Convergence of finite volume schemes for semilinear convection diffusion equations

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Abstract. The topic of this work is the discretization of semilinear elliptic problems in two space dimensions by the cell centered finite volume method. Non-homogeneous Dirichlet boundary conditions are considered here. A discrete Poincaré inequality is used, and estimates on the approximate solutions are proven. The convergence of the scheme without any assumption on the regularity of the exact solution is proven using some compactness results which are shown to hold for the approximate solutions. An approximate gradient may be constructed, which tends to the exact gradient as the size of the mesh tends to 0.

1 Introduction

We present here the discretization by the finite volume method of convection diffusion problems on general structured or non structured grids; these grids may consist of polygonal control volumes satisfying adequate geometrical conditions (which are stated in the sequel) and not necessarily ordered in a cartesian grid.

The finite volume method is a discretization method which is well suited for the numerical simulation of various types (elliptic, parabolic or hyperbolic, for instance) of conservation laws; it has been extensively used in several engineering fields, such as fluid mechanics, heat and mass transfer or petroleum engineering. Some of the important features of the finite volume method are that it may be used on arbitrary geometries, using structured or unstructured meshes, and it leads to robust schemes. An additional feature is the local conservativity of the numerical fluxes, that is the numerical flux is conserved from one discretization cell to its neighbour. This last feature makes the finite volume method quite attractive when modelling problems for which the flux is of importance, such as in fluid mechanics, semi-conductor device simulation, heat and mass transfer...

Finite volume methods for convection-diffusion equations were introduced as early as the early sixties by Tichonov and Samarskii, see [44], [39] and [40]. The convergence theory of such schemes in several space dimensions has only recently been undertaken. In the case of vertex-centered finite volume schemes, studies were carried out by [41] in the case of Cartesian meshes, [25], [3], [5], [6] and [45] in the case of unstructured meshes; see also [36], [43] and [33] in the case of quadrilateral meshes.
We are interested here by the so called "cell centered" approach, i.e. the discrete unknowns are located at some point in the control volumes. Cell-centered finite volume schemes are addressed in [34], [23], [50] and [30] in the case of Cartesian meshes and in [47], [26], [27], [29], [35] in the case of triangular or Voronoi meshes; let us also mention [9] and [7] where more general meshes are treated, with, however, a somewhat technical geometrical condition. In the pure diffusion case, the cell centered finite volume method has also been analyzed with finite element tools: [1], [4], [2]. The convergence analysis has also been performed in some cases of nonlinear convection-diffusion problems; see [19] with a combined finite element-finite volume method, [14] and [17] with a pure finite volume scheme.

A first order estimate for triangular meshes was obtained in [26] for a convection diffusion where the diffusion operator is the Laplacian under C^2 regularity assumptions of the solution. It generalizes easily to the case of Voronoï meshes, see [13], to the case of a diffusion operator involving discontinuous tensor diffusion coefficients and the time dependent case [27]. Error estimates assuming H^2 regularity of the solution may also be obtained for linear convection diffusion equations for Dirichlet boundary conditions [29], [30], [32], [13] and Neumann or Fourier boundary conditions [13]. Note also that the finite volume

scheme is well adapted to the discretization of hyperbolic systems (see e.g. [13] and references therein) and is therefore a good candidate for the discretization of systems of equations of different types, see e.g. [28], [48], [49].

We consider here a semilinear convection diffusion equation with non homogeneous Dirichlet boundary conditions, for which no regularity of the solution is known. The case of a semilinear time dependent convection diffusion equation is addressed in [19] where the convergence of a coupled finite element-finite volume scheme is proven (see also [13] for error estimates in the case of time-dependent linear equations and convergence results in the case of nonlinear equations). Here we consider a pure finite volume scheme and prove the existence and convergence of the approximate solutions in the stationary case. The proof uses the property of consistency of the numerical fluxes on regular test functions. This, together with a compactness result on the set of approximate solutions allows to prove a subsequence of approximate solutions (for which an *a priori* estimate is obtained) is a weak solution of the semilinear equation.

Let us consider the following semilinear elliptic equation:

$$-\Delta u(x) + \mathrm{div}(\mathbf{v}q(u))(x) = f(x,u(x)), \quad x \in \Omega, \tag{1}$$

with Dirichlet boundary condition:

$$u(x) = g(x), \quad x \in \partial\Omega, \tag{2}$$

where

Assumption 1

(i) Ω *is an open bounded polygonal subset of* $\mathbb{R}^2$,

(ii) $f : \Omega \to \mathbb{R}$ *is such that*

$$\begin{array}{c} f(x,s) \text{ is measurable with respect to } x \in \Omega \text{ for all } s \in \mathbb{R} \\ \text{and continuous with respect to } s \in \mathbb{R} \text{ for a.e. } x \in \Omega, \end{array} \tag{3}$$

$$\begin{array}{l} \text{There exist } \alpha < \dfrac{1}{\mathrm{diam}(\Omega)^2} \text{ and } \beta \in \mathbb{R} \text{ such that } f(x,s)s \le \alpha s^2 + \beta \\ \text{and } |f(x,s)| \le \beta|s|, \text{ for all } s \in \mathbb{R}, \text{ for a.e. } x \in \Omega. \end{array} \tag{4}$$

(iii) $\mathbf{v} \in C^1(\overline{\Omega}, \mathbb{R}^d), \mathrm{div}\mathbf{v} = 0$, *and let* $V = \max_{x\in\overline{\Omega}} |\mathbf{v}(x)|$.

(iv) $q \in C^1(\mathbb{R}, \mathbb{R})$ *is such that* $q' \ge 0$ *and there exists* $c_q \in \mathbb{R}_+$ *such that* $|q(s)| \le c_q|s|$.

(v) $g \in H^{1/2}(\partial\Omega, \mathbb{R})$; *let* $\tilde{g} \in H^1(\Omega)$ *such that* $\overline{\gamma}(\tilde{g}) = g$ *a.e. on* $\partial\Omega$.

Remark 1

(i)The Laplace operator is considered here for the sake of simplicity, but more general elliptic operators are possible to handle, for instance operators of the form $-div(a(u)\nabla u)$ *with adequate assumptions on* u.

(ii) Since $\mathrm{div}\mathbf{v} = 0$, *we can assume, without loss of generality, that* $q(0) = 0$. *This assumption will simplify the presentation of some proofs.*

Here, and in the sequel, $\overline{\gamma}$ denotes the trace operator from $H^1(\Omega)$ into $L^2(\partial\Omega)$. Note also that "a.e. on $\partial\Omega$" is a.e. for the one-dimensional Lebesgue measure on $\partial\Omega$.

Let us introduce the weak formulation of problem (1),(2). A weak solution of (1),(2) under Assumption 1 is a function $u = \tilde{u} + \tilde{g} \in H^1(\Omega)$ satisfying

$$\left\{\begin{array}{l} u = \tilde{u} + \tilde{g} \text{ where } \tilde{u} \in H_0^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega (\nabla u(x)\nabla\varphi(x) + \mathrm{div}(\mathbf{v}(x)q(u(x)))\varphi(x)dx = \\ \displaystyle\int_\Omega f(x,u(x))\varphi(x)dx,\ \forall\varphi \in H_0^1(\Omega). \end{array}\right. \tag{5}$$

Using Schauder's fixed point theorem (see e.g. [10]) or the convergence theorem 2 which is proved below, it is possible to prove that there exists at least one solution to (5).

2 The finite volume schemes

The finite volume scheme is found by integrating equation (1) on a given control volume of a discretization mesh and finding an approximation of the fluxes on the control volume boundary in terms of the discrete unknowns. Let us first give the assumptions which are needed on the mesh.

2.1 Meshes

Assume K and L to be two neighbouring control volumes of the mesh. A consistent discretization of the normal flux $-\nabla u \cdot \mathbf{n}$ over the interface of two control volumes K and L may be performed with a differential quotient involving values of the unknown located on the orthogonal line to the interface between K and L, on either side of this interface. This remark suggests the following definition of admissible finite volume meshes for the discretization of diffusion problems. We shall only consider here, for the sake of simplicity, the case of polygonal domains. The case of domains with a regular boundary does not introduce any supplementary difficulty other than complex notations.

Definition 1 (Admissible meshes) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$. An admissible finite volume mesh of Ω, denoted by $\mathcal{T}$, is given by a family of "control volumes", which are open polygonal convex subsets of Ω (with positive measure), a family of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^2$, denoted by $\mathcal{E}$ (these are the edges of the control volumes), with strictly positive one-dimensional measure, and a family of points of Ω denoted by $\mathcal{P}$ satisfying the following properties (in fact, we shall denote, somewhat incorrectly, by $\mathcal{T}$ the family of control volumes):*

(i) The closure of the union of all the control volumes is $\overline{\Omega}$;

(ii) For any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \overline{K} \setminus K = \cup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. Let $\mathcal{E} = \cup_{K\in\mathcal{T}} \mathcal{E}_K$.

(iii) For any $(K,L) \in \mathcal{T}^2$ with $K \neq L$, either the $(d-1)$-dimensional Lebesgue measure of $\overline{K} \cap \overline{L}$ is 0 or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$, which will then be denoted by $K|L$.

(iv) The family $\mathcal{P} = (x_K)_{K\in\mathcal{T}}$ is such that $x_K \in \overline{K}$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that $x_K \neq x_L$, and that the straight line $\mathcal{D}_{K,L}$ going through x_K and x_L is orthogonal to $K|L$.

In the sequel, the following notations are used. The mesh size is defined by: $\mathrm{size}(\mathcal{T}) = \sup\{\mathrm{diam}(K), K \in \mathcal{T}\}$. *For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$,* $\mathrm{m}(K)$ *is the area of K and* $\mathrm{m}(\sigma)$ *the length of σ. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\mathrm{int}}$ (resp. $\mathcal{E}_{\mathrm{ext}}$), that is $\mathcal{E}_{\mathrm{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\mathrm{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). The set of neighbours of K is denoted by $\mathcal{N}(K)$, that is $\mathcal{N}(K) = \{L \in \mathcal{T};\ \exists\sigma \in \mathcal{E}_K,\ \overline{\sigma} = \overline{K} \cap \overline{L}\}$. For any $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ we denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. For any $\sigma in \mathcal{E}$, we define $d_\sigma = d_{K,\sigma} + d_{L,\sigma}$ if $\sigma = K|L \in \mathcal{E}_{\mathrm{int}}$ (in which case d_σ is the Euclidean distance between x_K and x_L) and $d_\sigma = d_{K,\sigma}$ if $\sigma \in \mathcal{E}_{\mathrm{ext}} \cap \mathcal{E}_K$.*
For any $\sigma \in \mathcal{E}$; the "transmissibility" through σ is defined by $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$ if $d_\sigma \neq 0$ and $\tau_\sigma = 0$ if $d_\sigma = 0$. In some results and proofs given below, there are summations over $\sigma \in \mathcal{E}_0$, with $\mathcal{E}_0 = \{\sigma \in \mathcal{E};\ d_\sigma \neq 0\}$. For simplicity, (in these results and proofs) $\mathcal{E} = \mathcal{E}_0$ is assumed.

Note that admissible meshes include meshes which are made out of triangles, rectangles or Voronoï meshes (see [13] for more details).
Note that cell centered finite volume schemes may also be defined on meshes which are not admissible in the sense of the above definition [18], [9]. In this case, however, some more technical assumptions are needed on the mesh to show the convergence of the scheme.

Let us now introduce the space of piecewise constant functions associated with an admissible mesh and some "discrete H_0^1" norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

Definition 2 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ an admissible mesh. Define $X(\mathcal{T})$ as the set of functions from Ω to $\mathbb{R}$ which are constant over each control volume of the mesh.*

Definition 3 (Discrete norms) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, and $\mathcal{T}$ an admissible finite volume mesh in the sense of Definition 1.*
For $u \in X(\mathcal{T})$, define the discrete H_0^1 norm by

$$\|u\|_{1,\mathcal{T}} = \Big(\sum_{\sigma\in\mathcal{E}} \tau_\sigma (D_\sigma u)^2\Big)^{\frac{1}{2}} \tag{6}$$

where, for any $\sigma \in \mathcal{T}$, $\tau_\sigma = \mathrm{m}(\sigma)/d_\sigma$ and
$D_\sigma u = |u_K - u_L|$ if $\sigma \in \mathcal{E}_{\rm int}$, $\sigma = K|L$,
$D_\sigma u = |u_K|$ if $\sigma \in \mathcal{E}_{\rm ext} \cap \mathcal{E}_K$,
where u_K denotes the value taken by u on the control volume K and the sets $\mathcal{E}$, $\mathcal{E}_{\rm int}$, $\mathcal{E}_{\rm ext}$ and $\mathcal{E}_K$ are defined in Definition 1.

2.2 The schemes

Let $\mathcal{T}$ be an admissible mesh. Let us now define a finite volume scheme to discretize (1)-(2).
Let $(u_K)_{K\in\mathcal{T}}$ denote the discrete unknowns and let

$$f_K(u_K) = \frac{1}{\mathrm{m}(K)} \int_K f(x, u_K) dx, \forall K \in \mathcal{T}. \tag{7}$$

In order to describe the scheme in the most general way, one introduces some auxiliary unknowns namely the fluxes $F_{K,\sigma}$, for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, and some (expected) approximation of u on an edge σ, denoted by u_σ, for all $\sigma \in \mathcal{E}$. For $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, let $\mathbf{n}_{K,\sigma}$ denote the normal unit vector to σ outward to K and $v_{K,\sigma} = \int_\sigma \mathbf{v}(x) \cdot \mathbf{n}_{K,\sigma} d\gamma(x)$. Note that $d\gamma$ is the integration symbol for the $(d-1)$-dimensional Lebesgue measure on the considered hyperplane.

We may now write the finite volume scheme for the discretization of Problem (1)-(2) under assumptions 1 as the following set of equations:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} + \sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma} q(u_{\sigma,+}) = \mathrm{m}(K) f_K(u_K), \ \forall K \in \mathcal{T}, \tag{8}$$

where $u_{\sigma,+}$ is defined by

$$\begin{array}{l} \text{if } \sigma = K|L, \text{ then } u_{\sigma,+} = u_K \text{ if } v_{K,\sigma} \geq 0, \text{ and } u_{\sigma,+} = u_L \text{ otherwise;} \\ \text{if } \sigma \subset K \cap \partial\Omega, \text{ then } u_{\sigma,+} = u_K \text{ if } v_{K,\sigma} \geq 0 \text{ and } u_{\sigma,+} = u_\sigma \text{ otherwise,} \end{array} \tag{9}$$

and $F_{K,\sigma}$ is defined by

$$F_{K,\sigma} = -F_{L,\sigma}, \forall \sigma \in \mathcal{E}_{\rm int}, \text{ if } \sigma = K|L, \tag{10}$$

$$F_{K,\sigma} d_{K,\sigma} = -\mathrm{m}(\sigma)(u_\sigma - u_K),\ \forall \sigma \in \mathcal{E}_K,\ \forall K \in \mathcal{T}, \tag{11}$$

and

$$u_\sigma = \frac{1}{\mathrm{m}(\sigma)} \int_\sigma g(y) d\gamma(y),\ \ \forall \sigma \in \mathcal{E}_{\mathrm{ext}}. \tag{12}$$

Note that the values u_σ for $\sigma \in \mathcal{E}_{\mathrm{int}}$ are auxiliary values which may be eliminated so that (8)-(12) leads to a nonlinear system of N equations with N unknowns, namely the $(u_K)_{K\in\mathcal{T}}$, with $N = \mathrm{card}(\mathcal{T})$. This nonlinear system can be written, using some ordering of the unknowns and equations, as

$$AU + B(U) = C(U) + D(g), \tag{13}$$

where:
$U \in \mathbb{R}^N$ is the vector of discrete unknowns (that is the u_K, $K \in \mathcal{T}$), N being the number of cells of the mesh $\mathcal{T}$,
A is a linear application from $\mathbb{R}^N$ to $\mathbb{R}^N$ and AU corresponds to the discretization of $-\Delta u(x)$,
B is a continuous application from $\mathbb{R}^N$ to $\mathbb{R}^N$ and $B(U)$ corresponds to the discretization of $\mathrm{div}(\mathbf{v}q(u))(x)$,
C is a continuous application from $\mathbb{R}^N$ to $\mathbb{R}^N$ and $C(U)$ corresponds to the discretization of $f(x, u(x))$
and $D(g)$ is a vector of $\mathbb{R}^N$ which contains all the terms depending on g (note that D is and application from $L^1(\partial\Omega)$ into $\mathbb{R}^N$.

3 Discrete Poincaré inequalities and trace inequalities

We give in this section some inequalities for piecewise constant functions.

We recall here a discrete Poincaré inequality for the discrete H_0^1 norm of a piecewise constant function. The proof of this inequality may be found in [13] or [14]. Note that a "discrete mean value Poincaré inequality" may also be proven in order to deal with Neumann boundary conditions (see [13]).

Lemma 1 (Discrete Poincaré inequality) *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3, $\mathcal{T}$ an admissible finite volume mesh in the sense of Definition 1 and $u \in X(\mathcal{T})$ (see Definition 2), then*

$$\|u\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega) \|u\|_{1,\mathcal{T}}, \tag{14}$$

where $\|\cdot\|_{1,\mathcal{T}}$ is the discrete H_0^1 norm defined in Definition 3.

The following result will also be useful for getting estimates on the approximate solutions in the case of non homogeneous Dirichlet boundary conditions. We refer to [14] for the proof.

Lemma 2 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$, $\tilde{g} \in H^1(\Omega)$ and $g = \overline{\gamma}(\tilde{g})$ (recall that $\overline{\gamma}$ is the "trace" operator from $H_0^1(\Omega)$ to $H^{1/2}(\partial\Omega)$). Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) and let :*

$$\tilde{g}_K = \frac{1}{\mathrm{m}(K)} \int_K \tilde{g}(x) dx,\ \forall K \in \mathcal{T},$$

$$\tilde{g}_\sigma = \frac{1}{\mathrm{m}(\sigma)} \int_\sigma g(x) d\gamma(x)\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}},$$

and

$$|D\tilde{g}|_{\mathcal{T}} = \Big(\sum_{\sigma = K|L \in \mathcal{E}_{\mathrm{int}}} \tau_{K|L} (\tilde{g}_K - \tilde{g}_L)^2 + \sum_{\sigma \in \mathcal{E}_{\mathrm{ext}}} \tau_\sigma (\tilde{g}_{K(\sigma)} - \tilde{g}_\sigma)^2 \Big)^{\frac{1}{2}}.$$

Then there exists $C \in \mathbb{R}_+$, *only depending on* $\zeta = \min\{\frac{d_{K,\sigma}}{\mathrm{diam}(K)}, K \in \mathcal{T}, \sigma \in \mathcal{E}_K\}$ *and* $M = \max\{\mathrm{card}(\mathcal{E}_K), K \in \mathcal{T}\}$, *such that*

$$|D\tilde{g}|_{\mathcal{T}} \leq C\|\tilde{g}\|_{H^1(\Omega)}. \tag{15}$$

4 Existence and estimates for the approximate solution

Let us first prove the existence of the approximate solution and an estimate on this solution. This estimate will be obtained by using the discrete inequalities which were proved in the previous sections, and will yield convergence thanks to a compactness theorem given in the appendix.

Lemma 3 (Existence and estimate) *Under Assumptions 1, let* $\mathcal{T}$ *be an admissible mesh in the sense of Definition 1, and let:*

$$\zeta = \min\Big(\min_{K\in\mathcal{T}}\min_{\sigma\in K}\frac{d_{K,\sigma}}{\mathrm{diam}}(K), \min_{K\in\mathcal{T}}\min_{\sigma\in K}\frac{d_{K,\sigma}}{d_\sigma}\Big), \tag{16}$$

then there exists a solution $(u_K)_{K\in\mathcal{T}}$ *to the system of equations (8)-(12).*

Furthermore, let $u_{\mathcal{T}} \in X(\mathcal{T})$ *(see Definition 2) be defined by* $u_{\mathcal{T}}(x) = u_K$ *for a.e.* $x \in K$, *and for any* $K \in \mathcal{T}$; *there exists* $C \in \mathbb{R}$, *only depending on* Ω, $\|\tilde{g}\|_{H^1(\Omega)}$, ζ, $M = \max_{K\in\mathcal{T}}\mathrm{card}(\mathcal{E}_K)$, f *and* q, *such that*

$$\|\tilde{u}_{\mathcal{T}}\|_{1,\mathcal{T}} \leq C \text{ and } \|\tilde{u}_{\mathcal{T}}\|_{L^2(\Omega)} \leq C, \tag{17}$$

where

$$\tilde{u}_{\mathcal{T}}(x) = \tilde{u}_K = u_K - \frac{1}{\mathrm{m}(K)}\int_K \tilde{g}(y)dy \text{ for all } x \in K \text{ and all } K \in \mathcal{T}. \tag{18}$$

Remark 2 *In the case of homogeneous Dirichlet boundary conditions, the additional assumption (16) is not required. This technical assumption is essentially needed for the proof of Lemma 2 whereas the proof of the estimate (17) for homogeneous Dirichlet boundary conditions only requires the use of the discrete Poincaré inequality (1) (see [13]). Similarly, if there is no convection (i.e.* $q = 0$ *or* $\mathbf{v} = 0$*), then (16) is not needed. This latter assumption is used to obtain a bound on the convection terms in the estimate of the approximate solution.*

Proof of Lemma 3

Equations (8)-(12) lead, after an easy elimination of the auxiliary unknowns, to a nonlinear system of N equations with N unknowns, namely the $(u_K)_{K\in\mathcal{T}}$, with $N = \mathrm{card}(\mathcal{T})$.
We shall first prove the existence and uniqueness of the solution to the linearized system which is obtained from the numerical scheme. We shall then prove an estimate on any possible function $u_{\mathcal{T}}$ of (8)-(12). These two steps will allow to prove the existence of the solution to the numerical scheme by a topological degree argument.
We assume (without loss of generality) that $q(0) = 0$.

Step 1 (existence and uniqueness of the solution to the linear system)
Let $(r_K)_{K\in\mathcal{T}}$ be a given vector of $\mathbb{R}^N$ (with $N = \mathrm{card}(\mathcal{T})$). Let us introduce the linear systems of equations with unknowns $(u_K)_{K\in\mathcal{T}}$ consisting of the following equation:

$$\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = r_K \ \forall K \in \mathcal{T}, \tag{19}$$

where $F_{K,\sigma}$, for $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}$ is defined with respect to the unknowns $(u_K)_{K\in\mathcal{T}}$ by (10) and (11), and equation (12).
First assume that $(u_K)_{K\in\mathcal{T}}$ satisfies the linear system (19), (10), (11), (12) with $r_K = 0$ for all $K \in \mathcal{T}$, and $\int_\sigma g(y)d\gamma(y) = 0$ for any $\sigma \in \mathcal{E}_{\mathrm{ext}}$. Let us prove that in this case $(u_K) = 0$ for all $K \in \mathcal{T}$. This yields the uniqueness (and thus the existence) of the solution to the linear system (19), (10), (11), (12).

Multiplying (19) by u_K, summing over K, and using (10) and (11) leads to

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}u_K = 0. \tag{20}$$

Let us now perform a "discrete integration by parts", that is a reordering of the summations over the edges of the mesh. We obtain:

$$\sum_{\sigma\in\mathcal{E}}\tau_\sigma(D_\sigma u)^2 = 0, \tag{21}$$

where $|D_\sigma u| = |u_K - u_L|$, if $\sigma = K|L$ and $|D_\sigma u| = |u_K|$, if $\sigma \in \mathcal{E}_K \cap \mathcal{E}_{\rm ext}$.
Hence,

$$\|u_\mathcal{T}\|_{1,\mathcal{T}}^2 = 0, \tag{22}$$

which yields $u_K = 0$ for all $K \in \mathcal{T}$.
This proves the existence and the uniqueness of the solution to the linear system (19), (10), (11), (9), (12) for any $(w_K)_{K\in\mathcal{T}} \in \mathbb{R}^N$, and for any $\{\int_\sigma g(y)d\gamma(y),\ \sigma \in \mathcal{E}_{\rm ext}\}$.

Step 2 (Existence of a solution) Using the formulation (13) of the numerical scheme, let us prove the existence of U solution to (13). From step 1, A is invertible and (13) is therefore equivalent to:

$$U = -A^{-1}B(U) + A^{-1}C(U) + A^{-1}D(g). \tag{23}$$

In order to show that (23) admits at least one solution in $\mathbb{R}^N$, and therefore that (8)-(12) admits at least one solution, we are going to use a topological degree argument (see also Remark 3).
For $t \in [0,1]$ and $U \in \mathbb{R}^N$, let $F(t,U) = -A^{-1}B(tU) + tA^{-1}C(U) + A^{-1}D(tg)$, so that F is continuous from $[0,1]\times\mathbb{R}^N$ in $\mathbb{R}^N$.

Let us endow the space $\mathbb{R}^N$ with some norm; let us choose for instance the norm defined by $|U|^2 = \sum_{K\in\mathcal{T}} {\rm m}(K)u_K^2$, where the u_K, $K \in \mathcal{T}$, are the components of U.
One may show (see [14]) that for any $\varepsilon > 0$, there exists $C_\varepsilon \geq 0$ depending only on ε such that

$$\begin{aligned}&\|\tilde{u}_\mathcal{T}^{(t)}\|_{1,\mathcal{T}}^2 + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}q(tu_{\sigma,+})\tilde{u}_K^{(t)} \leq \alpha(1+\varepsilon)\|\tilde{u}_\mathcal{T}^{(t)}\|_{L^2(\Omega)}^2\\ &+C_\varepsilon t^2\|\tilde{g}\|_{L^2(\Omega)}^2 + \beta{\rm m}(\Omega) + \beta\|\tilde{u}_\mathcal{T}^{(t)}\|_{L^2(\Omega)}\|\tilde{g}\|_{L^2(\Omega)} + \beta\|\tilde{g}\|_{L^2(\Omega)}^2 + t|D\tilde{g}|_\mathcal{T}\|\tilde{u}_\mathcal{T}\|_{1,\mathcal{T}},\end{aligned} \tag{24}$$

and that

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} v_{K,\sigma}q(tu_{\sigma,+})\tilde{u}_K \leq Ct\|u_\mathcal{T}\|_{L^2(\Omega)}|D\tilde{g}|_\mathcal{T} \tag{25}$$

where C depends only on ζ, Ω, $\mathbf{v}$ and f. Therefore, from (24) (choosing ε small enough and (25), using Lemma 2 and the discrete Poincaré inequality (14), there exists $C_1 \in \mathbb{R}$, only depending on Ω, $\|\tilde{g}\|_{H^1(\Omega)}$, ζ, $\mathbf{v}$ and f, such that $\|\tilde{u}_\mathcal{T}\|_{1,\mathcal{T}} \leq C_1$ and $\|\tilde{u}_\mathcal{T}\|_{L^2(\Omega)} \leq C_1$.
This gives:

$$\exists R > 0 \text{ such that if } (t,U) \in [0,1]\times\mathbb{R}^N \text{ and } U = F(t,U) \text{ then } |U| \neq R. \tag{26}$$

with $R > C_1 + \|\tilde{g}\|_{L^2(\Omega)}$. One also deduces the estimates (17) on the solutions of (8)-(12). This concludes the proof of the lemma.

Since (26) is satisfied, it is possible to define for $t \in [0,1]$, the (Brouwer) topological degree of the application $Id - F(t,.)$ with respect to $B_R = \{U \in \mathbb{R}^N, |U| < R\}$ and 0, which is denoted by $d(Id - F(t,.), B_R, 0)$ (see e.g. [10] for the definition of the topological degree and its properties). Then, thanks to the homotopy invariance of the degree and since $F(0,U) = 0$ (for all $U \in \mathbb{R}^N$), one has

$$d(Id - F(1,.), B_R, 0) = d(Id, B_R, 0).$$

Since $d(Id, B_R, 0) = 1$, this leads to $d(Id - F(1,.), B_R, 0) \neq 0$ which proves the existence of $U \in B_R$ such that $U - F(1,U) = 0$ i.e. that U is a solution of (23). This proves the existence of a solution to (8)-(12).

Remark 3 *Note that the topological degree argument which was used in the above proof may also be used to show the existence of at least a solution to problem (1)-(2). In fact, the existence of a solution to problem (1)-(2) is also an immediate consequence of the convergence theorem 2 given hereafter.*

We now state a discrete maximum property of the scheme. Even though this property is not used in the sequel, it is a very important feature when dealing with problems where positivity of the solution must be ensured. This stability property is valid for the scheme (8)-(12) for any admissible mesh thanks to the upwind approximation of the convection term; note that a centered approximation of the convection term would still yield a convergent scheme, but that the scheme might become unstable for (that is the maximum principle will not hold) if the convection term is large with respect to the size of the mesh.

Proposition 1 *Under Assumption 1, let $\mathcal{T}$ be an admissible mesh in the sense of Definition 1. If $f \geq 0$ for all $K \in \mathcal{T}$, and if Dirichlet boundary conditions (2) hold with $g \geq 0$, then the solution $(u_K)_{K\in\mathcal{T}}$ of (8)-(12) satisfies $u_K \geq 0$ for all $K \in \mathcal{T}$.*

Proof A proof of this result is given in [26]. (see also [13]) in the linear case. It uses the strong formulation. We use here the weak formulation. For any $K \in \mathcal{T}$, denote by u_K^- the negative part of u_K, that is $u_K^- = \frac{1}{2}(|u_K| - u_K)$ (note that contrary to its name, the negative part of u_K is positive...) Multiplying (8) by u_K^-, summing over $K \in \mathcal{T}$ and using the fact that $f \geq 0$ and $g \geq 0$ yields that:

$$\sum_{\sigma\in\mathcal{E}} \tau_\sigma (u_{\sigma,+} - u_{\sigma,-})(u_{\sigma,+}^- - u_{\sigma,-}^-) + v_\sigma q(u_{\sigma,+})(u_{\sigma,+}^- - u_{\sigma,-}^-) \geq 0$$

Noting that $|u_{\sigma,+}^- - u_{\sigma,-}^-| \leq |u_{\sigma,+} - u_{\sigma,-}|$ and that $(u_{\sigma,+} - u_{\sigma,-})(u_{\sigma,+}^- - u_{\sigma,-}^-) \leq 0$ yields that

$$-\sum_{\sigma\in\mathcal{E}} \tau_\sigma (u_{\sigma,+}^- - u_{\sigma,-}^-)^2 + v_\sigma q(u_{\sigma,+})(u_{\sigma,+}^- - u_{\sigma,-}^-) \geq 0. \tag{27}$$

Let G be a primitive of q such that $G(0) = 0$, one may show that:

$$-\sum_{\sigma\in\mathcal{E}} v_\sigma q(u_{\sigma,+})(u_{\sigma,+}^- - u_{\sigma,-}^-) = \sum_{K\in\mathcal{T}} G(-u_K^-) \int_{\partial K} \mathbf{v}(x).\mathbf{n}_K(x) d\gamma(x) + \sum_{\sigma\in\mathcal{E}} v_\sigma \int_{-u_{\sigma,-}^-}^{-u_{\sigma,+}^-} (q(u_{\sigma,+}) - q(s)) ds \geq 0.$$

From (27), one obtains:

$$u_{\sigma,+}^- - u_{\sigma,-}^- = 0 \text{ for any } \sigma \in \mathcal{E}. \tag{28}$$

Hence $u_K \geq 0$ for any $K \in \mathcal{T}$. Indeed, for any $a \in \mathbb{R}$, let Ω_a be the interior of the union of the closures of the control volumes such that $u_K^- = a$. Let $\sigma \in \mathcal{E}$ such that $\sigma \subset \partial\Omega_a$. Then, from (28), for any $b \neq a$, $\sigma \not\subset \partial\Omega_b$, so that $\sigma \subset \partial\Omega$, which proves that $a = 0$ and concludes the proof of Proposition 1. ■

5 Convergence of the scheme

In order to show the convergence of the scheme, we shall first show that for any $C \geq 0$, the set $\{u_\mathcal{T} \in X(\mathcal{T})$ where $\mathcal{T}$ is an admissible mesh of $\Omega\}$ is relatively compact in $L^2(\Omega)$. In order to do so, we shall use the following compactness result, the proof of which may be found in [14] or [13].

Theorem 1 *Let Ω be an open bounded set of $\mathbb{R}^d$, $d \geq 1$, and $\{u_n,\, n \in \mathbb{N}\}$ a bounded sequence of $L^2(\Omega)$. For $n \in \mathbb{N}$, one defines $\overline{u}_n$ by $\overline{u}_n = u_n$ a.e. on Ω and $\overline{u}_n = 0$ a.e. on $\mathbb{R}^d \setminus \Omega$. Assume that there exist $C \in \mathbb{R}$ and $\{h_n,\, n \in \mathbb{N}\} \subset \mathbb{R}_+$ such that $h_n \to 0$ as $n \to \infty$ and*

$$\|\overline{u}_n(\cdot + \eta) - \overline{u}_n\|^2_{L^2(\mathbb{R}^d)} \leq C|\eta|(|\eta| + h_n), \forall n \in \mathbb{N},\ \forall \eta \in \mathbb{R}^d. \tag{29}$$

Then, $\{u_n,\, n \in \mathbb{N}\}$ is relatively compact in $L^2(\Omega)$. Furthermore, if $u_n \to u$ in $L^2(\Omega)$ as $n \to \infty$, then $u \in H^1_0(\Omega)$.

Lemma 4 *Let Ω be an open bounded set of $\mathbb{R}^d$, $d = 2$ or 3. Let $\mathcal{T}$ be an admissible mesh in the sense of Definition 1 and $u \in X(\mathcal{T})$ (see Definition 2). One defines $\overline{u}$ by $\overline{u} = u$ a.e. on Ω, and $\overline{u} = 0$ a.e. on $\mathbb{R}^d \setminus \Omega$. Then there exists $C > 0$, only depending on Ω, such that*

$$\|\overline{u}(\cdot + \eta) - \overline{u}\|^2_{L^2(\mathbb{R}^d)} \leq \|u\|^2_{1,\mathcal{T}}|\eta|(|\eta| + C\,\mathrm{size}(\mathcal{T})), \forall \eta \in \mathbb{R}^d. \tag{30}$$

We refer to [14] for the proof of this lemma. We may now state the convergence result:

Theorem 2 (Convergence)
Assume items 1, 2, 3 and 4 of Assumption 1 and $g \in H^{1/2}(\partial\Omega)$. Let $\zeta \in \mathbb{R}_+$ and $M \in \mathbb{N}$ be given values. Consider a family of admissible meshes of Ω (in the sense of Definition 1) such that $d_{K,\sigma} \geq \zeta \mathrm{diam}(K)$ for all control volume $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$, and $\mathrm{card}(\mathcal{E}_K) \leq M$ for all $K \in \mathcal{T}$. Assume that $\mathrm{size}(\mathcal{T})$ tends to 0 as n tends to infinity. Let $(u_K)_{K\in\mathcal{T}}$ be the solution of the system given by equations (8)-(12). For a given mesh $\mathcal{T}$, define $u_\mathcal{T} \in X(\mathcal{T})$ by $u_\mathcal{T}(x) = u_K$ for a.e. $x \in K$ and for any $K \in \mathcal{T}$. Then, there exists a subsequence of the sequence $u_\mathcal{T}$ which converges in $L^2(\Omega)$ to a function u as $\mathrm{size}(\mathcal{T}) \to 0$, where u satisfies (5). Moreover, if there exists only one solution to (5), then the whole sequence converges to u.

Proof of Theorem 2
For the sake of simplicity, we shall prove the above theorem in the case where $\mathbf{v} = 0$ and f depends only on x (and not on u) and we refer to [14] for the complete proof. For any mesh $\mathcal{T}$ of the family of admissible meshes which is considered here, let $\tilde{u} \in X(\mathcal{T})$ be defined by

$$\tilde{u}_\mathcal{T}(x) = \tilde{u}_K = u_K - \frac{1}{\mathrm{m}(K)}\int_K \tilde{g}(y)dy \text{ for all } x \in K \text{ and all } K \in \mathcal{T}. \tag{31}$$

Then $(\tilde{u}_K)_{K\in\mathcal{T}}$ satisfies

$$\sum_{\sigma\in\mathcal{E}_K} \tilde{F}_{K,\sigma} = \mathrm{m}(K)f_K - \sum_{\sigma\in\mathcal{E}_K} G_{K,\sigma},\ \forall K \in \mathcal{T}, \tag{32}$$

$$\tilde{F}_{K,\sigma} = -\tau_{K|L}(\tilde{u}_L - \tilde{u}_K),\ \forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \text{if } \sigma = K|L, \tag{33}$$

$$\tilde{F}_{K,\sigma} = \tau_\sigma(\tilde{u}_K),\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}} \text{ such that } \sigma \in \mathcal{E}_K. \tag{34}$$

$$G_{K,\sigma} = -\tau_{K|L}\Big(\frac{1}{\mathrm{m}(L)}\int_L \tilde{g}(y)dy - \frac{1}{\mathrm{m}(K)}\int_K \tilde{g}(y)dy\Big),\ \forall \sigma \in \mathcal{E}_{\mathrm{int}},\ \text{if } \sigma = K|L, \tag{35}$$

$$G_{K,\sigma} = -\tau_\sigma\Big(\frac{1}{\mathrm{m}(\sigma)}\int_\sigma g(x)d\gamma(x) - \frac{1}{\mathrm{m}(K)}\int_K \tilde{g}(y)dy\Big),\ \forall \sigma \in \mathcal{E}_{\mathrm{ext}} \text{ such that } \sigma \in \mathcal{E}_K. \tag{36}$$

Using Lemma 3 there exists $C_1 \in \mathbb{R}$, only depending on Ω, $\|\tilde{g}\|_{H^1(\Omega)}$, ζ, M and f, such that $\|\tilde{u}_\mathcal{T}\|_{1,\mathcal{T}} \leq C_1$ and $\|\tilde{u}_\mathcal{T}\|_{L^2(\Omega)} \leq C_1$. Furthermore, since $\mathrm{size}(\mathcal{T}) \to 0$, from Lemma 4 and Theorem 1, there exists a subsequence, still denoted by $\tilde{u}_\mathcal{T}$, and $\tilde{u} \in H^1_0(\Omega)$ such that $\tilde{u}_\mathcal{T}$ converges to $\tilde{u}$ in $L^2(\Omega)$. Let us now prove that $u = \tilde{u} + \tilde{g}$ satisfies (5). Since $\tilde{u} \in H^1_0(\Omega)$, there only remains to show that $\tilde{u}$ satisfies:

$$\int_\Omega (\nabla \tilde{u}(x) \nabla \varphi(x) = \int_\Omega f(x)\varphi(x)dx - \int_\Omega (\nabla \tilde{g}(x) \nabla \varphi(x))dx, \, \forall \varphi \in H_0^1(\Omega). \tag{37}$$

Let $\varphi \in C_c^\infty(\Omega)$ and let size($\mathcal{T}$) be small enough so that $\varphi(x) = 0$ if $x \in K$ and $K \in \mathcal{T}$ is such that $\partial K \cap \partial\Omega \neq \emptyset$. Taking $t = 1$ in (32), multiplying by $\varphi(x_K)$, and summing the result over $K \in \mathcal{T}$ yields

$$T_1 = T_2 + T_3, \tag{38}$$

with

$$T_1 = -\sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \tau_{K|L}(\tilde{u}_L - \tilde{u}_K)\varphi(x_K),$$

$$T_2 = \sum_{K\in\mathcal{T}} \mathrm{m}(K)\varphi(x_K) f_K\,.$$

$$T_3 = \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} \tau_{K|L}(\tilde{g}_L - \tilde{g}_K)\varphi(x_K),$$

where

$$\tilde{g}_K = \frac{1}{\mathrm{m}(K)} \int_K \tilde{g}(x)dx, \forall K \in \mathcal{T}.$$

First remark that, thanks to the assumptions on f,

$$T_2 \to \int_\Omega f(x)\varphi(x)dx \text{ as size}(\mathcal{T}) \to 0.$$

Let us now turn to the study of T_1;

$$T_1 = -\sum_{K|L\in\mathcal{E}_{\mathrm{int}}} \tau_{K|L}(\tilde{u}_L - \tilde{u}_K)(\varphi(x_K) - \varphi(x_L))\,.$$

Consider the following auxiliary expression:

$$\begin{aligned} T_1' &= \int_\Omega \tilde{u}_\mathcal{T}(x)\Delta\varphi(x)dx \\ &= \sum_{K\in\mathcal{T}} \tilde{u}_K \int_K \Delta\varphi(x)dx \\ &= \sum_{K|L\in\mathcal{E}_{\mathrm{int}}} (\tilde{u}_K - \tilde{u}_L) \int_{K|L} \nabla\varphi(x)\cdot \mathbf{n}_{K,L} d\gamma(x). \end{aligned}$$

Since $\tilde{u}_\mathcal{T}$ converges to $\tilde{u}$ in $L^2(\Omega)$, it is clear that T_1' tends to $\int_\Omega \tilde{u}(x)\Delta\varphi(x)\ dx$ as size($\mathcal{T}$) tends to 0. Define

$$R_{K,L} = \frac{1}{\mathrm{m}(K|L)} \int_{K|L} \nabla\varphi(x)\cdot \mathbf{n}_{K,L} d\gamma(x) - \frac{\varphi(x_L) - \varphi(x_K)}{d_{K|L}},$$

where $\mathbf{n}_{K,L}$ denotes the unit normal vector to $K|L$, outward to K, then

$$
\begin{aligned}
|T_1 + T_1'| &= |\sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)(\tilde{u}_K - \tilde{u}_L)R_{K,L}| \\
&\leq \Big[\sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)\frac{(\tilde{u}_K-\tilde{u}_L)^2}{d_{K|L}} \sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)d_{K|L}(R_{K,L})^2\Big]^{1/2},
\end{aligned}
$$

Regularity properties of the function φ give the existence of $C_1 \in \mathbb{R}$, only depending on φ, such that $|R_{K,L}| \leq C_1 {\rm size}(\mathcal{T})$. Therefore, since

$$
\sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)d_{K|L} \leq d{\rm m}(\Omega),
$$

using Estimate (17), we conclude that $T_1 + T_1' \to 0$ as ${\rm size}(\mathcal{T}) \to 0$.

The study of T_3 is similar. Let us introduce the function $\tilde{g}_\mathcal{T} \in X(\mathcal{T})$ by

$$
\tilde{g}_\mathcal{T}(x) = \frac{1}{{\rm m}(K)}\int_K \tilde{g}(y)dy,\ \forall x \in K,\ \forall K \in \mathcal{T},
$$

which converges to $\tilde{g}$ in $L^2(\Omega)$, as ${\rm size}(\mathcal{T}) \to 0$. Let

$$
T_3' = \int_\Omega \tilde{g}_\mathcal{T}(x)\Delta\varphi(x)dx.
$$

With computations similar to those carried out for T_1, we obtain that

$$
|T_3 + T_3'| \leq \Big[\sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)\frac{(\tilde{g}_K-\tilde{g}_L)^2}{d_{K|L}} \sum_{K|L\in\mathcal{E}_{\rm int}} {\rm m}(K|L)d_{K|L}(R_{K,L})^2\Big]^{1/2}.
$$

Hence, thanks to Lemma 2, and the fact that $|R_{K,L}| \leq C_1{\rm size}(\mathcal{T})$, one deduces that $T_3 + T_3' \to 0$ as ${\rm size}(\mathcal{T}) \to 0$, and since $\tilde{g}_\mathcal{T} \to \tilde{g}$ in $L^2(O)$ as ${\rm size}(\mathcal{T}) \to 0$,

$$
T_3 \to \int_\Omega \nabla\tilde{g}(x)\nabla\varphi(x)dx \text{ as } {\rm size}(\mathcal{T}) \to 0.
$$

Hence, letting ${\rm size}(\mathcal{T}) \to 0$ in (38) yields that the function $\tilde{u} \in H_0^1(\Omega)$ satisfies

$$
\int_\Omega \Big(-\tilde{u}(x)\Delta\varphi(x) - f(x)\varphi(x)\Big)dx = \int_\Omega \nabla\tilde{g}(x)\nabla\varphi(x)dx,\ \forall\varphi \in C_c^\infty(\Omega),
$$

which, in turn, yields (37) thanks to the fact that $\tilde{u} \in H_0^1(\Omega)$, and to the density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$.
Finally, the function $u_\mathcal{T}$ converges in $L^2(\Omega)$, as ${\rm size}(\mathcal{T}) \to 0$ to $u = \tilde{u} + \tilde{g} \in H^1(\Omega)$.
This concludes the proof of $u_\mathcal{T} \to u$ in $L^2(\Omega)$ as ${\rm size}(\mathcal{T}) \to 0$, where u satisfies (5).

■

6 Convergence of an approximate gradient

Since the approximate solution constructed with a classical cell-centered finite volume scheme is piecewise constant, an approximation of the gradient of the solution may be seen to be more complex than with a finite element method. Indeed, the convergence of a reconstructed gradient has been shown in [7], for certain quadrangular meshes using a nine point scheme. It has also been shown on certain meshes by rewriting the finite volume scheme as a finite element scheme [1], [46]. Here we show that one may

construct an approximate gradient on all admissible meshes by using some mesh functions which are very close to those used in mixed finite element theory (see e.g. [38]. For the sake of clarity, we shall assume here that $\mathbf{v} = 0$, $g = 0$, and that f does not depend on u. Hence the problem reduces to the Laplace equation

$$-\Delta u(x) = f(x), \quad \text{for a.e. } x \in \Omega, \tag{39}$$

with Dirichlet boundary condition:

$$u(x) = 0, \quad \text{for a.e. } x \in \partial\Omega, \tag{40}$$

6.1 Definition of an approximate gradient

In order to define an approximate gradient of u, we introduce, for $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$ the variational solution $\phi_{K,\sigma} \in H^1(K)$ of the following Neumann problem.

$$\Delta\phi_{K,\sigma}(x) = \frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)}, \quad \text{for a.e. } x \in K,$$

with

$$\int_K \phi_{K,\sigma}(x)dx = 0,$$

$$\nabla\phi_{K,\sigma}(y) \cdot \mathbf{n}_{K,\sigma} = 1, \text{ for a.e. } y \in \sigma,$$

and

$$\nabla\phi_{K,\sigma}(y) \cdot \mathbf{n}_{K,\tilde{\sigma}} = 0, \text{ for a.e. } y \in \tilde{\sigma}, \tilde{\sigma} \in \mathcal{E}_K, \tilde{\sigma} \neq \sigma.$$

Therefore, $\phi_{K,\sigma} \in H^1(K)$ is the unique solution with $\int_K \phi_{K,\sigma}(x)dx = 0$ of the variational formulation

$$\int_K \nabla\phi_{K,\sigma}(x)\nabla\psi(x)dx = -\frac{\mathrm{m}(\sigma)}{\mathrm{m}(K)}\int_K \psi(x)dx + \int_\sigma \psi(y)d\gamma(y), \quad \forall\psi \in H^1(K). \tag{41}$$

We define the function $\phi_K \in H^1(K)$ by

$$\phi_K = \sum_{\sigma\in\mathcal{E}_K} \frac{u_\sigma - u_K}{d_{K,\sigma}}\phi_{K,\sigma}, \tag{42}$$

where $(u_K)_{K\in\mathcal{T}}$, $(u_\sigma)_{\sigma\in\mathcal{E}}$ is the solution to (8)-(2) (with $\mathbf{v} = 0$, $g = 0$, and f independent of u) and we define the approximate gradient $G_\mathcal{T} \in H_{div}(\Omega)$ by

$$G_\mathcal{T}(x) = \nabla\phi_K(x), \forall x \in K, \forall K \in \mathcal{T}. \tag{43}$$

The result $G_\mathcal{T} \in H_{div}(\Omega)$ is a consequence of the principle of conservativity of the finite volume scheme.

6.2 Convergence of the approximate gradient

Theorem 3 *Let Ω be an open bounded polygonal domain of $\mathbb{R}^2$ and $f \in L^2(\Omega)$, let $\zeta > 0$ and $M > 0$ be given values and $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that the inequalities $d_{K,\sigma} \geq \zeta\mathrm{diam}(K)$ and $M \geq card(\mathcal{E}_K)$ hold for any control volume $K \in \mathcal{T}$ and for any $\sigma \in \mathcal{E}_K$. Let $G_\mathcal{T}$ be defined by equations (41)-(43). Then, for fixed values ζ and M, $G_\mathcal{T}$ converges in $H_{\mathrm{div}}(\Omega)$ to the gradient of the unique variational solution $u \in H_0^1(\Omega)$ of Problem (39), (40) as* $\mathrm{size}(\mathcal{T}) \to 0$.

Proof Let $\varepsilon > 0$ and $\varphi \in C_c^\infty(\Omega)$ be a function such that $\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx \leq \varepsilon$. We have, for all $K \in \mathcal{T}$,

$$\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(\varphi_\sigma - \varphi(x_K)) = \int_K \Delta\varphi(x)dx - \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) R_{K,\sigma}, \tag{44}$$

where, for all $\sigma \in \mathcal{E}$, the values φ_σ are defined by

$$\tau_{K,\sigma}(\varphi_\sigma - \varphi(x_K)) + \tau_{L,\sigma}(\varphi_\sigma - \varphi(x_L)) = 0, \, \forall \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L,$$

$$\varphi_\sigma = 0, \, \forall \sigma \in \mathcal{E}_{\rm ext},$$

and, for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$, the values $R_{K,\sigma}$ are defined by

$$R_{K,\sigma} = \frac{\varphi_\sigma - \varphi(x_K)}{d_{K,\sigma}} - \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \nabla\varphi(y) \cdot \mathbf{n}_{K,\sigma} d\gamma(y).$$

Thanks to the regularity of φ, it is easily seen that there exists $C_\varphi \geq 0$ depending only on φ such that

$$|R_{K,\sigma}| \leq C_\varphi \mathrm{size}(\mathcal{T}) \tag{45}$$

(Note that $R_{K,\sigma} + R_{L,\sigma} = 0$ if $\sigma = K|L$.) We set, for all $K \in \mathcal{T}$, $e_K = u_K - \varphi(x_K)$ and for all $\sigma \in \mathcal{E}$, $e_\sigma = u_\sigma - \varphi_\sigma$. We thus have

$$\tau_{K,\sigma}(e_\sigma - e_K) + \tau_{L,\sigma}(e_\sigma - e_L) = 0, \, \forall \sigma \in \mathcal{E}_{\rm int}, \sigma = K|L,$$

$$e_\sigma = 0, \, \forall \sigma \in \mathcal{E}_{\rm ext}.$$

We then get, adding (8) and (44),

$$-\sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K) = \mathrm{m}(K) f_K + \int_K \Delta\varphi(x)dx - \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) R_{K,\sigma}.$$

Multiplying the previous equation by e_K and summing on $K \in \mathcal{T}$ gives

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 = \sum_{K\in\mathcal{T}} \left(\mathrm{m}(K) f_K + \int_K \Delta\varphi(x)dx \right) e_K + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) R_{K,\sigma}(e_K - e_\sigma).$$

Since $R_{K,\sigma}(e_K - e_\sigma) \leq \frac{1}{2} d_{K,\sigma} R_{K,\sigma}^2 + \frac{1}{2d_{K,\sigma}} (e_K - e_\sigma)^2$, we get

$$\frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 \leq \sum_{K\in\mathcal{T}} \left(\mathrm{m}(K) f_K + \int_K \Delta\varphi(x)dx \right) e_K + \frac{1}{2} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) d_{K,\sigma} R_{K,\sigma}^2. \tag{46}$$

From the convergence result (2) we get that $\sum_{K\in\mathcal{T}} \left(\mathrm{m}(K) f_K + \int_K \Delta\varphi(x)dx) \right) e_K$ converges to $L = \int_\Omega (f(x) + \Delta\varphi(x))(u(x) - \varphi(x))dx$.

Hence one obains the existence of $F_1(\Omega, f, \varphi, \mathcal{T}) > 0$ which tends to 0 as $\mathrm{size}(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(e_\sigma - e_K)^2 \leq 2 \int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_1(\Omega, f, \varphi, \mathcal{T}).$$

Using the regularity of φ, we have, for $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, and $y \in \sigma$,

$$\left(\frac{u_\sigma - u_K}{d_{K,\sigma}} - \nabla\varphi(y) \cdot \mathbf{n}_{K,\sigma} \right)^2 \leq \frac{2}{{d_{K,\sigma}}^2} (e_\sigma - e_K)^2 + C_\varphi \mathrm{diam}(K)^2.$$

Using the two previous equations, we get the existence of $F_2(\Omega, f, \varphi, \mathcal{T}) > 0$ which tends to 0 as size$(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_K} d_{K,\sigma}\int_\sigma \left(\frac{u_\sigma - u_K}{d_{K,\sigma}} - \nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma}\right)^2 d\gamma(y) \le 4\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_2(\Omega, f, \varphi, \mathcal{T}). \quad (47)$$

We now study

$$A_K = \int_K (G_\mathcal{T}(x) - \nabla\varphi(x))^2 dx.$$

We define, for all $x \in K$, the function $w_K(x) = \varphi(x) - \dfrac{1}{\mathrm{m}(K)}\displaystyle\int_K \varphi(y)dy - v_K(x)$. Using the variational formulation defining $\phi_{K,\sigma}$, we have

$$A_K = \sum_{\sigma\in\mathcal{E}_K}\int_\sigma w_K(y)(\nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})d\gamma(y) - \int_K \Delta\varphi(x)w_K(x)dx.$$

Using the Cauchy-Schwarz inequality, we get, for some α and β which will be chosen later,

$$A_K \le \frac{\alpha}{2}B_K + \frac{1}{2\alpha}C_K + \frac{\beta}{2}D_K + \frac{1}{2\beta}E_K,$$

where

$$B_K = \sum_{\sigma\in\mathcal{E}_K}\int_\sigma w_K^2(y)d\gamma(y),$$

$$C_K = \sum_{\sigma\in\mathcal{E}_K}\int_\sigma (\nabla\varphi(y)\cdot\mathbf{n}_{K,\sigma} - \frac{u_\sigma - u_K}{d_{K,\sigma}})^2 d\gamma(y),$$

$$D_K = \int_K w_K^2(x)dx,$$

and

$$E_K = \int_K \Delta\varphi(x)^2 dx.$$

We now use the following lemma, which may be found in [15]:

Lemma 5 *Let Ω be an open bounded polygonal subset of $\mathbb{R}^d$. Let $\mathcal{T}$ be an admissible mesh (in the sense of Definition 1) such that, for some $\zeta > 0$, the inequality $d_{K,\sigma} \ge \zeta\mathrm{diam}(K)$ holds for all control volume $K \in \mathcal{T}$ and for all $\sigma \in \mathcal{E}_K$. Let $K \in \mathcal{T}$ be a given control volume and let $g \in H^1(K)$, such that $\int_K g(x)dx = 0$. Let us denote the trace of g on ∂K by g. Let $\sigma \in \mathcal{E}_K$. Then there exists $F_3(d,\zeta) \in \mathbb{R}_+$, only depending on d and ζ, such that*

$$\int_\sigma g^2(y)d\gamma(y) \le F_3(d,\zeta)\mathrm{diam}(K)\int_K (\nabla g(x))^2 dx, \quad (48)$$

and

$$\int_K g^2(x)dx \le F_3(d,\zeta)\mathrm{diam}(K)^2\int_K (\nabla g(x))^2 dx, \quad (49)$$

Using Lemma 5, there exists $C_{\zeta,d} \geq 0$ depending only on ζ and d and $M \geq \mathrm{card}\mathcal{E}_K$such that

$$B_K \leq MC_{\zeta,d}\mathrm{diam}(K)A_K,$$

and

$$D_K \leq C_{\zeta,d}\mathrm{diam}(K)^2 A_K.$$

We then choose $\alpha = \dfrac{1}{2MC_{\zeta,d}\mathrm{diam}(K)}$ and $\beta = \dfrac{1}{2C_{\zeta,d}\mathrm{diam}(K)^2}$. It leads to

$$\frac{1}{2}A_K \leq MC_\zeta \mathrm{diam}(K)C_K + C_\zeta \mathrm{diam}(K)^2 E_K.$$

Using (47) and $d_{K,\sigma} \geq \zeta \mathrm{diam}(K)$, we have

$$\sum_{K\in\mathcal{T}} \mathrm{diam}(K)C_K \leq \frac{4}{\zeta}\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + \frac{1}{\zeta}F_2(\Omega, f, \varphi, \mathcal{T}). \tag{50}$$

Using the two previous equations, we get the existence of $F_3(\Omega, f, \varphi, \zeta, M, \mathcal{T}) \geq 0$ which tends to 0 as $\mathrm{size}(\mathcal{T}) \to 0$ such that

$$\sum_{K\in\mathcal{T}} A_K = \int_\Omega (G_\mathcal{T}(x) - \nabla\varphi(x))^2 dx \leq \frac{8MC_{\zeta,d}}{\zeta}\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + F_3(\Omega, f, \varphi, \zeta, M, \mathcal{T}).$$

Using

$$(G_\mathcal{T}(x) - \nabla u(x))^2 \leq 2(G_\mathcal{T}(x) - \nabla\varphi(x))^2 + 2(\nabla\varphi(x) - \nabla u(x))^2$$

and setting $C_{M,\zeta,d} = \dfrac{16MC_{\zeta,d}}{\zeta} + 2$, we get

$$\int_\Omega (G_\mathcal{T}(x) - \nabla u(x))^2 dx \leq C_{M,\zeta}\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx + 2F_3(\Omega, f, \varphi, \zeta, M, \mathcal{T}).$$

Recall that we have chosen φ such that $\int_\Omega (\nabla u(x) - \nabla\varphi(x))^2 dx \leq \varepsilon$. We can now choose $\mathrm{size}(\mathcal{T})$ small enough such that $2F_3(\Omega, f, \varphi, \zeta, M, \mathcal{T}) \leq \varepsilon$. Then $\int_\Omega (G_\mathcal{T}(x) - \nabla u(x))^2 dx \leq (C_{M,\zeta} + 1)\varepsilon$, which concludes the proof of convergence of $G_\mathcal{T}$ to ∇u in $(L^2(\Omega))^2$.

Since $\mathrm{div}G_\mathcal{T}(x) = f_K$, for all $x \in K$, the convergence of $\mathrm{div}G_\mathcal{T}$ to Δu in $L^2(\Omega)$ is proved. This concludes the proof of the convergence of $G_\mathcal{T}$ to ∇u in $H_{\mathrm{div}}(\Omega)$. ∎

7 Conclusion

We showed here a proof of convergence of a cell-centered finite volume scheme for a semilinear convection diffusion equation with non-homogeneous Dirichlet boundary conditions. We also showed how one may construct an approximate gradient from the piecewise constant finite volume and its convergence to the exact gradient in the case of the Laplace equation with homogeneous Dirichlet boundary conditions. In the case where the exact solution is unique, one may derive an error estimate on the gradient (see [15]). Error estimates may be obtained in the case of other boundary conditions, see [24]. Convergence results also hold for time-dependent convection diffusion equations, see [13] for the linear case and [16] for the nonlinear case.

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