DDM Preprint
Editors: editor1, editor2, editor3, editor4

1 Non matching finite volume grids and the non overlapping Schwarz algorithm.

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Introduction

We consider the following diffusion-convection problem :

$$\left\{ \begin{array}{l} -\Delta u + \mathrm{div}(\mathbf{v}u) + bu = f \ \text{ on } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega, \end{array} \right. \tag{1}$$

where Ω is an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2, 3$, $\mathbf{v} \in C^1(\Omega, \mathbb{R}^d)$, $b \in L^\infty(\Omega)$, and $f \in L^2(,\Omega)$. The domain Ω is discretized with a grid which may feature some non-matching cells, such as described in Figure 1. Our purpose is first to study a finite volume scheme for Problem (1) on such a mesh and prove an error estimate under adequate assumptions on the unique weak solution to Problem (1). We only study here the case of homogeneous Dirichlet boundary conditions, but Neumann and Robin conditions may also be considered with the technical tools developed in [GHV00].

We then consider the decomposition of Ω in two non overlapping domains Ω_1 and Ω_2 and use a discrete version of the Lions adaptation [Lio90] of the Schwarz algorithm in order to solve Problem (1): for a given $\alpha \in \mathbb{R}_+$, choose $u^0 \in H^1_0(\Omega)$, and solve for each $n \geq 0$ and for $i = 1, 2$:

$$\left\{ \begin{array}{c} -\Delta u_i^{(n+1)} + \mathrm{div}(\mathbf{v}u_i^{(n+1)}) + bu_i^{(n+1)} = f_i \ \text{ on } \ \Omega_i, \\ u_i^{(n+1)} = 0 \ \text{ on } \ \Gamma_i, \\ -\dfrac{\partial u_i}{\partial n_i}^{(n+1)} + \alpha u_i^{(n+1)} = \dfrac{\partial u_j}{\partial n_j}^{(n)} + \alpha u_j^{(n)} \ \text{ on } \ \gamma, j = 1, 2, i \neq j, \end{array} \right. \tag{2}$$

where $\Gamma_i = \partial\Omega_i \cap \partial\Omega$, n_i is the normal unit vector to the interface $\gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$ outward to Ω_i and $f_i = f_{|\Omega_i}$.
We present a finite volume version of this algorithm to which the proof of convergence of P.L. Lions may be adapted.

The finite volume scheme

The finite volume method is known to be well adapted to the discretization of partial differential equations under conservative form. It yields a good approximation of the diffusive fluxes on the cell interfaces and it is easy to implement. Our aim here is to study how the method behaves in the presence of non-matching cells such as presented in Figure 1.

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Let us consider a family $\mathcal{T}$ of grid cells or "control volumes" K, which are open polygonal convex subsets of Ω such that the closure of the union of all the control volumes is $\overline{\Omega}$. In [EGH00], it is assumed that there exists a family $(x_K)_{K\in\mathcal{T}}$ (see Figure 1) such that for any two neighbouring cells K and L with common interface $K|L$, the line segment $x_K x_L$ is orthogonal to $K|L$. Here we shall relax this assumption on a number of "atypical cells", the set of which is denoted by $\mathcal{T}_a$. In the sequel, we shall use the following notations:

• for any $K \in \mathcal{T}$, the set of the edges of K is denoted by $\mathcal{E}_K$. The set of the edges of the control volumes of $\mathcal{T}$ is denoted by $\mathcal{E}$, and the set of "interior" (resp. "exterior") edges by $\mathcal{E}_{\rm int} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\rm ext} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$).

• for any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $\mathrm{m}(K)$ is the area (or volume in 3D) of K and $\mathrm{m}(\sigma)$ the length (or area in 3D) of σ. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ.

• for any $\sigma \in \mathcal{E}$, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{\rm int}$ and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$.

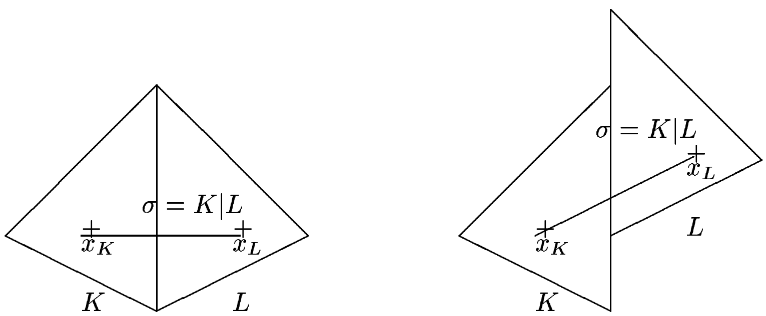

Figure 1: Example of "standard"(left) and ‘atypical" (right) control volumes in the 2D triangular case.

Let $X(\mathcal{T})$ be the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh. We define a "discrete‘ H_0^1" norm on $X(\mathcal{T})$ by:

$$\|u\|_{1,\mathcal{T}} = \left(\sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma) d_\sigma (\frac{D_\sigma u}{d_\sigma})^2\right)^{\frac{1}{2}}, \tag{3}$$

where, for any $\sigma \in \mathcal{T}$, $D_\sigma u = |u_K - u_L|$ if $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$, $D_\sigma u = |u_K|$ if $\sigma \in \mathcal{E}_{ext} \cap \mathcal{E}_K$, where u_K denotes the value taken by u on the control volume K.

Let $(u_K)_{K\in\mathcal{T}}$ be the discrete unknowns and let $(u_\sigma)_{\sigma\in\mathcal{E}}$ be a set of values which are expected to be approximations of u on edge σ, for all $\sigma \in \mathcal{E}$. The values u_σ are auxiliary since they may be eliminated from the resulting linear system.

The finite volume scheme is obtained by discretizing the balance equation associated to (1), which writes :

$$-\sum_{\sigma\in\mathcal{E}_K} \int_\sigma \nabla u \cdot \mathbf{n}_{\mathbf{K},\sigma} ds + \sum_{\sigma\in\mathcal{E}_K} \int_\sigma u\mathbf{v} \cdot \mathbf{n}_{\mathbf{K},\sigma} ds + \int_K b\, u dx = \int_K f dx$$

where $\mathbf{n}_{\mathbf{K},\sigma}$ denotes the unit normal vector to $\partial\Omega$ outward to Ω. Let us introduce a set of discrete unknowns $(u_k)_{K\in\mathcal{T}}$, and discrete fluxes $(F_{K,\sigma})_{K\in\mathcal{T}}$ which are the numerical approximations of $\int_\sigma -\nabla u \cdot \mathbf{n}_{\mathbf{K},\sigma} ds$ by a finite difference approximation. In order to discretize the convection term $\mathrm{div}(\mathbf{v}(x)u(x))$ in a stable way, let us define the upstream choice $u_{\sigma+}$ of u on an edge σ with respect to $\mathbf{v}$ in the following way. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, let $\mathbf{n}_{K,\sigma}$ denote the normal unit vector to σ outward to K and $\mathbf{v}_{K,\sigma} = \int_\sigma \mathbf{v} \cdot \mathbf{n}_{K,\sigma} ds$.

If $\mathbf{v}_{K,\sigma} \geq 0$ and $\sigma \in \mathcal{E}_K$ then $u_{\sigma+} = u_K$. If $\mathbf{v}_{K,\sigma} < 0$, $\sigma \in \mathcal{E}_{\mathrm{int}}$ and $\sigma = K|L$ then $u_{\sigma+} = u_L$. If $\mathbf{v}_{K,\sigma} < 0$ and $\sigma \in \mathcal{E}_{ext}$, then $u_{\sigma+} = u_\sigma$.

Let $f_K = \frac{1}{\mathrm{m}(K)} \int_K f dx$ and $b_K = \frac{1}{\mathrm{m}(K)} \int_K b dx$. Then with the notations defined above, a discretization by a cell centered finite volume method yields the following scheme:

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma \in \mathcal{E}_K} \mathbf{v}_{K,\sigma} u_{\sigma+} + b_K \ \mathrm{m}(K) u_K = \ \mathrm{m}(K) f_K, \ \forall K \in \mathcal{T}, \tag{4}$$

where:

$$F_{K,\sigma} \ d_{K,\sigma} = -\mathrm{m}(\sigma)(u_\sigma - u_K), \ \forall \sigma \in \mathcal{E}_K, \ \forall K \in \mathcal{T} \tag{5}$$

$$F_{K,\sigma} = -F_{L,\sigma}, \ \forall \sigma \in \mathcal{E}_{\mathrm{int}}, \quad \text{if } \sigma = K|L, \tag{6}$$

and

$$u_\sigma = 0, \forall \sigma \in \mathcal{E}_{ext}. \tag{7}$$

Note that the unknowns $(u_\sigma)_{\sigma \in \mathcal{E}}$ may be eliminated by using (6) and (5).

Error estimate

We now present error estimates in the discrete H_0^1 norm under some regularity assumptions on the solution to Problem (1). Some similar results are also in [ELV91] for rectangular meshes and some recent work of F. Nataf *et al* with a different computation of the diffusion fluxes on the atypical interfaces (see these proceedings). The analysis of the scheme is carried out under the following assumptions:

$$\left\{ \begin{array}{l} \Omega \text{ is a polygonal open bounded subset of } \mathbb{R}^d. \\ f \in L^2(\Omega). \\ \mathbf{v} \in C^1(\Omega, \mathbb{R}^d). \\ b \in L^\infty(\Omega). \\ \dfrac{1}{2}\mathrm{div}\mathbf{v}(x) + b(x) \geq 0, \ \text{a.e. } x \in \Omega. \end{array} \right. \tag{8}$$

Theorem 1 *Under Assumptions (8) , let $(u_K)_{K \in \mathcal{T}}$ be the solution to (6)-(4). Assume that the unique variational solution u of Problem (1) satisfies $u \in C^2(\overline{\Omega})$. Let $e_\mathcal{T} \in X(\mathcal{T})$ be defined by $e_\mathcal{T}(x) = e_K = u(x_K) - u_K$ a.e. $x \in K$, $K \in \mathcal{T}$. Then, there exists $C > 0$, only depending on u, $\mathbf{v}$, b, d and Ω, such that*

$$\|e_\mathcal{T}\|_{1,\mathcal{T}} \leq C \left(\mathrm{size}(\mathcal{T}) + \left(\sum_{K \in \mathcal{T}_a} \mathrm{m}(K) \right)^{\frac{1}{2}} \right), \tag{9}$$

where $\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K),\ K \in \mathcal{T}\}$. *Furthermore:*

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq C \left(\mathrm{size}(\mathcal{T}) + \left(\sum_{K \in \mathcal{T}_a} \mathrm{m}(K) \right)^{\frac{1}{2}} \right). \tag{10}$$

If we now assume that the unique variational solution u *to (1) only belongs to* $H^2(\Omega)$ *then (9) and (10) still hold with* C *only depending on* u, $\mathbf{v}$, b, Ω, d *and* $\zeta = \min_{K \in \mathcal{T}} \min_{\sigma \in \mathcal{E}_K} \dfrac{d_{K,\sigma}}{\mathrm{diam}(K)}$.

The proof of this theorem is an adaptation of the techniques used in the case of an admissible mesh [Her95] (see also [EGH00]) and will be presented in a forthcoming paper.

The main ingredients in the proof of convergence are the conservativity of the fluxes, i.e. $F_{K,\sigma} = -F_{L,\sigma}$ for two neighbouring cells (K, L), and the consistency of the approximation of the fluxes by finite differences. In the case of an atypical edge, the conservativity holds, but the consistency is lost on the diffusion flux because of the missing orthogonality condition. However, the approximation of the convective flux is still consistent.

If the number of "atypical" control volumes of $\mathcal{T}_a$ is of order $card(\mathcal{T})^{1/2}$ (this is the case for instance if the atypical cells neighbour a the interface between the subdomains of a given domain decomposition), then Inequality (9) (resp. (10)) yields an estimate of order $\frac{1}{2}$ for the discrete H_0^1 norm (resp. L^2 norm) of the error on the solution; numerical results (see section 5) seem to show that this estimate is not sharp. In fact for special examples of atypical cells, we have been able to obtain an order 1.

The discrete algorithm

We shall consider here a non-overlapping domain decomposition of Ω, under the following assumptions

$$\left\{ \begin{array}{l} \Omega_1 \text{ and } \Omega_2 \text{ are polygonal bounded connected open subsets of } \mathbb{R}^d. \\ \overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}. \\ \text{The interface } \gamma = \overline{\Omega_1} \cap \overline{\Omega_2} \text{ is polygonal and has a non zero} \\ \text{measure in } \mathbb{R}^{d-1}. \\ \Gamma_i = \partial\Omega \cap \partial\Omega_i, \text{ for } i = 1, 2. \\ \text{For } i = 1, 2, \text{ the mesh } \mathcal{T}_i \text{ is an admissible mesh of } \Omega_i \text{ which is the} \\ \text{restriction of the mesh } \mathcal{T} \text{ to } \Omega_i. \end{array} \right. \tag{11}$$

For $i = 1, 2$, the set of edges (resp. interior edges, resp. exterior edges) of the mesh $\mathcal{T}_i$ is denoted by $\mathcal{E}_i$ (resp. $\mathcal{E}_{i,ext}$, resp. $\mathcal{E}_{i,ext}$). We define $\mathcal{E}_{i,D} = \{\sigma \in \mathcal{E}_i,\ \sigma \subset \Gamma_i\}$ (Dirichlet edges) and $\mathcal{E}_\gamma = \{\sigma \in \mathcal{E}_i,\ \sigma \subset \gamma\}$ (interface edges), with $\mathcal{E}_{i,ext} = \mathcal{E}_{i,D} \cup \mathcal{E}_\gamma$.

The discrete version of the algorithm defined by equations (2) is then:

given $u_{\mathcal{T}}^{(0)} \in X(\mathcal{T})$ and assuming $u_{\mathcal{T}}^{(k)} \in X(\mathcal{T})$ for $1 \leq k \leq n$ to be known, let $u_{\mathcal{T}_i}^{(k)}$ be the element of $X(\mathcal{T}_i)$, defined for $i = 1, 2$ by: $u_{\mathcal{T}_i}^{(k)}(x) = u_{\mathcal{T}}^{(k)}|_{\Omega_i}(x)$, a.e. $x \in \Omega_i$ and $u_{i,K}^{(k)} = u_{\mathcal{T}_i}^{(k)}|_K$ for a.e. $x \in K$, for $K \in \mathcal{T}_i$. We compute $u_{\mathcal{T}}^{(n+1)} \in X(\mathcal{T})$ defined by

$u_{\mathcal{T}}^{(n+1)}(x) = u_{i,K}^{(n+1)}$, for a.e. $x \in K$, for any $K \in \mathcal{T}_i$, $i = 1, 2$, where $\left(u_{i,K}^{(n+1)}\right)_{K\in\mathcal{T}_i}$ is the unique solution to the following problem:

$$\sum_{\sigma\in\mathcal{E}_K} F_{i,K,\sigma}^{(n+1)} + \sum_{\sigma\in\mathcal{E}_K} \mathbf{v}_{K,\sigma} u_{i,\sigma+}^{(n+1)} + b_K \mathrm{m}(K) u_{i,K}^{(n+1)} = \mathrm{m}(K) f_K, \forall K \in \mathcal{T}_i, \tag{12}$$

with

$$F_{i,K,\sigma}^{(n+1)} d_{K,\sigma} = -\mathrm{m}(\sigma)(u_{i,\sigma}^{(n+1)} - u_{i,K}^{(n+1)}), \forall \sigma \in \mathcal{E}_K, \forall K \in \mathcal{T}_i, \tag{13}$$

$$F_{i,K,\sigma}^{(n+1)} = -F_{i,L,\sigma}^{(n+1)}, \forall \sigma \in \mathcal{E}_{i,int}, \text{ if } \sigma = K|L, \tag{14}$$

$$u_{i,\sigma}^{(n+1)} = 0, \forall \sigma \in \mathcal{E}_{i,D}, \tag{15}$$

and

$$-F_{i,K,\sigma}^{(n+1)} + \alpha\, u_{i,K}^{(n+1)} = F_{j,L,\sigma}^{(n)} + \alpha\, u_{j,L}^{(n)}, \forall \sigma \in \mathcal{E}_\gamma, \text{ for } j = 1, 2, j \neq i. \tag{16}$$

where for $K \in \mathcal{T}_i$ and $\sigma \in \mathcal{E}_K$, $F_{i,K,\sigma}^{(n)} = \mathrm{m}(\sigma)\dfrac{u_\sigma^{(n)} - u_K^{(n)}}{d_{K,\sigma}}$, and where $u_{i,\sigma+}^{(n+1)}$ is defined by the usual upstream scheme if σ is an interior edge, and by the following upstream choice if $\sigma \in \mathcal{E}_\gamma$ lies on the interface:

if $\sigma = K|L$ with $K \in \mathcal{T}_i$ and $L \in \mathcal{T}_j$, $j = 1, 2$, $j \neq i$, we choose $u_{i,\sigma+}^{(n+1)} = u_{i,K}^{(n+1)}$ if $\mathbf{v}_{K,\sigma} \geq 0$ and $u_{i,\sigma+}^{(n+1)} = u_{j,L}^{(n)}$ if $\mathbf{v}_{K,\sigma} < 0$.

Theorem 2 *Under Assumptions (8) and (11), the sequence* $\left(u_{\mathcal{T}}^{(n)}\right)_{n\in\mathbb{N}}$ *defined by the discrete algorithm (14)-(12) converges in* $L^2(\Omega)$ *towards* $u_{\mathcal{T}}$*, the unique solution to Problem (6)-(4)*

The proof of this theorem is an adaptation of the proof of Lions [Lio90] in a discrete finite volume setting.

Numerical results

Let us first study the convergence of the finite volume discretization for a set featuring some atypical cells. In Figure 2, the domain Ω is meshed with a coarse rectangular mesh on the left and a fine rectangular mesh on the right; the set $\mathcal{T}_a$ of atypical edges is such that $\sum_{K\in\mathcal{T}_a} m(K) \leq Ch$, thanks to assumptions on the mesh; hence for this case the result of Theorem 2 is an estimate of order $h^{1/2}$ where h is the maximum step size of the mesh. However, the numerical results show that when the mesh step decreases (with constant ratio between coarse and fine mesh), then the order of convergence behaves like 2 in the L^2 norm and 1 in the discrete H^1 norm; this shows that the error estimate is non optimal.

In Figure 3, we show the influence of the parameter α on the convergence of the Lions algorithm (12)-(15). The optimal parameter is roughly .85 and numerical results which are not shown here because of space limitations show that it is independent of the mesh size.

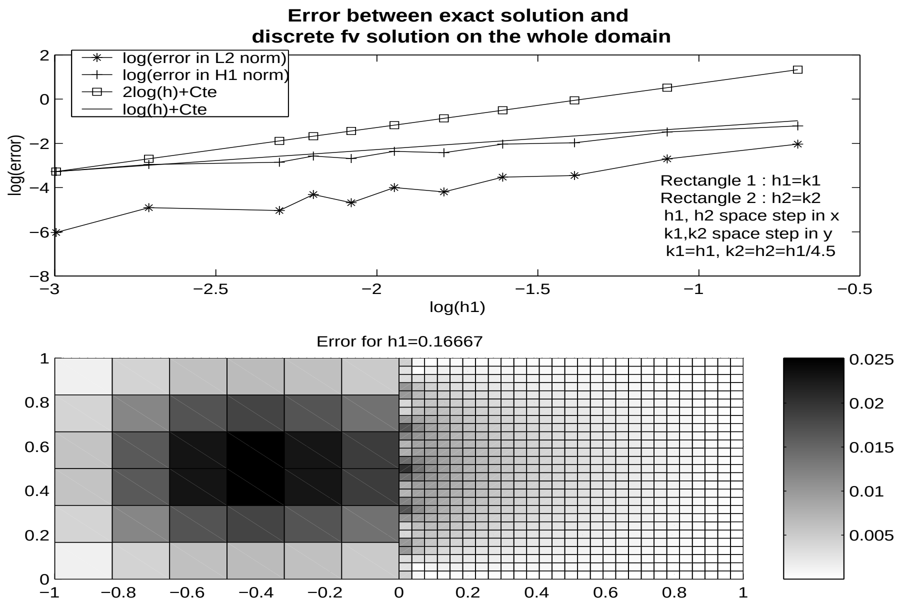

Figure 2: Convergence rate of the finite volume discretization

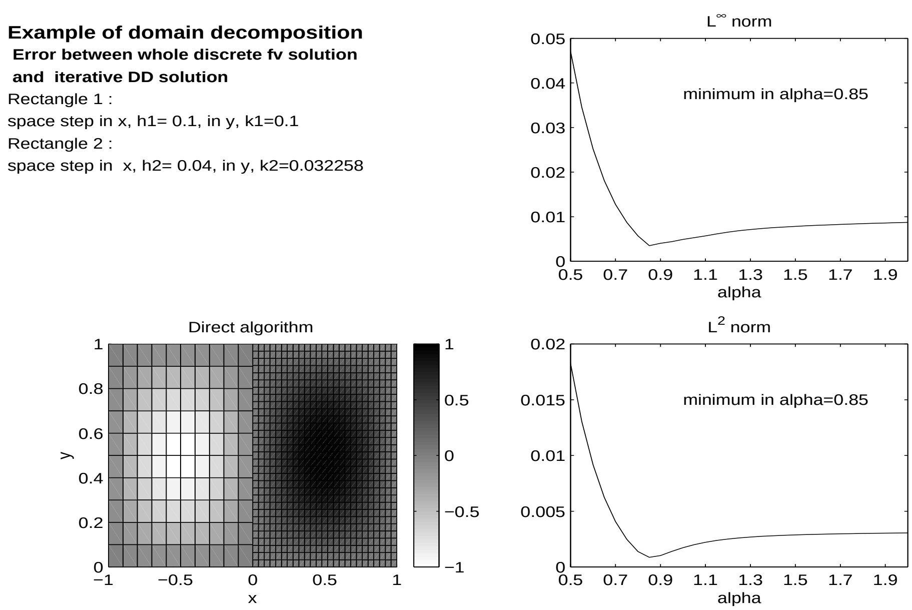

Figure 3: Optimal coefficient α

The solution by a direct solve of the finite volume system (4)-(7) is presented on the grid.

Finally, Figure 4 shows the difference between the solution of the direct solve of system refered to as "Direct Algorithm" and the solution by the domain decomposition algorithm for a relative maximum error of 10^{-3}. It is clear that the error is concentrated at the interface where the atypical meshes are located.

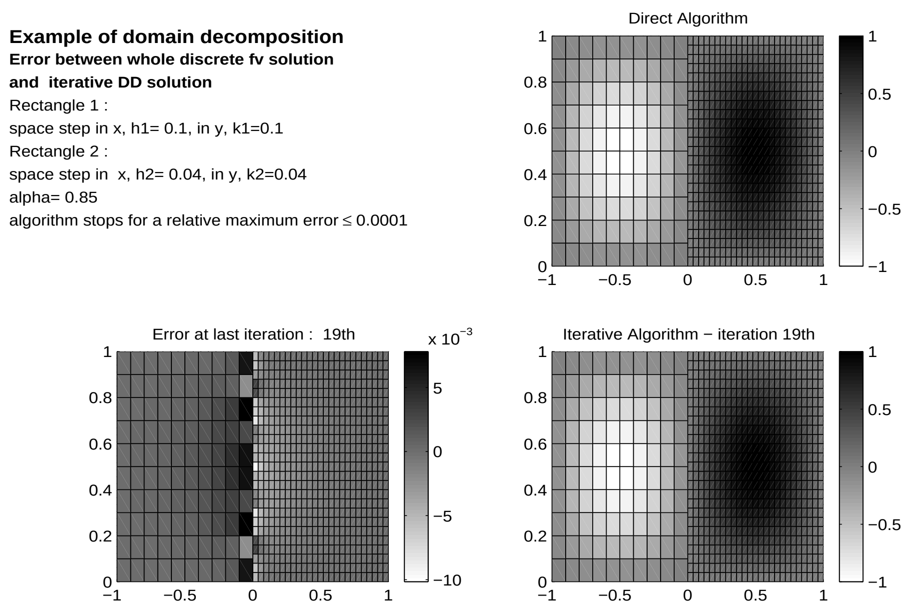

Figure 4: Error between domain decomposition and direct solve

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