauchy inequality and apply estimate (3.43). Similarly, from (1.6) we obtain (3.48). □

For our further considerations it is necessary to prove the family $\{u_h\}_{h\in(0,h_0)}$ is bounded in the space $L^2\left(0,T;\,H^1(\Omega)\right)$. The approximate solution $u_h \in C^1([0,T];\,V_h)$ satisfies the identity

$$\left(\frac{\partial u_h}{\partial t}, v_h\right) + a(u_h,v_h) + b_h(u_h,v_h) = (g,v_h), \quad \forall v_h \in V_h\,. \tag{3.49}$$

Substituting u_h for v_h in (3.49), we find that

$$\left(\frac{\partial u_h}{\partial t}, u_h\right) + a(u_h,u_h) + b_h(u_h,u_h) = (g,u_h)\,. \tag{3.50}$$

Since

$$\left(\frac{\partial u_h}{\partial t}, u_h\right) = \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\|u_h(t)\|^2_{L^2(\Omega)}, \tag{3.51}$$

$$a(u_h,u_h) = \varepsilon\,|u_h|^2_{H^1(\Omega)}, \tag{3.52}$$

$$|(g,u_h)| \le \| g\|_{L^2(\Omega)}\,\|u_h(t)\|_{L^2(\Omega)}, \tag{3.53}$$

we are able to prove the boundedness of the form b_h.

Lemma 3.9 *For the approximate solution u_h we have*

$$|b_h(u_h,u_h)| \le C\,\|u_h\|_{L^2(\Omega)}\,|u_h|_{H^1(\Omega)}, \quad h \in (0,h_0). \tag{3.54}$$

Proof. First we estimate $|L_h u_h|_{D_i}|$. From the definition (3.9) of the lumping operator, using the Cauchy inequality and the inequality $|D_i| \ge c\,h^2$, we get

$$|L_h u_h|_{D_i}| = \left|\frac{1}{|D_i|}\,\int_{D_i} u_h\,\mathrm{d}x\right| \le \frac{|D_i|^{1/2}}{|D_i|}\,\|u_h\|_{L^2(D_i)} \le \frac{C}{h}\,\|u_h\|_{L^2(D_i)}\,. \tag{3.55}$$

For a given $i \in J$ and $j \in s(i)$, when we exchange i and j, using (3.8) we get

$$\begin{aligned} &L_h u_h|_{D_i}\,H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})|\Gamma_{ij}| + L_h u_h|_{D_j}\,H(L_h u_h|_{D_j}, L_h u_h|_{D_i}, \boldsymbol{n}_{ji})|\Gamma_{ij}| \\ &= L_h u_h|_{D_i}\,H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})|\Gamma_{ij}| - L_h u_h|_{D_j}\,H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})|\Gamma_{ij}| \\ &= \left(L_h u_h|_{D_i} - L_h u_h|_{D_j}\right)\,H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})|\Gamma_{ij}|\,. \end{aligned} \tag{3.56}$$

Summation over all $i \in J$ and $j \in s(i)$ and the use of (2.13), (3.6), (3.30), (3.55) and the inequality $|\Gamma_{ij}| \le c\,h$ yield

$$|b_h(u_h,u_h)| = \left|\sum_{i\in J} L_h u_h|_{D_i}\,\sum_{j\in s(i)} H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij})\,|\Gamma_{ij}|\right| \tag{3.57}$$

$$
\begin{aligned}
&= \left| \frac{1}{2} \sum_{i\in J} \sum_{j\in s(i)} \left(L_h u_h|_{D_i} - L_h u_h|_{D_j}\right) \, H(L_h u_h|_{D_i}, L_h u_h|_{D_j}, \boldsymbol{n}_{ij}) |\Gamma_{ij}| \right| \\
&\le \frac{1}{2} \, C_H \, h \sum_{i\in J} \sum_{j\in s(i)} \left|L_h u_h|_{D_i} - L_h u_h|_{D_j}\right| \left(|L_h u_h|_{D_i}| + |L_h u_h|_{D_j}|\right) \\
&\le C \sum_{i\in J} \sum_{j\in s(i)} \left(|u_h|_{H^1(\omega(D_i))} + |u_h|_{H^1(\omega(D_j))}\right) \left(\|u_h\|_{L^2(D_i)} + \|u_h\|_{L^2(D_j)}\right) \\
&\le C \, \|u_h\|_{L^2(\Omega)} |u_h|_{H^1(\Omega)},
\end{aligned}
$$

which we wanted to prove. □

In the sequel, we shall apply the following version of Gronwall's lemma:

Lemma 3.10 *Let y, q, z, $r \in C([0,T])$ be nonnegative functions and*

$$y(t) + q(t) \le z(t) + \int_0^t r(s) \, y(s) \, \mathrm{d}s, \quad t \in [0,T] \, . \tag{3.58}$$

Then

$$y(t) + q(t) \le z(t) + \int_0^t r(\vartheta) \, z(\vartheta) \, \exp\left(\int_\vartheta^t r(s) \mathrm{d}s \right) \, \mathrm{d}\vartheta, \quad t \in [0,T]. \tag{3.59}$$

Proof. can be carried out similarly as in [11], Sect. 8.2.29

□

Lemma 3.11 *For all $t \in [0,T]$ it holds*

$$\|u_h(t)\|^2_{L^2(\Omega)} \le K(\varepsilon), \tag{3.60}$$

$$\varepsilon \int_0^{\mathrm{T}} |u_h(\vartheta)|^2_{H^1(\Omega)} \mathrm{d}\vartheta \le K(\varepsilon), \tag{3.61}$$

where

$$K(\varepsilon) = C \, \exp(CT/\varepsilon), \tag{3.62}$$

with a constant C independent of h and ε.

Proof. We start from identity (3.50) and use (3.51) – (3.54):

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u_h(t)\|^2_{L^2(\Omega)} + 2 \, \varepsilon |u_h(t)|^2_{H^1(\Omega)} \le 2 \, \|g(t)\|_{L^2(\Omega)} \, \|u_h(t)\|_{L^2(\Omega)} + 2 \, C \, \|u_h(t)\|_{L^2(\Omega)} \, |u_h(t)|_{H^1(\Omega)} \, . \tag{3.63}$$

Using Young's inequality

$$a \, b \le \tfrac{1}{2} \left(\varepsilon a^2 + \frac{b^2}{\varepsilon} \right), \tag{3.64}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \, \|u_h(t)\|^2_{L^2(\Omega)} + \varepsilon \, |u_h(t)|^2_{H^1(\Omega)} \le \|g(t)\|^2_{L^2(\Omega)} + \left(1 + \frac{C}{\varepsilon} \right) \, \|u_h(t)\|^2_{L^2(\Omega)} \, . \tag{3.65}$$

By (3.38) and (3.41), we have $\|u_h(0)\|_{L^2(\Omega)} \le C$ for all $h \in (0, h_0)$. The integration $\int_0^t$ of inequality (3.65) yields

$$\|u_h(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^t |u_h(\vartheta)|^2_{H^1(\Omega)} \mathrm{d}\vartheta$$

$$\leq \|u_h(0)\|^2_{L^2(\Omega)} + \int_0^t \|g(\vartheta)\|^2_{L^2(\Omega)} \mathrm{d}\vartheta + \int_0^t \left(1 + \frac{C}{\varepsilon}\right) \|u_h(\vartheta)\|^2_{L^2(\Omega)} \mathrm{d}\vartheta$$

$$\leq C + \left(1 + \frac{C}{\varepsilon}\right) \int_0^t \|u_h(\vartheta)\|^2_{L^2(\Omega)} \mathrm{d}\vartheta \,. \tag{3.66}$$

Now we shall apply Gronwall's lemma 3.10, where we set

$$y(t) = \|u_h(t)\|^2_{L^2(\Omega)} \,,$$

$$q(t) = \varepsilon \int_0^t |u_h(\vartheta)|^2_{H^1(\Omega)} \mathrm{d}\vartheta \,,$$

$$z(t) = C \,,$$

$$r(s) = 1 + \frac{C}{\varepsilon} \,.$$

By simple calculation we get

$$\|u_h(t)\|^2_{L^2(\Omega)} + \varepsilon \int_0^{\mathrm{T}} |u_h(\vartheta)|^2_{H^1(\Omega)} \,\mathrm{d}\vartheta \leq C \,\exp\left(\frac{CT}{\varepsilon}\right) =: K(\varepsilon) \,, \tag{3.67}$$

which we wanted to prove. □

Now we can already formulate the main result of our paper.

Theorem 3.12 *Let assumptions a) - d) on data be satisfied, let the numerical flux be Lipschitz continuous, consistent and conservative and let the triangulations have properties from* 3.1. *Then the error of the method* $e_h = u - u_h$*, where* u *is the exact solution of problem* (1.1) – (1.3) *satisfying* (3.37) *and* u_h *is the approximate solution defined by* (2.14)*, satisfies the estimates*

$$\max_{t\in[0,T]} \|e_h\|_{L^2(\Omega)} \leq C\,h \tag{3.68}$$

and

$$\sqrt{\varepsilon}\sqrt{\int_0^T |e_h(\vartheta)|^2_{H^1(\Omega)} \mathrm{d}\vartheta} \leq C\,h. \tag{3.69}$$

Proof. The error e_h is expressed in the form $e_h = \xi + \eta$, where

$$\xi = u_h - \Pi_h u \in V_h, \quad \eta = \Pi_h u - u \,. \tag{3.70}$$

On the basis of Lemmas 3.2, 3.5 and 3.8 we estimate the left-hand side of identity (3.16), for a.e. $t \in (0, T)$:

$$\left(\frac{\partial \xi}{\partial t}, \xi\right) + a(\xi,\xi) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2_{L^2(\Omega)} + \varepsilon\,|\xi|^2_{H^1(\Omega)} \tag{3.71}$$

$$\begin{aligned}
&\leq |\hat{\varepsilon}_h(u_h,v_h)| + |b(u,\xi) - b(u_h,\xi)| + \left|\left(\frac{\partial \eta}{\partial t}, \xi\right)\right| + |a(\eta,\xi)| \\
&\leq C\,h\,|u_h|_{H^1(\Omega)}\,|\xi|_{H^1(\Omega)} + C\,\|u - u_h\|_{L^2(\Omega)}\,|\xi|_{H^1(\Omega)} \\
&\quad + C\,h^2 \left|\frac{\partial u}{\partial t}\right|_{H^2(\Omega)} \|\xi\|_{L^2(\Omega)} + \varepsilon\,C\,h\,|u|_{H^2(\Omega)}\,|\xi|_{H^1(\Omega)} \\
&\leq C\,h\,|u_h|_{H^1(\Omega)}\,|\xi|_{H^1(\Omega)} + C\,\left(\|\xi\|_{L^2(\Omega)} + \|\eta\|_{L^2(\Omega)}\right)\,|\xi|_{H^1(\Omega)} \\
&\quad + C\,h^2 \left|\frac{\partial u}{\partial t}\right|_{H^2(\Omega)} \|\xi\|_{L^2(\Omega)} + \varepsilon\,C\,h\,|u|_{H^2(\Omega)}\,|\xi|_{H^1(\Omega)} \,.
\end{aligned} \tag{3.72}$$

Young's inequality and Lemma 3.7 imply that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\xi(t)\|^2_{L^2(\Omega)} + 2\,\varepsilon\,|\xi|^2_{H^1(\Omega)} \le \frac{C\,h^2}{\varepsilon}\,|u_h|^2_{H^1(\Omega)} + \frac{\varepsilon}{4}\,|\xi|^2_{H^1(\Omega)} + \frac{C}{\varepsilon}\,\|\xi\|^2_{L^2(\Omega)} + \frac{\varepsilon}{4}\,|\xi|^2_{H^1(\Omega)} \tag{3.73}$$

$$+\,\frac{C\,h^4}{\varepsilon}\,|u|^2_{H^2(\Omega)} + \frac{\varepsilon}{4}\,|\xi|^2_{H^1(\Omega)} + C\,h^4\,\left|\frac{\partial u}{\partial t}\right|^2_{H^2(\Omega)} + \|\xi\|^2_{L^2(\Omega)}$$

$$+\,\varepsilon\,C\,h^2\,|u|^2_{H^2(\Omega)} + \frac{\varepsilon}{4}\,|\xi|^2_{H^1(\Omega)}\,.$$

The integration $\int_0^t$, the use of Lemmas 3.6 and 3.11 and the fact that $\xi(0) = 0$ yield

$$\|\xi(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta \le \frac{C\,T\,h^2}{\varepsilon^2}\,\exp\left(\frac{C\,T}{\varepsilon}\right) + C\,h^4\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^2(\Omega))} \tag{3.74}$$

$$+\,C\,\left(\frac{h^4}{\varepsilon} + \varepsilon\,h^2\right)\,\|u\|^2_{L^2(0,T;H^2(\Omega))} + \left(1 + \frac{C}{\varepsilon}\right)\int_0^t \|\xi(\vartheta)\|^2_{L^2(\Omega)}\mathrm{d}\vartheta\,.$$

Now it is possible to apply Gronwall's Lemma 3.10, where we define the individual terms by

$$y(t) = \|\xi(t)\|^2_{L^2(\Omega)}\,,$$

$$q(t) = \varepsilon\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta\,,$$

$$z = C\,h^2\exp(C\,T/\varepsilon) + C\,h^4\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;H^2(\Omega))} + C\,(h^4/\varepsilon + \varepsilon\,h^2)\,\|u\|^2_{L^2(0,T;H^2(\Omega))}\,,$$

$$r = 1 + \frac{C}{\varepsilon}\,.$$

Then we have

$$\int_\vartheta^t r(s)\,\mathrm{d}s = \left(1 + \frac{C}{\varepsilon}\right)\,(t - \vartheta)\,, \tag{3.75}$$

$$\int_0^t r(\vartheta)\,z\,\exp\left(\int_\vartheta^t r(s)\,\mathrm{d}s\right)\mathrm{d}\vartheta = \int_0^t \left(1 + \frac{C}{\varepsilon}\right)\,z\,\exp\left(\left(1 + \frac{C}{\varepsilon}\right)\,(t - \vartheta)\right)\mathrm{d}\vartheta$$

$$= z\,\exp\left(\left(1 + \frac{C}{\varepsilon}\right)\,t\right) - z\,.$$

Hence,

$$\|\xi(t)\|^2_{L^2(\Omega)} + \varepsilon\int_0^t |\xi(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta \tag{3.76}$$

$$\le C\left(h^2\exp(CT/\varepsilon) + h^4\,\|\partial u/\partial t\|^2_{L^2(0,T;H^2(\Omega))} + (h^4/\varepsilon + \varepsilon\,h^2)\,\|u\|^2_{L^2(0,T;H^2(\Omega))}\right)\exp\left(\left(1 + \frac{C}{\varepsilon}\right)\,t\right)\,.$$

If we use the notation

$$Z(\varepsilon,h) := C\,h^2\,\left(\exp(CT/\varepsilon) + h^2\,\,||\partial u/\partial t||^2_{L^2(0,T;H^2(\Omega))}\right. \tag{3.77}$$

$$\left. +(h^2/\varepsilon + \varepsilon\,h^2)\,\|u\|^2_{L^2(0,T;H^2(\Omega))}\right)\exp\left(\left(1 + \frac{C}{\varepsilon}\right)\,t\right)\,,$$

then we have

$$\max_{t\in[0,T]} \|\xi(t)\|_{L^2(\Omega)} \le \sqrt{Z(\varepsilon,h)}\,.$$

The triangular and Young's inequalities imply that

$$\|e_h\|^2_{L^2(\Omega)} \le 2\,\|\xi\|^2_{L^2(\Omega)} + 2\,\|\eta\|^2_{L^2(\Omega)}, \qquad |e_h|^2_{H^1(\Omega)} \le 2\,|\xi|^2_{H^1(\Omega)} + 2\,|\eta|^2_{H^1(\Omega)}\,. \tag{3.78}$$

Now using (3.78) and assumption (3.38), we find that

$$\max_{t\in[0,T]} \|e_h(t)\|_{L^2(\Omega)} \le \sqrt{2\,Z(\varepsilon,h) + C\,h^2 \max_{t\in[0,T]} |u(t)|^2_{H^2(\Omega)}} \le C\,h\,. \tag{3.79}$$

Finally we shall prove the estimate for $\varepsilon \int_0^T |e_h(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta$. By (3.76) and (3.77),

$$\varepsilon \int_0^T |\xi(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta \le Z(\varepsilon,h)\,. \tag{3.80}$$

This, (3.78) and (3.42) imply that

$$\begin{aligned}\varepsilon \int_0^T |e_h(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta &\le 2\,\varepsilon \int_0^T |\xi(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta + 2\,\varepsilon \int_0^T |\eta(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta \\ &\le 2\,Z(\varepsilon,h) + C\,h^2\,\|u\|^2_{L^2(0,T;H^2(\Omega))} \le C\,h^2\,.\end{aligned} \tag{3.81}$$

Hence

$$\sqrt{\varepsilon}\sqrt{\int_0^T |e_h(\vartheta)|^2_{H^1(\Omega)}\mathrm{d}\vartheta} \le C\,h, \tag{3.82}$$

which concludes the proof. □

4 Numerical experiments

We shall verify our theoretical results by numerical experiments. Let us apply the combined FV-FE method to the *2D viscous Burgers equation*

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x_1} + u\frac{\partial u}{\partial x_2} - \varepsilon\Delta u = g\,, \tag{4.1}$$

considered in the space-time domain $Q_T = \Omega \times (0,1)$, $\Omega = (-1,1)^2$, equipped with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$, and initial condition $u|_{t=0} = 0$. The right-hand side g is chosen so that it conforms to the exact solution

$$u = (1 - e^{-2t})(1 - x_1^2)^2(1 - x_2^2)^2\,.$$

The time discretization is carried out by the semiimplicit Euler scheme

$$\left(\frac{u_h^k - u_h^{k-1}}{\tau}, v_h\right) + b_h(u_h^{k-1}, v_h) + a_h(u_h^k, v_h) = (g^{k-1}, v_h)\,, \tag{4.2}$$

which has better stability properties than a purely explicit scheme with no added computational cost, because the FE mass and stiffness matrices share their sparsity structure. In the definition (2.13) of the form b_h we use the numerical flux

$$H(u_1,u_2,\boldsymbol{n}) = \begin{cases} \sum_{s=1}^2 f_s(u_1)n_s, & \text{if } A > 0\,,\\ \sum_{s=1}^2 f_s(u_2)n_s, & \text{if } A \le 0\,,\end{cases} \tag{4.3}$$

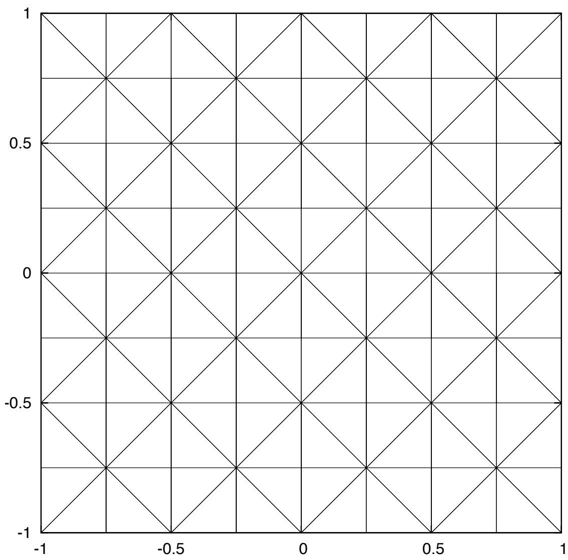

Fig. 3 Triangulation $\mathcal{T}_{h_1}$.

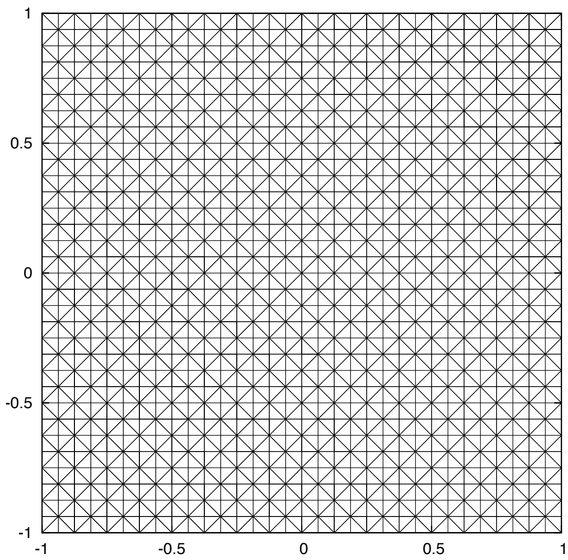

Fig. 4 Triangulation $\mathcal{T}_{h_3}$.

where

$$A = \sum_{s=1}^{2} f'_s(\bar{u}) n_s, \quad \bar{u} = \frac{1}{2}(u_1 + u_2) \quad \text{and} \quad \boldsymbol{n} = (n_1, n_2)\,. \tag{4.4}$$

As we want to examine the error of the space discretization, we *overkill* the time step so that the time discretization error is negligible.

In each computation we consider primary FE meshes $\mathcal{T}_{h_i},\ i = 1, \ldots, 6$, and construct secondary FV meshes $\mathcal{D}_{h_i},\ i = 1, \ldots, 6$. The FE and FV meshes are successively refined and for each refinement we evaluate the so-called *experimental order of convergence* (EOC) defined by

$$\mathrm{EOC} = \frac{\log e_{h_{i+1}} - \log e_{h_i}}{\log h_{i+1} - \log h_i}\,.$$

The symbol e_{h_i} denotes either the $L^\infty(L^2)$-norm or the $L^2(H^1)$-norm of the error of the approximate solution computed on the meshes $\mathcal{T}_{h_i}$ and $\mathcal{D}_{h_i}$.

We consider two different methods of deriving the secondary FV mesh $\mathcal{D}_h$. The first method (yielding the FV mesh $\mathcal{D}_h^1$) consists simply in copying the FE mesh $\mathcal{T}_h$. Hence, $\mathcal{D}_h^1 = \mathcal{T}_h$. In the second case (yielding the FV mesh $\mathcal{D}_h^2$) we create an interior FV node as the barycenter of each FE triangle from $\mathcal{T}_h$, add the FE boundary nodes and triangulate these nodes by means of the Delaunay triangulation.

In Figs. 3 and 4, the triangulations $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_3}$ are plotted. Figs. 5 and 6 show the Delaunay finite volume meshes $\mathcal{D}_{h_1}^2$ and $\mathcal{D}_{h_3}^2$.

The construction of the lumping operator gets tricky in the second case, as the FE and FV triangles can overlap in many different ways. Therefore, instead of covering all these possibilities, we evaluate the lumping operator approximately by a *Quasi Monte Carlo* approach and then scale the resulting matrix to *enforce conservativity* of constants - a constant function should be lumped to the same constant. Our numerical experiments show that this approach is acceptable.

In Tables 1 and 2 we show the computational results obtained with the aid of the FV meshes $\mathcal{D}_{h_i}^1$ and $\mathcal{D}_{h_i}^2$, respectively. By $e_{h,L^\infty(L^2)}$ and $e_{h,L^\infty(H^1)}$, computational errors evaluated in the $L^\infty(L^2)$- and $L^\infty(H^1)$-norms are denoted. The symbols $\mathrm{EOC}_{L^\infty(L^2)}$ and $\mathrm{EOC}_{L^\infty(L^2)}$ denote the corresponding experimental orders of convergence.

5 Conclusion and perspectives

In this paper we presented the error analysis of a combined triangular finite element - triangular finite volume scheme for nonlinear convection - diffusion problems. As shown in [6,12] and the above examples, the FV mesh $\mathcal{D}_h$ can be constructed in various ways. It is not clear what is the best combination of a FV and a FE mesh. From the point of view of algorithmization, the method with $\mathcal{D}_h = \mathcal{T}_h$ is particularly suitable. The obtained error estimates are optimal with respect to h. However, the

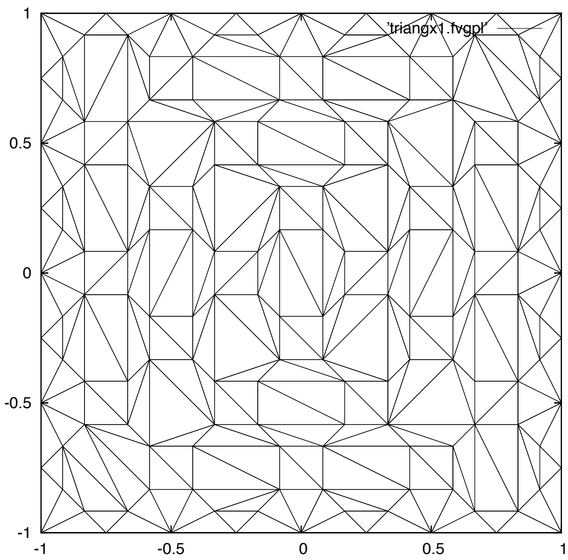

Fig. 5 Finite volume mesh $\mathcal{D}^2_{h_1}$.

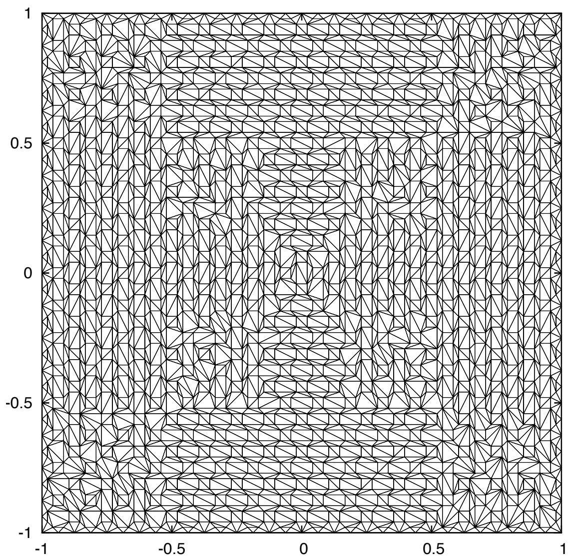

Fig. 6 Finite volume mesh $\mathcal{D}^2_{h_3}$.

Table 1 Method with the FV meshes $\mathcal{D}^1_h$.

$\#I$	h	$e_{h,L^\infty(L^2)}$	$\mathrm{EOC}_{L^\infty(L^2)}$	$e_{h,L^2(H^1)}$	$\mathrm{EOC}_{L^2(H^1)}$
128	3.54E-01	6.57E-02	–	1.09E-01	–
512	1.77E-01	2.95E-02	1.16	5.58E-02	0.97
2048	8.84E-02	1.40E-02	1.08	2.81E-02	0.99
8192	4.42E-02	6.87E-03	1.03	1.41E-02	0.99
32768	2.21E-02	3.40E-03	1.02	7.05E-03	1.00
131072	1.11E-02	1.69E-03	1.01	3.53E-03	1.00
Average			1.06		0.99

Table 2 Method with the FV meshes $\mathcal{D}^2_h$.

$\#I$	h	$e_{h,L^\infty(L^2)}$	$\mathrm{EOC}_{L^\infty(L^2)}$	$e_{h,L^2(H^1)}$	$\mathrm{EOC}_{L^2(H^1)}$
128	3.54E-01	7.50E-02	–	1.13E-01	–
512	1.77E-01	4.57E-02	0.71	6.18E-02	0.87
2048	8.84E-02	1.78E-02	1.36	3.01E-02	1.04
8192	4.42E-02	1.18E-02	0.59	1.62E-02	0.89
32768	2.21E-02	4.37E-03	1.43	7.56E-03	1.10
131072	1.11E-02	2.99E-03	0.55	4.12E-03	0.88
Average			0.93		0.96

constants in the resulting error estimates depend on the diffusion parameter ε and blow up as ε tends to 0. It is still an open problem, to our knowledge, to find error estimates which would be independent of ε, and which would cover the degenerate nonlinear parabolic case. Moreover, in the light of the use of adaptive mesh refinement procedures, the inverse assumption is restrictive. An open question is its weakening. Another issue is the study of the fully discrete scheme. In particular, the influence of a mass lumping technique for this scheme should be evaluated.

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