# Unstructured cell centred schemes for the Navier-Stokes equations

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ABSTRACT. *This paper presents a review of recent finite volume schemes which were developed for the numerical simulation of the incompressible Navier-Stokes equations on unstructured meshes. Both a staggered scheme and a colocated scheme were analysed. The colocated scheme, where the unknowns $u, v$ and $p$ are all located within the discretization cell, is easy to implement ; moreover, it generalizes to 3D. A cluster-type stabilization technique is used, which avoids a mass redistribution over the whole domain. We conclude with the discretization of the full viscous tensor for constant viscosity in the compressible case.*

RÉSUMÉ. *Cet article présente des schémas volumes finis récemment développés pour la résolution numérique des équations de Navier–Stokes incompressibles. Un schéma sur maillage décalé et un maillage de type localisé ont été étudiés. Le schéma colocalisé, où les inconnues $u, v$ et $p$ sont toutes localisées dans les mailles s'est révélé plus facile à implanter ; de plus, il se généralise à 3 dimensions d'espace. Une stabilisation par agrégats de cellules a été introduite, qui permet d'éviter une trop grande redistribution de masse. Enfin, on termine avec une proposition de discrétisation du tenseur visqueux, dans le cas compressible avec coefficients de viscosité constants.*

KEYWORDS: *cell centered finite volumes, incompressible Navier-Stokes, stabilisation*

MOTS-CLÉS : *Navier-Stokes incompressible, volumes finis*

## 1. Introduction

Numerical schemes for the incompressible Navier-Stokes equations have been extensively studied: see [GIR 86, GUN 89, GLO 03, KWA 05] and references therein. An advantage of the finite volume schemes is that the unknowns are approximated by piecewise constant functions: this makes it easy to take into account additional nonlinear phenomena or the coupling with algebraic or differential equations, for instance in the case of reactive flows; in particular, one can find in [PAT 80] the presentation of the classical finite volume scheme on rectangular meshes, which has been the basis of many industrial applications. Proofs of the convergence of the so-called “MAC scheme" [HAR 65] were performed, for the Stokes equations, see [BLA 05a] and references therein. However, the use of rectangular grids makes an important limitation to the type of domain which can be gridded and more recently, finite volume schemes for the Navier-Stokes equations on triangular grids have been presented, either staggered [GUN 93], or colocated [BOI 00] where primal variables are used with a Chorin type projection method to ensure the divergence condition.

## 2. Staggered schemes on unstructured meshes

In [EYM 00], we introduced a finite volume scheme using the “classical" FV4 scheme for the Laplace equation. Since staggered schemes have the reputation of being the most stable schemes for incompressible flows, our idea was to generalise the MAC scheme to triangular meshes. Hence we considered a scheme where the velocity unknowns were associated to the control volumes of the mesh, and the ”classical" four points cell-centred scheme was applied to discretize the Laplacian of the velocities, while a Galerkin expansion was introduced for the pressure, with the pressure unknowns associated to the vertices of the mesh. But in fact, we were able to prove the convergence [EYM 00] and some error estimates [BLA 04], only in the case of equilateral triangles, making use of the fact that the center of gravity is also the circumcenter.

When trying to prove the convergence of this scheme on general triangular meshes, it became clear that the approximation of the divergence operator is an important property. This motivated a new staggered scheme [EYM 03], obtained from the previous one by modifying both the coefficients of the discrete divergence operator and the discrete gradient (hence the discrete operators are dual to one another) in order to obtain a consistent approximation of the divergence. Let us emphasise that the coefficients of this new staggered scheme are equal to those of the former scheme in the case of equilateral triangles, for which we could prove convergence in the first place. In order to obtain a well posed scheme, we use a penalisation term of the form involving the pressure in the mass equation, which is the key to obtaining an $L^2$ estimate on the pressure, thus permitting the proof of the weak $L^2$ convergence of the pressure. We also prove existence and uniqueness to the penalised and non penalised scheme. In [BLA 05b], we also obtained an error estimate of order $1/4$ , which was proved to be non sharp by the numerical experiments. Indeed the numerical order of convergence were found to be equal to (roughly) 2 for the velocities and 1 for the pressure. Note that we also found in these experiments that the former Galerkin-based scheme did not converge well with respect to the pressure when refining general triangular meshes (see [BLA 05b] for further details). This scheme was extended to the Navier-Stokes equations [EYM 04b], [EYM 05a]. However, our attempts to generalise this scheme to the three-dimensional case have not been successful yet. Hence we also derived and studied a colocated scheme,

## 3. The colocated scheme

In the three-dimensional case, it is much easier to deal with non staggered schemes, since the geometrical conditions on the meshes are easier to comply with. Hence we turn to a colocated scheme, where all unknowns (velocities and pressure) are located in the control volumes. Note that this also makes it easy to add passive convected terms such as a concentration, or thermodynamics.

The colocated finite volume scheme is based on three basic ingredients. First, a stabilization technique *à la* Brezzi-Pikäranta [BRE 84] is used to cope with the instability of colocated velocity/pressure approximation spaces. The penalisation term involves $\Delta p$, leading to a $H^1$ estimate on the pressure, and therefore strong convergence in $L^2$ (rather than the weak convergence which was obtained in the case of the staggered scheme). Second, the discretization of the pressure gradient in the momentum balance equation is performed to ensure, by construction, that it is the transpose of the divergence term of the continuity constraint. Finally, the contribution of the discrete nonlinear advection term to the kinetic energy balance vanishes for discrete divergence free velocity fields, as in the continuous case. These features appear to be essential in the proof of convergence.

We are then able to prove the stability of the scheme and the convergence of discrete solutions towards a solution of the continuous problem when the size of the mesh tends to zero, for the steady linear case (generalized Stokes problem), the stationary and the transient Navier-Stokes equations, in 2D and 3D. Our results are valid for general meshes, do not require any assumption on the regularity of the continuous solution nor, in the nonlinear case, any small data condition. We emphasise that the convergence of the fully discrete (time and space) approximation is proven here, using an original estimate on the time translates, which yields, combined with a classical estimate on the space translates, a sufficient relative compactness property.

For the sake of simplicity, we start with the scheme for the Stokes equations; the discretization of the convective term is briefly stated in section 3.4. Consider the generalised stationary Stokes equations on $\Omega \subset \mathbb{R}^d$, which read:

$$\left\{ \begin{array}{lcl} \eta u - \nu \Delta u + \nabla p &=& f, \\ \nabla \cdot u &=& 0. \end{array} \right. \tag{1}$$

where $\nu$ is the kinematic viscosity, $u = (u_i)_{i=1,\ldots,d}$ denotes the velocity field, $d$ is the space dimension, and $p$ the pressure field, with the following assumptions:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^d,\ d = 2 \text{ or } 3 \tag{2}$$

$$\nu \in (0, +\infty),\ \eta \in [0, +\infty), \tag{3}$$

$$f \in L^2(\Omega)^d. \tag{4}$$

If we consider homogeneous boundary conditions for $u$ $\partial\Omega$, we have the following weak formulation of this problem:

$$\begin{aligned}&\bar u \in E(\Omega),\ \bar p \in L^2(\Omega) \text{ with } \textstyle\int_\Omega \bar p(x)\mathrm{d}x = 0,\\ &\eta \int_\Omega \bar u(x)\cdot \bar v(x)\mathrm{d}x + \nu \int_\Omega \nabla \bar u(x) : \nabla \bar v(x)\mathrm{d}x -\\ &\int_\Omega \bar p(x)\mathrm{div}\bar v(x)\mathrm{d}x = \int_\Omega f(x)\cdot \bar v(x)\mathrm{d}x,\ \forall \bar v \in H_0^1(\Omega)^d.\end{aligned} \tag{5}$$

### 3.1. *Discrete spaces and operators*

Let $\Omega$ be an open bounded polygonal subset or $\mathbb{R}^d$. We consider admissible meshes $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ in the sense of [EYM 00]: $\mathcal{M}$ is a partition of $\Omega$ by polygonal (or polyhedral) convex subsets of $\mathbb{R}^d$, $\mathcal{E}$ the set of edges of these subsets, and $\mathcal{P} = (x_K)_{K\in\mathcal{M}}$ a set of points satisfying the usual orthogonality condition (see Figure 1). The following notations are used. The size of the discretization is defined by:

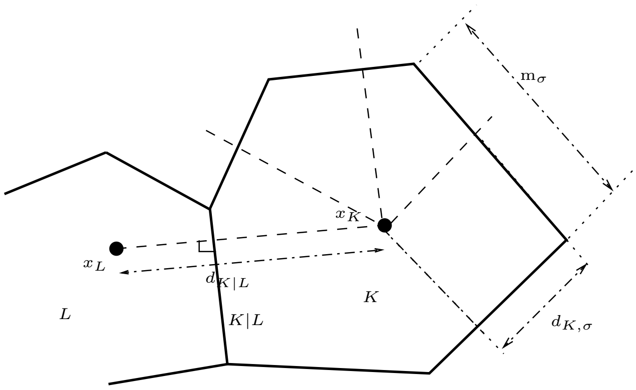


**Figure 1.** *Notations for an admissible mesh*

$\mathrm{size}(\mathcal{D}) = \sup\{\mathrm{diam}(K), K \in \mathcal{M}\}$. For all $K \in \mathcal{M}$, and $\sigma \in \mathcal{E}_K$ (the set of edges of $K$), we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to $\sigma$ outward to $K$. We denote by $d_{K,\sigma}$ the Euclidean distance between $x_K$ and $\sigma$. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For all $K \in \mathcal{M}$, we denote by $\mathcal{N}_K$ the subset of $\mathcal{M}$ of the neighbouring control volumes. For all $K \in \mathcal{M}$ and $L \in \mathcal{N}_K$, we set $\mathbf{n}_{KL} = \mathbf{n}_{K,K|L}$, we denote by $d_{KL}$ the Euclidean distance between $x_K$ and $x_L$. We shall measure the regularity of the mesh through the function $\mathrm{regul}(\mathcal{D})$ defined by

$$\begin{aligned}\mathrm{regul}(\mathcal{D}) = \inf \quad &\left\{\tfrac{d_{K,\sigma}}{\mathrm{diam}(K)},\ K \in \mathcal{M},\ \sigma \in \mathcal{E}_K\right\}\\ &\cup \left\{\tfrac{d_{K,K|L}}{d_{KL}},\ K \in \mathcal{M},\ L \in \mathcal{N}_K\right\} \cup \left\{\tfrac{1}{\mathrm{card}(\mathcal{E}_K)},\ K \in \mathcal{M}\right\}.\end{aligned} \tag{6}$$

Next we define some functional spaces and norms:

DEFINITION 3.1 Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, with $d \in \mathbb{N}^*$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$ be an admissible finite volume discretization of $\Omega$. We denote by $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $w \in H_{\mathcal{D}}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by $w_K$ the constant value of $w$ in $K$. The space $H_{\mathcal{D}}(\Omega)$ is equipped with the following Euclidean structure: for $(v, w) \in (H_{\mathcal{D}}(\Omega))^2$, we first define the following inner product (corresponding to Neumann boundary conditions)

$$\langle v, w \rangle_{\mathcal{D}} = \frac{1}{2} \sum_{K \in \mathcal{M}} \sum_{L \in \mathcal{N}_K} \tau_{KL}(v_L - v_K)(w_L - w_K). \tag{7}$$

and then another inner product (corresponding to Dirichlet boundary conditions)

$$[v, w]_{\mathcal{D}} = \langle v, w \rangle_{\mathcal{D}} + \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}} \tau_{K\sigma} v_K w_K. \tag{8}$$

The corresponding semi-norm and norm in $H_{\mathcal{D}}(\Omega)$ (thanks to the discrete Poincaré inequality (10) given below) are

$$|w|_{\mathcal{D}} = (\langle w, w \rangle_{\mathcal{D}})^{1/2}, \qquad \|w\|_{\mathcal{D}} = ([w, w]_{\mathcal{D}})^{1/2}.$$

Let us define an interpolation operator $P_{\mathcal{D}} : C(\Omega) \to H_{\mathcal{D}}(\Omega)$ by

$$(P_{\mathcal{D}} \varphi)_K = \varphi(x_K), \text{ for all } K \in \mathcal{M}, \text{ for all } \varphi \in C(\Omega). \tag{9}$$

Similarly, for $u = (u^{(i)})_{i=1,\ldots,d} \in (H_{\mathcal{D}}(\Omega))^d$, $v = (v^{(i)})_{i=1,\ldots,d} \in (H_{\mathcal{D}}(\Omega))^d$ and $w = (w^{(i)})_{i=1,\ldots,d} \in (H_{\mathcal{D}}(\Omega))^d$, we define:

$$\|u\|_{\mathcal{D}} = \left( \sum_{i=1}^{d} [u^{(i)}, u^{(i)}]_{\mathcal{D}} \right)^{1/2}, \qquad [v, w]_{\mathcal{D}} = \sum_{i=1}^{d} [v^{(i)}, w^{(i)}]_{\mathcal{D}},$$

and $P_{\mathcal{D}} : C(\Omega)^d \to H_{\mathcal{D}}(\Omega)^d$ by $(P_{\mathcal{D}} \varphi)_K = \varphi(x_K)$, for all $K \in \mathcal{M}$, for all $\varphi \in C(\Omega)^d$.

We recall the discrete Poincaré inequality (see e.g. [EYM 00]):

$$\|u\|_{L^2(\Omega)} \le \operatorname{diam}(\Omega) \|u\|_{\mathcal{D}}, \forall u \in H_{\mathcal{D}}. \tag{10}$$

Let us note that for $u \in H_{\mathcal{D}}(\Omega)$, one has:

$$\|u\|_{\mathcal{D}} = \Big( \sum_{\sigma \in \mathcal{E}} \tau_\sigma (D_\sigma u)^2 \Big)^{\frac{1}{2}}, \tag{11}$$

where $\tau_\sigma = \mathrm{m}_\sigma / d_\sigma$ and $D_\sigma u = |u_K - u_L|$ if $\sigma \in \mathcal{E}_{\text{int}}$, $\sigma = \sigma_{KL}$, $D_\sigma u = |u_K|$ if $\sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K$, where $u_K$ denotes the value taken by $u$ on the control volume $K$.

A discrete divergence operator $\widetilde{\operatorname{div}}_{\mathcal{D}} : (H_{\mathcal{D}}(\Omega))^d \to H_{\mathcal{D}}(\Omega)$, is defined by:

$$\widetilde{\operatorname{div}}_{\mathcal{D}} u|_K = \frac{1}{\mathrm{m}_K} \sum_{L \in \mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL} \cdot (u_K + u_L), \tag{12}$$

and set $E_{\mathcal{D}}(\Omega) = \{u \in (H_{\mathcal{D}}(\Omega))^d, \operatorname{div}_{\mathcal{D}}(u) = 0\}$.

Note that in (12), the factor 1/2 may be replaced by some coefficient $a_{KL} \geq 0$ such that $a_{KL} + a_{LK} = 1$. Such a choice, combined with the definition

$$\widetilde{\operatorname{div}}_{\mathcal{D}}(u)(x) = \frac{1}{\mathrm{m}_K} \sum_{L \in \mathcal{N}_K} (a_{KL}\, \mathrm{m}_{K|L} \mathbf{n}_{KL} \cdot u_K - a_{LK}\, \mathrm{m}_{K|L}\, \mathbf{n}_{LK} \cdot u_L),$$

produces the same results of convergence as those which are proven in this paper. In fact, on particular meshes, one can prove a better error estimate, choosing $a_{KL} = d(x_L, \sigma_{KL})/d_{KL}$ (see [EYM 05b]).

Let us now define a discrete gradient as the adjoint of the discrete divergence: $\widetilde{\nabla}_{\mathcal{D}} : H_{\mathcal{D}}(\Omega) \to (H_{\mathcal{D}}(\Omega))^d$:

$$(\widetilde{\nabla}_{\mathcal{D}} u)_K = \frac{1}{\mathrm{m}_K} \sum_{L \in \mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL}(u_L - u_K),\ \forall K \in \mathcal{M},\ \forall u \in H_{\mathcal{D}}(\Omega). \tag{13}$$

The operator $\widetilde{\nabla}_{\mathcal{D}}$ then satisfies the following properties.

LEMMA 3.2 Let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of $\Omega$, such that $\lim\limits_{m\to\infty} h_{\mathcal{D}^m} = 0$. Let us assume that there exists $C > 0$ and $\alpha \in [0, 2)$ and a sequence $(u^{(m)})_{m\in\mathbb{N}}$ such that $u^{(m)} \in H_{\mathcal{D}^{(m)}}(\Omega)$ and $|u^{(m)}|^2_{\mathcal{D}_m} \leq C\ h^{-\alpha}_{\mathcal{D}^m}$, for all $m \in \mathbb{N}$.

Then the following property holds:

$$\lim_{m\to+\infty} \int_\Omega \Big( P_{\mathcal{D}_m}\varphi(x) \widetilde{\nabla}_{\mathcal{D}_m} u^{(m)}(x) + u^{(m)}(x)\nabla\varphi(x) \Big)\, dx = 0,\ \forall \varphi \in C_c^\infty(\Omega), \tag{14}$$

and therefore:

$$\lim_{m\to+\infty} \int_\Omega \widetilde{\nabla}_{\mathcal{D}_m} u^{(m)}(x) \cdot P_{\mathcal{D}_m}\psi(x) \mathrm{d}x = 0,\ \forall \psi \in C_c^\infty(\Omega)^d \cap E(\Omega). \tag{15}$$

LEMMA 3.3 (DISCRETE RELLICH THEOREM) Let $(\mathcal{D}^{(m)})_{m\in\mathbb{N}}$ be a sequence of admissible discretizations of $\Omega$, such that $\lim\limits_{m\to\infty} h_{\mathcal{D}^m} = 0$. Let us assume that there exists $C > 0$ and a sequence $(u^{(m)})_{m\in\mathbb{N}}$ such that $u^{(m)} \in H_{\mathcal{D}^{(m)}}(\Omega)$ and $\|u^{(m)}\|_{\mathcal{D}_m} \leq C$ for all $m \in \mathbb{N}$.

Then, there exists $u \in H^1_0(\Omega)$ and a subsequence of $(u^{(m)})_{m\in\mathbb{N}}$, again denoted $(u^{(m)})_{m\in\mathbb{N}}$, such that the sequence $(u^{(m)})_{m\in\mathbb{N}}$ converges in $L^2(\Omega)$ to $u$ as $m \to +\infty$; the sequence also satisfies: $\widetilde{\nabla}_{\mathcal{D}_m} u^{(m)}$ weakly converges to $\nabla\bar{u}$ in $L^2(\Omega)^d$ as $m \to +\infty$ and (14) holds.

### 3.2. *The finite volume scheme in the linear case*

Let $\mathcal{D}$ be an admissible discretization of $\Omega$. It is then natural to write an approximate problem to the Stokes problem (5) in the following way.

$$\left\{\begin{array}{l} u \in E_{\mathcal{D}}(\Omega),\ p \in H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0 \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} \\ \qquad\qquad - \displaystyle\int_\Omega p(x)\widetilde{\mathrm{div}}_{\mathcal{D}}(v)(x)\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x, \quad \forall v \in H_{\mathcal{D}}(\Omega)^d. \end{array}\right. \tag{16}$$

As we use a colocated approximation for the velocity and the pressure fields, the scheme must be stabilised. Using a non-consistent stabilization *à la* Brezzi-Pitkäranta [BRE 84], we then look for $(u,p)$ such that

$$\left\{\begin{array}{l} (u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega) \text{ with } \displaystyle\int_\Omega p(x)\mathrm{d}x = 0, \\ \eta \displaystyle\int_\Omega u(x)\cdot v(x)\mathrm{d}x + \nu[u,v]_{\mathcal{D}} \\ \qquad\qquad - \displaystyle\int_\Omega p(x)\widetilde{\mathrm{div}}_{\mathcal{D}}(v)(x)\mathrm{d}x = \int_\Omega f(x)\cdot v(x)\mathrm{d}x,\ \forall v \in H_{\mathcal{D}}(\Omega)^d, \\ \displaystyle\int_\Omega \widetilde{\mathrm{div}}_{\mathcal{D}}(u)(x)q(x)\mathrm{d}x = -\lambda\, h_{\mathcal{D}}^\alpha\, \langle p,q\rangle_{\mathcal{D}},\ \forall q \in H_{\mathcal{D}}(\Omega), \end{array}\right. \tag{17}$$

where $\lambda > 0$ and $\alpha \in (0,2)$ are adjustable parameters of the scheme which will have to be tuned in order to make a balance between accuracy and stability.

System (17) is equivalent to finding the family of vectors $(u_K)_{K\in\mathcal{M}} \subset \mathbb{R}^d$, and scalars $(p_K)_{K\in\mathcal{M}} \subset \mathbb{R}$ solution of the system of equations obtained by writing for each control volume $K$ of $\mathcal{M}$:

$$\begin{array}{l} \eta\, \mathrm{m}_K\, u_K - \nu \displaystyle\sum_{L\in\mathcal{N}_K} \tau_{KL}(u_L - u_K) - \nu \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\mathrm{ext}}} \tau_{K\sigma}(0 - u_K) \\ \qquad\qquad\qquad + \displaystyle\sum_{L\in\mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL}\, (p_L - p_K) = \int_K f(x)\mathrm{d}x \\ \displaystyle\sum_{L\in\mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL}\cdot(u_K + u_L) - \lambda\, h_{\mathcal{D}}^\alpha \sum_{L\in\mathcal{N}_K} \tau_{KL}(p_L - p_K) = 0, \end{array} \tag{18}$$

supplemented by the relation

$$\sum_{K\in\mathcal{M}} \mathrm{m}_K\, p_K = 0. \tag{19}$$

Defining $p_\sigma = (p_K + p_L)/2$ if $\sigma = \sigma_{KL}$, and $p_\sigma = p_K$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$, and using the fact that $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma \mathbf{n}_{K,\sigma} = 0$, one notices that: $\sum_{L\in\mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL}\, (p_L - p_K)$ is in fact equal to $\sum_{\sigma\in\mathcal{E}_K} \mathrm{m}_\sigma p_\sigma \mathbf{n}_{K,\sigma}$, thus yielding a conservative form, which shows that (18) is indeed a finite volume scheme.

### 3.3. *Convergence analysis*

It is easily shown (see [EYM 04a] for details) that if $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ is a solution to (17) (for a given $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$), then the following inequalities hold:

$$\nu \|u\|_{\mathcal{D}} \leq \mathrm{diam}(\Omega) \|f\|_{(L^2(\Omega))^d}, \tag{20}$$

and

$$\nu\, \lambda\, h_{\mathcal{D}}^{\alpha}\, |p|_{\mathcal{D}}^2 \leq \mathrm{diam}(\Omega)^2 \|f\|_{(L^2(\Omega))^d}^2. \tag{21}$$

From these *a priori* estimates, one immediately gets the uniqueness of a discrete solution to (17).

Thanks to the penalisation term, using the Nečas technique, we then get the following strong estimate on the pressure:

$$\|p\|_{L^2(\Omega)} \leq C_1 \|f\|_{(L^2(\Omega))^d}, \tag{22}$$

where $C_1$ only depends on $d$, $\Omega$, $\eta$, $\nu$, $\lambda$, $\alpha$ and $\theta$, and not on $h_{\mathcal{D}}$.

From these estimates, and thanks to some estimates on the translates which may be obtained in a similar way as in [EYM 00], we obtain compactness properties of sequences of approximate solutions, as the mesh size tends to 0. Passing to the limit in the scheme, we then get the following convergence result (see [EYM 04a] for details).

THEOREM 3.4 (CONVERGENCE) Under hypotheses (2)-(4), let $(\bar{u},\bar{p})$ be the unique weak solution of the Stokes problem (1). Let $\lambda \in (0,+\infty)$, $\alpha \in (0,2)$ and $\theta > 0$ be given and let $\mathcal{D}$ be an admissible discretization of $\Omega$ such that $\mathrm{regul}(\mathcal{D}) \geq \theta$. Let $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ be the unique solution to (17). Then $u$ converges to $\bar{u}$ in $(L^2(\Omega))^d$ and $p$ weakly converges to $\bar{p}$ in $L^2(\Omega)$ as $h_{\mathcal{D}}$ tends to $0$.

Note that the proof of the strong convergence of $p$ to $\bar{p}$ is also a straightforward consequence of the error estimate which we now state, under additional regularity hypotheses.

THEOREM 3.5 (ERROR ESTIMATE) Under hypotheses (2)-(4), we assume that the weak solution $(\bar{u},\bar{p})$ of the Stokes problem (1) *i.e.* satisfying (5) is such that $(\bar{u},\bar{p}) \in H^2(\Omega)^d \times H^1(\Omega)$. Let $\lambda \in (0,+\infty)$ and $\alpha \in (0,2)$ be given, let $\mathcal{D}$ be an admissible discretization of $\Omega$ and let $\theta > 0$ such that $\mathrm{regul}(\mathcal{D}^{(m)}) \geq \theta$. Let $(u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega)$ be the solution to (17). Then there exists $C_2$, which only depends on $d$, $\Omega$, $\nu$, $\eta$ and $\theta$ such that

$$\|u-\bar{u}\|_{L^2(\Omega)}^2 \leq C_2 \epsilon(\lambda, h_{\mathcal{D}}, \bar{p}, \bar{u}), \tag{23}$$

$$\lambda\, h_{\mathcal{D}}^{\alpha}\, |p|_{\mathcal{D}}^2 \leq C_2 \epsilon(\lambda, h_{\mathcal{D}}, \bar{p}, \bar{u}) \tag{24}$$

$$\|p-\bar{p}\|_{L^2(\Omega)}^2 \leq C_2 \epsilon(\lambda, h_{\mathcal{D}}, \bar{p}, \bar{u}). \tag{25}$$

where

$$\begin{aligned}\epsilon(\lambda, h_{\mathcal{D}}, \bar{p}, \bar{u}) = \quad & \max\left(\lambda h_{\mathcal{D}}^{\alpha}, \tfrac{1}{\lambda} h_{\mathcal{D}}^{2-\alpha}\right) \\ & \times \left( \|\bar{p}\|_{H^1(\Omega)}^2 + \|\bar{u}\|_{H^2(\Omega)}^2 \right).\end{aligned} \tag{26}$$

Hence, for $\alpha = 1$ we get an order $1/2$ for the convergence of the scheme. In fact, this result is not sharp, and the numerical results show a much better order of convergence.

### 3.4. *Discretization of the nonlinear convective term*

A colocated scheme for the Navier-Stokes equations is analysed in [EYM 04a]. The integration of the non linear convective term $(u \cdot \nabla)u$ over a control volume $K$ yields the term:

$$\sum_{\sigma \in \mathcal{E}_K} \int_{\sigma} (u(x) \cdot \mathbf{n}_{K,\sigma}) u(x) ds(x).$$

Each integral over an edge $\sigma_{KL}$ is approximated by the expression:

$$m_{KL} \frac{u_K + u_L}{2} \cdot \mathbf{n}_{KL} \frac{u_L - u_K}{2},$$

Using an adequate penalisation term, convergence was proven in [EYM 04a]. Now if an $H^1$ penalisation term $\lambda_{KL} \tau_{KL}(p_L - p_K)$ (with $\lambda_{KL} > 0$) is added to the discretization of the convective flux, then this is equivalent to the conservative formulation

$$m_{KL} \frac{u_K + u_L}{2} \cdot \mathbf{n}_{KL} \frac{u_L + u_K}{2} + \lambda_{KL} \tau_{KL}(p_L - p_K).$$

Indeed, this latter discretization was implemented together with the cluster technique described below, and found to be very effective [CHE 05]. The proof of convergence in this case is the object of ongoing work.

### 3.5. *Stabilization by clusters*

The drawback of the stabilisation used in system (18) is that it yields some redistribution of the fluid mass over the whole domain. Moreover, in order to obtain convergence, one needs to let the stabilisation parameter tend to 0. For both reasons, we replace here the system (18) by:

$$\begin{aligned}\eta\, \mathrm{m}_K\, u_K - \nu \sum_{L \in \mathcal{N}_K} \tau_{KL}(u_L - u_K) - \nu \sum_{\sigma \in (\mathcal{E})_K \cap \mathcal{E}_{ext}} \tau_{K\sigma}(0 - u_K) & \\ + \sum_{L \in \mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL}\, (p_L - p_K) = \int_K f(x) \mathrm{d}x & \\ \sum_{L \in \mathcal{N}_K} \frac{1}{2}\, \mathrm{m}_{K|L}\, \mathbf{n}_{KL} \cdot (u_K + u_L) - \sum_{L \in \mathcal{N}_K} \lambda_{KL} \tau_{KL}(p_L - p_K) = 0, &\end{aligned} \tag{27}$$

where the parameters $\lambda_{KL}$ are chosen according to the following method. The family of control volumes $\mathcal{M}$ is partitioned into disjoint clusters of neighbouring control volumes. These clusters are chosen such that the distance between two control volumes belonging to the same cluster is bounded by $Ch_{\mathcal{D}}$, where $C$ is a given constant. Then the stabilising parameter $\lambda_{KL}$ is chosen equal to some $\lambda > 0$ for any pair of neighbouring control volumes $K$ and $L$ belonging to the same cluster, $0$ otherwise. The value $\lambda$ is chosen large enough to prevent instabilities.

An example of such an algorithm for partitioning $\mathcal{M}$ is the following:

– select an order for the control volumes $K_i$, $i = 1, M$;

– for $i = 1$ to $M$, if $K_i$ and all its neighbours do not yet belong to a cluster, initialise a new one by $K_i$ and all its neighbours;

– for $i = 1$ to $M$, if $K_i$ does not yet belong to one of the clusters, one of its neighbour does: include $K_i$ in this cluster.

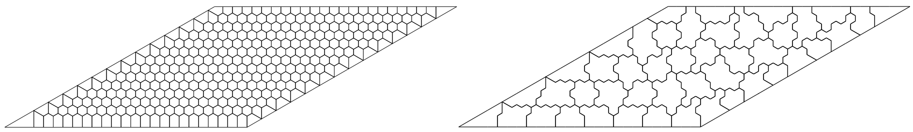

**Figure 2.** *Voronoï mesh and clusters of controls volumes.*

This stabilisation technique has been implemented with success for the transient Navier Stokes equations with or without the energy equation [CHE 05]. Note that a crucial difference with the stabilisation of [BRE 84] is that there is no need to let $\lambda$ tend to $0$ with the size of the mesh, which means that the presence of a finite stabilization does not decrease the quality of the approximation.

## 4. Discretization of the full viscous tensor

Let us write the conservation of mass and momentum for a general compressible viscous flow:

$$\begin{aligned}&\partial_t \rho + \operatorname{div}(\rho u) = 0, \text{ in } \Omega \times (0,T),\\ &\partial_t(\rho u) - \operatorname{div}\tau + \nabla p + \operatorname{div}(\rho u \otimes u) = \rho g \text{ in } \Omega \times (0,T),\end{aligned} \tag{28}$$

with an energy equation, which we do not state here, and where the viscous stress tensor $\tau$ is defined by:

$$\tau = \mu(\nabla u + (\nabla u)^t - \frac{2}{3}\operatorname{div}u\, I_d) + \xi \operatorname{div}u I_d,$$

where $\mu$ and $\xi$ are respectively the shear (resp. bulk) viscosities.

If the coefficients $\mu$ and $\xi$ are assumed constant, then the divergence of the viscous stress tensor maybe written as:

$$\operatorname{div}\tau = \mu \Delta u + (\xi + \frac{\mu}{3})\nabla(\operatorname{div}u). \tag{29}$$

Let us then give a finite volume discretization of $\tau$ with the discrete derivatives introduced in [EYM 04c]. For a given function $v \in (H_{\mathcal{D}})^2$, let $\widehat{\mathrm{div}}_{\mathcal{D}} \in H_{\mathcal{D}}$ be the function defined by:

$$(\widehat{\mathrm{div}}_{\mathcal{D}} v)|_K = \frac{1}{\mathrm{m}_K} \sum_{L \in \mathcal{N}_K} \tau_{KL}(v_L - v_K) \cdot (x_\sigma - x_K), \text{ for any } K \in \mathcal{M}. \qquad (30)$$

We may then define the discrete operator $\nabla_{\mathcal{D}}(\widehat{\mathrm{div}}_{\mathcal{D}})$ by:

$$(\nabla_{\mathcal{D}} \widehat{\mathrm{div}}_{\mathcal{D}} v)|_K = \frac{1}{\mathrm{m}_K} \sum_{L \in \mathcal{N}_K} \tau_{KL} \big( \widehat{\mathrm{div}}_{\mathcal{D}} v)|_K (x_{\sigma_{KL}} - x_K) + \widehat{\mathrm{div}}_{\mathcal{D}} v)|_K (x_{\sigma_{KL}} - x_K) \big).$$

The complete discretization of (29) is then obtained by using the classical cell centred finite volume scheme described in [EYM 00] for the Laplace operator, and the above formula for the discretization of the term $\nabla(\mathrm{div} u)$. We thus have an easy way of discretizing the full viscous tensor, using the discrete gradient introduced in [EYM 04c]. This scheme has been implemented successfully on test cases, and its convergence analysis is the object of ongoing work.

## 5. Conclusions and perspectives

Staggered and colocated schemes for the incompressible Navier-Stokes equations were analysed and implemented on unstructured meshes. The colocated scheme generalises to the 3D case, while this seems not so clear for the staggered scheme. However, the colocated scheme was found to be more sensitive to high Reynolds numbers on tests such as the backward step. The cluster stabilisation was then implemented and improved dramatically the stability of the colocated scheme. Note that in fact, this clustering is a sort of large scale staggering of the pressure... Work is in progress concerning the convergence analysis of the scheme with clustering, and of the viscous tensor term in the compressible case. The case of non constant viscosity also needs to be addressed.

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