Existence of a solution to a coupled elliptic system

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Abstract We study here an elliptic system. Using the L^1 theory for elliptic operators, we prove existence of a solution by a fixed point technique.

Keywords : Nonlinear elliptic system, Schauder Theorem, L^1 data, Electrochemical modelling.

1 Introduction

While studying the modelling of an electrochemical engineering problem [7], [8], we came across a coupled system between temperature and electrical potential. One of the coupling terms arises from the "Joule effect", i.e. the production of heat by electrical current. This term may be encountered in other physical problems, see for instance [5]. Here, in an attempt to prove existence of the solution to the electrochemical problem, we study the following simplified problem:

$$
\begin{aligned}
-\mathrm{div}(\sigma(x,u(x))D\phi(x)) &= f(x,u(x)),\ x \in \Omega, && (1)\\
\phi(x) &= 0,\ x \in \partial\Omega, && (2)\\
-\mathrm{div}(\lambda(x,u(x))Du(x)) &= \sigma(x,u(x))D\phi(x).D\phi(x),\ x \in \Omega, && (3)\\
u(x) &= 0,\ x \in \partial\Omega, && (4)
\end{aligned}
$$

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where Ω is a bounded regular domain of $\mathbb{R}^N$, $N \in \mathbb{N}, N > 1$, $\partial\Omega$ its boundary, σ, λ and f are bounded functions from $\Omega \times \mathbb{R}$ to $\mathbb{R}$, continuous with respect to $y \in \mathbb{R}$ for a.e. $x \in \Omega$, and measurable with respect to $x \in \Omega$ for any $y \in \mathbb{R}$, and such that:

$$\exists \alpha > 0; \alpha \leq \sigma(.,y) \text{ and } \alpha \leq \lambda(.,y), \forall y \in \mathbb{R}, \text{ for a.e. } x \in \Omega. \tag{5}$$

The aim of this work is to prove the following existence theorem:

Theorem 1 *Under the above assumptions, there exists a solution (u,ϕ) satisfying :*

$$\left\{ \begin{array}{l} u \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), \phi \in H_0^1(\Omega), \\ \int_\Omega (\sigma(x,u(x))D\phi(x)D\psi(x)dx = \int_\Omega f(x,u(x))\psi(x)dx, \forall \psi \in H_0^1(\Omega), \\ \int_\Omega \lambda(x,u(x))Du(x)Dv(x)dx = \int_\Omega \sigma(x,u(x))D\phi(x).D\phi(x)v(x)dx, \\ \forall v \in \cup_{q>N} W_0^{1,q}(\Omega). \end{array} \right. \tag{6}$$

Let us remark that, since for $q > N$, $W_0^{1,q}(\Omega) \subset C(\overline{\Omega}, \mathbb{R})$, all the terms in (6) make sense.
The proof of this theorem is detailed in the following sections. We begin by the construction of an operator for the application of Schauder's fixed point theorem. Note that, for other physical problems, Clain [4] gives, with a slightly different technique, existence results for parabolic problems.

2 Construction of the fixed point operator

For a given $u \in L^1(\Omega)$, let $\phi_u \in H_0^1(\Omega)$ denote the unique solution (by Lax-Milgram's theorem) to the problem:

$$\left\{ \begin{array}{l} \phi_u \in H_0^1(\Omega), \\ \int_\Omega (\sigma(x,u(x))D\phi_u(x)D\psi(x)dx = \int_\Omega f(x,u(x))\psi(x)dx, \forall \psi \in H_0^1(\Omega); \end{array} \right. \tag{7}$$

denoting by g_u the application: $x \in \Omega \mapsto \sigma(x,u(x))D\phi_u(x).D\phi_u(x)$, we consider next the following problem :

$$\left\{ \begin{array}{l} v \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), \\ \int_\Omega \lambda(x,u(x))Dv(x)Dw(x)dx = \int_\Omega g_u(x)w(x)dx, \forall w \in \cup_{q>N} W_0^{1,q}(\Omega). \end{array} \right. \tag{8}$$

Since $\sigma \in L^\infty(\Omega \times \mathbb{R})$ and $\phi_u \in H_0^1(\Omega)$, we have $g_u \in L^1(\Omega)$. Hence there is no "classical" variational formulation for this problem. We shall distinguish here the cases $N = 2$ and $N \geq 3$. We shall prove in the following section Theorem 2 for $N = 2$, the existence part of which is due to Boccardo and Gallouët [3]. For $N > 2$, we shall use the results of [1] (see Theorem 3).

Theorem 2 *Let $N=2$, and $\mu \in M(\Omega)$, where $M(\Omega)$ is the set of bounded Radon measures (that is, the dual of $C(\overline{\Omega}, \mathbb{R})$ with its usual topology), let $\Lambda = (\lambda_{i,j})_{i,j=1,\ldots,N}$ such that $\lambda_{i,j} \in L^\infty(\Omega)$, for all $i,j = 1,\ldots,N$, and such that there exists $\beta > 0$ such that $\beta|\xi|^2 \le \sum_{i,j=1}^N \lambda_{i,j}\xi_i\xi_j$, for all $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$. Then there exists a unique function u such that:*

$$\left\{ \begin{array}{l} u \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), \\ \int_\Omega \Lambda(x)Du(x)D\varphi(x)dx = \int_\Omega \varphi(x)d\mu(x), \forall \varphi \in \cup_{q>N} W_0^{1,q}(\Omega). \end{array} \right. \tag{9}$$

Moreover, for any $p < \frac{N}{N-1}$, there exists $C_p \in \mathbb{R}_+$ depending on p, β and Ω such that $||u||_{W_0^{1,p}(\Omega)} \le C_p ||\mu||_{M(\Omega)}$. (If $\mu \in L^1(\Omega)$, i.e. there exists $g \in L^1(\Omega)$ such that $\int_\Omega \varphi(x)d\mu(x) = \int_\Omega g(x)\varphi(x)dx$, then $||\mu||_{M(\Omega)} = ||g||_{L^1(\Omega)}$.)

For $N > 2$, the uniqueness part of Theorem 2 does not hold (see [10]). In the case $\mu \in L^1(\Omega)$, a uniqueness result can be obtained by adding some "entropy conditions" [1], as in Theorem 3 below.

Theorem 3 *Let $g \in L^1(\Omega)$, let $\Lambda = (\lambda_{i,j})_{i,j=1,\ldots,N}$ such that $\lambda_{i,j} \in L^\infty(\Omega)$, for all $i,j = 1,\ldots,N$, and such that there exists $\beta > 0$ such that $\beta|\xi|^2 \le \sum_{i,j=1}^N \lambda_{i,j}\xi_i\xi_j$, for all $\xi \in \mathbb{R}^N$, for a.e. $x \in \Omega$. Then there exists a unique function u such that:*

$$\left\{ \begin{array}{lcl} u \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), & & \\ \int_\Omega \Lambda(x)Du(x)D\varphi(x)dx & = & \int_\Omega g(x)\varphi(x)dx, \forall \varphi \in \cup_{q>N} W_0^{1,q}(\Omega), \end{array} \right. \tag{10}$$

and

$$\left\{ \begin{array}{lcl} u \in L^1(\Omega), T_k(u) \in H_0^1(\Omega), & & \\ \int_\Omega \Lambda(x)Du(x)D(T_k(u-\varphi))(x)dx & \le & \int_\Omega g(x)T_k(u-\varphi)(x)dx, \\ & & \forall \varphi \in C_c^\infty(\Omega), \forall k \in \mathbb{R}_+, \end{array} \right. \tag{11}$$

where $T_k(s) = \max(-k, \min(s,k))$, $s \in \mathbb{R}$.
Moreover, the application $g \mapsto u$ is linear, and, for any $p < \frac{N}{N-1}$, there exists $C_p \in \mathbb{R}_+$ depending on p, β and Ω such that $||u||_{W_0^{1,p}(\Omega)} \le C_p ||g||_1$.

Remark In fact, in [1], the coefficient Λ does not depend on x. However, this is an easy generalization with the the above assumptions on Λ.

Remark In the case of a measure μ (with $g(x)dx$ replaced by $d\mu(x)$ in the right hand sides of (10) and (11)), the above characterization is not a convenient way to select a unique solution to (10), in particular, the right hand side of (11) is not defined in this case. It is however possible to prove that the "solution of (10) obtained by approximation" is unique (see [6] and [9]).

Let $u \in L^1(\Omega)$, ϕ_u the solution to (7) and $\overline{u}$ be the solution to problem (8) given by theorems 2 and 3 (with $\mu = g = g_u$, $\lambda_{i,j}(x) = \lambda(x, u(x))\delta_{i,j}$, and $\beta = \alpha$).

Since $\overline{u} \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega)$, $F : u \mapsto \overline{u}$ is an application from $L^1(\Omega)$ into $L^1(\Omega)$. Hence, if we prove that F is continuous and compact from $L^1(\Omega)$ into a ball of $L^1(\Omega)$, Theorem 1 will be proven by Schauder's theorem.

3 Proof of theorems 2 and 3

We begin with the case $N = 2$, i.e. Theorem 2. The existence of u satisfying (9) was proven in [3]. In order to prove uniqueness, we shall prove that if v satisfies:

$$\left\{ \begin{array}{l} v \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), \\ \int_\Omega \Lambda(x) Dv(x) D\varphi(x) dx = 0, \forall \varphi \in \cup_{q>N} W_0^{1,q}(\Omega), \end{array} \right. \tag{12}$$

then $v \equiv 0$.
Indeed, let v satisfy (12). Let $\Lambda^* = (\lambda_{ji})_{i,j=1,\dots,N}$, $B \subset \Omega$ be a measurable set and ψ_B solution to:

$$\left\{ \begin{array}{l} \psi_B \in H_0^1(\Omega), \\ \int_\Omega \Lambda^*(x) D\psi_B(x) D\varphi(x) dx = \int_B \varphi(x) dx, \forall \varphi \in H_0^1(\Omega). \end{array} \right. \tag{13}$$

In fact, ψ_B is the variational solution to the Dirichlet problem :

$$\left\{ \begin{array}{rcl} -\mathrm{div}(\Lambda^*(x) D\psi_B(x)) &=& 1_B(x),\ x \in \Omega, \\ \psi_B(x) = 0,\ x \in \partial\Omega. && \end{array} \right. \tag{14}$$

Using a regularity result of Meyers (see [2]), since $1_B \in L^\infty(\Omega)$, there exists $\overline{q} > 2$ (depending only on Λ, Ω and not on B) such that $\psi_B \in W_0^{1,\overline{q}}(\Omega)$. Therefore, taking $\varphi = \psi_B$ in (12) (recall that $N = 2$) yields:

$$\int_\Omega \Lambda(x) Dv(x) D\psi_B(x) dx = 0 \tag{15}$$

Since $\overline{q}' = \frac{\overline{q}}{\overline{q}-1} < 2$, we have $v \in W_0^{1,\overline{q}'}(\Omega)$; therefore, there exists $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty(\Omega, \mathbb{R})$ such that $\varphi_n \to v$ as $n \to +\infty$, in $W_0^{1,\overline{q}'}(\Omega)$. Now, taking $\varphi = \varphi_n$ in (13) yields:

$$\int_\Omega \Lambda^*(x) D\psi_B(x) D\varphi_n(x) dx = \int_B \varphi_n(x) dx \tag{16}$$

Letting $n \to +\infty$, we obtain:

$$\int_\Omega \Lambda^*(x) D\psi_B(x) Dv(x) dx = \int_B v(x) dx \tag{17}$$

Using (15) yields $\int_B v(x) dx = 0$, and, since B is arbitrary, $v = 0$ a. e..
Hence we have shown that for any $\mu \in M(\Omega)$, there exists a unique u_μ solution to (9). The application $u \mapsto u_\mu$ is clearly linear from $M(\Omega)$ into $\cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega)$.

Moreover, it is proven in [3] that for any $p < \frac{N}{N-1}$, there exists c_p depending on p, β and Ω such that if $||\mu||_{M(\Omega)} \leq 1$, then $||u_\mu||_{W_0^{1,p}(\Omega)} \leq c_p$. Therefore, by linearity, $||u_\mu||_{W_0^{1,p}(\Omega)} \leq c_p||\mu||_{M(\Omega)}$. This completes the proof of Theorem 2.

In the case $N > 2$, the method developped in [3] gives the existence of the solution u to:

$$\begin{cases} u \in \cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega), \\ \int_\Omega \Lambda(x)Du(x)D\varphi(x)dx = \int_\Omega \varphi(x)d\mu(x), \forall \varphi \in \cup_{q>N} W_0^{1,q}, \end{cases} \tag{18}$$

The solution to (18) is not unique (see [10]). However, in the case $\mu = g$ with $g \in L^1(\Omega)$, the solution constructed in [3] also satisfies (see [1]):

$$\begin{cases} u \in L^1(\Omega), T_k(u) \in H_0^1(\Omega), \forall k \in \mathbb{R}_+, \\ \quad \int_\Omega (\Lambda(x)Du(x)D(T_k(u-\varphi))(x)dx \quad \leq \quad \int_\Omega g(x)T_k(u-\varphi)(x)dx, \\ \qquad \forall \varphi \in C_c^\infty(\Omega, \mathbb{R}), \forall k \in \mathbb{R}_+, \end{cases} \tag{19}$$

where $T_k(s) = \max(-k, \min(s,k)), s \in \mathbb{R}$. This gives the existence part of Theorem 3. Moreover, we still have (see [3]) :

$$\begin{array}{l} \forall p < \frac{N}{N-1}, \exists c_p \text{ depending on } p, \beta \text{ and } \Omega \text{ such that } ||\mu||_{M(\Omega)} \leq 1 \Rightarrow \\ ||u||_{W_0^{1,p}(\Omega)} \leq c_p, \text{ where } u \text{ is a solution obtained by [3].} \end{array} \tag{20}$$

In fact, (19) implies (18), and the solution to (19) is unique [1]. This gives the uniqueness part of Theorem 3. The linearity of the application $g \mapsto u$ from $L^1(\Omega)$ to $\cap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega)$, where u is the solution to (19), is a straightforward consequence of the approximation technique used in [3] to construct the solution to (18). Finally, the fact that $||u||_{W_0^{1,p}} \leq c_p||g||_1, \forall p < \frac{N}{N-1}$, where c_p depends on p, β and Ω is a consequence of the linearity of the application and (20).

4 Proof of theorem 1

In order to prove that F is continuous from $L^1(\Omega)$ to $L^1(\Omega)$, we first show the following lemma:

Lemma 1 *Let $(u_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$; let $\phi_{u_n} \in H_0^1(\Omega)$ be the solution to the following problem:*

$$\begin{cases} \phi_{u_n} \in H_0^1(\Omega), \\ \int_\Omega (\sigma(x, u_n(x))D\phi_{u_n}(x)D\varphi(x)dx = \int_\Omega f(x, u_n(x))\varphi(x)dx, \\ \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{21}$$

Then $\phi_{u_n} \to \phi_u$ in $H_0^1(\Omega)$ for the strong topology (recall that ϕ_u is the solution to problem (7).).

Proof: Since $f \in L^\infty(\Omega \times \mathbb{R})$, from (21) we get that $\|D\phi_{u_n}\|_{L^2(\Omega)} \leq \frac{C}{\alpha}\|f\|_\infty$ $\text{meas}(\Omega)^{\frac{1}{2}}$, where C is the Poincaré constant. The sequence $(\phi_{u_n})_{n\in\mathbb{N}}$ is therefore bounded in $H_0^1(\Omega)$. Hence there exists a subsequence $(\phi_{u_{n_k}})_{k\in\mathbb{N}}$ which converges to a limit, say ϕ, in $H_0^1(\Omega)$ for the weak topology.
Let us first show that $\phi = \phi_u$. Since $\sigma(x,.)$ is a continuous function, there exists a subsequence of $(u_{n_k})_{k\in\mathbb{N}}$ which, for the sake of simplicity, will still be denoted by $(u_{n_k})_{k\in\mathbb{N}}$, such that $\sigma(x, u_{n_k}(x)) \rightarrow \sigma(x, u(x))$ for a.e. $x \in \Omega$ as $k \rightarrow +\infty$. Hence, for $\psi \in H_0^1(\Omega)$, by Lebesgue's theorem, $\sigma(., u_{n_k}(.))D\psi \rightarrow \sigma(., u(.))D\psi$ in $L^2(\Omega)$ for the strong topology, as $k \rightarrow +\infty$, so that :

$$\begin{array}{l} \int_\Omega \sigma(x, u_{n_k}(x))D\phi_{u_{n_k}}(x)D\psi(x)dx \rightarrow \int_\Omega \sigma(x, u(x))D\phi(x)D\psi(x)dx, \\ \text{as } k \rightarrow +\infty. \end{array} \tag{22}$$

Now, by definition of $\phi_{u_{n_k}}$:
$\int_\Omega \sigma(x, u_{n_k}(x))D\phi_{u_{n_k}}(x)D\psi(x)dx = \int_\Omega f(x, u_{n_k}(x))\ \psi(x)dx.$
Since f is bounded and continuous with respect to its second variable, we have, for a subsequence of $(u_{n_k})_{k\in\mathbb{N}}$, (still denoted $(u_{n_k})_{k\in\mathbb{N}}$):

$$\int_\Omega f(x, u_{n_k}(x))\psi(x)dx \rightarrow \int_\Omega f(x, u(x))\psi(x)dx \text{ as } k \rightarrow +\infty. \tag{23}$$

Therefore, because of the uniqueness of the solution to problem (7), we obtain $\phi = \phi_u$. Note also that, by a classical argument, the whole sequence $(\phi_{u_n})_{n\in\mathbb{N}}$ converges to ϕ_u in $H_0^1(\Omega)$ for the weak topology.
Let us now show that the sequence $(\phi_{u_n})_{n\in\mathbb{N}}$ converges to ϕ_u in $H_0^1(\Omega)$ for the strong topology; to this purpose, we show that: $I_n = \int_\Omega \sigma(x, u_n(x))D(\phi_{u_n} - \phi_u)(x)D(\phi_{u_n} - \phi_u)(x)dx \rightarrow 0$ as $n \rightarrow +\infty$. Therefore, since $I_n \geq \alpha\|D\phi_{u_n} - D\phi_u\|^2_{L^2(\Omega)}$, Lemma 1 is proven.
The integral I_n can be written as the sum of three terms: $I_n = I_n^1 + I_n^2 + I_n^3$, with:

$$\begin{aligned} I_n^1 &= \int_\Omega \sigma(x, u_n(x))D\phi_{u_n}(x)D\phi_{u_n}(x)dx, & (24)\\ I_n^2 &= -2\int_\Omega \sigma(x, u_n(x))D\phi_{u_n}(x)D\phi_u(x)dx, & (25)\\ I_n^3 &= \int_\Omega \sigma(x, u_n(x))D\phi_u(x)D\phi_u(x)dx; & (26) \end{aligned}$$

Since $I_n^1 = \int_\Omega f(x, u_n(x))\phi_{u_n}(x)dx$, $I_n^1 \rightarrow \int_\Omega f(x, u(x))\phi_u(x)dx$ as $n \rightarrow +\infty$. Using the same arguments as in the proof of (22), it is easy to show that:

$$I_n^2 \rightarrow -2\int_\Omega \sigma(x, u(x))D\phi_u(x)D\phi_u(x)dx, \text{ as } n \rightarrow +\infty,$$

so that: $I_n^2 \to -2\int_\Omega f(x,u(x))\phi_u(x)dx$. Finally, by continuity of σ with respect to the second variable, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that: $I_{n_k}^3 \to \int_\Omega f(x,u(x))\phi_u(x)dx$ as $k \to +\infty$. Therefore, by a classical argument, the whole sequence I_n^3 tends to $\int_\Omega f(x,u(x))\phi_u(x)dx$ as $n \to +\infty$. Therefore, $I_n \to 0$ as $n \to +\infty$, so that $(\phi_{u_n})_{n\in\mathbb{N}}$ converges to ϕ_u in $H_0^1(\Omega)$ for the strong topology. The proof of Lemma 1 is now complete.

In order to prove Theorem 1, let $(u_n)_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ as $n \to +\infty$, we shall now prove that $\overline{u}_n = F(u_n) \to \overline{u} = F(u)$ in $L^1(\Omega)$. Thanks to lemma 1, we may construct a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\sigma(x,u_{n_k}(x)) \to \sigma(x,u(x))$, $D\phi_{u_{n_k}}(x) \to D\phi_u(x)$ for a.e $x \in \Omega$, as $k \to +\infty$ and $\sigma(x,u_{n_k}(x))D\phi_{u_{n_k}}(x)D\phi_{u_{n_k}}(x) \le \varphi(x)$ for a.e. $x \in \Omega$, where φ is some function of $L^1(\Omega)$. Therefore, by Lebesque's theorem, $g_{u_{n_k}} \to g_u$ in $L^1(\Omega)$, as $k \to +\infty$, and, by a classical argument, $g_{u_n} \to g_u$ in $L^1(\Omega)$, as $n \to +\infty$.
From theorems 2 and 3, we know that $(\overline{u}_n)_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$, $\forall p < \frac{N}{N-1}$. Hence there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and $w \in \bigcap_{p<\frac{N}{N-1}} W_0^{1,p}(\Omega)$ such that:

$$\begin{aligned}
&u_{n_k} \to u \text{ a.e.}, \ \overline{u}_{n_k} \to w \text{ a.e.}, \ g_{u_{n_k}} \to g_u \text{ a.e.},\\
&|g_{u_{n_k}}| \le F \text{ a.e.}, \forall k \in \mathbb{N}, \text{ with } F \in L^1(\Omega),\\
&D\overline{u}_{n_k} \to Dw \text{ in } L^p(\Omega) \text{ for the weak topology } \forall p < \frac{N}{N-1}.
\end{aligned}$$

Let $\varphi \in W_0^{1,q}(\Omega), q > N$; by definition of u_n,

$$\int_\Omega \lambda(x,u_{n_k}(x))D\overline{u}_{n_k}(x)D\varphi(x)dx = \int_\Omega g_{u_{n_k}}(x)\varphi(x)dx;$$

therefore, since $\varphi \in L^\infty(\Omega)$ and $\frac{q}{q-1} < \frac{N}{N-1}$, by the dominated convergence theorem,

$$\int_\Omega \lambda(x,u(x))Dw(x)D\varphi(x)dx = \int_\Omega g_u(x)\varphi(x)dx.$$

In the case $N = 2$, by Theorem 2, this is sufficient to prove that $w = \overline{u}$ and that the whole sequence $(\overline{u}_n)_{n\in\mathbb{N}}$ converges to $\overline{u}$ (in $W_0^{1,p}(\Omega)$ for the weak topology, for all $p < \frac{N}{N-1}$)). Hence, thanks to the compactness of the injection of $W^{1,1}(\Omega)$ in $L^1(\Omega)$, F is continuous (and compact) from $L^1(\Omega)$ to $L^1(\Omega)$.
In the case $N > 2$, we still have to prove that w satisfies the entropy inequalities (11) in order to write that $w = \overline{u}$.
Let $m \in \mathbb{R}_+$, from [3], we have $\|DT_m(\overline{u}_n)\|_2^2 \le \frac{m}{\beta}\|g_{u_n}\|_1$, so that the sequence $(T_m(\overline{u}_n))_{n\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. Hence $T_m(\overline{u}_{n_k}) \to T_m(w)$ in $H_0^1(\Omega)$ for the weak topology, for all $m \in \mathbb{R}_+$. Now, by definition of $\overline{u}_n$,

$$\left\{ \begin{array}{l} \int_\Omega \lambda(x,u_{n_k}(x))1_{\{|\overline{u}_{n_k}-\varphi|\le m\}}D\overline{u}_{n_k}(x)D(\overline{u}_{n_k}-\varphi)(x)dx \le \\ \int_\Omega g_{u_{n_k}}(x)T_m(\overline{u}_{n_k}-\varphi)(x)dx, \ \forall\varphi \in C_c^\infty(\Omega), \forall m \in \mathbb{R}_+, \end{array} \right. \tag{27}$$

For given $\varphi \in C_c^\infty(\Omega)$ and $m \in \mathbb{R}_+$, taking $M \geq m + ||\varphi||_\infty$, (27) yields :

$$\left\{ \begin{array}{l} \int_\Omega \lambda(x, u_{n_k}(x)) 1_{\{|\overline{u}_{n_k} - \varphi| \leq m\}} D(T_M(\overline{u}_{n_k}))(x) D(T_M(\overline{u}_{n_k}))(x) dx - \\ \int_\Omega \lambda(x, u_{n_k}(x)) 1_{\{|\overline{u}_{n_k} - \varphi| \leq m\}} D(T_M(\overline{u}_{n_k}))(x) D\varphi(x) dx \leq \\ \int_\Omega g_{u_{n_k}}(x) T_m(\overline{u}_{n_k} - \varphi)(x) dx. \end{array} \right. \tag{28}$$

Since $g_{u_{n_k}} \to g_u$ a.e., $|g_{u_{n_k}}| \leq F$ with $F \in L^1(\Omega)$, and $\overline{u}_{n_k} \to w$ a.e., the right hand side of (28) tends to $\int_\Omega g_u(x) T_m(w - \varphi)(x) dx$ as $k \to +\infty$, while the second term of the left hand side of (28) tends to $\int_\Omega \lambda(x, u(x))\, 1_{\{|w-\varphi| \leq m\}} D(T_M(w))(x)$ $D\varphi(x) dx$. Now remark that the first term of the left hand side of (28) can be written $||f_{n_k}||_2^2$ with $f_n = \Big(\lambda(., u_n)\Big)^{\frac{1}{2}}\, 1_{\{|\overline{u}_n - \varphi| \leq m\}} D(T_M(\overline{u}_n))$; since $f_{n_k} \to f = \Big(\lambda(., u)\Big)^{\frac{1}{2}} 1_{\{|w-\varphi| \leq m\}} D(T_M(w))$ in $L^2(\Omega)$ for the weak topology, as $k \to +\infty$, we have $||f||_2 \leq \liminf_{k \to +\infty} ||f_{n_k}||_2$, so that, finally:

$$\int_\Omega \lambda(., u) 1_{\{|w-\varphi| \leq m\}} Dw D(w - \varphi) \leq \int_\Omega g_u(x) T_m(w - \varphi)(x) dx.$$

Hence $w = \overline{u}$, and the whole sequence $(\overline{u}_n)_{n \in \mathbb{N}}$ converges to $\overline{u}$ (in $W_0^{1,p}(\Omega)$ for the weak topology, for all $p < \frac{N}{N-1}$). Therefore, thanks to the compactness of the injection of $W^{1,1}(\Omega)$ in $L^1(\Omega)$, F is continuous and compact from $L^1(\Omega)$ into $L^1(\Omega)$.
Now, in order to complete the proof of Theorem 1, we remark that, since ϕ_u si bounded in $H_0^1(\Omega)$, g_u is bounded in $L^1(\Omega)$ independently of $u \in L^1(\Omega)$. Hence, since theorems 2 and 3 give that $||\overline{u}||_{W_0^{1,p}(\Omega)} \leq C_p ||g_u||_{L^1(\Omega)}$, we have, for some $R > 0$, $F(L^1(\Omega)) \subset B(0, R)$, where $B(0, R)$ is the ball of $L^1(\Omega)$, of radius R. Therefore, we may apply Schauder's theorem so that Theorem 1 is proven.

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