DDM Preprint
Editors: editor1, editor2, editor3, editor4
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1 A Neumann-Neumann method using a finite volume discretization.

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Introduction

In this work, we present a non-overlapping domain decomposition method which is well adapted to the discretization of convection-diffusion equations by the finite volume scheme. The method which we shall consider is closely related to the so-called Neumann-Neumann relaxation operator which was studied in the finite element framework by several researchers among whom Glowinski et al. [BGLTV89] Dryja and Widlund [DW95] and Quarteroni and Marini [MQ89].
The algorithm is first written in the continuous case and then its discrete counterpart is presented in the framework of a finite volume discretization.
For the sake of simplicity, we shall only consider here the classical Laplace equation:

$$\begin{cases} -\Delta u = f \ \text{ on } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$, $d = 2$ or 3, whose boundary $\partial\Omega$ is Lipschitz-continuous, $f \in L^2(\Omega)$. The generalization of the method to convection-diffusion equations seems possible since the convection term is easily handled in the finite volume scheme. This is the object of on-going work.

The Neumann-Neumann method

For the sake of simplicity, we shall consider here a non-overlapping domain decomposition which is defined by two subdomains Ω_1 and Ω_2 of Ω, which are bounded open subsets of $\mathbb{R}^d$ with Lipschitz-continuous boundaries such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, and the interface $\gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$ has a non zero $(d-1)$-dimensional measure.
For $i = 1, 2$, we denote by $\Gamma_i = \partial\Omega \bigcap \partial\Omega_i$, and $f_i = f_{|\Omega_i}$. We consider for $i = 1, 2$ the Hilbert spaces $V_i = \{\varphi_i \in H^1(\Omega_i), \ \exists\, \varphi \in H^1_0(\Omega), \ \varphi_i = \varphi_{|\Omega_i}\}$ equipped with the L^2 norm of the gradient. Let $H^{\frac{1}{2}}_{00}(\gamma)$ be the space of traces of elements of $H^1_0(\Omega)$ (or V_i) on γ. This space may be endowed with the norms of the harmonic lift in V_i on Ω_i , $\|\cdot\|_{i,H^{\frac{1}{2}}_{00}}$, for $i = 1$ or $i = 2$. It is well known that these norms are equivalent (see [QV99]). Hence there exist α and $\beta \in \mathbb{R}^*_+$ which only depend on Ω, Ω_1 and Ω_2, such that

$$\alpha \|\lambda\|_{1,H^{\frac{1}{2}}_{00}} \leq \|\lambda\|_{2,H^{\frac{1}{2}}_{00}} \leq \beta \|\lambda\|_{1,H^{\frac{1}{2}}_{00}}. \tag{2}$$

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Definition 1
For any $\lambda \in H_{00}^{\frac{1}{2}}(\gamma)$, let $u_i^{(\lambda)}$, for $i = 1, 2$ be the unique weak solution to the following problem

$$\left\{ \begin{array}{c} -\Delta u_i = f_i \ \text{ on } \ \Omega_i, \\ u_i = 0 \text{ on } \Gamma_i, \\ u_i = \lambda \ \text{ on } \ \gamma, \end{array} \right. \tag{3}$$

and $\Phi(\lambda)$ be the jump between the normal fluxes of the solutions $u_1^{(\lambda)}$ and $u_2^{(\lambda)}$, namely

$$\Phi(\lambda) = \sum_{i=1}^{2} \nabla u_i^{(\lambda)} \cdot n_i, \tag{4}$$

where n_i denotes the unit normal vector to the interface γ outward to Ω_i. We then define $v_1^{(\lambda)}$ as the unique weak solution to the following problem

$$\left\{ \begin{array}{c} -\Delta v_1 = 0 \ \text{ on } \ \Omega_1, \\ v_1 = 0 \ \text{ on } \ \Gamma_1, \\ \nabla v_1 \cdot n_1 = \Phi(\lambda) \ \text{ on } \ \gamma, \end{array} \right. \tag{5}$$

and $S_1(\lambda)$ as the trace of $v_1^{(\lambda)}$ on γ. Finally, for $\rho > 0$, let $T_{1,\rho}$ be defined from $H_{00}^{\frac{1}{2}}(\gamma)$ to $H_{00}^{\frac{1}{2}}(\gamma)$ by $T_{1,\rho}(\lambda) = \lambda - \rho S_1(\lambda)$. Let us now present the Neumann-Neumann type domain decomposition method. Let $\lambda^{(0)}$ be a given function of $H_{00}^{\frac{1}{2}}(\gamma)$. Assume that $\lambda^{(i)} \in H_{00}^{\frac{1}{2}}$ is known for $i \leq n$. Then iteration n consists in :

$$\left\{ \begin{array}{ll} (1)_n & \text{Let } u_i^{(\lambda^{(n)})} \in H_0^1(\Omega_i) \text{ for } i = 1, 2 \text{ be the solution of (3) with } \lambda = \\ & \lambda^{(n)}, \text{and let } \Phi(\lambda^{(n)}) \in (H_{00}^{\frac{1}{2}}(\gamma))' \text{ be defined by Formula (4).} \\ (2)_n & \text{Let } v_1^{(\lambda^{(n)})} \text{ be the solution to (5) with } \lambda = \lambda^{(n)} \text{ and let} \\ & S_1(\lambda^{(n)}) \in H_{00}^{\frac{1}{2}}(\gamma) \text{ be the trace of } v_1^{(\lambda^{(n)})} \text{ on } \gamma. \\ (3)_n & \text{Set } \lambda^{(n+1)} = T_{1,\rho}(\lambda^{(n)}). \end{array} \right. \tag{6}$$

The following convergence result holds (the proof of which can be performed by a fixed point theorem applied to the operator $T_{1,\rho}$) :

Theorem 1 *There exists $\rho_0 > 0$ such that if $0 < \rho < \rho_0$ then the sequence $(\lambda^{(n)})_{n\in\mathbb{N}}$ converges in $H_{00}^{\frac{1}{2}}(\gamma)$ towards $\lambda \in H_{00}^{\frac{1}{2}}(\gamma)$ as n tends to infinity, where λ is the trace of the unique weak solution u to Problem (1) on the interface γ.*
Furthermore if $u^{(n)}$ denotes the element of $H_0^1(\Omega)$ such that $u^{(n)}_{|\Omega_i} = u_i^{(\lambda^{(n)})}$, for $i = 1, 2$, the sequence $(u^{(n)})_{n\in\mathbb{N}}$ converges to u in $H_0^1(\Omega)$ as n tends to infinity.

The cell centered finite volume scheme

We now assume that Ω_1 and Ω_2 are polygonal bounded open subsets of $\mathbb{R}^d$, $d = 2, 3$ and the interface $\gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$ is polygonal. The basic principle of the finite volume method is to write the balance equation associated with (1) over each discretization cell (or "control volume") of the mesh, and use the Stokes formula to obtain:

$\int_{\partial K} -\nabla u.\mathbf{n}(s)d\gamma(s) = \int_K f(x)dx$ for any cell K ($\mathbf{n}$ denotes the outward normal unit vector to ∂K).

The finite volume method is known to be well adapted to the discretization of partial differential equations under conservative form. It yields a good approximation of the diffusion fluxes on the cell boundaries and it is quite easy to write and implement, thanks to the balanced form of the equations which is used. Moreover it is well adapted for convection-diffusion equations since the discrete solution satisfies the maximum principle with no condition on the mesh size, see [GHV00]. Since we use here a cell centered scheme, we want to approximate the fluxes $-\nabla u.\mathbf{n}$ on each edge (or face in 3D) of the mesh using the discrete unknowns $(u_K)_{K\in\mathcal{T}}$ associated to the cells. Let $\mathcal{T}$ be a finite volume admissible mesh of Ω in the sense of [EGH00], that is roughly speaking (see [EGH00] for a precise definition), a set of non intersecting convex polygonals $\{K \in \mathcal{T}\}$ which is such that there exists an associate family of points $\{x_K, K \in \mathcal{T}\}$ such that for any two neighbours K and L the edge (or face) between K and L is orthogonal to the line segment $x_K x_L$. This condition is needed in order to define a consistent approximation of the normal flux $-\nabla u.\mathbf{n}$ through any edge. Meshes satisfying this condition include rectangular and triangular meshes satisfying the Delaunay condition, Voronoï meshes, and mixed meshes with triangular and rectangular cells of this type (see [EGH00] or [GHV00]).

Let $\mathcal{T}_i$ be an admissible mesh of Ω_i, for $i = 1, 2$, such that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$. We denote by $\mathcal{E}_i$ the edges of control volumes of $\mathcal{T}_i$, for $i = 1, 2$. $\Gamma_i = \partial\Omega \bigcap \partial\Omega_i$. We denote by $\mathcal{E}_{i,D}$ the Dirichlet edges of $\mathcal{E}_i$ which are included in $\Gamma_i = \partial\Omega \cap \partial\Omega_i$, for $i = 1, 2$. Since $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is an admissible mesh of Ω, one has $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. We denote by $\mathcal{E}_\gamma$ the edges of $\mathcal{E}$ which are included in γ and by $\mathcal{E}_{i,int}$ the edges of control volumes of $\mathcal{T}_i$ which are not included in $\partial\Omega_i$, for $i = 1, 2$.
For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. For any $\sigma \in \mathcal{E}$, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{int}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$. For any $\sigma \in \mathcal{E}$, let $\tau_\sigma = meas(\sigma)/d_\sigma$ if $d_\sigma \neq 0$ and $\tau_\sigma = 0$ if $d_\sigma = 0$.

Let $X(\mathcal{T})$ be the set of functions from Ω to $\mathbb{R}$ which are a.e. constant over each control volume of the mesh, and $Y(\gamma)$ the set of functions from γ to $\mathbb{R}$ which are a.e. constant over each edge of the interface γ. Let us denote by λ_σ the value on the edge σ of $\mathcal{E}_\gamma$ of an element λ of $Y(\gamma)$. For a given set of values $(u_K)_{K\in\mathcal{T}}$, we shall denote by $u_\mathcal{T}$ the corresponding piecewise constant function of $X(\mathcal{T})$ defined a. e. by $u_\mathcal{T}(x) = u_K$ if $x \in K$. For all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, we introduce some auxiliary unknowns, namely the numerical fluxes, $F_{K,\sigma}(u_\mathcal{T})$ and for all $\sigma \in \mathcal{E}$ some approximation of u on edge σ, denoted by u_σ. The cell centered finite volume scheme for the approximation of Problem (1) writes:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u_\mathcal{T}) = meas(K)f_K, \ \forall K \in \mathcal{T}, \tag{7}$$

where the discrete fluxes $F_{K,\sigma}$ are defined with respect to the discrete unknowns as follows:

$$F_{K,\sigma}(u_\mathcal{T}) = -F_{L,\sigma}(u_\mathcal{T}), \forall\sigma \in \mathcal{E}_{int}, \text{ if } \sigma = K|L, \tag{8}$$

$$F_{K,\sigma}(u_{calT})\ d_{K,\sigma} = -meas(\sigma)(u_\sigma - u_K), \forall\sigma \in \mathcal{E}_K, \forall K \in \mathcal{T}, \tag{9}$$

$$u_\sigma = 0, \forall\sigma \in \mathcal{E}_{ext}, \tag{10}$$

and or all $K \in \mathcal{T}$, $f_K = \frac{1}{meas(K)} \int_K f(x)dx$.

For $i = 1, 2$, let $Z(\mathcal{T}_i, \gamma)$ be the space of functions defined a.e. on $\Omega_i \cup \gamma$ which are constant on the control volumes of $\mathcal{T}_i$ and on the edges of $\mathcal{E}_\gamma$. We then define on $Z(\mathcal{T}_i, \gamma)$ the following bilinear form

$$\begin{array}{rl} (z_{i,\mathcal{T}_i}^{(\lambda)}, z_{i,\mathcal{T}_i}^{(\mu)})_{Z(\mathcal{T}_i,\gamma)} = & \sum_{\sigma \in \mathcal{E}_{i,int},\, \sigma = K|L} \tau_\sigma (w_{i,K}^{(\lambda)} - w_{i,L}^{(\lambda)})(w_{i,K}^{(\mu)} - w_{i,L}^{(\mu)}) \\ & + \sum_{\sigma \in \mathcal{E}_{i,D}} \tau_\sigma w_{i,K_\sigma}^{(\lambda)}\, w_{i,K_\sigma}^{(\mu)} + \\ & \sum_{\sigma \in \mathcal{E}_\gamma} \tau_{i,\sigma} (\lambda_\sigma - w_{i,K_{i,\sigma}}^{(\lambda)})(\mu_\sigma - w_{i,K_{i,\sigma}}^{(\mu)}), \end{array} \tag{11}$$

where $z_{i,\mathcal{T}_i}^{(\lambda)}$ (respectively $z_{i,\mathcal{T}_i}^{(\mu)}$) is the element of $Z(\mathcal{T}_i, \gamma)$ defined a.e. on each $K \in \mathcal{T}_i$ and each $\sigma \in \mathcal{E}_\gamma$ by $z_{i,\mathcal{T}_i}^{(\lambda)}(x) = w_{i,K}^{(\lambda)}$ if $x \in K$, $z_{i,\mathcal{T}_i}^{(\lambda)}(x) = \lambda_\sigma$(resp. μ_σ) if $x \in \sigma$.

The space $Y(\gamma)$ is then endowed with the following inner products:

$$(\lambda, \mu)_{i,Y(\gamma)} = (z_{i,\mathcal{T}_i}^{(\lambda)}, z_{i,\mathcal{T}_i}^{(\mu)})_{Z(\mathcal{T}_i,\gamma)},\ \forall \lambda \in Y(\gamma),\ \forall \mu \in Y(\gamma),\ \text{for } i = 1, 2, \tag{12}$$

where $z_{i,\mathcal{T}_i}^{(\lambda)}$ (respectively $z_{i,\mathcal{T}_i}^{(\mu)}$) is the element of $Z(\mathcal{T}_i, \gamma)$ such that $z_{i,\mathcal{T}_i}^{(\lambda)}(x) = w_{i,K}^{(\lambda)}$ if $x \in K$, $z_{i,\mathcal{T}_i}^{(\lambda)}(x) = \lambda_\sigma$(resp. μ_σ) if $x \in \sigma$, and $(w_{i,K}^{(\lambda)})_{K \in \mathcal{T}_i}$, $(w_{i,\sigma}^{(\lambda)})_{\sigma \in \mathcal{E}_\gamma}$ is the unique solution of the following problem :

$$w_{i,\sigma} = 0\ \forall \sigma \in \mathcal{E}_{i,D}. \tag{13}$$

$$w_{i,\sigma} = \lambda_\sigma\ (\text{resp. } w_{i,\sigma} = \mu_\sigma),\ \forall \sigma \in \mathcal{E}_\gamma. \tag{14}$$

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(w_{\mathcal{T}_i}) = 0,\ \forall K \in \mathcal{T}, \tag{15}$$

where the numerical fluxes $F_{K,\sigma}(w_{\mathcal{T}_i})$ are defined as in (8)-(9).

The Euclidean norms $\|\cdot\|_{1,Y(\gamma)}$ and $\|\cdot\|_{2,Y(\gamma)}$ are equivalent on the finite dimensional space $Y(\gamma)$. Hence there exist $\alpha_\mathcal{T}$ and $\beta_\mathcal{T} \in \mathbb{R}_+^*$ depending on the open bounded subsets Ω, Ω_i and on the meshes $\mathcal{T}_i$, for $i = 1, 2$ such that $\alpha_\mathcal{T} \|\lambda\|_{1,Y(\gamma)} \leq \|\lambda\|_{2,Y(\gamma)} \leq \beta_\mathcal{T} \|\lambda\|_{1,Y(\gamma)}$ for all λ in $Y(\gamma)$.

Let us now define the discrete counterparts of the continuous operators of Definition 1.

Definition 2

For any $\lambda \in Y(\gamma)$, let us define $u_{i,\mathcal{T}_i}^{(\lambda)} \in Z(\mathcal{T}_i, \gamma)$ for $i = 1, 2$ such that $u_{i,K}^{(\lambda)})_{K \in \mathcal{T}_i}$ $(u_{i,\sigma}^{(\lambda)})_{\sigma \in \mathcal{E}_\gamma}$ is the unique solution to the following problem:

$$u_{i,\sigma} = 0,\ \forall \sigma \in \mathcal{E}_{i,D}. \tag{16}$$

$$u_{i,\sigma} = \lambda_\sigma,\ \forall \sigma \in \mathcal{E}_\gamma. \tag{17}$$

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(u_{\mathcal{T}_i}) = meas(K) f_K,\ \forall K \in \mathcal{T}_i, \tag{18}$$

Let us denote by $\Phi_\mathcal{T}(\lambda) \in Y(\gamma)$ the jump between the numerical normal fluxes of the discrete solutions $u_{\mathcal{T}_1}^{(\lambda)}$ and $u_{\mathcal{T}_2}^{(\lambda)}$, that is

$$\Phi_\mathcal{T}(\lambda)_\sigma = \sum_{i=1}^{2} \tau_{i,\sigma} (\lambda_\sigma - u_{i,K_{i,\sigma}}^{(\lambda)}),\ \forall \sigma \in \mathcal{E}_\gamma. \tag{19}$$

Let $v_{\mathcal{T}_1}^{(\lambda)} \in Z(\mathcal{T}_i,\gamma)$ such that $(v_{1,K}^{(\lambda)})_{K\in\mathcal{T}_1}$, $(v_{1,\sigma}^{(\lambda)})_{\sigma\in\mathcal{E}_\gamma}$ is the unique solution of the following problem

$$v_{1,\sigma} = 0,\ \forall\sigma \in \mathcal{E}_{1,D}. \tag{20}$$

$$F_{K,\sigma}(v_{\mathcal{T}_1}) = -\Phi_{\mathcal{T}}(\lambda)_\sigma,\ \forall\sigma\in\mathcal{E}_\gamma. \tag{21}$$

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(v_{\mathcal{T}_1}) = 0,\ \forall K\in\mathcal{T}_1. \tag{22}$$

We define the discrete trace of $v_{\mathcal{T}_1}^{(\lambda)}$ a.e. on the interface γ by

$$v_1^{(\lambda)}|_\gamma(x) = v_{1,\sigma}^{(\lambda)},\ \forall x\in\sigma,\ \forall\sigma\in\mathcal{E}_\gamma, \tag{23}$$

where $v_{1,\sigma}^{(\lambda)}$ is defined by the equations

$$\tau_{1,\sigma}(v_{1,\sigma}^{(\lambda)} - v_{1,K_{1,\sigma}}^{(\lambda)}) = -F_{K,\sigma}(v_{\mathcal{T}_1}^{(\lambda)}) = \Phi_{\mathcal{T}}(\lambda)_\sigma,\ \forall\sigma\in\mathcal{E}_\gamma. \tag{24}$$

We define the function $S_{\mathcal{T}_1}$ from $Y(\gamma)$ to $Y(\gamma)$ by

$$S_{\mathcal{T}_1}(\lambda) = v_{\mathcal{T}_1|\sigma}^{(\lambda)}. \tag{25}$$

Finally, for $\rho>0$, let $T_{\mathcal{T}_1,\rho}$ be defined from $Y(\gamma)$ to $Y(\gamma)$ by

$$T_{\mathcal{T}_1,\rho}(\lambda) = \lambda - \rho S_{\mathcal{T}_1}(\lambda). \tag{26}$$

Let us now describe the discrete counterpart of the domain decomposition iteration (6). Let $\lambda^{(0)}$ be a given function of $Y(\gamma)$. Assume that $\lambda^{(i)}\in Y(\gamma)$ is known for $i\le n$.

$$\left\{\begin{array}{ll}
(1)_n & \text{Let } u_{\mathcal{T}_i}^{(\lambda^{(n)})} \in Z(\mathcal{T}_i,\gamma) \text{ for } i=1,2 \text{ solution of (16)-(18) with } \lambda_\sigma = \\
& \lambda_\sigma^{(n)}, \sigma\in\mathcal{E} \text{ and let } \Phi_{\mathcal{T}}(\lambda^{(n)})\in Y(\gamma) \text{ be defined by Formula (19).} \\
(2)_n & \text{Let } v_1^{(\lambda^{(n)})} \text{ be the solution to (20)-22) with } \lambda=\lambda^{(n)} \text{ and let} \\
& S_1(\lambda^{(n)}) \in H_{00}^{\frac{1}{2}}(\gamma) \text{ be the trace of } v_1^{(\lambda^{(n)})} \text{ on } \gamma. \\
(3)_n & \text{Set } \lambda^{(n+1)} = T_{\mathcal{T}_1,\rho}(\lambda^{(n)}).
\end{array}\right. \tag{27}$$

Theorem 2 *Let $\mathcal{T}$ be any admissible mesh of Ω, and $\mathcal{T}_i, i=1,2$ be an admissible mesh of $\Omega_i, i=1,2$ such that $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_1$. Let $(\lambda^{(n)})_{n\in\mathbb{N}}$ and $(u_{\mathcal{T}_i}^{(\lambda^{(n)})})_{n\in\mathbb{N}}$ be the sequences defined by (27). Let $u_{\mathcal{T}}\in X(\mathcal{T})$ be such that $(u_K)_{K\in\mathcal{T}}$,$(u_\sigma)_{\sigma\in\mathcal{E}}$ is the unique solution to Problem (8)-(7). Let $u_{\mathcal{T}}|_\gamma\in Y(\gamma)$ be defined a.e on γ by $u_{\mathcal{T}}|_\gamma(x)=u_\sigma, \forall\sigma\in\mathcal{E}_\gamma$. Let $u_{\mathcal{T}}^{(n)}\in X(\mathcal{T})$ be defined a.e. on Ω by $u_{\mathcal{T}}^{(n)}(x) = u_{i,\mathcal{T}_i}^{(\lambda^{(n)})}(x)$ if $x\in\Omega_i$ for $i=1,2$.*

There exists $\rho_0^{(\mathcal{T})}>0$ such that if $0<\rho<\rho_0^{(\mathcal{T})}$, the sequence $(\lambda^{(n)})_{n\in\mathbb{N}}$ converges in $Y(\gamma)$ with either of the norms defined by (12), towards $u_{\mathcal{T}|\gamma}$ as $n\longrightarrow\infty$ and the sequence $(u_{\mathcal{T}}^{(n)})_{n\in\mathbb{N}}$ converges to $u_{\mathcal{T}}$ in $L^2(\Omega)$ as $n\longrightarrow\infty$.

We do not give here the details of the proof of this theorem for reasons of space limitation; we only mention that it is a discrete adaptation of the proof of Theorem 1 and refer to a forthcoming paper for the details.

Let us mention that in a finite element discretisation, by using the inverse inequality, one can prove that the convergence rate does not depend of the mesh size. This result is known as the *finite element uniform extension theorem* (see, for instance [QV99] pp 105-106 and the references therein). Such a result in the finite volume framework is not yet known. Nevertheless, numerical results show that the convergence rate is still independent of the mesh size. It is the goal of on-going work to prove this fact.

Numerical results

Let $\Omega = (-1,1) \times (0,1) = \Omega_1 \bigcup \Omega_2$, $\Omega_1 = (-1,0.5) \times (0,1)$. We consider a 20×20 rectangular regular mesh. We choose the right hand side of Problem (1) so that the exact solution is: $u(x,y) = sin(\pi\ x) \sin(\pi\ y)$. Let:

$$\mathcal{K}_\rho(n) = \frac{\|\lambda^{n+1} - \lambda^n\|}{\|\lambda^n - \lambda^{n-1}\|} \forall n = 1, nit, \text{ and } \mathcal{K}(\rho) = \left(\prod_{n=1}^{nit} \mathcal{K}_\rho(n) \right)^{\frac{1}{nit}}$$

where nit is the total number of iterations performed. The optimal parameter ρ minimizes the Lipschitz constant $\kappa(\rho)$ of the discrete operator $T_{\mathcal{T}_1,\rho}$ defined by (26). In the proof of the Theorem 2, we show that this Lipschitz constant is a polynomial of degree 2 in the variable ρ. In order to automatically compute the optimal parameter, we use the golden section method and approximate $\kappa(\rho)$ by $\mathcal{K}_\rho(n)$ at iteration n.

We present a comparison between this "relaxation" procedure and the method consisting in solving the trace equation by a conjuguate gradient method presented in [LT94] which we shall call "the Schur complement method" int the sequel. Since our relaxation method has a computational cost by iteration greater than the Schur complement method (because of the number of unknowns), we present the error versus the CPU time in seconds, rather than the number of iterations.

The plotted error is defined by the discrete L^2 norm of the difference between λ^n and λ_{exact} . First, one can observe from Figure 1 that the relaxation method behaves as welle as the Schur complement method. Moreover, we can remark from Figure 2 that as in the finite element discretization, the convergence rate does not depend on the mesh size.

We now consider a 80×80 rectangular regular mesh and in Table 1 we present results on a decomposition featuring more that 2 subdomains.

It is quite clear from Table 1 that even for this sequential experiment, the CPU time decreases very fast with respect to the mesh size, thanks to the fact that the local systems to be solved decrease in size.

We finally present some results of a parallel implementation which was set up on a Sun Ultra using up to 32 processors, using the PVM communication protocol. One processor is assigned to one subdomain.

We give in Table 2 the parallel efficiency, i.e. the ratio of the CPU time for n processors over the CPU time using 1 processor, using in both cases the n subdomains decomposition method. One may observe a decrease of the efficiency due to fact that the communication cost between processors increases faster than the CPU time decreases with the number of subdomains.

Conclusion

We have shown that the Neumann-Neumann method works well as a relaxation method in the finite volume setting. It would then also be interesting to apply it as a preconditioner in a conjugate gradient iteration. There also remains to prove the independence of the convergence rate of the method with respect to the mesh, and to adapt the proof of convergence for the case of a convection-diffusion equation.

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# subdomains	2	4	8	16	32
mesh by subdomains	80× 40	40× 40	40× 20	20× 20	10× 20
cpu (s)	359.2	221.4	89.4	29.8	22.4

Table 1: CPU time for different numbers of subdomains

# of processors	2	4	8	16	32
mesh by sub-domain	80×40	40×40	40×20	20×20	10×20
CPU	185.64	60.48	16.77	3.18	1.31
Speed-up	1.9348	3.66	5.3334	9.3572	17.102
Efficiency, E_p (%)	96.74	91.50	66.66	58.48	53.44

Table 2: CPU time and efficiency for several decompositions

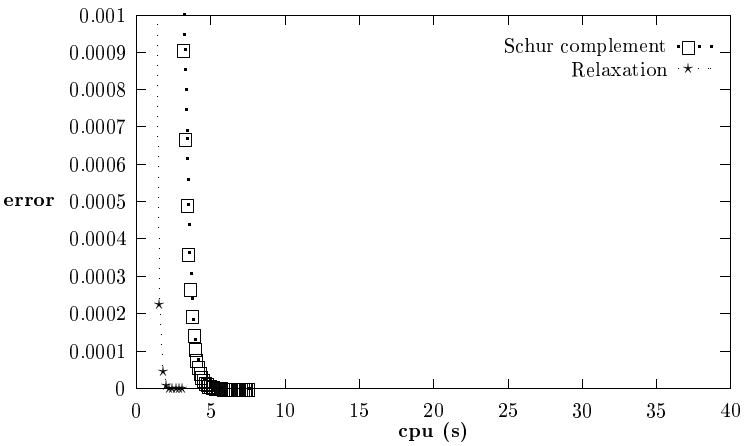

Figure 1: Comparison of the Neumann-Neumann method and the Schur complement method

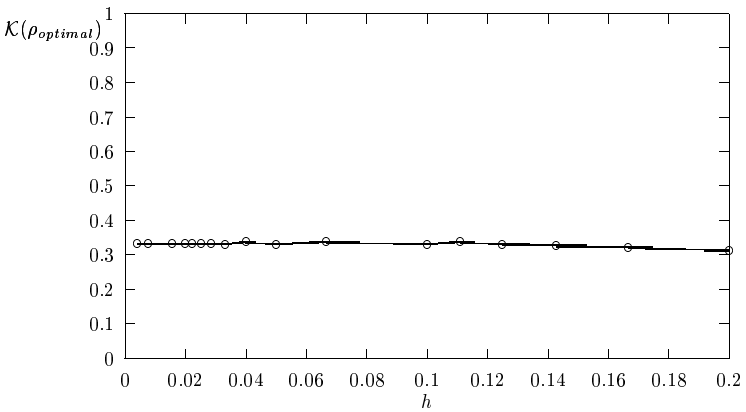

Figure 2: Convergence rate $\mathcal{K}(\rho_{optimal})$ as a function of the discretization step h