AN ERROR ESTIMATE FOR FINITE VOLUME METHODS FOR THE STOKES EQUATIONS ON EQUILATERAL TRIANGULAR MESHES

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Abstract. We give here an error estimate for a finite volume discretization of the Stokes equations in two space dimensions on equilateral triangular meshes. This work was initiated by an analogous result presented in [AA 02] for general triangular meshes. However, in this latter article, the result is not actually proven. We state here the restricting assumptions (namely equilateral triangles) under which the error estimate holds, using the tools which were introduced in [EGH 00] and used in [AA 02].

1. Introduction

Consider the flow of an incompressible fluid in a two-dimensional domain Ω, with velocity $u = (u^{(1)}, u^{(2)})^t$ and pressure p. Recall that the Stokes equations may be written:

$$-\nu \Delta u^{(i)}(x) + \frac{\partial p}{\partial x_i}(x) = f^{(i)}(x),\ x \in \Omega,\ \forall i = 1, 2, \tag{1.1}$$

$$\text{div } u(x) = \sum_{i=1}^{2} \frac{\partial u^{(i)}}{\partial x_i}(x) = 0,\ x \in \Omega, \tag{1.2}$$

with Dirichlet boundary condition

$$u^{(i)}(x) = 0,\ x \in \partial\Omega,\ \forall i = 1, 2, \tag{1.3}$$

under the following assumption:

Assumption 1.

(i) Ω is an open bounded connected polygonal subset of $\mathbb{R}^2$,
(ii) $\nu > 0$,
(iii) $f^{(i)} \in L^2(\Omega)$, $\forall i = 1, 2$.

In order to ensure uniqueness of the pressure, we also impose the following condition:

$$\int_\Omega p(x)dx = 0. \tag{1.4}$$

For the sake of completeness, let us recall the definition of an admissible finite element triangular mesh denoted by $\mathcal{T}$, see e.g. [CIA 78], with the following notations, see also Fig. 1.1.

The set of edges of the mesh is denoted by $\mathcal{E}$; for any triangle K of the mesh, one denotes by x_K its circumcenter and by $\mathcal{E}_K$ the set of the three edges of the triangle. For two neighbouring triangles K and L, one denotes by $\sigma = K|L$ their common edge. For any $\sigma \in \mathcal{E}_K$, one denotes by z_σ the orthogonal projection of x_K on σ.

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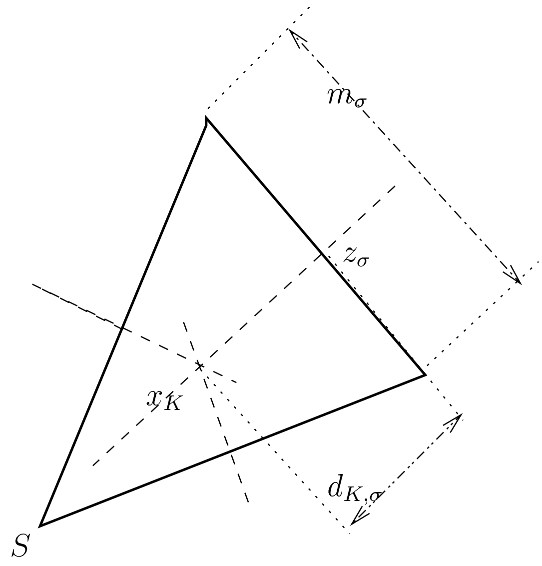

Figure 1.1: Notations for a control volume K

The mesh size is defined by: $\text{size}(\mathcal{T}) = \sup\{\text{diam}(K),\, K \in \mathcal{T}\}$.

For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$, $\text{m}(K)$ is the area of K and $\text{m}(\sigma)$ the length of σ.

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), that is $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E};\, \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E};\, \sigma \subset \partial\Omega\}$).

The set of neighbours of K is denoted by $\mathcal{N}_K$, that is $\mathcal{N}_K = \{L \in \mathcal{T};\, \exists \sigma \in \mathcal{E}_K,\, \overline{\sigma} = \overline{K} \cap \overline{L}\}$.

If $\sigma = K|L$, we denote by d_σ or $d_{K|L}$ the Euclidean distance between x_K and x_L (which is positive) and by $d_{K,\sigma,}$ the distance from x_K to σ.

If $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$, let d_σ denote the Euclidean distance between x_K and z_σ (then, $d_\sigma = d_{K,\sigma,}$).

For any $\sigma \in \mathcal{E}$; the "transmissibility" through σ is defined by $\tau_\sigma = \text{m}(\sigma)/d_\sigma$ if $d_\sigma \neq 0$.

The set of vertices of $\mathcal{T}$ is denoted by $\mathcal{S}_\mathcal{T}$. For $S \in \mathcal{S}_\mathcal{T}$, let ϕ_S be the shape function associated to S in the piecewise linear finite element method for the mesh $\mathcal{T}$. For all $K \in \mathcal{T}$, let $\mathcal{S}_K \subset \mathcal{S}_\mathcal{T}$ be the set of the vertices of K.

The finite volume scheme given in [EGH 00], which is cell-centered for the velocities and uses a Galerkin expansion for the pressure, reads:

$$\nu \sum_{\sigma \in \mathcal{E}_K} F^{(i)}_{K,\sigma} + \sum_{S \in \mathcal{S}_K} p_S \int_K \frac{\partial \phi_S}{\partial x_i}(x)dx = \text{m}(K) f^{(i)}_K, \quad \forall K \in \mathcal{T},\, \forall i = 1, 2, \tag{1.5}$$

$$\begin{aligned} F^{(i)}_{K,\sigma} &= \tau_\sigma (u^{(i)}_K - u^{(i)}_L), \text{ if } \sigma \in \mathcal{E}_{\text{int}},\, \sigma = K|L,\, i = 1, 2, \\ F^{(i)}_{K,\sigma} &= \tau_\sigma u^{(i)}_K, \text{ if } \sigma \in \mathcal{E}_{\text{ext}} \cap \mathcal{E}_K,\, i = 1, 2, \end{aligned} \tag{1.6}$$

$$\sum_{K \in \mathcal{T}} \sum_{i=1}^{2} u^{(i)}_K \int_K \frac{\partial \phi_S}{\partial x_i}(x)dx = 0,\, \forall S \in \mathcal{S}_\mathcal{T}, \tag{1.7}$$

$$\int_\Omega \sum_{S\in\mathcal{S}_\mathcal{T}} p_S\phi_S(x)dx = 0, \tag{1.8}$$

$$f_K^{(i)} = \frac{1}{\mathrm{m}(K)}\int_K f(x)dx,\ \forall K\in\mathcal{T}. \tag{1.9}$$

The discrete unknowns of (1.5)-(1.9) are $u_K^{(i)}$, $K\in\mathcal{T}$, $i=1,2$ and p_S, $S\in\mathcal{S}_\mathcal{T}$.

In the next section, we give some results which were proved in [EGH 00] for this scheme, and explain why the results of [AA 02] are not satisfying. We then give in Section 3 an error estimate under restrictive assumptions.

2. Previous results

Existence and uniqueness of $u_K^{(i)}$, $K\in\mathcal{T}$, $i=1,2$ and p_S, $S\in\mathcal{S}_\mathcal{T}$ were proven in [EGH 00] and therefore the approximate solution may be defined by

$$p_\mathcal{T} = \sum_{S\in\mathcal{S}_\mathcal{T}} p_S\phi_S, \tag{2.10}$$

$$u_\mathcal{T}^{(i)}(x) = u_K^{(i)},\ \text{ a.e. } x\in K,\ \forall K\in\mathcal{T},\ \forall i=1,2. \tag{2.11}$$

The proof of the convergence of the scheme is not straightforward in the general case. In [EGH 00], the convergence of the discrete velocities given by the finite volume scheme (1.5)-(1.9) is proven in the case of a mesh consisting of equilateral triangles.

In [AA 02], the authors state a theorem (Theorem 4.1. page 5) which gives an error estimate with no restriction on the admissible mesh nor on the regularity of the exact solution. The statement of their theorem seems to be at least incomplete (the property of consistency of the fluxes [AA 02](4.5) cannot be obtained without some regularity hypotheses on u) and the conclusion of their theorem remains, in our opinion, unproved. Let us focus on Relation [AA 02](4.10), which writes:

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\left(\int_K \frac{\partial p}{\partial x_i}(x)dx - \sum_{S\in S_K} p_S\int_K \frac{\partial \Phi_S}{\partial x_i}(x)dx\right)e_K^{(i)} = 0.$$

We do not believe that the brief argumentation of this equality, given in [AA 02], which states:

"Using div $(u)=0$ *and the relation (3.4), we deduce that:"*

(relation [AA 02](3.4) is Relation (1.7) of the present paper) holds. First, the term

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\left(\int_K \frac{\partial p}{\partial x_i}(x)dx\right)e_K^{(i)}$$

is not in general equal to 0 and should be adequately controled by the interpolation error on the pressure.

Secondly, the use of Relation [AA 02](3.4) does not seem to be sufficient in order to yield:

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\left(\sum_{S\in S_K} p_S\int_K \frac{\partial\Phi_S}{\partial x_i}(x)dx\right) e_K^{(i)} = 0, \tag{2.12}$$

with the definition $e_K^{(i)} = u_K^{(i)} - u^{(i)}(x_K)$ given in the statement of Theorem [AA 02] 4.1 (which is also that of [H 95], [EGH 00] or [GHV 00], recall that the family of points $(x_K)_{K\in\mathcal{T}}$ is such that if the control volumes K and L have a common edge σ, than the line segment x_Kx_L is orthogonal to σ). Indeed, Relation [AA 02](3.4) is only sufficient to prove that

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\left(\sum_{S\in S_K} p_S\int_K \frac{\partial\Phi_S}{\partial x_i}(x)dx\right) u_K^{(i)} = 0,$$

but not that

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\left(\sum_{S\in S_K} p_S\int_K \frac{\partial\Phi_S}{\partial x_i}(x)dx\right) u^{(i)}(x_K) = 0.$$

Note that in fact, (2.12) can be obtained under the following sufficient conditions:

(1) $e_K^{(i)} = u_K^{(i)} - \frac{1}{\mathrm{m}(K)}\int_K u^{(i)}(x)dx$ instead of $e_K^{(i)} = u_K^{(i)} - u^{(i)}(x_K)$,

(2) the discretization mesh is made of triangles, so that the following equality is satisfied:

$$\int_K \frac{\partial\phi_S}{\partial x_i}(x)dx\frac{1}{\mathrm{m}(K)}\int_K u^{(i)}(x)dx = \int_K \frac{\partial\phi_S}{\partial x_i}(x)u^{(i)}(x)dx. \tag{2.13}$$

But, if $e_K^{(i)}$ is chosen to be as $u_K^{(i)} - \frac{1}{\mathrm{m}(K)}\int_K u^{(i)}(x)dx$ rather than $u_K^{(i)} - u^{(i)}(x_K)$, then the consistency of the fluxes condition [AA 02](4.5) (which remains a necessary step of the proof) is no longer true in the general case: this is why the meshes were restricted to the equilateral case in the convergence theorem of [EGH 00].

In the next section, we state the error estimate and give its proof according to the previous remarks.

3. Error estimate

Theorem 3.1. *Under assumptions 1, we assume that Ω is such that there exists an admissible triangular mesh on Ω such that all triangles are equilateral. Let $\mathcal{T}$ be such a mesh. Furthermore we assume that there exists $(u,p)\in(H^2(\Omega))^2\times H^1(\Omega)$ solution to (1.1)-(1.3),(1.4). Let (u,p) be such a solution. Define*

$$\tilde{u}_K^{(i)} = \frac{1}{m(K)}\int_K u^{(i)}(x)dx,\ \textit{for } i=1,2, \tag{3.14}$$

and

$$e_K^{(i)} = \tilde{u}_K^{(i)} - u_K^{(i)},\ \textit{for } i=1,2, \tag{3.15}$$

and let e be the piecewise constant function defined a.e. from Ω to $\mathbb{R}^2$ by $e_{|K} = e_K$. Then there exists a constant $C\in\mathbb{R}$ (independent of all the data and of the diameter h of the control volumes of the mesh) such that:

$$\|e\|_{1,\mathcal{T}} \le Ch\left(\|u\|_{(H^2(\Omega))^2} + \frac{1}{\nu}\|p\|_{H^1(\Omega)}\right),$$

where:

$$\|e\|_{1,\mathcal{T}} = \Big(\sum_{i=1,2} \|e^{(i)}\|^2_{1,\mathcal{T}}\Big)^{\frac{1}{2}} \text{ and } \|e^{(i)}\|_{1,\mathcal{T}} = \Big(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma e^{(i)})^2\Big)^{\frac{1}{2}}, \text{ for } i = 1, 2, \tag{3.16}$$

with $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$, $D_\sigma e^{(i)} = |e^{(i)}_K - e^{(i)}_L|$ *if* $\sigma \in \mathcal{E}_{\mathrm{int}}$, $\sigma = K|L$, *and* $D_\sigma e^{(i)} = |e^{(i)}_K|$ *if* $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$*; we recall that* $\mathcal{E}_{\mathrm{int}}$ *(resp.* $\mathcal{E}_{\mathrm{ext}}$, $\mathcal{E}_K$*) denotes the set of interior edges (resp. edges of the boundary* $\partial\Omega$*, edges of the boundary* ∂K*).*

Proof

Let $K \in \mathcal{T}$ be a given control volume. Let

$$\tilde{F}^{(i)}_{K,\sigma} = \begin{cases} -\dfrac{m(\sigma)}{d_{K,L}}(\tilde{u}^{(i)}_L - \tilde{u}^{(i)}_K) & \text{if } \sigma = K|L, \\ \dfrac{m(\sigma)}{d_{K,\sigma}}\tilde{u}^{(i)}_K & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}. \end{cases}, \text{ for } i = 1, 2,$$

Integrating (1.1) over K for $i = 1, 2$, substracting off (1.5) and introducing $\tilde{F}^{(i)}_{K,\sigma}$ yields:

$$\nu \sum_{\sigma\in\varepsilon_K} E^{(i)}_{K,\sigma} - \int_K \frac{\partial p_{\mathcal{T}}}{\partial x_i}(x)dx = \nu \sum_{\sigma\in\varepsilon_K} m(\sigma)\varphi^{(i)}_{K,\sigma}(u) - \int_K \frac{\partial p}{\partial x_i}(x)dx, \tag{3.17}$$

with:

$$E^{(i)}_{K,\sigma} = \tilde{F}^{(i)}_{K,\sigma} - F^{(i)}_{K,\sigma},$$

$$\varphi^{(i)}_{K,\sigma}(u) = \begin{cases} \dfrac{1}{d_{K,L}}\left(\tilde{u}^{(i)}_K - \tilde{u}^{(i)}_L\right) + \dfrac{1}{\mathrm{m}(\sigma)}\displaystyle\int_\sigma \nabla u^{(i)}(x)\cdot \mathbf{n}_{K,\sigma} d\gamma(x) & \text{if } \sigma = K|L, \\ \dfrac{1}{d_{K,\sigma}}\tilde{u}^{(i)}_K + \dfrac{1}{\mathrm{m}(\sigma)}\displaystyle\int_\sigma \nabla u^{(i)}(x)\cdot \mathbf{n}_{K,\sigma} d\gamma(x) & \text{if } \sigma \in \mathcal{E}_{\mathrm{ext}}. \end{cases}$$

Multiplying (3.17) by $e^{(i)}_K$ and summing over i and K leads to:

$$\nu\|e\|^2_{1,\mathcal{T}} + X = \nu A + B, \tag{3.18}$$

with

$$X = -\sum_{i=1,2}\sum_{K\in\mathcal{T}} e^{(i)}_K \int_K \frac{\partial p_{\mathcal{T}}}{\partial x_i}(x)dx,$$

$$A = \sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\sum_{\sigma\in\varepsilon_K} m(\sigma)\varphi^{(i)}_{K,\sigma}(u)e^{(i)}_K$$

and

$$B = -\sum_{i=1}^{2}\sum_{K\in\mathcal{T}} \int_K \frac{\partial p}{\partial x_i}(x)dx\, e^{(i)}_K.$$

Let us first show that

$$X = 0 \tag{3.19}$$

holds (note that (3.19) is exactly Relation (2.12) presented in the previous section). Indeed, by definition of $p_{\mathcal{T}}$, one has

$$\int_K \frac{\partial p_{\mathcal{T}}}{\partial x_i}(x)dx = \sum_{S\in\mathcal{S}_{\mathcal{T}}} \alpha^{(i)}_{K,S} p_S, \text{ with } \alpha^{(i)}_{K,S} = \int_K \frac{\partial \Phi_S}{\partial x_i}(x)dx.$$

Hence

$$X = \sum_{S\in\mathcal{S}_{\mathcal{T}}} \Big(\sum_{K\in\mathcal{T}} \sum_{i=1,2} \alpha^{(i)}_{K,S} u^{(i)}_K - \sum_{K\in\mathcal{T}} \sum_{i=1,2} \alpha^{(i)}_{K,S} \tilde{u}^{(i)}_K \Big) p_S.$$

Now by (1.7), $\sum_{i=1,2} \alpha^{(i)}_{K,S} u^{(i)}_K = 0$ for all $S \in \mathcal{S}_{\mathcal{T}}$. Furthermore, since $\frac{\partial \Phi_S}{\partial x_i}$ is constant on each triangle K, (2.13) holds and therefore, one has:

$$\sum_{K\in\mathcal{T}} \sum_{i=1,2} \alpha^{(i)}_{K,S} \tilde{u}^{(i)}_K = \sum_{K\in\mathcal{T}} \int_K \nabla \Phi_S(x) \cdot u(x) dx.$$

Thanks to this last equality and to (1.2)-(1.3), one then gets:

$$\sum_{K\in\mathcal{T}} \sum_{i=1,2} \alpha^{(i)}_{K,S} \tilde{u}^{(i)}_K = -\int_\Omega \operatorname{div}\, u(x)\, \Phi_S(x) dx + \int_{\partial\Omega} \Phi_S(x)\, u(x) \cdot \mathbf{n}(x) d\gamma(x) = 0.$$

Hence (3.19) is proven.

Let us now study the term A. Reordering the summation over the edges of the mesh, one obtains:

$$A \le \sum_{i=1}^{2} \sum_{\sigma\in\mathcal{E}} m(\sigma) \varphi^{(i)}_\sigma(u)\, D_\sigma e^{(i)},$$

where $\varphi^{(i)}_\sigma(u) = |\varphi^{(i)}_{K,\sigma}(u)|$ for $\sigma = K|L \in \mathcal{E}_{\text{int}}$ or $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$. Therefore, by the Cauchy-Schwarz inequality,

$$A \le \left(\sum_{i=1}^{2} \sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma (\varphi^{(i)}_\sigma(u))^2 \right)^{1/2} \|e\|_{1,\mathcal{T}}.$$

Now by Lemma 3.2 given below, there exists a constant $C_0 \in \mathbb{R}$ such that:

$$\sum_{i=1}^{2} \sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma (\varphi^{(i)}_\sigma(u))^2 \le C_0 h^2 \sum_{i=1}^{2} \left(\sum_{\sigma\in\mathcal{E}_{\text{int}}, \sigma=K|L} |u^{(i)}|^2_{H^2(K\cup L)} + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} |u^{(i)}|^2_{H^2(K)} \right)$$

that is

$$\sum_{i=1}^{2} \sum_{\sigma\in\mathcal{E}} m(\sigma) d_\sigma (\varphi^{(i)}_\sigma(u))^2 \le 3C_0 h^2 \left(\sum_{i=1}^{2} \sum_{K\in\mathcal{T}} |u^{(i)}|^2_{H^2(K)} \right) \le 3C_0 h^2 \|u\|^2_{(H^2(\Omega))^2}$$

Hence there exists a constant $C_1 \in \mathbb{R}$ such that

$$A \le C_1 \|u\|_{(H^2(\Omega))^2} h \|e\|_{1,\mathcal{T}}. \tag{3.20}$$

Let us now turn to the term B. Let

$$\bar{p}_{\mathcal{T}} = \sum_{S\in\mathcal{S}_{\mathcal{T}}} \tilde{p}_S \Phi_S \tag{3.21}$$

be an interpolation of the exact pressure p defined by

$$\tilde{p}_S = \frac{1}{\int_\Omega \Phi_S(x) dx} \int_\Omega p(x) \Phi_S(x) dx. \tag{3.22}$$

Again, thanks to (1.7) (the computations are similar to those in the proof of (3.19)) one has

$$\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\int_K \frac{\partial \tilde p_{\mathcal{T}}}{\partial x_i}(x)dx\, e_K^{(i)} = 0,$$

and therefore

$$B = \sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\int_K \frac{\partial (\tilde p_{\mathcal{T}} - p)}{\partial x_i}(x)dx\, e_K^{(i)} = \sum_{i=1}^{2}\sum_{K\in\mathcal{T}} e_K^{(i)} \sum_{\sigma\in\mathcal{E}_K}\int_\sigma (\tilde p_{\mathcal{T}}(x) - p(x))c_{K,\sigma}^{(i)} d\gamma(x),$$

where $c_{K,\sigma}^{(i)}$ is the dot product between the normal unit vector to σ outward to K and the i-th vector of the orthonormal basis of $\mathbb{R}^2$. Reordering the summation over the edges of the mesh, one obtains:

$$B \le \sum_{i=1}^{2}\left(\sum_{\sigma\in\mathcal{E}} |\chi_\sigma(p)|\, D_\sigma e^{(i)}\right),$$

where

$$\chi_\sigma(p) = \int_\sigma (\tilde p_{\mathcal{T}}(x) - p(x))\, d\gamma(x).$$

By the Cauchy-Schwarz inequality, one obtains that:

$$B^2 \le 2\|e\|_{1,\mathcal{T}}^2 \sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{m(\sigma)}(\chi_\sigma(p))^2.$$

Thanks to Lemma 3.3 given below, there exists a constant $C_2\in\mathbb{R}$ such that:

$$\sum_{\sigma\in\mathcal{E}} \frac{d_\sigma}{m(\sigma)}(\chi_\sigma(p))^2 \le C_2 h^2 \sum_{\sigma\in\mathcal{E}} \|p\|_{H^1(V_\sigma)}^2, \tag{3.23}$$

where V_σ denotes the set of triangles $K\in\mathcal{T}$ to which at least one of the vertices of σ is a vertex.

Noting that a given control volume $K\in\mathcal{T}$ is a subset of at most 15 different V_σ sets, there exists a constant $C_3\in\mathbb{R}$ such that:

$$B \le C_3 h \|e\|_{1,\mathcal{T}} \|p\|_{H^1(\Omega)}. \tag{3.24}$$

From (3.18), (3.19), (3.20), (3.24), one concludes the proof of the theorem.

□

We now give the proof of the consistency results which are used in the above proof. In Lemma 3.2, we first control the error on the discretization of the flux when considering the mean value of the exact solution, which was used in order to obtain (3.19). Note that in [EGH 00], the consistency was proven with respect to the value of the solution at the points x_K and not with respect to the mean value. It is because of this mean value that we have to restrict to equilateral triangles here.

Lemma 3.2 (Consistency of the flux). *Under the hypotheses of Theorem 3.1, let $K\in\mathcal{T}$, $\sigma\in\mathcal{E}_K$, there exists a constant $\hat C$ such that the following inequality holds:*

$$\begin{aligned}&\Big|\tfrac{1}{d_{K,\sigma}}\Big(\tfrac{1}{\mathrm{m}(K)}\textstyle\int_K w(x)dx - \tfrac{1}{\mathrm{m}(\sigma)}\int_\sigma w(x)d\gamma(x)\Big) + \\ &\quad \tfrac{1}{\mathrm{m}(\sigma)}\textstyle\int_\sigma \nabla w(x)\cdot \mathbf{n}_{K,\sigma} d\gamma(x)\Big| \le \hat C |w|_{H^2(K)},\end{aligned} \tag{3.25}$$

where

$$|w|_{H^2(K)} = \left(\sum_{i,j=1}^{2} \int_K |\frac{\partial^2 w}{\partial x_i \partial x_j}(x)|^2 dx \right)^{\frac{1}{2}}.$$

Proof: Let $\varphi \in (H^2(K))'$ (dual space of $H^2(K)$) be defined for any $w \in H^2(K)$ by:

$$\varphi(w) = \frac{h}{d_{K,\sigma}} \left(\frac{1}{\mathrm{m}(K)} \int_K w(x) dx - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma w(x) d\gamma(x) \right) - \frac{h}{\mathrm{m}(\sigma)} \int_\sigma \nabla w(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x).$$

Let $\hat{K}$ be a given equilateral triangle with diameter 1, let ψ be the one to one mapping from K to $\hat{K}$, and let $\hat{\varphi} \in (H^2(\hat{K}))'$ be defined for any $w \in H^2(K)$ by:

$$\begin{array}{rl} \hat{\varphi}(\hat{w}) = & \dfrac{h}{d_{K,\sigma}} \left(\dfrac{1}{\mathrm{m}(K)} \displaystyle\int_K \hat{w}(\psi(x)) dx - \dfrac{1}{\mathrm{m}(\sigma)} \displaystyle\int_\sigma \hat{w}(\psi(x)) d\gamma(x) \right) \\ & + \dfrac{h}{\mathrm{m}(\sigma)} \displaystyle\int_\sigma \nabla(\hat{w} \circ \psi)(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x) \end{array},$$

Performing the change of variable $x = \psi^{-1}(\hat{x})$, and using the fact that $\nabla(\hat{w} \circ \psi)(\psi^{-1}(\hat{x})) = \frac{1}{h} \nabla \hat{w}(\hat{x})$, we obtain:

$$\begin{array}{rl} \hat{\varphi}(\hat{w}) = & \dfrac{h}{d_{K,\sigma}} \left(\dfrac{h^2}{\mathrm{m}(K)} \displaystyle\int_{\hat{K}} \hat{w}(\hat{x}) d\hat{x} - \dfrac{h}{\mathrm{m}(\sigma)} \displaystyle\int_{\hat{\sigma}} \hat{w}(\hat{x}) d\gamma(\hat{x}) \right) \\ & + \dfrac{h}{\mathrm{m}(\sigma)} \displaystyle\int_{\hat{\sigma}} \nabla \hat{w}(\hat{x}) \cdot \mathbf{n}_{\hat{K},\hat{\sigma}} d\gamma(\hat{x}), \end{array}$$

and therefore,

$$\hat{\varphi}(\hat{w}) = \frac{1}{d_{\hat{K},\hat{\sigma}}} \left(\frac{1}{m_{\hat{K}}} \int_{\hat{K}} \hat{w}(\hat{x}) d\hat{x} - \frac{1}{m_{\hat{\sigma}}} \int_{\hat{\sigma}} \hat{w}(\hat{x}) d\gamma(\hat{x}) \right) - \frac{1}{m_{\hat{\sigma}}} \int_{\hat{\sigma}} \nabla \hat{w}(\hat{x}) \cdot \mathbf{n}_{\hat{K},\hat{\sigma}} d\gamma(\hat{x}).$$

Since $\hat{\varphi} \in (H^2(\hat{K}))'$ and $\hat{\varphi}$ vanishes on the set of polynomials of degree 1 on $\hat{K}$, by the Bramble-Hilbert lemma, there exists $\hat{C} \in \mathbb{R}$ independent of h such that:

$$|\hat{\varphi}(\hat{w})| \leq \hat{C} |\hat{w}|_{H^2(\hat{K})}.$$

Hence we get, with $\hat{w} = w \circ \psi^{-1}$,

$$|\varphi(w)| \leq \hat{C} h |w|_{H^2(\hat{K})}, \quad \forall w \in H^2(K).$$

This yields (3.25) and concludes the proof. □

Next, we give in Lemma 3.3 the consistency result needed to establish (3.23).

Lemma 3.3. *Under the assumptions of Theorem 3.1, let $\tilde{p}_{\mathcal{T}}$ be the interpolation of p defined by (3.21), (3.22), then there exists a constant $C \in \mathbb{R}$ such that, for any $\sigma \in \mathcal{E}$,*

$$\int_\sigma (p(x) - \tilde{p}_{\mathcal{T}}(x)) d\gamma(x) \leq Ch|p|_{H^1(V_\sigma)},$$

where V_σ denotes the set of triangles $K \in \mathcal{T}$ to which at least one of the vertices of σ is a vertex, and $|p|_{H^1(V_\sigma)} = \|\nabla p\|_{(L^2(V_\sigma))^2}$.

Proof:

Let $\sigma \in \mathcal{E}$. We first remark V_σ is chosen such that the restriction of $\tilde{p}_{\mathcal{T}}$ to σ only depends on the values of p on V_σ. Let us then remark that there exists a finite number of possible geometries for V_σ, which we number from 1 to P.

For each $k = 1, \dots, P$, let $\hat{V}_\sigma^{(k)}$ denote a given such set corresponding to $h = 1$.

Let ψ be the one-to-one mapping from V_σ to $\hat{V}_\sigma^{(k)}$, and let $\hat{\chi} \in (H^1(\hat{V}_\sigma^{(k)}))'$ be defined by:

$$\hat{\chi}(\hat{p}) = \frac{1}{h} \int_\sigma (\hat{p}(\psi(x)) - \tilde{\hat{p}}(\psi(x))) d\gamma(x).$$

where $\tilde{\hat{p}}$ denotes the piecewise linear interpolation, defined as in (3.21), (3.22), of $\hat{p}$ on $\hat{V}_\sigma^{(k)}$. Performing the change of variable $x = \psi^{-1}(\hat{x})$, we obtain:

$$\hat{\chi}(\hat{p}) = \int_{\hat{\sigma}} (\hat{p}(\hat{x}) - \tilde{\hat{p}}(\hat{x})) d\gamma(\hat{x}).$$

The linear form $\hat{\chi}$ vanishes for constant functions and therefore by the Bramble-Hilbert lemma, there exists $c^{(k)}$ depending only on the geometry of $\hat{V}_\sigma^{(k)}$ such that $\hat{\chi}(\hat{p}) \leq c^{(k)} |p|_{H^1(\hat{V}_\sigma^{(k)})}$. Then, noting that if $\hat{p} = p \circ \psi^{-1}$ one has $\int_{\hat{\sigma}} (\hat{p}(\hat{x}) - \tilde{\hat{p}}(\hat{x})) d\gamma(\hat{x}) = \frac{1}{h} \int_\sigma (p(x) - \tilde{p}_{\mathcal{T}}(x)) dx$, one concludes the proof of the lemma with $C = \max_{k=1,\dots,P} c^{(k)}$.

□

4. Conclusion

We underlined here some restrictive assumptions under which an error estimate for the discrete velocities may be obtained by the finite volume scheme which was introduced in [EGH 00]. Indeed, if the mesh is made of equilateral triangles, then an error estimate holds. It is easily seen that the error estimate still holds for meshes which are "close to" equilateral, that is if the distance between the circumcenter and the center of gravity is of order two with respect to the size of the mesh: in this case, the consistency error will remain of order one if one takes x_K to be the center of gravity rather than the circumcenter, and the rest of the above proof remains unchanged. We may note, however that both equilateral or "close to" equilateral meshes are rather hard to use in practice. To our knowledge, it is still an open question to know if the result stated in [AA 02] (i.e. without the assumption of equilateral triangles) holds. In fact, we recently obtained some numerical results on general meshes which seem to indicate poor convergence of the present scheme. One might conclude that the finite volume based on piecewise constant velocities and piecewise linear pressures is not a good idea; we wish to emphasize, however, that on the one hand, the piecewise constant velocities are equipped with a discrete H^1 norm. On the other hand, the piecewise linear approximation of the pressure must be viewed as a tool to discretize the divergence of the velocity at the vertices in an adequate way. In fact, in [EGH 03a], we introduce a modified scheme, for which we may obtain a property of consistency of the discrete divergence, along with a "weak" estimate on the discrete pressure, and for which we are thus able to prove an error estimate on general triangular meshes, see also [EGH 03b]. We further note that this modified scheme is equivalent to the scheme studied here in the case of equilateral triangles.

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