

# Finite volume approximation of a class of variational inequalities

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*appeared in IMA Journal of Numerical Analysis (2001) 21, 553-585*

**Abstract :** We prove here convergence results for the approximate finite volume solutions of a diffusion problem with mixed Dirichlet, Neumann and Signorini boundary conditions which is formulated as a variational inequality. The convergence result is also shown to be easily adapted to the case of the obstacle problem which also writes as a variational inequality. An error estimate of order one with respect to the mesh size is given when the solutions to the continuous problems belong to  $H^2(\Omega)$ .

**AMS subject classification** 65N15,

**Keywords:** Finite volume methods, Unstructured meshes, Variational inequalities, Diffusion equations, Signorini conditions, Obstacle Problem.

## 1 Introduction

The first problem which we deal with here arises from the modelling of the so called "triple point" of an electrochemical reaction [26]. A diffusion equation is written for the concentration of oxygen vacancies on a square domain. The upper part of the domain is assumed to be insulated, yielding a homogeneous Neumann condition; on the left side of the domain, symmetry considerations also yield a homogeneous Neumann condition; on the bottom side the concentration is fixed and therefore a non-homogeneous Dirichlet boundary condition is considered. Finally, unilateral constraints of the Signorini type appear on the right side. This type of constraints also appears in friction mechanics, see e.g. [7].

We generalize the problem to the case where  $\Omega$  is a polygonal domain of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , so that the problem under consideration writes:

$$-\Delta u(x) = 0, \quad x \in \Omega, \quad (1)$$

$$u(x) = 0, \quad x \in \Gamma^1, \quad (2)$$

$$\nabla u(x) \cdot \mathbf{n} = 0, \quad x \in \Gamma^2, \quad (3)$$

with the following Signorini boundary condition:

$$\left. \begin{aligned} u(x) &\geq a, \\ \nabla u(x) \cdot \mathbf{n} &\geq b, \\ (u(x) - a)(\nabla u(x) \cdot \mathbf{n} - b) &= 0, \end{aligned} \right\} \quad x \in \Gamma^3, \quad (4)$$

under the following assumptions:

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### Assumption 1.1

1.  $\Omega$  is a bounded open polygonal subset of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ .
2.  $\partial\Omega$ , the boundary of  $\Omega$ , is composed of three non empty, non intersecting connected sets  $\Gamma^1$ ,  $\Gamma^2$  and  $\Gamma^3$ , such that  $\overline{\Gamma^1} \cup \overline{\Gamma^2} \cup \overline{\Gamma^3} = \partial\Omega$ ,
3.  $a \in \mathbb{R}_-$  and  $b \in \mathbb{R}$ ,

and where  $\mathbf{n}$  is the unit normal vector to  $\partial\Omega$  outward to the domain  $\Omega$ . Under some regularity assumptions, it may be shown [17] that Problem (1)-(4) is equivalent to the following variational problem:

$$\left\{ \begin{array}{l} u \in \mathcal{K} = \{v \in H^1(\Omega), v|_{\Gamma^1} = 0, v|_{\Gamma^3} \geq a \text{ a.e}\}, \text{ satisfying :} \\ \int_{\Omega} \nabla u(x) \cdot \nabla (v - u)(x) dx \geq \int_{\Gamma^3} b(\gamma(v) - \gamma(u))(s) ds, \quad \forall v \in \mathcal{K}, \end{array} \right. \quad (5)$$

with  $v|_{\Gamma^i} = \gamma(v)|_{\Gamma^i}$  ( $i = 1, 3$ ), where  $\gamma$  is the trace operator from  $H^1(\Omega)$  to  $L^2(\partial\Omega)$ . By Stampacchia's Theorem, Problem (5) has a unique solution. We refer to [2], [29] [32] and references therein for results concerning existence, uniqueness and regularity of the solution to this type of problem.

In order to demonstrate the generality of our methodology, we shall also consider the well-known “obstacle problem”; using the tools developped for the Signorini problem we shall show that an approximation by the finite volume scheme converges to the exact solution. Let us then consider the problem :

$$(-\Delta u(x) - f(x))(\psi(x) - u(x)) = 0, \quad x \in \Omega, \quad (6)$$

$$u(x) \leq \psi(x), \quad x \in \Omega, \quad (7)$$

$$u(x) = 0, \quad x \in \partial\Omega, \quad (8)$$

$$-\Delta u(x) - f(x) \geq 0, \quad x \in \partial\Omega, \quad (9)$$

under the following assumptions:

### Assumption 1.2

1.  $\Omega$  is a bounded open polygonal subset of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ .
2.  $f \in L^2(\Omega)$ ,
3.  $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ ,
4.  $\psi \geq 0$  a.e. in a neighborhood of  $\partial\Omega$ ,

A weak form of the obstacle problem (6)-(8) yields the following variational inequality:

$$\left\{ \begin{array}{l} u \in \tilde{\mathcal{K}} = \{v \in H_0^1(\Omega), v \leq \psi \text{ on } \Omega\}, \text{ satisfying :} \\ \int_{\Omega} \nabla u(x) \cdot \nabla (v - u)(x) dx \geq \int_{\Omega} f(x)(v(x) - u(x)) dx, \quad \forall v \in \tilde{\mathcal{K}}, \end{array} \right. \quad (10)$$

By Stampacchia's theorem, there exists a unique solution of (10) in  $H_0^1(\Omega)$ .

Variational inequalities arising from problems with unilateral constraints such as (5) and (10) are classically approximated by the finite element method, [25], [18] and [19]; error estimates have been established in Falk [13], Mosco-Strang [31], Glowinski-Lions-Trémolières [18], Ciarlet [5]. Glowinski [19], Brezzi-Hager-Raviart [3], [20], [4] and Falk-Mercier [14]. For the particular Signorini problem (i.e. Signorini boundary condition on the whole boundary), Brezzi-Hager-Raviart [3] use a piecewise linear finite element approximation and prove a  $0(h)$  convergence under optimal  $H^2$  regularity conditions.

In this work, we choose to discretize the problem by the finite volume method rather than the finite element method for two reasons:

- in electrochemical modelling, it is crucial to obtain a good approximation of the diffusion fluxes at the cell interfaces; it is however impossible to obtain a precise approximation of the flux at cell interfaces with the piecewise linear finite element; one way around this is to use mixed finite elements. We prefer to use the finite volume method because we find it to be computationally cheaper and also easier to implement (see e.g. [15]). Also note that the finite volume method is widely used in the related area of semi-conductor modelling.
- we treat the nonlinearity due to the Signorini condition by a monotonous method which was proved to be quite efficient [24], and whose convergence is proven in a forthcoming paper [23]; this method requires the approximate unknown and the approximate normal derivative to be defined at the same location on the Signorini boundary. This is indeed the case with the finite volume method. This monotonous method can also be used for the obstacle problem; however for this latter problem, both the finite volume of the finite element method may be used since the unilateral constraint is on the whole domain  $\Omega$  and not on the boundary (see [23] for details).

These reasons motivate our interest in proving the convergence of the finite volume method for the discretization of Problem (5) (and of Problem (10)). We shall use here a classical cell-centered finite volume discretization using a finite difference approximation of the fluxes at the interfaces. Recently, error estimates and convergence results for the cell-centered finite volume approximations on structured or unstructured meshes were obtained for linear convection diffusion equations for Dirichlet boundary conditions (see [27], [28] [21], [30], [22], [9]) and Neumann or Fourier boundary conditions [9], and a convergence result (without regularity assumption) for semilinear convection diffusion equations [10].

In the following section, we introduce the meshes and some discrete functional spaces, norms and tools for these spaces which we use in our convergence proofs. In particular, we prove in Lemma 2.2 a lower bound for the lower limit of the “discrete  $H^1$  norm” (see Definition 2.4) of piecewise constant functions converging weakly in  $L^2(\Omega)$  which is needed in the proof of convergence.

We then state in Section 3 the finite volume scheme for both problems. Because of the unilateral constraint (on the boundary in the case of the Signorini problem and on the domain  $\Omega$  itself in the case of the obstacle problem), the discrete system is nonlinear. We prove the existence and uniqueness of its solution.

Section 4 is devoted to the proof of convergence in  $L^2(\Omega)$  of the approximate finite volume solution to the exact one. This proof uses a compactness result which is obtained thanks to an estimate on the space translates of the approximate solutions. This compactness result is adapted from one which was obtained for linear or semi-linear convection diffusion equations [10], [9]. The two main original points which have to be introduced here for the proof of convergence are

1. the use of a lower bound for the lower limit of the “discrete  $H^1$  norm” obtained in Lemma 2.2 in order to obtain the term “ $-\int_{\Omega} \nabla u(x) \nabla u(x) dx$ ” of the variational inequalities (5) and (10) when passing to the limit on the numerical scheme as the mesh size tends to 0.
2. the use of the trace inequality (Lemma 2.4) in order to prove the convergence of the approximate solution on the Signorini boundary  $\Gamma_3$  towards the trace on  $\Gamma_3$  of the solution of (10) (this is not needed for the obstacle problem, in which case the compactness result also yields that the possible limit lies in  $H_0^1(\Omega)$ ).

Note that this convergence proof also yields as a by-product the existence of the solution to (1)-(4) and (6)-(8).

In Section 5, under (optimal)  $H^2$  regularity assumptions on the exact solution, we give an estimate of order 1 for the “discrete”  $H^1$  norm and  $L^2$  norm of the error on the solution.

Finally, numerical tests are shown in Section 6.

## 2 Discretization meshes and discrete functional spaces

In order to obtain a numerical approximation of the solution to (5), let us first define a discretization mesh over  $\Omega$  which is assumed (following [9]) to be admissible in the following sense:

**Definition 2.1 (Admissible meshes)** *Let  $\Omega$  be an open bounded polygonal domain of  $\mathbb{R}^d$ ,  $d = 2, 3$ . An admissible finite volume mesh of  $\Omega$ , denoted by  $\mathcal{T}$ , is given by a finite family of “control volumes”, which are non intersecting open polygonal convex subsets of  $\Omega$ , a finite family of non-intersecting subsets of  $\overline{\Omega}$  contained in hyperplanes of  $\mathbb{R}^d$ , denoted by  $\mathcal{E}$  (these are the “sides” of the control volumes), with strictly positive  $(d - 1)$ -dimensional measure, and a family of points of  $\Omega$  denoted by  $\mathcal{P}$  satisfying the following properties (in fact, we shall denote, somewhat incorrectly, by  $\mathcal{T}$  the family of control volumes):*

- (i) *The closure of the union of all the control volumes is  $\overline{\Omega}$ .*
- (ii) *For any  $K \in \mathcal{T}$ , there exists a subset  $\mathcal{E}_K$  of  $\mathcal{E}$  such that  $\partial K = \overline{K} \setminus K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$  and  $\cup_{K \in \mathcal{T}} \mathcal{E}_K = \mathcal{E}$ .*

- (iii) For any  $(K, L) \in \mathcal{T}^2$  with  $K \neq L$ , either the  $(d-1)$ -dimensional Lebesgue measure of  $\overline{K} \cap \overline{L}$  is 0 or  $\overline{K} \cap \overline{L} = \overline{\sigma}$  for some  $\sigma \in \mathcal{E}$ , which will then be denoted by  $K|L$ .
- (iv) The family  $\mathcal{P} = (x_K)_{K \in \mathcal{T}}$  is such that  $x_K \in K$  (for all  $K \in \mathcal{T}$ ) and, if  $K$  and  $L$  are two neighbouring control volumes, it is assumed that  $x_K \neq x_L$ , and the straight line  $\mathcal{D}_{K,L}$  going through  $x_K$  and  $x_L$  is assumed to be orthogonal to  $K|L$ .
- (v) For any  $\sigma \in \mathcal{E}$  such that  $\sigma \subset \partial\Omega$ , there exists  $i \in \{1, 2, 3\}$  such that  $\sigma \subset \Gamma^i$ .
- (vi) For any  $\sigma \in \mathcal{E}$  such that  $\sigma \subset \partial\Omega$ , let  $K$  be the control volume such that  $\sigma \in \mathcal{E}_K$  and  $\mathcal{D}_{K,\sigma}$  be the straight line going through  $x_K$  and orthogonal to  $\sigma$ ; then  $y_\sigma = \mathcal{D}_{K,\sigma} \cap \sigma$ .

In the sequel, the following notations are used. Let  $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\}$ . For any  $K \in \mathcal{T}$  and  $\sigma \in \mathcal{E}$ ,  $m(K)$  is the  $d$ -dimensional Lebesgue measure of  $K$  and  $m(\sigma)$  the  $(d-1)$ -dimensional measure of  $\sigma$ . The set of interior (resp. boundary) edges is denoted by  $\mathcal{E}_{\text{int}}$  (resp.  $\mathcal{E}_{\text{ext}}$ ), that is  $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$  (resp.  $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E}; \sigma \subset \partial\Omega\}$ ). The set of neighbors of  $K$  is denoted by  $\mathcal{N}(K)$ , that is  $\mathcal{N}(K) = \{L \in \mathcal{T}; \exists \sigma \in \mathcal{E}_K \sigma = \overline{K} \cap \overline{L}\}$ . If  $\sigma = K|L$ , we denote by  $d_\sigma$  or  $d_{K|L}$  the Euclidean distance between  $x_K$  and  $x_L$  (which is positive). If  $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$ , let  $d_\sigma$  denote the Euclidean distance between  $x_K$  and  $y_\sigma$ . For any  $\sigma \in \mathcal{E}$ ; the transmissivity through  $\sigma$  is defined by  $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$ . For any control volume  $K$  and any edge  $\sigma \in \mathcal{E}_K$ , we shall denote by  $d_{K,\sigma}$  the distance between  $x_K$  and  $\sigma$ .

### Remark 2.1

1. The condition  $x_K \neq x_L$  if  $\sigma = K|L$ , is in fact quite easy to satisfy: two neighbouring control volumes  $K, L$  which do not satisfy it just have to be collapsed into a new control volume  $M$  with  $x_M = x_K = x_L$ , and the edge  $K|L$  removed from the set of edges. The new mesh thus obtained is admissible.
2. The finite volume scheme which we introduce in the next section for the above mesh may be easily generalized to meshes which are more general than the above admissible meshes, by using some interpolation techniques (see e.g. the “nine-point” scheme in [9] or [11], [12] or the scheme in [6]); these schemes have proven to be efficient in practice; however we shall use here the admissible meshes because it is not yet known how to prove theoretical convergence (even for linear convection diffusion problems) on general meshes.

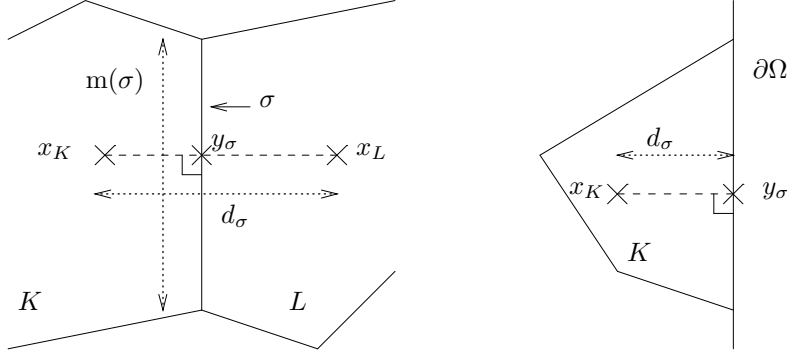


Figure 1: admissible meshes, case  $d = 2$

Let us now define a "discrete" functional space, a "discrete" trace operator and "discrete" norms.

**Definition 2.2** (*Discrete functional space*) Let  $\Omega$  be an open bounded polygonal domain of  $\mathbb{R}^d$ , and  $\mathcal{T}$  be an admissible mesh in the sense of Definition 2.1. Let  $X(\mathcal{T})$  be the set of functions defined from  $\Omega_{\mathcal{T}} = \cup_{K \in \mathcal{T}} K \cup \cup_{\sigma \in \Gamma^3} \sigma$  to  $\mathbb{R}$  which are constant over each control volume of the mesh, and which are constant over each edge of  $\mathcal{E}_{\text{ext}}$  which is included in  $\Gamma^3$ . Let  $\tilde{X}(\mathcal{T})$  be the set of functions defined from  $\tilde{\Omega}_{\mathcal{T}} = \cup_{K \in \mathcal{T}} K$  and which are constant over each control volume of the mesh. We shall denote by  $u_K$  the value taken by  $u \in X(\mathcal{T})$  or  $\tilde{X}(\mathcal{T})$  on the control volume  $K$ , and by  $u_{\sigma}$  the value taken by  $u \in \tilde{X}(\mathcal{T})$  on the edge  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ .

**Definition 2.3** (*Discrete trace operator*)

Let  $\Omega$  be an open bounded polygonal domain of  $\mathbb{R}^d$ , and  $\mathcal{T}$  be an admissible mesh in the sense of Definition 2.1. Define the operator  $\overline{\gamma}$  from  $X(\mathcal{T})$  to  $L^2(\partial\Omega)$ , in the following way: let  $u \in X(\mathcal{T})$ , let  $u_K$  be the value of  $u$  in the control volume  $K$  and  $u_{\sigma}$  be the value of  $u$  on the edge  $\sigma$ , for  $\sigma \subset \Gamma^3$ ; let us define

$$\begin{aligned} \overline{\gamma}(u) &= u_{\sigma} & \text{on } \sigma, & \text{if } \sigma \subset \Gamma^3, \\ \overline{\gamma}(u) &= u_K & \text{on } \sigma & \text{if } \sigma \subset \Gamma^2 \text{ and } \sigma \in \mathcal{E}_K, \\ \overline{\gamma}(u) &= 0 & \text{on } \sigma, & \text{if } \sigma \subset \Gamma^1. \end{aligned} \quad (11)$$

**Definition 2.4** (*Discrete norm and seminorm*)

Let  $\Omega$  be an open bounded polygonal domain of  $\mathbb{R}^d$ , and  $\mathcal{T}$  be an admissible mesh in the sense of Definition 2.1. For  $u \in X(\mathcal{T})$  define the discrete  $H_{0,\Gamma^1}^1$  norm by

$$\|u\|_{1,\mathcal{T}}^2 = \sum_{\sigma \in \mathcal{E}} \tau_{\sigma} (D_{\sigma} u)^2, \quad (12)$$

and the discrete  $H^1$  seminorm by

$$|u|_{1,\mathcal{T}}^2 = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \tau_{\sigma} (D_{\sigma} u)^2, \quad (13)$$

with

$$|D_{\sigma} u| = |u_K - u_L| \text{ if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K/L, \quad (14)$$

$$D_\sigma u = -u_K \text{ if } \sigma \in \Gamma^1, \sigma \in \mathcal{E}_K, \quad (15)$$

$$D_\sigma u = 0 \text{ if } \sigma \in \Gamma^2, \quad (16)$$

$$D_\sigma u = u_\sigma - u_K \text{ if } \sigma \in \Gamma^3, \sigma \in \mathcal{E}_K. \quad (17)$$

Similarly, for  $u \in \tilde{X}(\mathcal{T})$ , define the discrete  $H^1$  seminorm by (13) and the discrete  $H_0^1$  norm by

$$\|u\|_{1,\mathcal{T}}^2 = \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u)^2, \quad (18)$$

where

$$|D_\sigma u| = |u_K - u_L| \text{ if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \quad (19)$$

$$D_\sigma u = -u_K \text{ if } \sigma \in \mathcal{E}_{\text{ext}}, \sigma \in \mathcal{E}_K, \quad (20)$$

Let us now turn to some functional results on the discrete spaces  $X(\mathcal{T})$  and  $\tilde{X}(\mathcal{T})$ . The first result which we state is a discrete equivalent of the Poincaré inequality; it is crucial in order to obtain a priori estimates on the approximate finite volume solution in the  $L^2$  norm; it was already used (and proved) for linear and semi-linear convection-diffusion equations [21], [10], [16] and [9] (see also [33] for other applications).

**Lemma 2.1 (Discrete Poincaré inequality)** *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^d$ ,  $\mathcal{T}$  an admissible finite volume mesh in the sense of Definition 2.1 and  $u \in X(\mathcal{T})$  or  $\tilde{X}(\mathcal{T})$  (see Definition 2.2), then:*

$$\|u\|_{L^2(\Omega)} \leq \text{diam}(\Omega) \|u\|_{1,\mathcal{T}}. \quad (21)$$

where  $\|\cdot\|_{1,\mathcal{T}}$  is defined in Definition 2.4.

The next result gives a lower bound of the discrete  $H^1$  norm of piecewise constant functions which converge weakly to a function in  $L^2(\Omega)$ . This result is new and is needed for the convergence of the finite volume scheme for variational inequalities in order to deal with the discrete  $H^1$  seminorm of the approximate solution; when passing to the (lower) limit in the numerical scheme (see the proof of Theorem 4.2), we need to make the term  $-\int_\Omega \nabla u(x) \nabla u(x) dx$  appear so that the variational inequality (5) or (10) is obtained for the limit (which we shall show to exist thanks to a compactness result, see Theorem 4.1) on the approximate solutions.

## Lemma 2.2

*Let  $\Omega$  be a convex polygonal subset of  $\mathbb{R}^d$ , with  $d \geq 1$ , let  $u \in H^1(\Omega)$  and let  $\zeta > 0$ . We consider here admissible meshes  $\mathcal{T}$  in the sense of Definition 2.1 which satisfy*

$$d_{K,\sigma} \geq \zeta \text{diam}(K), \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K, \quad (22)$$

*Let  $u_\mathcal{T} \in X(\mathcal{T})$  or  $\tilde{X}(\mathcal{T})$  (see Definition 2.2) such that  $u_\mathcal{T} \rightarrow u$  weakly in  $L^2(\Omega)$  as  $\text{size}(\mathcal{T}) \rightarrow 0$ , then the following inequality holds*

$$\liminf_{\text{size}(\mathcal{T}) \rightarrow 0} |u_\mathcal{T}|_{1,\mathcal{T}}^2 \geq \int_\Omega |\nabla u(x)|^2 dx. \quad (23)$$

Proof

First note that for any  $u \in H^1(\Omega)$ , one has:

$$\int_{\Omega} |\nabla u(x)|^2 dx = \sup_{\varphi \in (C_c^\infty(\Omega))^d} \frac{1}{\|\varphi\|_{(L^2(\Omega))^d}} \int_{\Omega} \nabla u(x) \cdot \varphi(x) dx. \quad (24)$$

Let  $\varphi \in (C_c^\infty(\Omega))^d$ , by Lemma 2.3 given below, there exist  $\psi \in H^2(\Omega)$  and  $\tilde{\varphi} \in (L^2(\Omega))^d$  such that

$$\varphi = \nabla \psi + \tilde{\varphi}$$

and

$$\int_{\Omega} \tilde{\varphi}(x) \cdot \nabla v(x) dx = 0, \forall v \in H^1(\Omega).$$

Therefore,

$$\int_{\Omega} \nabla \psi(x) \cdot \nabla v(x) dx = \int_{\Omega} \varphi(x) \cdot \nabla v(x) dx, \forall v \in H^1(\Omega). \quad (25)$$

Furthermore since  $\varphi \in (C_c^\infty(\Omega))^d$  it is obvious that  $\nabla \psi \cdot \mathbf{n} = 0$  on  $\partial\Omega$  and  $\|\nabla \psi\|_{L^2(\Omega)} \leq \|\varphi\|_{(L^2(\Omega))^d}$ .

Since  $u_{\mathcal{T}}$  tends to  $u$  weakly in  $L^2(\Omega)$ , (25) implies that

$$\int_{\Omega} \nabla u(x) \cdot \varphi(x) dx = - \int_{\Omega} u(x) \Delta \psi(x) dx = - \lim_{\text{size}(\mathcal{T}) \rightarrow 0} \int_{\Omega} u_{\mathcal{T}}(x) \Delta \psi(x) dx.$$

Now let  $u_K$  denote the value of  $u_{\mathcal{T}}$  on the control volume  $K$ . One has

$$\int_{\Omega} u_{\mathcal{T}}(x) \Delta \psi(x) dx = \sum_{K \in \mathcal{T}} u_K \sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x),$$

where  $\mathbf{n}_{K,\sigma}$  denotes the unit normal vector to  $\sigma$  outward to  $K$ . Performing a discrete integration by part, this yields, since  $\nabla \psi \cdot \mathbf{n} = 0$  on  $\partial\Omega$ :

$$- \int_{\Omega} u_{\mathcal{T}}(x) \Delta \psi(x) dx \leq \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_{\sigma} u| \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{\sigma} d\gamma(x);$$

where  $\mathbf{n}_{\sigma}$  denotes a unit normal vector to  $\sigma$ . By the Cauchy-Schwarz inequality, this in turn yields

$$- \int_{\Omega} u_{\mathcal{T}}(x) \Delta \psi(x) dx \leq |u_{\mathcal{T}}|_{1,\mathcal{T}} \left[ \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left( \frac{1}{m(\sigma)} \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{\sigma} d\gamma(x) \right)^2 m(\sigma) d\sigma \right]^{\frac{1}{2}}. \quad (26)$$

Let us assume for a moment that

$$\lim_{\text{size}(\mathcal{T}) \rightarrow 0} \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left( \frac{1}{m(\sigma)} \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{\sigma} d\gamma(x) \right)^2 m(\sigma) d\sigma = \int_{\Omega} |\nabla \psi(x)|^2 dx. \quad (27)$$

Then from (26),

$$- \int_{\Omega} u(x) \Delta \psi(x) dx \leq \liminf_{\text{size}(\mathcal{T}) \rightarrow 0} |u_{\mathcal{T}}|_{1,\mathcal{T}} \left( \int_{\Omega} |\nabla \psi(x)|^2 dx \right)^{\frac{1}{2}} \leq \liminf_{\text{size}(\mathcal{T}) \rightarrow 0} |u_{\mathcal{T}}|_{1,\mathcal{T}} \|\varphi\|_{(L^2(\Omega))^d},$$



and (23) follows from (24).

There now remains to show that (27) holds true.

Let  $\sigma \in \mathcal{E}_{\text{int}}$  such that  $\sigma = K|L$ ; since  $\psi \in H^2(\Omega)$ , the approximation of  $\nabla \psi \cdot \mathbf{n}_{K,\sigma}$  by  $\frac{\psi_L - \psi_K}{d_\sigma}$  is consistent of order one, see e.g. [16] or [9]; hence

$$-\frac{1}{\mathfrak{m}(\sigma)} \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) = \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} + R_{K,L}$$

with  $|R_{K,L}| = |R_{L,K}| = |R_\sigma|$  and

$$R_\sigma^2 \leq \frac{(\text{size}(\mathcal{T}))^2}{\mathfrak{m}(\sigma)d_\sigma} \int_{\mathcal{V}_\sigma} |D^2\psi(x)|^2 dx, \quad (28)$$

with  $\mathcal{V}_\sigma = \mathcal{V}_{K,\sigma} \cup \mathcal{V}_{L,\sigma}$  where  $K$  and  $L$  are the control volumes such that  $\sigma = K|L$  and  $\mathcal{V}_{K,\sigma} = \{tx_K + (1-t)x, x \in \sigma, t \in [0, 1]\}$ . Hence

$$A_{\mathcal{T}} = \sum_{\sigma \in \mathcal{E}_{\text{int}}} \left( \frac{\int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_\sigma d\gamma(x)}{\mathfrak{m}(\sigma)} \right)^2 \mathfrak{m}(\sigma) d_\sigma = \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} \left( \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} + R_\sigma \right)^2 \mathfrak{m}(\sigma) d_\sigma.$$

We may then decompose  $A_{\mathcal{T}}$  as  $A_{\mathcal{T}} = B_{\mathcal{T}} + C_{\mathcal{T}} + D_{\mathcal{T}}$  with

$$B_{\mathcal{T}} = \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} \left( \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} \right)^2 \mathfrak{m}(\sigma) d_\sigma,$$

$$D_{\mathcal{T}} = \sum_{\sigma \in \mathcal{E}_{\text{int}}} R_\sigma^2 \mathfrak{m}(\sigma) d_\sigma \leq (\text{size}(\mathcal{T}))^2 \sum_{\sigma \in \mathcal{E}_{\text{int}}} \int_{\mathcal{V}_\sigma} |D^2\psi(x)|^2 dx \rightarrow 0 \text{ as } \text{size}(\mathcal{T}) \rightarrow 0$$

and

$$C_{\mathcal{T}} = 2 \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} R_\sigma \mathfrak{m}(\sigma) d_\sigma \leq \sqrt{B_{\mathcal{T}}} \sqrt{D_{\mathcal{T}}}, \quad (29)$$

so that  $C_{\mathcal{T}} \rightarrow 0$  as  $\text{size}(\mathcal{T}) \rightarrow 0$  if  $B_{\mathcal{T}}$  is bounded. Let us then turn to  $B_{\mathcal{T}}$ .

$$B_{\mathcal{T}} = \sum_{K \in \mathcal{T}} \left( \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} \mathfrak{m}(\sigma) - \sum_{\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K} \int_{\sigma} \nabla \psi(x) \cdot \mathbf{n}_{K,\sigma}(x) d\gamma(x) \right) \psi(x_K) = E_{\mathcal{T}} + R,$$

where

$$E_{\mathcal{T}} = \sum_{K \in \mathcal{T}} \left( - \int_K \Delta \psi(x) dx \right) \psi(x_K),$$

and

$$|R| \leq \sum_{K \in \mathcal{T}} \left( \sum_{\sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K} |R_{K,L}| \mathfrak{m}(\sigma) \right) \psi(x_K) \leq \sum_{\sigma \in \mathcal{E}_{\text{int}}} |R_\sigma| \left| \frac{\psi(x_K) - \psi(x_L)}{d_\sigma} \right| \mathfrak{m}(\sigma) d_\sigma \leq \sqrt{B_{\mathcal{T}}} \sqrt{D_{\mathcal{T}}}. \quad (30)$$

Now

$$E_{\mathcal{T}} = \int_{\Omega} |\nabla \psi(x)|^2 dx + \overline{R},$$

with

$$\overline{R} \leq \sup_{|x-y| \leq \text{size}(\mathcal{T})} |\psi(x) - \psi(y)| \int_{\Omega} |\Delta \psi(x)| dx \rightarrow 0 \text{ as } \text{size}(\mathcal{T}) \rightarrow 0.$$

Remarking that  $E_{\mathcal{T}} \leq \|\psi\|_{\infty} \|\Delta \psi\|_1$  and by (30) one has:  $|B_{\mathcal{T}}| \leq \|\psi\|_{\infty} \|\Delta \psi\|_1 + \sqrt{B_{\mathcal{T}}} \sqrt{D_{\mathcal{T}}}$  and therefore there exists  $C > 0$  depending only on  $\psi$  (and not on  $\mathcal{T}$ ) such that  $B_{\mathcal{T}} \leq C$ . This, in turns yields from (30) that  $R \rightarrow 0$  as  $\text{size}(\mathcal{T}) \rightarrow 0$ . Hence

$$\lim_{\text{size}(\mathcal{T}) \rightarrow 0} B_{\mathcal{T}} = \lim_{\text{size}(\mathcal{T}) \rightarrow 0} E_{\mathcal{T}} = \int_{\Omega} |\nabla \psi(x)|^2 dx,$$

and by (29),

$$\lim_{\text{size}(\mathcal{T}) \rightarrow 0} A_{\mathcal{T}} = \int_{\Omega} |\nabla \psi(x)|^2 dx,$$

which proves (27) and concludes the proof of Lemma 2.2.  $\blacksquare$

Let us now give for the sake of completeness the proof of the Hodge decomposition which we used in the above lemma:

**Lemma 2.3** *Let  $\Omega$  be an open bounded connected subset of  $\mathbb{R}^d$ ,  $d \geq 1$ . Let  $\varphi \in (L^2(\Omega))^d$ , then there exist  $\psi \in H^1(\Omega)$  and  $\tilde{\varphi} \in (L^2(\Omega))^d$  such that*

$$\varphi = \nabla \psi + \tilde{\varphi} \quad (31)$$

and

$$\int_{\Omega} \tilde{\varphi}(x) \cdot \nabla v(x) dx = 0, \forall v \in H^1(\Omega). \quad (32)$$

If  $\varphi \in (H^1(\Omega))^d$  and  $\varphi \cdot \mathbf{n} = 0$  on  $\partial\Omega$  and if  $\Omega$  is of class  $C^2$  or  $\Omega$  is convex, then there exist  $\psi \in H^2(\Omega)$  and  $\tilde{\varphi} \in (H^1(\Omega))^d$  such that (31) and (32) hold.

**Proof**

Let  $\varphi \in (L^2(\Omega))^d$  and let  $\psi$  be the solution to

$$\begin{cases} \psi \in H^1(\Omega), \int_{\Omega} \psi(x) dx = 0, \\ \int_{\Omega} \nabla \psi(x) \cdot \nabla v(x) dx = \int_{\Omega} \varphi(x) \cdot \nabla v(x) dx \forall v \in H^1(\Omega), \end{cases} \quad (33)$$

which exists and is unique thanks to the Lax-Milgram lemma, then (31) and (32) hold true with  $\tilde{\varphi} = \varphi - \nabla \psi$ . Furthermore, if  $\varphi \in (H^1(\Omega))^d$  and  $\varphi \cdot \mathbf{n} = 0$  on  $\partial\Omega$  and if  $\Omega$  is of class  $C^2$  or  $\Omega$  is convex, thanks to the regularity of the solution to (33),  $\psi$  belongs to  $H^2(\Omega)$ .  $\blacksquare$

**Remark 2.2** *Recently (that is after submission of this paper), J. Droniou [8] showed that if  $\Omega$  is a polygonal or regular open set in  $\mathbb{R}^d$  then the set  $\{\varphi \in C^\infty(\overline{\Omega}), \nabla \varphi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$  is dense in  $W^{1,p}(\Omega)$  for any  $p \in [1, +\infty[$  ( $\mathbf{n}$  denotes the unit outward normal to  $\Omega$ ); note that this result does not hold if the boundary  $\partial\Omega$  is only Lipschitz, see [8] for a counter example.*

We may use this result to generalize Lemma 2.2 to non convex polygonal open subsets of  $\mathbb{R}^d$ . Indeed, we may then prove (23) directly by density instead of using the Hodge decomposition of Lemma 2.3.

Let us now give a discrete trace inequality which was used and proved in [9] and [16] in the case of Neumann boundary conditions. We shall need it here in order to obtain the Signorini boundary condition (4) when passing to the limit on the numerical scheme (see the proof of Theorem 4.2). This trace inequality is not needed for the obstacle problem because the compactness result of Theorem 4.1 also yields that any possible limit of approximate finite volume solutions lies in  $H_0^1(\Omega)$ .

**Lemma 2.4 (Trace inequality)** *Let  $\Omega$  be an open polygonal bounded open set of  $\mathbb{R}^d$ . Let  $\mathcal{T}$  be an admissible mesh, in the sense of Definition 2.1, and  $u \in X(\mathcal{T})$  (see Definition 2.2). Let  $u_K$  be the value of  $u$  in the control volume  $K$  and  $u_\sigma$  be the value of  $u$  on the edge  $\sigma$ , for  $\sigma \subset \Gamma^3$ . Let  $\bar{\gamma}(u) \in L^2(\partial\Omega)$  be defined by Definition 2.3. Then, there exists  $C$ , depending only on  $\Omega$ , such that*

$$\|\bar{\gamma}(u)\|_{L^2(\partial\Omega)} \leq C(\|u\|_{1,\mathcal{T}} + \|u\|_{L^2(\Omega)}). \quad (34)$$

### 3 The finite volume scheme

A discretization by a "classical" finite volume method, that is integrating Equation (1) over each control volume  $K$  and approximating the normal derivative on each cell boundary by finite differences yields, taking into account the boundary conditions (3), (2), (4) the following "discrete problem":

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = 0 \quad \forall K \in \mathcal{T}, \quad (35)$$

with

$$F_{K,\sigma} = -\tau_\sigma(u_L - u_K) \quad \forall \sigma \in \mathcal{E}_{\text{int}} \text{ if } \sigma = K/L, \quad (36)$$

$$F_{K,\sigma} = \tau_\sigma u_K \quad \forall \sigma \subset \Gamma^1, \sigma \in \mathcal{E}_K \quad (37)$$

$$F_{K,\sigma} = 0 \quad \forall \sigma \subset \Gamma^2, \sigma \in \mathcal{E}_K \quad (38)$$

$$F_{K,\sigma} = -\tau_\sigma(u_\sigma - u_K) \quad \forall \sigma \subset \Gamma^3, \sigma \in \mathcal{E}_K \quad (39)$$

and on the Signorini boundary:

$$u_\sigma \geq a \quad \forall \sigma \subset \Gamma^3, \quad (40)$$

$$-F_{K,\sigma} \geq m(\sigma)b \quad \forall \sigma \subset \Gamma^3, \quad (41)$$

$$(u_\sigma - a) \left( \frac{F_{K,\sigma}}{m(\sigma)} + b \right) = 0 \quad \forall \sigma \subset \Gamma^3. \quad (42)$$

In a similar way, we introduce a finite volume discretization of the obstacle problem (6)-(6)

$$\left( \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} \right) (u_K - \psi_K) = 0 \quad \forall K \in \mathcal{T}, \quad (43)$$

$$u_K \leq \psi_K \quad \forall K \in \mathcal{T}, \quad (44)$$

with

$$F_{K,\sigma} = -\tau_\sigma(u_L - u_K) \quad \forall \sigma \in \mathcal{E}_{\text{int}} \text{ if } \sigma = K/L, \quad (45)$$

$$F_{K,\sigma} = \tau_\sigma u_K \quad \forall \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K, \quad (46)$$

where  $\psi_K = \psi(x_K)$ , for any  $K \in \mathcal{T}$ .

In order to show the existence and uniqueness of  $U = ((u_K)_{K \in \mathcal{T}}, (u_\sigma)_{\sigma \in \Gamma^3})$  where  $(u_K)_{K \in \mathcal{T}}$  and  $(u_\sigma)_{\sigma \in \Gamma^3}$  satisfy (35)-(42), (resp. of  $U = (u_K)_{K \in \mathcal{T}}$  satisfying (43)-(46)), let us give an equivalent "variational" formulation to (35)-(42):

**Lemma 3.1** *Under Assumptions 1.1, let  $\mathcal{T}$  be an admissible finite volume mesh in the sense of Definition 2.1; and  $u_{\mathcal{T}} \in X(\mathcal{T})$  (see Definition 2.2) defined by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $u_{\mathcal{T}}(x) = u_\sigma$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3$ .*

*Then  $(u_K)_{K \in \mathcal{T}}, (u_\sigma)_{\sigma \in \Gamma^3}$  is solution to Problem (35)-(42) if and only if  $u_{\mathcal{T}}$  is solution to the following Problem:*

$$\begin{cases} u_{\mathcal{T}} \in \mathcal{K}_{\mathcal{T}} = \{v \in X(\mathcal{T}), \text{ s.t. } v_\sigma \geq a \quad \forall \sigma \subset \Gamma^3\} \text{ such that :} \\ \mathcal{A}(u_{\mathcal{T}}, v - u_{\mathcal{T}}) \geq \mathcal{L}(v - u_{\mathcal{T}}) \quad \forall v \in \mathcal{K}_{\mathcal{T}}, \end{cases} \quad (47)$$

with :

$$\mathcal{A}(u, v) = \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \tau_{K|L}(u_K - u_L)(v_K - v_L) + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} \tau_\sigma(D_\sigma u)(D_\sigma v) \quad \forall u, v \in \mathcal{K}_{\mathcal{T}} \text{ and } \quad (48)$$

$$\mathcal{L}(u) = \sum_{\sigma \in \Gamma^3} b u_\sigma m(\sigma) \quad \forall u \in \mathcal{K}_{\mathcal{T}}, \quad (49)$$

with  $D_\sigma u$  be defined by (15)-(17).

Similarly, under assumptions 1.2,  $(u_K)_{K \in \mathcal{T}}$  is a solution to Problem (35)-(42) if and only if the function  $u_{\mathcal{T}}$  defined from  $\cup_{K \in \mathcal{T}} K$  to  $\mathbb{R}$  by  $u_{\mathcal{T}}(x) = u_K$  if  $x \in K$  satisfies

$$\begin{cases} u_{\mathcal{T}} \in \tilde{\mathcal{K}}_{\mathcal{T}} = \{v \in \tilde{X}(\mathcal{T}), \text{ s.t. } v_K \leq \psi_K \quad \forall K \in \mathcal{T}\} \\ \mathcal{A}(u_{\mathcal{T}}, v - u_{\mathcal{T}}) \geq \tilde{\mathcal{L}}(v - u_{\mathcal{T}}) \quad \forall v \in \tilde{\mathcal{K}}_{\mathcal{T}} \end{cases} \quad (50)$$

where  $\mathcal{A}$  is defined by (48) and

$$\tilde{\mathcal{L}}(v) = \sum_{K \in \mathcal{T}} m(K) f_K v_K. \quad (51)$$

Proof of Lemma 3.1: We only prove here the first part of the lemma, i.e. the part concerning the Signorini problem, since the proof in the case of the obstacle case is similar and easier.

Let  $u_{\mathcal{T}} \in X(\mathcal{T})$  (see Definition 2.2) defined by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $u_{\mathcal{T}}(x) = u_\sigma$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3$ .

Let us assume that  $(u_K)_{K \in \mathcal{T}}, (u_\sigma)_{\sigma \in \Gamma^3}$  satisfy (35)-(42), and let us show that  $u_{\mathcal{T}}$  satisfies Problem (47). From (40),  $u_{\mathcal{T}}$  is clearly in  $\mathcal{K}_{\mathcal{T}}$ . Let  $v \in \mathcal{K}_{\mathcal{T}}$ , multiplying (35) by  $v_K - u_K$  and summing over  $K$  leads to:

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} (v_K - u_K) = 0,$$

which gives, reordering the summation over the set of edges and using (36)-(39):

$$\mathcal{A}(u_{\mathcal{T}}, v - u_{\mathcal{T}}) = \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} \tau_{\sigma}(D_{\sigma}u_{\mathcal{T}})(v_{\sigma} - u_{\sigma}). \quad (52)$$

Let us now show that:

$$\sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} \tau_{\sigma}(D_{\sigma}u_{\mathcal{T}})(v_{\sigma} - u_{\sigma}) \geq \mathcal{L}(v - u_{\mathcal{T}}). \quad (53)$$

Let  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$  and  $K \in \mathcal{T}$  such that  $\sigma \in \mathcal{E}_K$ , then:

$$-F_{K,\sigma}(v_{\sigma} - u_{\sigma}) = (-F_{K,\sigma} - m(\sigma)b)(v_{\sigma} - a) + (-F_{K,\sigma} - m(\sigma)b)(a - u_{\sigma}) + m(\sigma)b(v_{\sigma} - u_{\sigma}), \quad (54)$$

from (42), we obtain:

$$(-F_{K,\sigma} - m(\sigma)b)(a - u_{\sigma}) = 0, \quad (55)$$

Since  $v \in \mathcal{K}_{\mathcal{T}}$ ,  $v_{\sigma} - a \geq 0$ , hence from (41), we obtain:

$$(-F_{K,\sigma} - m(\sigma)b)(v_{\sigma} - a) \geq 0. \quad (56)$$

Then from (54)-(56),  $\forall \sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ , we have:

$$\tau_{\sigma}(D_{\sigma}u_{\mathcal{T}})(v_{\sigma} - u_{\sigma}) \geq m(\sigma)b(v_{\sigma} - u_{\sigma}),$$

hence, the inequality (53) is proven; then from (52) and (53),  $u_{\mathcal{T}}$  satisfies Problem (47).

Conversely assume now that  $u_{\mathcal{T}} \in X(\mathcal{T})$  satisfies Problem (47); let us prove that  $(u_K)_{K \in \mathcal{T}}$ ,  $(u_{\sigma})_{\sigma \subset \Gamma^3}$  is solution to Problem (35)-(42).

Let  $K_0 \in \mathcal{T}$ , let us check that:

$$\sum_{\sigma \in \mathcal{E}_{K_0}} F_{K_0,\sigma} = 0,$$

where  $F_{K_0,\sigma}$  is defined by (36)-(39). Let  $v_1 = u_{\mathcal{T}} + w$  and  $v_2 = u_{\mathcal{T}} - w$  with  $w \in X(\mathcal{T})$  such that  $w_{K_0} = 1$ ,  $w_K = 0 \forall K \in \mathcal{T}$  such that  $K \neq K_0$ , and  $w_{\sigma} = 0 \forall \sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ ; it is easily seen that  $v_1$  and  $v_2 \in \mathcal{K}_{\mathcal{T}}$ . Taking  $v_1$  (respectively  $v_2$ ) in (47), we obtain  $\mathcal{A}(u_{\mathcal{T}}, w) \geq 0$  (respectively  $\mathcal{A}(u_{\mathcal{T}}, w) \leq 0$ ). Hence,  $\mathcal{A}(u_{\mathcal{T}}, w) = 0$ , with  $\mathcal{A}(u_{\mathcal{T}}, w) = \sum_{\sigma \in \mathcal{E}_{K_0}} F_{K_0,\sigma}$ , so that  $u_{\mathcal{T}}$  satisfies the equations (35)-(39). Furthermore, since  $u_{\mathcal{T}} \in \mathcal{K}_{\mathcal{T}}$ ,  $u_{\mathcal{T}}$  clearly satisfies (40).

Let us now show that  $u_{\mathcal{T}}$  satisfies (41). Let  $\sigma_0 \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$ , where  $K \in \mathcal{T}$ , and  $\sigma_0 \subset \Gamma^3$ ; let  $v = u_{\mathcal{T}} + w$ , with  $w \in X(\mathcal{T})$  such that  $w_K = 0$  for any  $K \in \mathcal{T}$ ,  $w_{\sigma_0} = 1$  and  $w_{\sigma} = 0 \forall \sigma \subset \Gamma^3$ ,  $\sigma \neq \sigma_0$ ; since  $v \in \mathcal{K}_{\mathcal{T}}$  one may take  $v$  in (47) which yields

$$\tau_{\sigma_0}(u_{\sigma_0} - u_K) \geq bm(\sigma_0).$$

There remains to show that  $u_{\mathcal{T}}$  satisfies (42). (Our proof is inspired of one written for the continuous Problem see [17]). Let  $\Gamma_a = \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3 \text{ such that } u_{\sigma} > a\}$  and  $\sigma_0 \in \Gamma_a$ , let us prove that:

$$\tau_{\sigma_0}(u_{\sigma_0} - u_K) = bm(\sigma_0). \quad (57)$$

Let  $v_1 = u_{\mathcal{T}} + \mu w$  and  $v_2 = u_{\mathcal{T}} - \mu w$  with  $w \in X(\mathcal{T})$  such that  $w_K = 0 \forall K \in \mathcal{T}$ ,  $w_{\sigma_0} = 1$  and  $w_{\sigma} = 0 \forall \sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ ,  $\sigma \neq \sigma_0$ , and  $\mu = u_{\sigma_0} - a$ ; since  $v_1$  and  $v_2 \in \mathcal{K}_{\mathcal{T}}$ , one may take  $v_1$  (respectively  $v_2$ ) in (47) and obtain  $\tau_{\sigma_0}(u_{\sigma_0} - u_K) \geq bm(\sigma_0)$  (respectively  $\tau_{\sigma_0}(u_{\sigma_0} - u_K) \leq bm(\sigma_0)$ ), which proves (57). This concludes the proof of Lemma 3.1. ■

Lemma 3.1 is used to obtain the following result:

**Proposition 3.1 (Existence and estimate)** *Under Assumptions 1.1 (resp. 1.2), let  $\mathcal{T}$  be an admissible finite volume mesh in the sense of Definition 2.1; there exists a unique solution  $(u_K)_{K \in \mathcal{T}}$ ,  $(u_\sigma)_{\sigma \in \Gamma^3}$  to Problem (35)-(42) (resp. a unique solution  $(u_K)_{K \in \mathcal{T}}$  to Problem (43)-(46)). We may then define  $u_{\mathcal{T}} \in X(\mathcal{T})$  (resp.  $u_{\mathcal{T}} \in \tilde{X}(\mathcal{T})$ ) (see Definition 2.2) by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$  and for any  $K \in \mathcal{T}$ , and by  $u_{\mathcal{T}}(x) = u_\sigma$  for  $x \in \sigma$  and for any  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$  (resp.  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ ). There exists  $C > 0$  only depending on  $\Omega$  and  $b$  such that:*

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C \text{ and } \|u_{\mathcal{T}}\|_{L^2(\Omega)} \leq C \quad (58)$$

where  $\|\cdot\|_{1,\mathcal{T}}$  denotes the discrete  $H_{0,\Gamma^1}^1$  norm (resp. the discrete  $H_0^1$  norm) defined in Definition 2.4.

**Proof of Proposition 3.1:**

*Step 1 (existence and uniqueness)*

The space  $X(\mathcal{T})$  (resp.  $\tilde{X}(\mathcal{T})$ ), endowed with the discrete norm  $H_{0,\Gamma^1}^1$  defined by (12) (resp. the discrete  $H_0^1$  norm defined by (18)) is a Hilbert space and  $\mathcal{K}_{\mathcal{T}}$  (resp.  $\tilde{\mathcal{K}}_{\mathcal{T}}$ ) is a non empty closed convex subset of  $X(\mathcal{T})$  (resp.  $\tilde{X}(\mathcal{T})$ ); the bilinear form  $\mathcal{A}$  (defined by (48)) and the linear form (resp.  $\tilde{\mathcal{L}}$ )  $\mathcal{L}$  defined by (49) (resp. by (51)) are continuous on  $X(\mathcal{T})$  (resp.  $\tilde{X}(\mathcal{T})$ ). Furthermore,  $\mathcal{A}(u_{\mathcal{T}}, u_{\mathcal{T}}) = \|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \forall u_{\mathcal{T}} \in X(\mathcal{T})$  or  $\tilde{X}(\mathcal{T})$ . Hence the assumptions for Stampacchia's Theorem (see [1]) are satisfied, and therefore there exists a unique solution  $u_{\mathcal{T}}$  to Problem (47) (resp. (50)); hence by Lemma 3.1, there exists a unique solution  $(u_K)_{K \in \mathcal{T}}$ ,  $(u_\sigma)_{\sigma \in \Gamma^3}$  to Problem (35)-(42) (resp.  $(u_K)_{K \in \mathcal{T}}$  to Problem (43)-(46)).

*Step 2 (estimate)*

Let us now prove (58). Let  $u_{\mathcal{T}}$  be a solution to (47) or (50). Then taking  $v = 0$  one has:

$$\|u_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \mathcal{L}(u_{\mathcal{T}}) \text{ (or } \tilde{\mathcal{L}}(u_{\mathcal{T}})). \quad (59)$$

In the case of Problem (47),  $\mathcal{L}(u_{\mathcal{T}}) = \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} b u_\sigma m(\sigma)$ , and using the Cauchy-Schwarz inequality, we obtain:

$$|\mathcal{L}(u_{\mathcal{T}})| \leq |b| \left( \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma^2 m(\sigma) \right)^{\frac{1}{2}} \left( \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} m(\sigma) \right)^{\frac{1}{2}},$$

since  $\sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} m(\sigma) = m(\Gamma^3)$  and  $\sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma^2 m(\sigma) = \|\bar{\gamma}(u_{\mathcal{T}})\|_{L^2(\Gamma^3)}^2$  (where  $\bar{\gamma}(u_{\mathcal{T}})$  is defined in Definition 2.3), using lemmas 2.1 and 2.4, we deduce the existence of  $C \in \mathbb{R}_+$  depending only on  $\Omega$  and  $\Gamma_3$  such that  $|\mathcal{L}(u_{\mathcal{T}})| \leq C \|u_{\mathcal{T}}\|_{1,\mathcal{T}}$ . Hence from (59) we obtain the first inequality of (58).

In the case of Problem (50),  $\tilde{\mathcal{L}}(u_{\mathcal{T}}) = \sum_{K \in \mathcal{T}} m(K) f_K u_K \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$  and from (59) and the Poincaré inequality, (58) also holds.

In both cases, the second inequality follows immediately by the discrete Poincaré inequality stated in Lemma 2.1. ■

## 4 Convergence

We begin by a compactness result the proof of which is a straightforward adaptation of the proof of the compactness result presented in [10] and [9]; note that this adaptation is needed in the case of the Signorini boundary condition. In the case of the obstacle problem,  $\Gamma^1 = \partial\Omega$  and  $\Gamma^2 \cup \Gamma^3 = \emptyset$  and the original form of the theorem as given in [10] is used.

**Theorem 4.1** *Let  $\Omega$  and  $\tilde{\Omega}$  be bounded open sets of  $\mathbb{R}^d$ , such that  $\Omega$  satisfies Assumption (1.1) and  $\tilde{\Omega} = \mathbb{R}^d \setminus (\Gamma_2 \cup \Gamma_3)$ . Let  $\{u_n, n \in \mathbb{N}\}$  be a bounded sequence of  $L^2(\Omega)$ . For  $n \in \mathbb{N}$ , one defines  $\tilde{u}_n$  by  $\tilde{u}_n = u_n$  a.e. on  $\Omega$  and  $\tilde{u}_n = 0$  a.e. on  $\mathbb{R}^d \setminus \Omega$ . One assumes that there exist  $C \in \mathbb{R}$  and  $\{h_n, n \in \mathbb{N}\} \subset \mathbb{R}_+$  such that  $h_n \rightarrow 0$  as  $n \rightarrow \infty$  and such that*

$$\|\tilde{u}_n(\cdot + \eta) - \tilde{u}_n\|_{L^2(\mathbb{R}^d)}^2 \leq C|\eta|, \forall n \in \mathbb{N}, \forall \eta \in \mathbb{R}^d. \quad (60)$$

and that for all compact subset  $\bar{\omega}$  of  $\tilde{\Omega}$ ,

$$\|\tilde{u}_n(\cdot + \eta) - \tilde{u}_n\|_{L^2(\bar{\omega})}^2 \leq C|\eta|(|\eta| + h_n), \forall n \in \mathbb{N}, \forall \eta \in \mathbb{R}^d, |\eta| < d(\bar{\omega}, \tilde{\Omega}^c), \quad (61)$$

where  $d(\bar{\omega}, \tilde{\Omega}^c)$  denotes the distance between  $\bar{\omega}$  and  $\mathbb{R}^d \setminus \tilde{\Omega}$ .

Then  $\{u_n, n \in \mathbb{N}\}$  is relatively compact in  $L^2(\Omega)$ . Furthermore if  $u_n \rightarrow u$ , in  $L^2(\Omega)$ , as  $n \rightarrow \infty$ , then  $u \in H^1(\Omega)$  and  $u = 0$  a.e. on  $\Gamma^1$ .

In order to use Theorem 4.1 in the proof of convergence, one needs the following lemma:

**Lemma 4.1** *Let  $\Omega$  and  $\tilde{\Omega}$  be bounded open sets of  $\mathbb{R}^d$ , such that  $\Omega$  satisfies Assumption (1.1) and  $\tilde{\Omega} = \mathbb{R}^d \setminus (\Gamma_2 \cup \Gamma_3)$ . Let  $\mathcal{T}$  be an admissible mesh in the sense of Definition 2.1 and  $u \in X(\mathcal{T})$  (see Definition 2.2). One defines  $\tilde{u}$  by  $\tilde{u} = u$  a.e. on  $\Omega$ , and  $\tilde{u} = 0$  a.e. on  $\mathbb{R}^d \setminus \Omega$ . Then there exists  $C$  and  $\tilde{C} > 0$ , only depending on  $\Omega$ , such that*

$$\|\tilde{u}(\cdot + \eta) - \tilde{u}\|_{L^2(\mathbb{R}^d)}^2 \leq \tilde{C}\|u\|_{1,\mathcal{T}}^2|\eta|, \text{ for any admissible mesh } \mathcal{T} \text{ and for any } \eta \in \mathbb{R}^d, |\eta| \leq 1, \quad (62)$$

and that, for any compact subset  $\bar{\omega}$  of  $\tilde{\Omega}$ ,

$$\begin{aligned} \|u(\cdot + \eta) - u\|_{L^2(\bar{\omega})}^2 &\leq \|u\|_{1,\mathcal{T}}^2(|\eta| + C\text{size}(\mathcal{T})), \quad \text{for any admissible mesh } \mathcal{T}, \text{ and} \\ &\quad \text{for any } \eta \in \mathbb{R}^d \text{ such that } |\eta| < d(\bar{\omega}, \tilde{\Omega}^c). \end{aligned} \quad (63)$$

Proof of Lemma 4.1:

The proof we give here is an adaptation of the proof of similar results which hold in the case of Dirichlet boundary conditions (i.e.  $\Gamma_1 = \partial\Omega$ ,  $\Gamma_2 = \Gamma_3 = \emptyset$ ), see [10] or [9], and Neumann boundary conditions which may be found in [9]. We give it here for the sake of completeness. Note that again, in the case of the obstacle problem, no adaptation is needed since  $\Gamma_1 = \partial\Omega$ ,  $\Gamma_2 = \Gamma_3 = \emptyset$ .

Following [9] and [10] let us define for  $\sigma \in \mathcal{E}$  the function  $\chi_\sigma$  from  $\mathbb{R}^d \times \mathbb{R}^d$  to  $\{0,1\}$  by  $\chi_\sigma(x,y) = 1$  if  $[x,y] \cap \sigma \neq \emptyset$  and  $\chi_\sigma(x,y) = 0$  if  $[x,y] \cap \sigma = \emptyset$ .

Let  $\eta \in \mathbb{R}^d \setminus \{0\}$ . Then:

$$|\tilde{u}(x+\eta) - \tilde{u}(x)| \leq \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) |D_\sigma u| + \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^1} \chi_\sigma(x, x+\eta) |u_\sigma|, \quad \text{for a.e. } x \in \mathbb{R}^d, \quad (64)$$

where, for  $\sigma \subset \Gamma^2$ ,  $u_\sigma = u_{K_\sigma}$ , and  $K_\sigma$  is the control volume such that  $\sigma \in \mathcal{E}_K$  and  $D_\sigma u$  is defined in Definition 2.4.

In order to prove (62), remark that the number of non zero terms in the second term of the right hand side of (64) is, for a.e.  $x \in \Omega$ , bounded by some real positive number, which only depends on  $\Omega$ , which can be taken, for instance, as the number of sides of  $\Omega$ , denoted by  $N$ . Hence, with  $C_1 = (N+1)^2$  (which only depends on  $\Omega$ , Indeed, if  $\Omega$  is convex,  $N = 2$  is also convenient), one has

$$|\tilde{u}(x+\eta) - \tilde{u}(x)|^2 \leq C_1 \left( \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) |D_\sigma u|^2 \right) + C_1 \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^1} \chi_\sigma(x, x+\eta) u_\sigma^2, \quad \text{for a.e. } x \in \mathbb{R}^d, \quad (65)$$

Let us prove, first, that there exists  $C > 0$ , depending only on  $\Omega$  such that:

$$\int_{\mathbb{R}^d} \left( \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) |D_\sigma u|^2 \right) \leq \left( \sum_{\sigma \in \mathcal{E}} \frac{m(\sigma)}{d_\sigma} |D_\sigma u|^2 \right) |\eta| (|\eta| + C \text{size}(\mathcal{T})). \quad (66)$$

Using the Cauchy Schwarz inequality, one has:

$$\left( \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) |D_\sigma u|^2 \right) \leq \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) \frac{|D_\sigma u|^2}{d_\sigma c_\sigma} \sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) d_\sigma c_\sigma, \quad \text{for a.e. } x \in \mathbb{R}^d, \quad (67)$$

where  $c_\sigma = |\mathbf{n}_\sigma \cdot \frac{\eta}{|\eta|}|$ , and  $\mathbf{n}_\sigma$  denotes a unit normal vector to  $\sigma$ .

Let us check that there exists  $C > 0$ , only depending on  $\Omega$ , such that

$$\sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(x, x+\eta) d_\sigma c_\sigma \leq |\eta| + C \text{size}(\mathcal{T}), \quad \text{for a.e. } x \in \mathbb{R}^d. \quad (68)$$

Let  $x \in \mathbb{R}^d$  such that  $\sigma \cap [x, x+\eta]$  contains at most one point, for all  $\sigma \in \mathcal{E}$ ,  $\sigma \subset \Omega \cup \Gamma^1$ , and  $[x, x+\eta]$  does not contain any vertex of  $\mathcal{T}$  (proving (68) for such points  $x$  gives (68) for a.e.  $x \in \mathbb{R}^d$ , since  $\eta$  is fixed). Since  $\Omega$  is not assumed to be convex, it may happen that the line segment  $[x, x+\eta]$  intersects  $\Gamma^1$  several times. In order to deal with this, let  $y, z \in [x, x+\eta]$  such that  $y \neq z$  and  $[y, z] \subset \Omega \cup \Gamma^1$ ; there exist  $K, L \in \mathcal{T}$  such that  $y \in \overline{K}$  and  $z \in \overline{L}$ . Hence,

$$\sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(y, z) d_\sigma c_\sigma = |(y_1 - z_1) \cdot \frac{\eta}{|\eta|}|,$$

where  $y_1 = x_K$  or  $y_\sigma$  with  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^1$  and  $z_1 = x_L$  or  $y_{\tilde{\sigma}}$  with  $\tilde{\sigma} \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^1$ , depending on the position of  $y$  and  $z$  in  $\overline{K}$  or  $\overline{L}$  respectively.



Since  $y_1 = y + y_2$ , with  $|y_2| \leq \text{size}(\mathcal{T})$ , and  $z_1 = z + z_2$ , with  $|z_2| \leq \text{size}(\mathcal{T})$ , one has

$$|(y_1 - z_1) \cdot \frac{\eta}{|\eta|}| \leq |y - z| + |y_2| + |z_2| \leq |y - z| + 2 \text{size}(\mathcal{T})$$

and

$$\sum_{\substack{\sigma \in \mathcal{E} \\ \sigma \subset \Omega \cup \Gamma^1}} \chi_\sigma(y, z) d_\sigma c_\sigma \leq |y - z| + 2 \text{size}(\mathcal{T}). \quad (69)$$

Note that this yields (68) with  $C = 2$  if  $[x, x + \eta] \subset \overline{\Omega}$ .

Since  $\Omega$  has a finite number of sides, the line segment  $[x, x + \eta]$  intersects  $\partial\Omega$  a finite number of times; hence there exist  $t_1, \dots, t_n$  such that  $0 \leq t_1 < t_2 < \dots < t_n \leq 1$ ,  $n \leq N$ , where  $N$  only depends on  $\Omega$  (indeed, it is possible to take  $N = 2$  if  $\Omega$  is convex and  $N$  equal to the number of sides of  $\Omega$  for a general  $\Omega$ ) and such that

$$\sum_{\sigma \in \mathcal{E}} \chi_\sigma(x, x + \eta) d_\sigma c_\sigma = \sum_{\substack{i=1, n-1 \\ \text{odd } i}} \sum_{\sigma \in \mathcal{E}} \chi_\sigma(x_i, x_{i+1}) d_\sigma c_\sigma,$$

with  $x_i = x + t_i \eta$ , for  $i = 1, \dots, n$ ,  $x_i \in \partial\Omega$  if  $t_i \notin \{0, 1\}$  and  $[x_i, x_{i+1}] \subset \overline{\Omega}$  if  $i$  is odd.

Then, using (69) with  $y = x_i$  and  $z = x_{i+1}$ , for  $i = 1, \dots, n-1$ , yields (68) with  $C = 2(N-1)$  (in particular, if  $\Omega$  is convex,  $C = 2$  is convenient for (68)), then, (66) is proved.

In order to prove (62), remark that, for all  $\sigma \in \mathcal{E}$ ,  $\sigma \subset \Omega \cup \Gamma^1$ ,

$$\int_{\mathbb{R}^d} \chi_\sigma(x, x + \eta) dx \leq m(\sigma) c_\sigma |\eta|.$$

and then, with (67) and (68), (66) holds.

Let us now turn to the second term of the right hand side of (65) integrated over  $\mathbb{R}^d$ ;

$$\begin{aligned} \int_{\mathbb{R}^d} \left( \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^1} \chi_\sigma(x, x + \eta) u_\sigma^2 \right) dx &\leq \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^1} m(\sigma) |\eta| u_\sigma^2 \\ &\leq \|\overline{\gamma}(u_{\mathcal{T}})\|_{L^2(\partial\Omega)}^2 |\eta|; \end{aligned}$$

with  $\overline{\gamma}(u_{\mathcal{T}})$  be defined in Definition 2.3. Therefore, thanks to lemmas 2.4 and 2.1, there exists a real positive number  $\tilde{C}$ , depending only on  $\Omega$ , such that

$$\int_{\mathbb{R}^d} \left( \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^1} \chi_{\sigma, \eta}(x) u_\sigma^2 \right) dx \leq \tilde{C} \|u\|_{1, \mathcal{T}}^2 |\eta|. \quad (70)$$

Hence from (65), (66) and (70), (62) is proven for some real positive number  $C$  depending only on  $\Omega$ .

Let us prove now (63). Let  $\overline{\omega}$  be a compact subset of  $\Omega \cup \Omega^c \cup \Gamma^1$ . If  $x \in \overline{\omega}$  and  $|\eta| < d(\overline{\omega}, \Omega^c)$ , the second term of the right hand side of (64) is 0, and integrating over  $\overline{\omega}$  instead of  $\mathbb{R}^d$ , (66) is still valid. Then, there exists  $C \geq 0$  such that:

$$\|u(\cdot + \eta) - u(\cdot)\|_{L^2(\overline{\omega})}^2 \leq \|u\|_{1, \mathcal{T}}^2 |\eta| (|\eta| + C \text{size}(\mathcal{T})), \quad (71)$$

which concludes the proof of Lemma 4.1. ■

**Theorem 4.2 (Convergence)** *Under assumptions 1.1, let  $u$  be the unique solution to (5); let  $\zeta \in \mathbb{R}_+^*$ ; let  $\mathcal{T}$  be some admissible mesh such that  $d_{K,\sigma} \geq \zeta \text{diam}(K)$  for any  $K \in \mathcal{T}$  and any  $\sigma \in \mathcal{E}_K$ ; let  $(u_K)_{K \in \mathcal{T}}$ ,  $(u_\sigma)_{\sigma \in \Gamma^3}$  be the unique solution to (35)-(42) (Recall that the existence and uniqueness of this solution is given in Proposition 3.1.) Define  $u_{\mathcal{T}} \in X(\mathcal{T})$  (see Definition 2.2 by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$ , and  $u_{\mathcal{T}}(x) = u_\sigma$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}} \cap \Gamma^3$ . Then*

$$u_{\mathcal{T}} \rightarrow u, \text{ in } L^2(\Omega), \text{ as } \text{size}(\mathcal{T}) \rightarrow 0, \text{ and} \quad (72)$$

$$\overline{\gamma}(u_{\mathcal{T}}) \rightarrow \gamma(u) \text{ in } L^2(\partial\Omega), \text{ for the weak topology as } \text{size}(\mathcal{T}) \rightarrow 0, \quad (73)$$

where  $\overline{\gamma}$  is the "discrete" trace operator from  $X(\mathcal{T})$  to  $L^2(\partial\Omega)$  defined in Definition 2.3 and  $\gamma$  is the trace operator from  $H^1(\Omega)$  to  $L^2(\partial\Omega)$ .

Similarly, if  $u \in H_0^1(\Omega)$  is the unique solution to (10) and  $(u_K)_{K \in \mathcal{T}}$  is the unique solution to (43)-(46), one defines  $u_{\mathcal{T}} \in \tilde{X}(\mathcal{T})$  by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$ , then:

$$u_{\mathcal{T}} \rightarrow u, \text{ in } L^2(\Omega), \text{ as } \text{size}(\mathcal{T}) \rightarrow 0. \quad (74)$$

**Proof**

We shall only prove the convergence results (72) and (73): the proof of (74) is easier and uses the same tools as the proof of (72). Since we use here the result of Lemma 25, we shall assume  $\Omega$  to be convex. However this assumption may be relaxed, see Remark 2.2.

Let  $(\mathcal{T}_n)_{n \in \mathbb{N}}$  be a sequence of admissible meshes of  $\Omega$  in the sense of Definition 2.1 and  $(u_{\mathcal{T}_n})_{n \in \mathbb{N}}$  be a sequence of approximate solutions such that  $u_{\mathcal{T}_n} \in X(\mathcal{T}_n)$  is the solution to (47), with  $\mathcal{T} = \mathcal{T}_n$ . Thanks to Proposition 3.1, there exists  $C_1 \in \mathbb{R}$ , only depending on  $\Omega$  and  $b$ , such that  $\|u_{\mathcal{T}_n}\|_{1,\mathcal{T}} \leq C_1$  for all  $n \in \mathbb{N}$ . Then, thanks to Lemma 4.1 and to Theorem 4.1, there exists a subsequence of  $(u_{\mathcal{T}_n})_{n \in \mathbb{N}}$ , still denoted by  $(u_{\mathcal{T}_n})_{n \in \mathbb{N}}$ , such that  $u_{\mathcal{T}_n}$  converges to some  $u$  in  $L^2(\Omega)$  as  $n \rightarrow +\infty$ , where  $u$  belongs to  $H^1(\Omega)$ .

There remains to prove that  $u$  is the (unique) solution to (5) (indeed if it is, then by uniqueness of the limit, the whole sequence converges to  $u$ ) and that  $\gamma(u_{\mathcal{T}_n}) \rightarrow \gamma(u)$  weakly in  $L^2(\partial\Omega)$ .

Thanks to Lemma 2.4, Lemma 2.1 and Proposition 3.1, the sequence  $(\overline{\gamma}(u_{\mathcal{T}_n}))_{n \in \mathbb{N}}$  is bounded in  $L^2(\partial\Omega)$  (see Definition 2.3 for the definition of  $\overline{\gamma}(u_{\mathcal{T}_n})$ ). Hence, one may assume that there exists a subsequence still denoted by  $(\overline{\gamma}(u_{\mathcal{T}_n}))_{n \in \mathbb{N}}$  and  $g \in L^2(\partial\Omega)$  such that  $\overline{\gamma}(u_{\mathcal{T}_n})$  converges to  $g$  weakly in  $L^2(\partial\Omega)$  as  $n \rightarrow +\infty$ .

Let us show that  $\gamma(u) = g$  a.e. on  $\partial\Omega$ . By definition,  $\overline{\gamma}(u_{\mathcal{T}_n}) = 0$  on  $\cup_{\sigma \in \mathcal{E}, \sigma \subset \Gamma^1}$ , and thanks to Theorem 4.1, one has  $\gamma(u) = 0$  a.e. on  $\Gamma^1$  for the  $(d-1)$ -dimensional Hausdorff measure on  $\Gamma^1$  and hence  $\gamma(u) = g$  a.e. on  $\Gamma^1$ . For the sake of simplicity, let us momentarily drop the index  $n$  in the notations in order to show that  $\gamma(u) = g$  a.e. on  $\Gamma^i$  ( $i = 2, 3$ ) for the  $(d-1)$ -dimensional Hausdorff measure (i.e. denote  $\mathcal{T}_n$  by  $\mathcal{T}$  and  $u_{\mathcal{T}_n}$  by  $u_{\mathcal{T}}$ ). Let  $\varphi \in C^2(\overline{\Omega})$ , such that  $\varphi|_{\Gamma^1} = 0$  and  $\varphi|_{\Gamma^3} \geq 0$ . Let  $\varphi_K = \varphi(x_K)$  for  $K \in \mathcal{T}$  and  $\varphi_\sigma = \varphi(y_\sigma)$  for  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \not\subset \Gamma^2$  (see Definition 2.1, for the definition of  $y_\sigma$ ), define  $\varphi_{\mathcal{T}} \in X(\mathcal{T})$  by  $\varphi_{\mathcal{T}}(x) = \varphi_K$ , for  $x \in K$  and for any control volume  $K$ , and  $\varphi_{\mathcal{T}}(x) = \varphi_\sigma$  for  $x \in \sigma$ , for any  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ .

Let  $\bar{v} = u_T + \varphi_T \in \mathcal{K}_T$ , taking  $v = \bar{v}$  in (47) we obtain:

$$\mathcal{A}(u_T, \varphi_T) \geq \mathcal{L}(\varphi_T), \quad (75)$$

Reordering the terms in  $\mathcal{A}(u_T, \varphi_T)$ , yields

$$\begin{aligned} \mathcal{A}(u_T, \varphi_T) = & - \sum_{K \in \mathcal{T}} u_K \sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \tau_{K|L}(\varphi_L - \varphi_K) - \sum_{K \in \mathcal{T}} u_K \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}, \sigma \not\subset \Gamma^2} \tau_\sigma(\varphi_\sigma - \varphi_K) \\ & - \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma \tau_\sigma(\varphi_{K_\sigma} - \varphi_\sigma), \end{aligned} \quad (76)$$

where  $K_\sigma$  denotes the control volume such that  $\sigma \in \mathcal{E}_{K_\sigma}$ . Following the argument of [10], we use the consistency of the approximation of the normal fluxes and the fact that  $\varphi \in C^2(\bar{\Omega})$  to remark that there exists  $C_1$  depending only on  $\varphi$ , such that

$$\begin{aligned} \sum_{L \in \mathcal{N}(K)} \tau_{K|L}(\varphi_L - \varphi_K) - \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}, \\ \sigma \not\subset \Gamma^2}} \tau_\sigma(\varphi_\sigma - \varphi_K) = & \int_K \Delta \varphi(x) dx - \int_{\Gamma^2 \cap \partial K} \nabla \varphi \cdot \mathbf{n}(s) ds \\ & + \sum_{L \in \mathcal{N}(K)} R_{K,L} + \sum_{\substack{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}, \\ \sigma \not\subset \Gamma^2}} R_{K,\sigma}, \end{aligned} \quad (77)$$

$$- \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma \tau_\sigma(\varphi_{K_\sigma} - \varphi_\sigma) = \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma \int_{\partial K_\sigma \cap \Gamma^3} \nabla \varphi \cdot \mathbf{n}(s) ds - \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} u_\sigma R_{K,\sigma}, \quad (78)$$

and

$$R_{K,\sigma} = -R_{L,\sigma} \text{ for all } \sigma = K|L \in \mathcal{E}_{\text{int}}, \quad (79)$$

with

$$|R_{K,\sigma}| \leq C_1 m(\sigma) \text{size}(\mathcal{T}) \text{ for any } K \in \mathcal{T} \text{ and any } \sigma \in \mathcal{E}_{\text{int}} \text{ or } \sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^2 \cup \Gamma^3. \quad (80)$$

Thanks to (77)-(80), (76) becomes:

$$\mathcal{A}(u_T, \varphi_T) = - \int_{\Omega} u_T(x) \Delta \varphi(x) dx + \int_{\Gamma^2 \cup \Gamma^3} \nabla \varphi \cdot \mathbf{n}(s) \bar{\gamma}(u_T)(s) ds + r_1(\varphi, \mathcal{T}) \quad (81)$$

where

$$\begin{aligned} |r_1(\varphi, \mathcal{T})| & \leq C_1 \sum_{\sigma \in \mathcal{E}} |D_\sigma u_T| m(\sigma) \text{size}(\mathcal{T}) \\ & \leq C_1 \left( \sum_{\sigma \in \mathcal{E}} |D_\sigma u_T|^2 \frac{m(\sigma)}{d_\sigma} \right)^{\frac{1}{2}} \left( \sum_{\sigma \in \mathcal{E}} m(\sigma) d_\sigma \right)^{\frac{1}{2}} \text{size}(\mathcal{T}) \\ & \leq C_2 \text{size}(\mathcal{T}), \end{aligned}$$

where  $C_2$  is a real positive number depending only on  $\varphi$ ,  $b$  and  $\Omega$  (thanks to Proposition 3.1 and the fact that  $\sum_{\sigma \in \mathcal{E}} m(\sigma) d_\sigma = m(\Omega)$ ).

Remark that:

$$\mathcal{L}(\varphi_T) = \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} b m(\sigma) \varphi_\sigma = \int_{\Gamma^3} b \varphi_T(s) ds.$$

Since  $\varphi \in C^1(\overline{\Omega})$ , by Taylor's formula there exists  $C_3 > 0$  such that:

$$\int_{\Gamma^3} b\varphi_{\mathcal{T}}(s)ds = \int_{\Gamma^3} b\gamma(\varphi)(s)ds + r_2(\varphi, \mathcal{T}),$$

where  $|r_2(\varphi, \mathcal{T})| \leq C_3 \text{size}(\mathcal{T})$ , where  $C_3$  depends only on  $b$  and  $\Gamma^3$ .

Let us now go back now to the sequence of admissible meshes  $(\mathcal{T}_n)_{n \in \mathbb{N}}$  and write (75) with  $\mathcal{T} = \mathcal{T}_n$ ; passing to the limit as  $n$  tends to infinity yields

$$-\int_{\Omega} u(x) \Delta \varphi(x) dx + \int_{\Gamma^2 \cup \Gamma^3} \nabla \varphi \cdot \mathbf{n}(s) g(s) ds \geq \int_{\Gamma^3} b\gamma(\varphi)(s) ds \quad (82)$$

Since  $u \in H^1(\Omega)$  and  $\gamma(u)|_{\Gamma^1} = 0$ , an integration by parts in (82) yields

$$\int_{\Omega} \nabla u(x) \cdot \nabla \varphi(x) dx + \int_{\Gamma^2 \cup \Gamma^3} \nabla \varphi \cdot \mathbf{n}(s) (g(s) - \gamma(u)(s)) ds \geq \int_{\Gamma^3} b\gamma(\varphi)(s) ds \quad (83)$$

for any  $\varphi \in C^2(\overline{\Omega})$  such that  $\varphi|_{\Gamma^1} = 0$  and  $\varphi|_{\Gamma^3} \geq 0$  a.e.

Taking  $\varphi \in C_c^\infty(\Omega)$  (respectively  $-\varphi$ ) in (83), one has  $\int_{\Omega} \nabla u(x) \cdot \nabla \varphi(x) dx \geq 0$  (respectively

$\int_{\Omega} \nabla u(x) \cdot \nabla \varphi(x) dx \leq 0$ ), hence,

$\int_{\Omega} \nabla u(x) \cdot \nabla \varphi(x) dx = 0$  for any  $\varphi \in H_0^1(\Omega)$ , thanks to the density of  $C_c^\infty(\Omega)$  in  $H_0^1(\Omega)$ . From (83), one has

$$\int_{\Gamma^2 \cup \Gamma^3} \nabla \varphi \cdot \mathbf{n}(s) (g(s) - \gamma(u)(s)) ds = 0, \text{ for any } \varphi \in C^2(\overline{\Omega}) \text{ such that } \varphi|_{\partial\Omega} = 0. \quad (84)$$

The wide choice of  $\varphi$  in (84) allows to conclude  $g = \gamma(u)$  a.e. on  $\Gamma^2 \cup \Gamma^3$ . We conclude that  $\overline{\gamma}(u_{\mathcal{T}_n}) \rightarrow \gamma(u)$  weakly in  $L^2(\partial\Omega)$  and  $\text{size}(\mathcal{T}_n) \rightarrow 0$  as  $n \rightarrow +\infty$  and (73) is proven. Furthermore, since  $\overline{\gamma}(u_{\mathcal{T}_n}) \geq a$  a.e. on  $\Gamma^3$ , one has  $\gamma(u) \geq a$  a.e. on  $\Gamma^3$  (here a.e. stands for the  $(d-1)$ -dimensional Hausdorff measure on  $\partial\Omega$ ); hence  $u \in \mathcal{K}$ .

In order to prove that  $u$  is the (unique) solution to (5), there only remains to show that

$$\int_{\Omega} \nabla u(x) \cdot \nabla (v - u)(x) dx \geq \int_{\Gamma^3} b(\gamma(v) - \gamma(u))(s) ds, \quad \forall v \in \mathcal{K}. \quad (85)$$

Let  $\varphi \in C^2(\overline{\Omega})$ , such that  $\varphi|_{\Gamma^1} = 0$  and  $\varphi|_{\Gamma^3} \geq a$ . Let  $\varphi_K = \varphi(x_K)$  and  $\varphi_\sigma = \varphi(y_\sigma)$  for  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \not\subset \Gamma^2$  (see Definition 2.1, for the definition of  $y_\sigma$ ), define  $\varphi_{\mathcal{T}} \in X_{\mathcal{T}}$  by  $\varphi_{\mathcal{T}}(x) = \varphi_K$ , for  $x \in K$  and for any control volume  $K$ , and  $\varphi_{\mathcal{T}}(x) = \varphi_\sigma$  for  $x \in \sigma$ , for any  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ .

As  $\varphi_{\mathcal{T}} \in \mathcal{K}_{\mathcal{T}}$ , we may take  $v = \varphi_{\mathcal{T}}$  in (47) with  $\mathcal{T} = \mathcal{T}_n$ , which yields

$$\mathcal{A}(u_{\mathcal{T}_n}, \varphi_{\mathcal{T}_n}) - \mathcal{A}(u_{\mathcal{T}_n}, u_{\mathcal{T}_n}) \geq \mathcal{L}(\varphi_{\mathcal{T}_n} - u_{\mathcal{T}_n}), \quad (86)$$

Writing (86) with  $\mathcal{T} = \mathcal{T}_n$  and passing to the lower limit as  $n$  tends to infinity yields using previous calculations and Lemma 2.2,

$$\int_{\Omega} \nabla u(x) \cdot \nabla \psi(x) dx - \int_{\Omega} \nabla u(x) \cdot \nabla u(x) dx \geq \int_{\Gamma^3} b(\gamma(\psi)(s) - \gamma(u)(s)) ds, \quad (87)$$

for any  $\psi \in C^2(\overline{\Omega})$  such that  $\psi|_{\Gamma^1} = 0$ , and  $\psi|_{\Gamma^3} \geq a$ .

By a density result (see e.g. [19]), we deduce from (87) that (85) holds true for any  $\psi \in \mathcal{K}$ . Since  $u \in \mathcal{K}$   $u$  is the (unique) solution to (5), this completes the proof of Theorem 4.2.  $\blacksquare$

## 5 Error estimate

Under regularity assumptions on the exact solution, namely  $u \in H^2(\Omega)$ , we give an error estimate for the "discrete"  $H^1$  norm and  $L^2$  norm of the error on the solution to (35)-(42).

**Theorem 5.1 (Error estimate)** *Under Assumption 1.1, let  $\mathcal{T}$  be an admissible mesh as defined in Definition 2.1, let  $\Omega$  be an open bounded polygonal subset of  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ .*

*Let  $\zeta = \min_{K \in \mathcal{T}} \min_{\sigma \in \mathcal{E}_K} \frac{d_{K,\sigma}}{\text{diam}(K)}$  and  $u_{\mathcal{T}} \in X(\mathcal{T})$  be defined by  $u_{\mathcal{T}}(x) = u_K$  for a.e. (for the  $d$ -dimensional-Lebesgue measure)  $x \in K$ , for all  $K \in \mathcal{T}$ , and  $u_{\mathcal{T}}(x) = u_{\sigma}$  for a.e. (for the  $(d-1)$ -dimensional-Lebesgue measure)  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$  where  $(u_K)_{K \in \mathcal{T}}$  and  $(u_{\sigma})_{\sigma \subset \Gamma^3}$  is the solution to (35)-(42). Assume that the unique variational solution  $u$  of Problem (1)-(4) satisfies  $u \in H^2(\Omega)$ . For each  $K \in \mathcal{T}$ , let  $e_K = u(x_K) - u_K$  and for each  $\sigma \in \mathcal{E}_{\text{ext}}$  such that  $\sigma \subset \Gamma^3$  let  $e_{\sigma} = u(y_{\sigma}) - u_{\sigma}$  and  $e_{\mathcal{T}} \in X(\mathcal{T})$  be defined by  $e_{\mathcal{T}}(x) = e_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $e_{\mathcal{T}}(x) = e_{\sigma}$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ .*

*Then there exists  $C \in \mathbb{R}$  depending only on  $u$ ,  $b$ ,  $\zeta$  and  $\Omega$  such that*

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C \text{size}(\mathcal{T}), \quad (88)$$

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq C \text{size}(\mathcal{T}), \quad (89)$$

where  $\|\cdot\|_{1,\mathcal{T}}$  is the discrete  $H_{0,\Gamma^1}^1$  norm defined in Definition 2.4 and  $\text{size}(\mathcal{T}) = \sup_{K \in \mathcal{T}} \text{diam}(K)$ .

Similarly, if the solution  $u$  to (10) is assumed to be in  $H^2$ , and if  $u_{\mathcal{T}}$  denotes the approximate solution given by (43)-(46) then let  $e_K = u(x_K) - u_K$  for any  $K \in \mathcal{T}$  and  $e_{\mathcal{T}} \in \tilde{X}(\mathcal{T})$  be defined by  $e_{\mathcal{T}}(x) = e_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$ . Then there exists  $C \in \mathbb{R}$  depending only on  $u$ ,  $\psi$ ,  $\zeta$  and  $\Omega$  such that (88) and (89) hold.

### Remark 5.1

1. Inequality (88) (resp.(89)) yields an estimate of order 1 for the discrete  $H_{0,\Gamma^1}^1$  norm (resp.  $L^2$  norm) of the error on the solution. Note also that, using  $u \in C^1(\overline{\Omega})$ , one deduces, from (89), the existence of  $C$  only depending on  $u$ ,  $b$  and  $\Omega$  such that  $\|u - u_{\mathcal{T}}\|_{L^2(\Omega)} \leq C \text{size}(\mathcal{T})$ .

2. The assumption  $u \in H^2(\Omega)$  is realistic under adequate assumptions on  $a$  and  $b$  and  $\Omega$  (see e.g. [17]).

3. Theorem 5.1 is still valid for  $a \in C^2(\Gamma^3, \mathbb{R}_-)$  instead of constant  $a$ .

Proof of Theorem 5.1:

Again we shall only prove the first part of the theorem, that is estimates (88) and (89) for the Signorini problem, since the proof of the estimates for the obstacle problem use the same technique and are somewhat easier.

Let  $u_{\mathcal{T}} \in X(\mathcal{T})$  be defined in  $\Omega$  by  $u_{\mathcal{T}}(x) = u_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$ , and  $u_{\mathcal{T}}(x) = u_{\sigma}$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$  where  $(u_K)_{K \in \mathcal{T}}$ ,  $(u_{\sigma})_{\sigma \subset \Gamma^3}$  is the solution to (35)-(42). Let us write the flux balance for any  $K \in \mathcal{T}$ :

$$\sum_{\sigma \in \mathcal{E}_K} \bar{F}_{K,\sigma} = 0, \quad (90)$$

where  $\bar{F}_{K,\sigma} = - \int_{\sigma} \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x)$  is the exact diffusion flux through  $\sigma$  outward to  $K$ .  
Let  $F_{K,\sigma}^*$  be defined by

$$\begin{aligned} F_{K,\sigma}^* &= -\tau_{\sigma}(u(x_L) - u(x_K)) \quad \forall \sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K \text{ if } \sigma = K|L, \\ F_{K,\sigma}^* &= \tau_{\sigma}u(x_K) \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^1, \\ F_{K,\sigma}^* &= 0 \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^2, \\ F_{K,\sigma}^* &= -\tau_{\sigma}(u(y_{\sigma}) - u(x_K)) \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^3. \end{aligned}$$

Then the consistency error on the diffusion flux may be defined as:

$$R_{K,\sigma} = \frac{1}{m(\sigma)} (\bar{F}_{K,\sigma} - F_{K,\sigma}^*). \quad (91)$$

Thanks to the regularity of  $u$ , one may prove (see [9] or [16]) that there exists  $C_1 \in \mathbb{R}$ , depending only on  $\|D^2u\|_{L^2(\Omega)}$  and  $\zeta$ , such that

$$m(\sigma)d_{\sigma}|R_{K,\sigma}|^2 \leq C_1(\text{size}(\mathcal{T}))^2, \quad \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K. \quad (92)$$

Subtracting (35) from (90) and using (91) and the regularity of  $u$  yields

$$\sum_{\sigma \in \mathcal{E}_K} G_{K,\sigma} = - \sum_{\sigma \in \mathcal{E}_K} m(\sigma) R_{K,\sigma}, \quad (93)$$

where  $G_{K,\sigma} = F_{K,\sigma}^* - F_{K,\sigma}$  is such that:

$$\begin{aligned} G_{K,\sigma} &= -\tau_{\sigma}(e_L - e_K) \quad \forall \sigma \in \mathcal{E}_{\text{int}} \cap \mathcal{E}_K \text{ if } \sigma = K|L, \\ G_{K,\sigma} &= \tau_{\sigma}e_K \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^1, \\ G_{K,\sigma} &= 0 \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^2, \\ G_{K,\sigma} &= -\tau_{\sigma}(e_{\sigma} - e_K) \quad \forall \sigma \in \mathcal{E}_K, \sigma \subset \Gamma^3, \end{aligned}$$

with  $e_K = u(x_K) - u_K$  for  $K \in \mathcal{T}$  and  $e_{\sigma} = u(y_{\sigma}) - u_{\sigma}$  for  $\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3$ . Multiplying (93) by  $e_K$ , summing for  $K \in \mathcal{T}$ , and reordering the terms thanks to the property of conservativity  $G_{K,\sigma} = -G_{L,\sigma}$  for any  $\sigma \in \mathcal{E}_{\text{int}}$  such that  $\sigma = K|L$ , (93) yields:

$$\sum_{\sigma \in \mathcal{E}} \tau_{\sigma} (D_{\sigma} e_{\mathcal{T}})^2 + \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} G_{K,\sigma} e_{\sigma} = - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) R_{K,\sigma} e_K,$$

and then,

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 = - \sum_{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3} G_{K,\sigma} e_{\sigma} - \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} m(\sigma) R_{K,\sigma} e_K, \quad (94)$$

with  $D_{\sigma} e_{\mathcal{T}}$  defined by (14)-(17) where  $e_{\mathcal{T}} \in X(\mathcal{T})$  is defined by  $e_{\mathcal{T}}(x) = e_K$  for a.e.  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $e_{\mathcal{T}}(x) = e_{\sigma}$  for a.e.  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3$ .

Reordering the terms in the last summation, using the fact that  $R_{K,\sigma} = -R_{L,\sigma}$  for any  $\sigma \in \mathcal{E}_{\text{int}}$  such that  $\sigma = K|L$ , (94) becomes:

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K, \sigma \subset \Gamma^3} (-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} + \sum_{\sigma \in \mathcal{E}} m(\sigma)R_{\sigma}|D_{\sigma}e_{\mathcal{T}}|, \quad (95)$$

with  $R_{\sigma} = |R_{K,\sigma}| = |R_{L,\sigma}|$  for  $\sigma = K|L$  and  $R_{\sigma} = |R_{K,\sigma}|$  for  $\sigma \subset \partial K$  and  $\sigma \in \mathcal{E}_{\text{ext}}$ .

Let  $K \in \mathcal{T}$  and  $\sigma$  such that  $\sigma \in \mathcal{E}_K$  and  $\sigma \subset \Gamma^3$ . From the definitions of  $D_{\sigma}e_{\mathcal{T}}$  (see Definition 2.4),  $G_{K,\sigma}$  and  $R_{K,\sigma}$ , we obtain that:

$$(-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} = (F_{K,\sigma} - \bar{F}_{K,\sigma})e_{\sigma},$$

Introducing  $a$ ,  $m(\sigma)b$ , since  $\bar{F}_{K,\sigma} = -\int_{\sigma} \nabla u(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x)$  and thanks to the last equation in (4) and to (42), we deduce that

$$\begin{aligned} (-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} &= (F_{K,\sigma} + m(\sigma)b)(u(y_{\sigma}) - a) + (F_{K,\sigma} + m(\sigma)b)(a - u_{\sigma}) \\ &\quad + \int_{\sigma} (\nabla u(x) \cdot \mathbf{n}_{K,\sigma} - b)(u(y_{\sigma}) - a) d\gamma(x) \\ &\quad + \int_{\sigma} (\nabla u(x) \cdot \mathbf{n}_{K,\sigma} - b)(a - u_{\sigma}) d\gamma(x). \end{aligned} \quad (96)$$

Thanks to (4) and to (40)-(42), the first and the last term in the right hand side are non-positive, the second one is equal to zero and the third one is non-negative. Hence, one has

$$(-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} \leq \int_{\sigma} (\nabla u(x) \cdot \mathbf{n}_{K,\sigma} - b)(u(y_{\sigma}) - a) d\gamma(x), \quad (97)$$

For a given edge  $\sigma$ , if  $u \neq a$  on  $\sigma$ , then from (4), one deduces that  $\nabla u \cdot \mathbf{n}_{K,\sigma} - b = 0$  on  $\sigma$  so that

$$(-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} \leq 0.$$

Now if  $u = a$  on a subset of  $\sigma$  whose measure is non zero, then  $\nabla u \cdot \tau = 0$  on the same subset, where  $\tau$  denotes the unit tangent vector to  $\sigma$ ; hence, by regularity of  $u$ , there exists  $C \geq 0$  only depending on  $D^2u$  such that

$$u(y_{\sigma}) - a \leq C(\text{size}(\mathcal{T}))^2,$$

in this case, one has

$$(-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} \leq \tilde{C}m(\sigma)(\text{size}(\mathcal{T}))^2,$$

where  $\tilde{C} > 0$  only depends on  $u$  and  $b$ , so that

$$\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K, \sigma \subset \Gamma^3} (-G_{K,\sigma} - m(\sigma)R_{K,\sigma})e_{\sigma} \leq \tilde{C}m(\Gamma^3)(\text{size}(\mathcal{T}))^2. \quad (98)$$

Hence from (98) and using the Cauchy-Schwarz inequality, (95) yields that

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq \left( \sum_{\sigma \in \mathcal{E}} m(\sigma) d_{\sigma} R_{\sigma}^2 \right)^{\frac{1}{2}} \left( \sum_{\sigma \in \mathcal{E}} \frac{m(\sigma)}{d_{\sigma}} (D_{\sigma}e_{\mathcal{T}})^2 \right)^{\frac{1}{2}} + C_2 \text{size}(\mathcal{T}).$$

Using (92) we deduce from this inequality that there exists  $C_3 \in \mathbb{R}_+$  only depending on  $u$ ,  $\zeta$  and  $\Omega$  such that

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}}^2 \leq C_3^2 \text{size}(\mathcal{T}) \|e_{\mathcal{T}}\|_{1,\mathcal{T}} + C_2 (\text{size}(\mathcal{T}))^2. \quad (99)$$

Hence by Young's Inequality, there exists  $C \in \mathbb{R}_+$  only depending on  $u$ ,  $b$  and  $\Omega$  such that

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C \text{size}(\mathcal{T}). \quad (100)$$

The estimate of the discrete  $L^2$  norm (89) follows by Lemma 2.1.

## 6 Numerical tests

The aim of this section is to determine the numerical performance of the discretization by the finite volume scheme (35)-(42).

In order to do so we study the following nonhomogeneous version of Problem (1)-(4):

$$-\Delta(u(\mathbf{x})) = f, \quad \mathbf{x} \in \Omega, \quad (101)$$

$$u(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Gamma^1, \quad (102)$$

$$\nabla u(\mathbf{x}) \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \Gamma^2, \quad (103)$$

$$\left. \begin{aligned} u(\mathbf{x}) &\geq a, \\ \nabla u(\mathbf{x}) \cdot \mathbf{n} &\geq b, \\ (u(\mathbf{x}) - a)(\nabla u(\mathbf{x}) \cdot \mathbf{n} - b) &= 0, \end{aligned} \right\} \quad \mathbf{x} \in \Gamma^3, \quad (104)$$

with  $\Omega = ]0, x_m[ \times ]0, y_m[$ , the boundary  $\partial\Omega$  of  $\Omega$  is composed of three non empty, non intersecting connected sets  $\Gamma^1$ ,  $\Gamma^2$  and  $\Gamma^3$ , such that :

$$\begin{aligned} \Gamma^1 &= \{(x, y) \in \overline{\Omega} \text{ s.t. } x = 0\}, \\ \Gamma^2 &= \{(x, y) \in \overline{\Omega} \text{ s.t. } y = y_m\} \cup \{(x, y) \in \overline{\Omega} \text{ s.t. } x = x_m\}, \\ \Gamma^3 &= \{(x, y) \in \overline{\Omega} \text{ s.t. } y = 0\}, \end{aligned}$$

$a, b \in \mathbb{R}$ ,  $f \in L^2(\Omega)$ ,  $g \in H^{1/2}(\Gamma^1) \cap C(\Gamma^1)$ ,  $\mathbf{n}$  is the unit normal vector to  $\partial\Omega$  outward to the domain  $\Omega$ . Hence for some  $a, b < 0$ ,  $u$  satisfies Problem (101)-(104), with  $f \in C(\overline{\Omega})$  and  $g \in C^2(\overline{\Gamma^1})$ .

The functions  $f$  and  $g$  are chosen such that Problem (101)- (104) has a solution  $u$  which satisfies:

1.  $u \in C^2(\overline{\Omega})$ ,
2.  $u(x, 0) \geq a$ ,  $\nabla u(x, 0) \cdot \mathbf{n} = b$ ,  $\forall x \in [0, \frac{x_m}{2}]$ ,
3.  $u(x, 0) = a$ ,  $\nabla u(x, 0) \cdot \mathbf{n} \geq b$ ,  $\forall x \in [\frac{x_m}{2}, x_m]$ ,
4.  $\nabla u(\mathbf{x}) \cdot \mathbf{n} = 0$ ,  $\forall \mathbf{x} \in \Gamma^2$ .



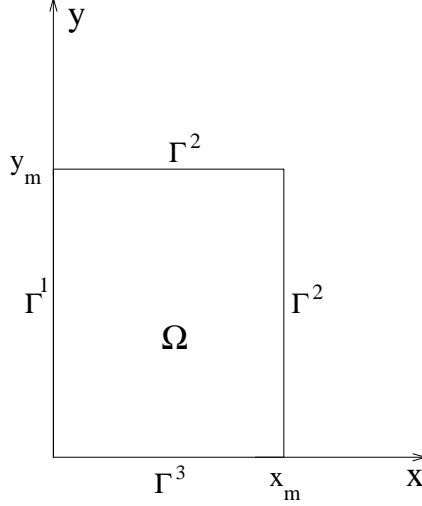


Figure 2: Domain of study

We define an admissible mesh  $\mathcal{T}$  on  $\Omega$ , in the sense of Definition 2.1: each cell  $K \in \mathcal{T}$  is a rectangle and we choose  $x_K$  at the center of the cell  $K$ .

A discretization by a finite volume method gives the following discrete problem :

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} = m(K)f_K \quad \forall K \in \mathcal{T}, \quad (105)$$

where  $F_{K,\sigma}$  is defined by

$$F_{K,\sigma} = -\tau_\sigma(u_L - u_K) \quad \forall \sigma \in \mathcal{E}_{\text{int}} \text{ if } \sigma = K/L, \quad (106)$$

$$F_{K,\sigma} = -\tau_\sigma(g_{\text{m}(\sigma)} - u_K) \quad \forall \sigma \subset \Gamma^1, \sigma \in \mathcal{E}_K \quad (107)$$

$$F_{K,\sigma} = 0 \quad \forall \sigma \subset \Gamma^2, \sigma \in \mathcal{E}_K \quad (108)$$

$$F_{K,\sigma} = -\tau_\sigma(u_\sigma - u_K) \quad \forall \sigma \subset \Gamma^3, \sigma \in \mathcal{E}_K \quad (109)$$

and on the Signorini boundary:

$$u_\sigma \geq a \quad \forall \sigma \subset \Gamma^3, \quad (110)$$

$$-F_{K,\sigma} \geq m(\sigma)b \quad \forall \sigma \subset \Gamma^3, \quad (111)$$

$$(u_\sigma - a) \left( \frac{F_{K,\sigma}}{m(\sigma)} + b \right) = 0 \quad \forall \sigma \subset \Gamma^3, \quad (112)$$

where  $f_K = \frac{1}{m(K)} \int_K f(x)dx$  and  $g_{\text{m}(\sigma)}$  is the value of the fonction  $g$  taken at the center of the edge  $\sigma$ .

We calculate the solution  $U = ((u_K)_{K \in \mathcal{T}}, (u_\sigma)_{\sigma \subset \Gamma^3})$  of Problem (105)-(112) thanks to the monotony algorithm introduced in [24] and whose convergence is proven in a forthcoming paper [23]. We recall it here for the sake of completeness.

## Monotony algorithm

• Initialization: Let  $\mathcal{E}_a^{(0)}$  and  $\mathcal{E}_b^{(0)} \subset \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\}$  such that  $\mathcal{E}_a^{(0)} \cap \mathcal{E}_b^{(0)} = \emptyset$  and  $\mathcal{E}_a^{(0)} \cup \mathcal{E}_b^{(0)} = \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\}$ .

• step  $(j)$ : Assume the sets  $\mathcal{E}_a^{(j)}$  and  $\mathcal{E}_b^{(j)}$  known such that  $\mathcal{E}_a^{(j)} \cap \mathcal{E}_b^{(j)} = \emptyset$  and  $\mathcal{E}_a^{(j)} \cup \mathcal{E}_b^{(j)} = \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\}$ .

Let  $u_{\mathcal{T}}^{(j)} \in X(\mathcal{T})$  be defined by  $u_{\mathcal{T}}^{(j)}(x) = u_K^{(j)}$  for  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $u_{\mathcal{T}}^{(j)}(x) = u_{\sigma}^{(j)}$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3$  where  $(u_K^{(j)}, K \in \mathcal{T}, u_{\sigma}, \sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3)$  is the solution to the set of equations (105)-(109) and :

$$u_{\sigma}^{(j)} = a \quad \forall \sigma \in \mathcal{E}_a^{(j)}, \quad (113)$$

$$F_{K,\sigma}^{(j)} = -m(\sigma) b \quad \forall \sigma \in \mathcal{E}_b^{(j)}, \quad (114)$$

Let  $\mathcal{E}_a^{(j+1)}$  and  $\mathcal{E}_b^{(j+1)}$  be defined in the following way:

$$\begin{aligned} \mathcal{E}_{a,0}^{(j)} &= \{\sigma \in \mathcal{E}_a^{(j)}; -F_{K,\sigma}^{(j)} \geq m(\sigma) b\}, & \mathcal{E}_{a,1}^{(j)} &= \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\} \setminus \mathcal{E}_{a,0}^{(j)}, \\ \mathcal{E}_{b,0}^{(j)} &= \{\sigma \in \mathcal{E}_b^{(j)}; u_{\sigma}^{(j)} \geq a\}, & \mathcal{E}_{b,1}^{(j)} &= \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\} \setminus \mathcal{E}_{b,0}^{(j)}, \\ \mathcal{E}_a^{(j+1)} &= \mathcal{E}_{a,0}^{(j)} \cup \mathcal{E}_{b,1}^{(j)}, & \mathcal{E}_b^{(j+1)} &= \{\sigma \in \mathcal{E}_{\text{ext}}, \sigma \subset \Gamma^3\} \setminus \mathcal{E}_b^{(j+1)}, \end{aligned} \quad (115)$$

• The algorithm stops if there exists a step  $(J)$  such that  $\mathcal{E}_a^{(Jo)} = \mathcal{E}_a^{(J+1)}$  and  $\mathcal{E}_b^{(Jo)} = \mathcal{E}_b^{(J+1)}$ .

We give below a graphical representation of  $u$ , of the trace of  $u$  on the Signorini boundary  $\Gamma^3$ , of  $z = \nabla u \cdot \mathbf{n}(x, 0) \forall x \in [0, x_m]$  and the trace of  $u$  on  $\Gamma^1$ , with the following data are used:

$$x_m = 1, \quad y_m = 1, \quad a = -1, \quad b = -2. \quad (116)$$

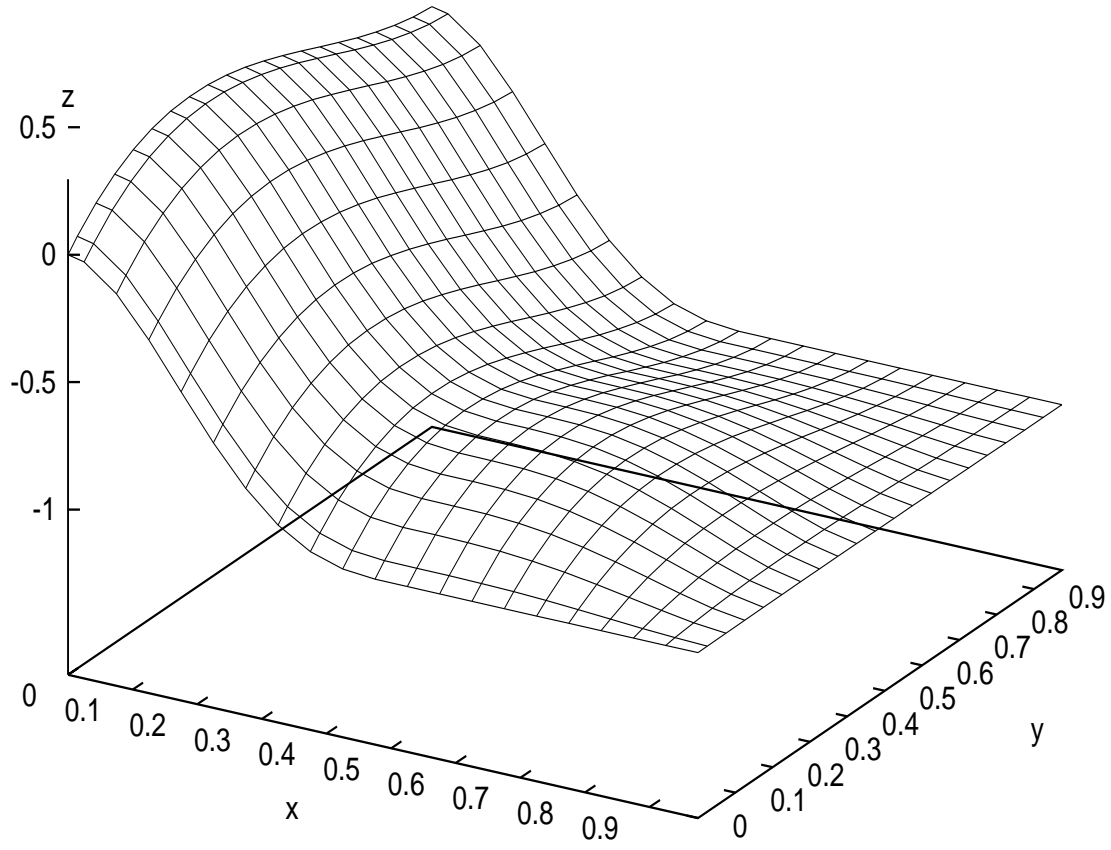


Figure 3:  $z = u(x, y) \forall (x, y) \in \Omega$

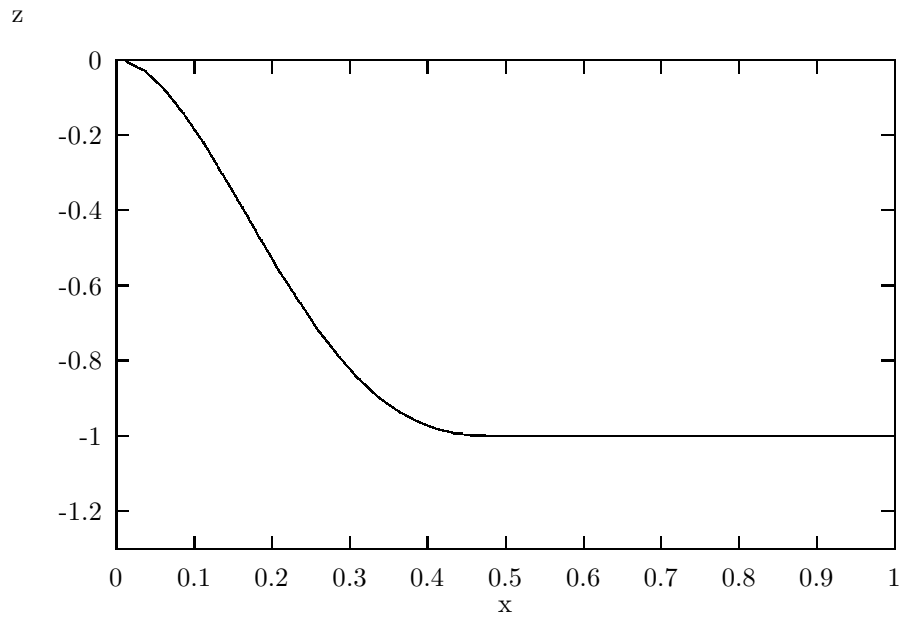


Figure 4:  $z = u(x, 0) \forall x \in [0, x_m]$  (on  $\Gamma^3$  : Signorini boundary)

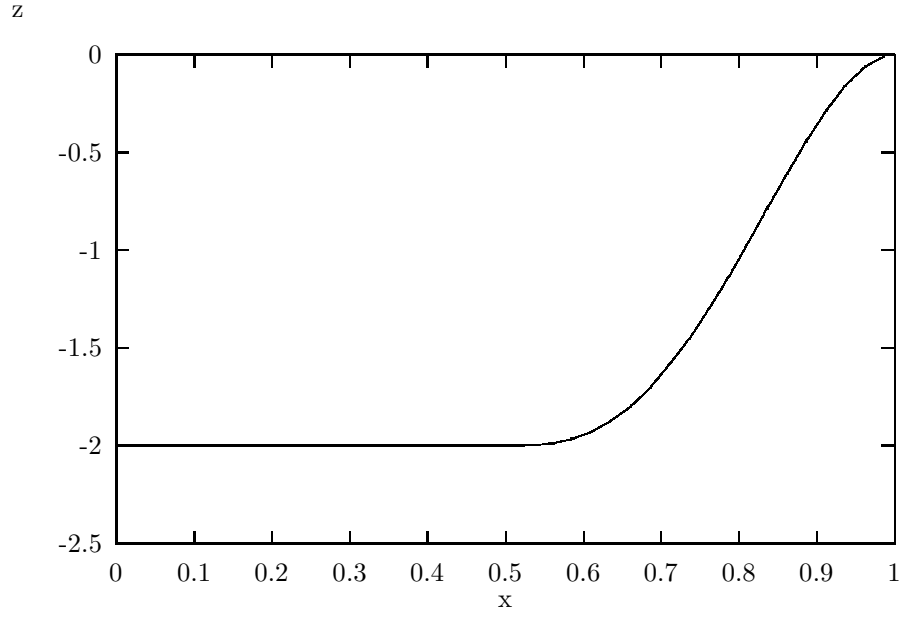


Figure 5:  $z = \nabla u \cdot \mathbf{n}(x, 0) \forall x \in [0, x_m]$  (on  $\Gamma^3$  : Signorini boundary)

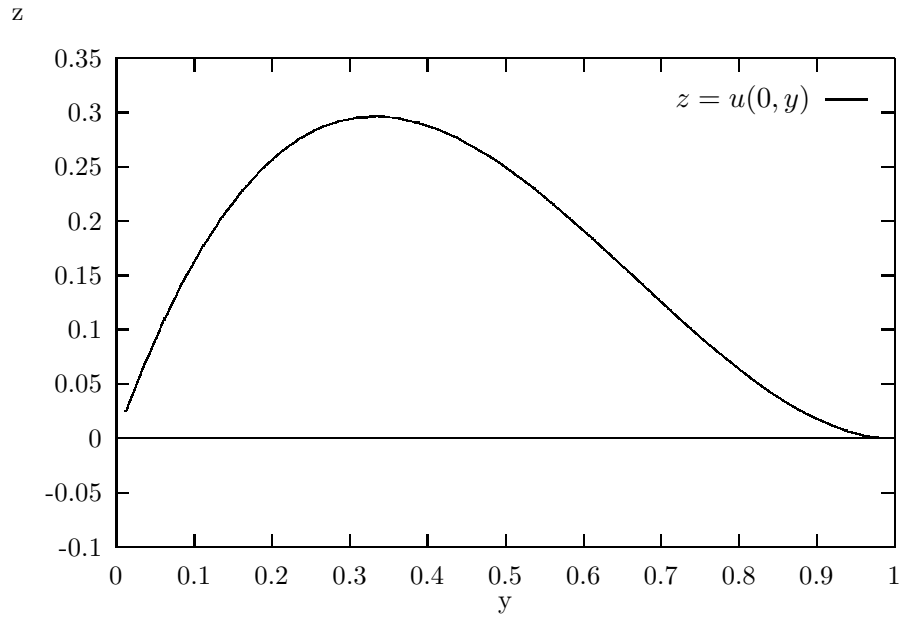


Figure 6:  $z = u(0, y) \forall y \in [0, y_m]$

In order to evaluate the error between the exact solution and the approximate solution, we define for each  $K \in \mathcal{T}$ ,  $e_K = u(x_K) - u_K$ , for each  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ ,  $e_\sigma = u(y_\sigma) - u_\sigma$ , with  $(u_K)_{K \in \mathcal{T}}$  and  $(u_\sigma)_{\sigma \in \Gamma^3}$  the (unique) solution to (105)-(112). Let us define again  $e_{\mathcal{T}} \in X(\mathcal{T})$  (see Definition 2.2) by  $e_{\mathcal{T}}(x) = e_K$  for  $x \in K$ , for all  $K \in \mathcal{T}$  and by  $e_{\mathcal{T}}(x) = e_\sigma$  for  $x \in \sigma$ , for all  $\sigma \in \mathcal{E}_{\text{ext}}$ ,  $\sigma \subset \Gamma^3$ .

The study of the logarithm of the norm of the error with respect to the logarithm of the mesh size allows us to determine an approximate order of convergence. With data defined in (116) using uniform meshes from  $900 = 30 \times 30$  to  $1600 = 40 \times 40$  cells, numerical tests give the following relations, as shown in figures 7, 8 and 9:

$$\|e_{\mathcal{T}}\|_{L^\infty(\Omega)} = C_\infty h^{1.95} \quad ,$$

$$\|e_{\mathcal{T}}\|_{1,\mathcal{T}} = C_1 h^{1.46} \quad \text{and}$$

$$\|e_{\mathcal{T}}\|_{L^2(\Omega)} = C_2 h^2,$$

with  $h = \text{size}(\mathcal{T})$  and  $\|\cdot\|_{1,\mathcal{T}}$  the discrete norm  $H_{0,\Gamma^1}^1$  defined in Definition 2.4.

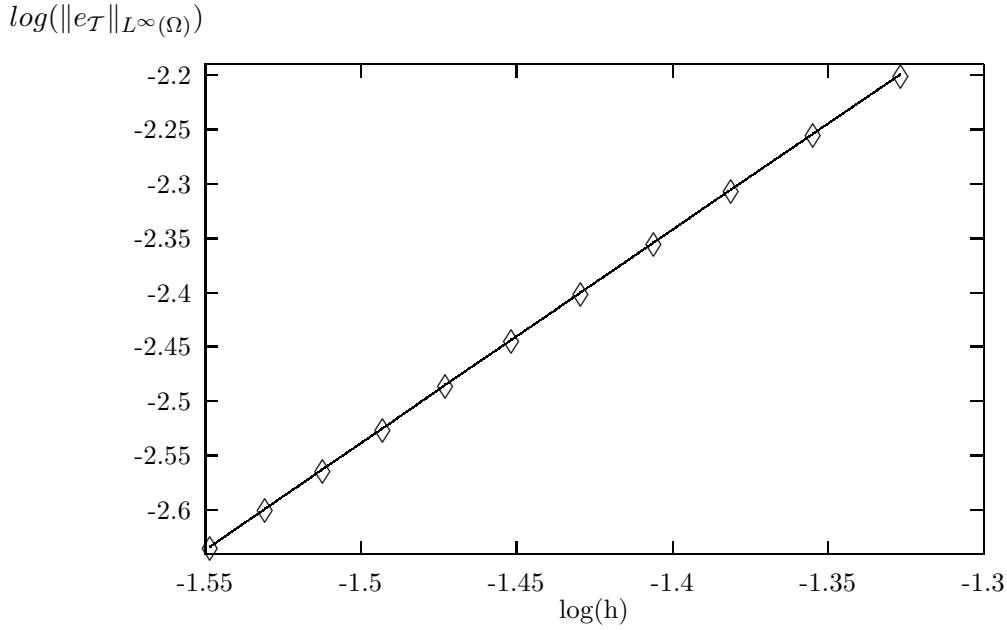


Figure 7:  $\log(\|e_{\mathcal{T}}\|_{L^\infty(\Omega)}) = \log(C_\infty) + 1.95\log(h)$

$\log(\|e_{\mathcal{T}}\|_{1,\mathcal{T}})$

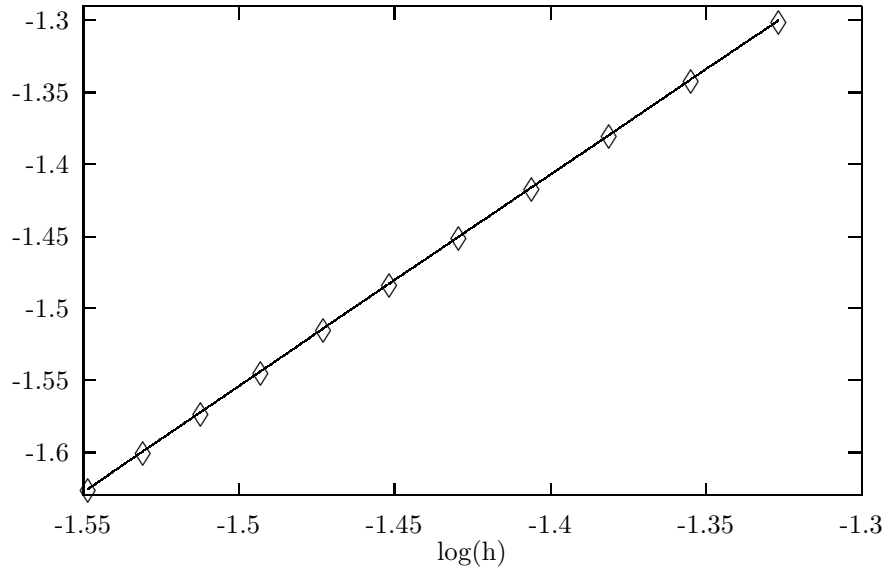


Figure 8:  $\log(\|e_{\mathcal{T}}\|_{1,\mathcal{T}}) = \log(C_1) + 1.46\log(h)$

$\log(\|e_{\mathcal{T}}\|_{L^2(\Omega)})$

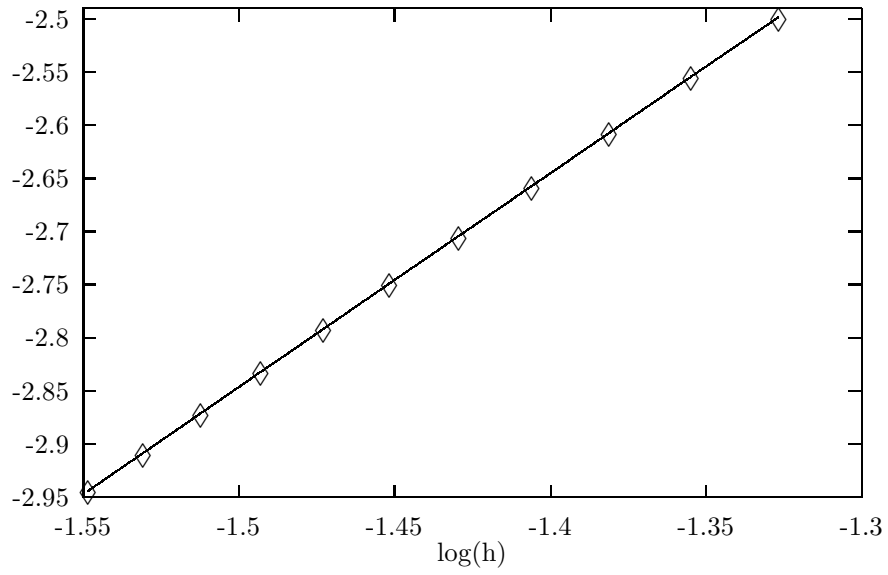


Figure 9:  $\log(\|e_{\mathcal{T}}\|_{L^2(\Omega)}) = \log(C_2) + 2\log(h)$

Recall that in Theorem 5.1, we found  $\|e_{\mathcal{T}}\|_{1,\mathcal{T}} \leq Ch$  and  $\|e_{\mathcal{T}}\|_{L^2(\Omega)} \leq Ch$ . The higher orders which we find here are due to the fact that we are using a second order approximation of the fluxes thanks to the fact that the mesh is uniform. This second order approximation of the fluxes would also be obtained for any Voronoï mesh (not necessarily uniform), for which the edges are equidistant from the points  $x_K$  defined in Definition 2.1. This is obvious by a Taylor expansion if the solution  $u$  is of class  $C^2$  in the one-dimensional case, and follows from an adaptation of the proof of the consistency of the flux in the case  $u \in H^2(\Omega)$  which is given in [9] or [16].

**Remark 6.1** *It is worthwhile mentioning that the monotonicity algorithm which is used here for the solution of the discrete Signorini problem has also been successfully adapted in the case of the obstacle problem [23].*

**Acknowledgements** The authors are grateful to Jérôme Droniou, Robert Eymard and Thierry Gallouët for numerous enlightening discussions.

## References

- [1] H. Brezis, Analyse fonctionnelle, Masson, 1980.
- [2] H. Brezis *Problèmes unilatéraux*, J. Maths pures et appl., 51, 1-168, (1972).
- [3] F. Brezzi, W.W. Hager, P.A. Raviart, *Error estimates for the finite element solution of variational inequalities* Part I - Primal theory. Numer. Math. 28, 431-443 (1977).
- [4] C. Carstensen J. Gwinner, Error estimates for the approximation of a class of variational inequalities, FEM and BEM coupling for a nonlinear transmission problem with Signorini contact. SIAM J. Numer. Anal. 34 (1997), no. 5, 1845–1864.
- [5] P.G. Ciarlet, The Finite Element Method, North-Holland, Amsterdam, 1978.
- [6] Y. Coudière, J.P. Vila and P. Villedieu (1999), Convergence rate of a finite volume scheme for a two-dimensional convection diffusion problem, Math. Model. Numer. Anal. 33 (1999), no. 3, 493–516.
- [7] G. Duvaut and J.L. Lions, Inequalities in mechanics and physics, Springer-Verlag, Berlin, 1976.
- [8] J. Droniou, *Un résultat de densité dans les espaces de Sobolev*, submitted.
- [9] R. Eymard, T. Gallouët and R. Herbin, *Finite Volume methods*, Preprint 97-19 of LATP, UMR 6632, Marseille, to appear in Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds, North Holland, see also <http://www-gm3.univ-mrs.fr/>.
- [10] R. Eymard, T. Gallouët and R. Herbin, *Convergence of finite Volume schemes for semilinear convection diffusion equations*, Numer. Math. 82:91-116, 1999.
- [11] I. Faille, Modélisation bidimensionnelle de la genèse et la migration des hydrocarbures dans un bassin sédimentaire, Thesis, Université de Grenoble, (1992).

- [12] I. Faille, A control volume method to solve an elliptic equation on a two-dimensional irregular meshing, *Comp. Meth. Appl. Mech. Engrg.*, 100, 275-290, (1992).
- [13] R. Falk, *Error estimates for the approximation of a class of variational inequalities*, Math. Comput, 28, 963-971 (1974).
- [14] R. Falk and B. Mercier, *Error estimates for elatoplastic problems*, Rev. Française Automat. Informat. Rech. Operationnelle . Anal Num. 11, 135 -144 (1977).
- [15] J.M. Fiard and R. Herbin, *Comparison between finite volume finite element methods for the numerical simulation of an elliptic problem arising in electrochemical engineering*, Comput. Meth. Appl. Mech. Engin., 115, 315-338, 1994.
- [16] T. Gallouët, R. Herbin and M.H. Vignal, *Error estimate for the approximate finite volume solutions of convection diffusion equations with general boundary conditions*, preprint 99-2, LATP, Marseille, to appear in SIAM J. Numer. Anal. 2000.
- [17] S. Gerbi, R. Herbin and E. Marchand, *Existence of a solution to a coupled elliptic system with a Signorini condition*, Advances in Differential Equations, 4,2, 225-250, 1999.
- [18] R. Glowinski, J.L. Lions and R. Tremolière, Analyse numérique des inéquations variationnelles, 2 vols., Dunod, Paris, 1976.
- [19] R. Glowinski, *Numerical methods for non-linear variational problems*, Tata Institute of fundamental research, Bombay, 1980.
- [20] J. Gwinner, A discretization theory for monotone semicoercive problems and finite element convergence for  $p$ -harmonic Signorini problems. Z. Angew. Math. Mech. 74 (1994), no. 9, 417-427.
- [21] R. Herbin, *An error estimate for a finite volume scheme for a diffusion-convection problem on a triangular mesh*, Num. Meth. P.D.E., **11**, 165-173, 1995.
- [22] R. Herbin, *Finite volume methods for diffusion convection equations on general meshes*, in Finite volumes for complex applications, Problems and Perspectives, F. Benkhaldoun and R. Vilsmeier eds, Hermes, 153-160, 1996.
- [23] R. Herbin, *A monotonicity method for the numerical solution of some free boundary value problems*, submitted.
- [24] R. Herbin and E. Marchand, *Numerical approximation of a nonlinear problem with a Signorini condition*, Proc. of Third IMACS International Symposium on Iterative Methods in Scientific Computation, 1997.
- [25] N. Kikuchi and J.T. Oden, Contact problems in elasticity, SIAM, philadelphia, 1981.
- [26] M. Kleitz, L.Dessemond, R. Jimenez, F. Petitbon, R. Herbin et E. Marchand, *Micro-modelling of the cathode and experimental approaches*, Proc. of the second European Solid Oxide Fuel Cell Forum, Oslo, Norway, 1996.



- [27] R.D. Lazarov and I.D. Mishev, *Finite volume methods for reaction diffusion problems* in: F. Benkhaldoun and R. Vilsmeier eds, *Finite volumes for complex applications, Problems and Perspectives* (Hermes, Paris), 233-240, 1996.
- [28] R.D. Lazarov, I.D. Mishev and P.S Vassilevski, *Finite volume methods for convection-diffusion problems*, SIAM J. Numer. Anal., 33, 31-55, 1996.
- [29] J.L. Lions, *Quelques méthodes de résolution de problèmes aux limites non linéaires*, Dunod, Paris, (1969).
- [30] I.D. Mishev, *Finite volume methods on Voronoï meshes*, Num. Meth. P.D.E. 14, 2, 193-212, 1998.
- [31] U. Mosco, G. Strang (1974) *One sided approximation and variational inequalities* , Bull. Amer. Math. Soc. 80, 308-312.
- [32] J.F. Rodrigues, Obstacle problems in mathematical physics, Mathematics studies, North Holland, 1987.
- [33] R. Temam, Navier-Stokes Equations North Holland, Amsterdam, 1977.