A finite volume scheme on general meshes for the steady Navier-Stokes equations in two space dimensions

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1 The incompressible Navier-Stokes equations

Numerical schemes for the Navier-Stokes equations (1) have been extensively studied: see [7, 11, 12, 13, 8, 6, 15] and references therein. Among different schemes, finite element schemes and finite volume schemes are frequently used for mathematical or engineering studies. An advantage of the finite volume schemes is that the unknowns are approximated by piecewise constant functions: this makes it easy to take into account additional nonlinear phenomena or the coupling with algebraic or differential equations, for instance in the case of reactive flows; in particular, one can find in [11] the presentation of the classical finite volume scheme on rectangular meshes, which has been the basis of many industrial applications. A convergence proof of the so-called MAC scheme is given in [10] in the case of a uniform rectangular grid. However, the use of rectangular grids makes an important limitation to the type of domain which can be gridded and more recently, finite volume schemes for the Navier-Stokes equations on triangular grids have been presented: see for example [9] where the vorticity formulation is used,and [2] where primal variables are used with a Chorin type projection method to ensure the divergence condition (but no proof of convergence is known). In this paper, we propose a new method which uses the primitive variables and enforces the divergence condition directly, using quite general meshes such as mixed rectangular-triangular or Voronoï meshes, and for which we are able to prove convergence under general conditions (in particular, no regularity of the exact solution is required). Note that an error estimate for this scheme in the case of the linear Stokes equations is presented in [4] in the case of the linear Stokes equations.

Our aim is thus to find an approximation of $u = (u^{(1)}, u^{(2)})^t \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $p \in L^2(\Omega)$, weak solution to the incompressible generalized Navier-Stokes equations, which write:

$$\begin{aligned} &\eta u^{(i)} - \nu \Delta u^{(i)} + \partial_i p + u^{(1)} \partial_1 u^{(i)} + u^{(2)} \partial_2 u^{(i)} = f^{(i)} \text{ in } \Omega, \quad \text{for } i = 1, 2, \\ &\partial_1 u^{(1)} + \partial_2 u^{(2)} = 0 \text{ in } \Omega. \end{aligned} \tag{1}$$

where $\eta \geq 0$, $u^{(1)}$ and $u^{(2)}$ are the two components of the velocity, p denotes the pressure, ν the viscosity of the fluid, under the following assumptions:

$$\Omega \text{ is a polygonal open bounded connected subset of } \mathbb{R}^2, \tag{2}$$

$$\nu \in (0, +\infty), \ \eta \in [0, +\infty), \tag{3}$$

$$f^{(i)} \in L^2(\Omega), \quad \text{for } i = 1, 2. \tag{4}$$

The terms $\eta u^{(i)}$ appear when considering an implicit time discretization of the unsteady Stokes or Navier-Stokes equations (with η as the inverse of the time step). The case $\eta = 0$ yields the usual steady-state equations.

For the simplicity of this presentation, we prescribe for both problems a homogeneous Dirichlet boundary condition on the velocity $(u^{(1)}, u^{(2)})$. In all this paper, we denote by $x = (x^{(1)}, x^{(2)})$ any point of Ω and by dx the 2-dimensional Lebesgue measure $dx = dx^{(1)}dx^{(2)}$.

Definition 1 (Weak solution). *Under hypotheses (2)-(4), $u = (u^{(1)}, u^{(2)})^t$ is called a weak solution of (1) if and only if*

$$\begin{cases} u = (u^{(1)}, u^{(2)})^t \in E(\Omega), \\ \eta \displaystyle\sum_{i=1,2} \int_\Omega u^{(i)}(x) v^{(i)}(x) dx + \nu \sum_{i=1,2} \int_\Omega \nabla u^{(i)} \cdot \nabla v^{(i)}(x) dx + b(u,u,v) = \\ \displaystyle\sum_{i=1,2} \int_\Omega f^{(i)}(x) v^{(i)}(x) dx, \quad \forall v = (v^{(1)}, v^{(2)})^t \in E(\Omega), \end{cases} \tag{5}$$

where the trilinear form b is defined for all $u, v, w \in (H_0^1(\Omega))^2$ by

$$b(u,v,w) = \sum_{k=1,2} \sum_{i=1,2} \int_\Omega u^{(i)}(x) \partial_i v^{(k)}(x) w^{(k)}(x) dx, \tag{6}$$

which classically satisfies, for all $u \in E(\Omega)$,

$$b(u,v,w) = \sum_{k=1,2} \sum_{i=1,2} \int_\Omega \partial_i (u^{(i)} v^{(k)})(x) w^{(k)}(x) dx.$$

2 The finite volume scheme

Let us then present the following notion of admissible discretization.

Definition 2. [Admissible discretization] *Let Ω be an open bounded polygonal subset of $\mathbb{R}^2$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. An admissible finite volume discretization of Ω, denoted by $\mathcal{D}$, is given by $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$, where:*

- *$\mathcal{M}$ is a finite family of non empty open polygonal convex disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K and $\mathrm{m}(K) > 0$ denote the area of K.*
- *$\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathcal{E}$, there exists a hyperplane E of $\mathbb{R}^2$ and $K \in \mathcal{M}$ with $\overline{\sigma} = \partial K \cap E$ and σ is a non empty open subset of E. We then denote by $m_\sigma > 0$ the 1-dimensional measure of σ. We assume that, for all $K \in \mathcal{M}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \cup_{\sigma \in \mathcal{E}_K} \overline{\sigma}$. It then results from the previous hypotheses that, for all $\sigma \in \mathcal{E}$, either $\sigma \subset \partial\Omega$ or there exists $(K, L) \in \mathcal{M}^2$ with $K \neq L$ such that $\overline{K} \cap \overline{L} = \overline{\sigma}$; we denote in the latter case $\sigma = K|L$.*
- *$\mathcal{P}$ is a family of points of Ω indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K \in \mathcal{M}}$. The coordinates of x_K are denoted by $x_K^{(i)}$, $i = 1, 2$. The family $\mathcal{P}$ is such that, for all $K \in \mathcal{M}$, $x_K \in K$. Furthermore, for all $\sigma \in \mathcal{E}$ such that there exists $(K, L) \in \mathcal{M}^2$ with $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) going through x_K and x_L is orthogonal to $K|L$. For all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{E}_K$, let z_σ be the orthogonal projection of x_K on σ. We suppose that $z_\sigma \in \sigma$.*

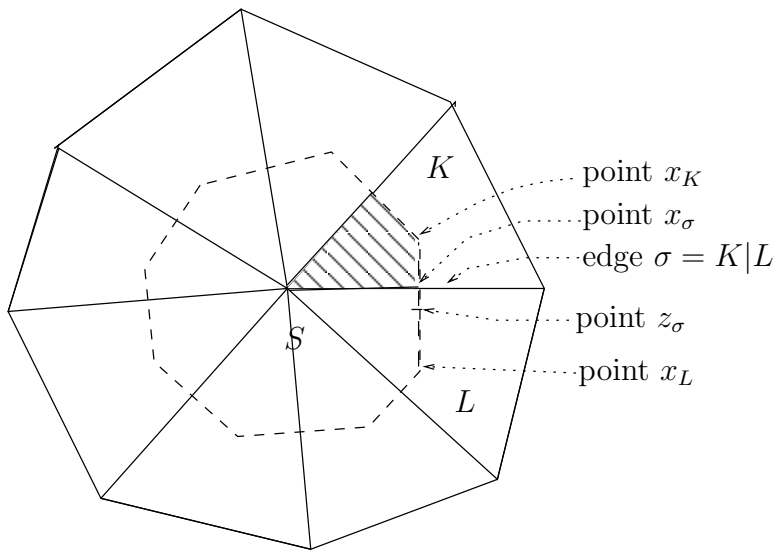

Fig. 1. Example of an admissible triangular discretization

– *$\mathcal{V}$ is a finite family of non empty open polygonal disjoint subsets of Ω (constituting the "dual mesh" of $\mathcal{M}$), which are centered around the vertices $(x_s)_{s=1,N_V}$ in the following way (N_V is the number of vertices):*
for $1 \leq s \leq N_V$, let $\mathcal{M}_s \subset \mathcal{M}$ be the set of control volumes to which x_s is a vertex. For $K \in \mathcal{M}_s$, denote by $\sigma_{K,s,1}$ and $\sigma_{K,s,2} \in \mathcal{E}_K$ the two edges of K with vertex x_s. Define K_s as the convex hull of the four points

$$\left(x_s, x_K, z_{\sigma_{K,s,1}}, z_{\sigma_{K,s,2}}\right).$$

The dual cell around x_s, denoted by S, is then defined as (also see Figure 1):

$$\overline{S} = \cup_{K \in \mathcal{M}_s} K_s.$$

Since there is a one-to-one mapping between the set $\{1, \ldots, N_V\} \subset \mathbb{N}$ and the set $\mathcal{V}$, we shall replace all subscripts s by S when dealing with the dual mesh. Let $\mathcal{V}_K$ denote the set of vertices of a given control volume K. Note that:

$$\overline{K} = \cup_{x_s \in \mathcal{V}_K} K_s, \quad \text{and } K_s = \overline{K \cap S}.$$

The following notations are used. The size of the discretization is defined by:

$$\text{size}(\mathcal{D}) = \sup\{\text{diam}(K), K \in \mathcal{M}\}.$$

We shall measure the regularity of the mesh through the function angle($\mathcal{D}$) defined by

$$\text{angle}(\mathcal{D}) = \inf\left\{|\widehat{z_\sigma x_K x_S}|,\ |\widehat{z_\sigma x_S x_K}|, K \in \mathcal{M},\ S \in \mathcal{V}_K,\ \sigma \in \mathcal{E}_K \cap \mathcal{E}_S\right\}, \tag{7}$$

where $|\widehat{xyz}|$ designates the absolute value of the measure of the angle $\widehat{xyz}$ (note that $\widehat{z_\sigma x_K x_S} = \frac{\pi}{2} - \widehat{z_\sigma x_S x_K}$.).

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K. We denote by $d_{K,\sigma}$ the Euclidean distance between x_K and σ. We then define

$$\tau_{K,\sigma} = \frac{m_\sigma}{d_{K,\sigma}}.$$

The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_{\text{int}}$ (resp. $\mathcal{E}_{\text{ext}}$), that is $\mathcal{E}_{\text{int}} = \{\sigma \in \mathcal{E};\ \sigma \not\subset \partial\Omega\}$ (resp. $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E};\ \sigma \subset \partial\Omega\}$). For any $\sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L$ (resp. $\mathcal{E}_{\text{ext}}$, $\sigma \in \mathcal{E}_K$), let x_σ be the center point of the line segment $[x_K x_L]$ (resp. $[x_K z_\sigma]$), and $x_\sigma^{(1)}$ and $x_\sigma^{(2)}$ its coordinates.

For all $K \in \mathcal{M}$ and all $S \in \mathcal{V}_K$, let σ_1 and $\sigma_2 \in \mathcal{E}_K \cap \mathcal{E}_S$ numbered such that $(x_{\sigma_1}^{(2)} - x_S^{(2)})(x_{\sigma_2}^{(1)} - x_S^{(1)}) - (x_{\sigma_2}^{(2)} - x_S^{(2)})(x_{\sigma_1}^{(1)} - x_S^{(1)}) > 0$.

We then define the coefficients

$$\begin{aligned} A_{K,S}^{(1)} &= x_{\sigma_1}^{(2)} - x_{\sigma_2}^{(2)} \\ A_{K,S}^{(2)} &= x_{\sigma_2}^{(1)} - x_{\sigma_1}^{(1)}. \end{aligned} \tag{8}$$

Definition 3. *Let Ω be an open bounded polygonal subset of $\mathbb{R}^N$, with $N \in \mathbb{N}_*$. Let $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}, \mathcal{V})$ be an admissible finite volume discretization of Ω in the sense of Definition 2. We denote by $H_\mathcal{D}(\Omega) \subset L^2(\Omega)$ the space of functions which are piecewise constant on each control volume $K \in \mathcal{M}$. For all $w \in H_\mathcal{D}(\Omega)$ and for all $K \in \mathcal{M}$, we denote by w_K the constant value of w in K and we define $(w_\sigma)_{\sigma\in\mathcal{E}}$ by:*

$$w_\sigma = 0,\ \forall \sigma \in \mathcal{E}_{\text{ext}} \tag{9}$$

and

$$\tau_{K,\sigma}(w_\sigma - w_K) + \tau_{L,\sigma}(w_\sigma - w_L) = 0,\ \forall \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L. \tag{10}$$

We denote by $L_\mathcal{D}(\Omega)$ the space of functions which are piecewise constant on the domains S, for all $S \in \mathcal{V}$. We then define the discrete divergence $\text{div}_\mathcal{D} : (H_\mathcal{D}(\Omega))^2 \to L_\mathcal{D}(\Omega)$, by:

$$\text{div}_\mathcal{D}(u)(x) = \frac{1}{\text{m}(S)} \sum_{K\in\mathcal{M}_S} \sum_{i=1,2} A_{K,S}^{(i)}\ u_K^{(i)}, \quad \textit{for a.e. } x \in S, \forall S \in \mathcal{V}.$$

We then set $E_\mathcal{D}(\Omega) = \{u \in (H_\mathcal{D}(\Omega))^2, \text{div}_\mathcal{D}(u) = 0\}$. For $(v, w) \in (H_\mathcal{D}(\Omega))^2$, we denote by

$$[v, w]_\mathcal{D} = \sum_{K\in\mathcal{M}} \sum_{\sigma\in\mathcal{E}_K} \tau_{K,\sigma}(v_\sigma - v_K)(w_\sigma - w_K), \tag{11}$$

Remark that thanks to (10), one has:

$$[v, w]_\mathcal{D} = \sum_{\sigma\in\mathcal{E}_{\text{int}}, \sigma=K|L} \tau_\sigma(v_K - v_L)(w_K - w_L) + \sum_{\sigma\in\mathcal{E}_{\text{ext}}} \tau_\sigma v_{K_\sigma} w_{K_\sigma},$$

where K_σ denotes the control volume to which σ is an edge. We define a norm in $H_\mathcal{D}(\Omega)$ (thanks to the discrete Poincaré inequality (12) given below) by

$$|w|_\mathcal{D} = ([w, w]_\mathcal{D})^{1/2}.$$

Similarly, for $u = (u^{(1)}, u^{(2)})^t \in (H_{\mathcal{D}}(\Omega))^2$, $v = (v^{(1)}, v^{(2)})^t \in (H_{\mathcal{D}}(\Omega))^2$ *and* $w = (w^{(1)}, w^{(2)})^t \in (H_{\mathcal{D}}(\Omega))^2$, *we define:*

$$|u|_{\mathcal{D}} = \left(\sum_{i=1,2} [u^{(i)}, u^{(i)}]_{\mathcal{D}} \right)^{1/2}.$$

and

$$[v, w]_{\mathcal{D}} = \sum_{i=1,2} [v^{(i)}, w^{(i)}]_{\mathcal{D}}.$$

The discrete Poincaré inequality (see [3]) writes:

$$\|w\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega)|w|_{\mathcal{D}}, \ \forall w \in H_{\mathcal{D}}(\Omega). \tag{12}$$

Because of space limitations, we shall only present here a centered finite volume scheme, and refer to [5] for the upstream version of the finite volume scheme. Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2. Let $\lambda \in (0, +\infty)$ be given. The finite volume scheme for the approximation of the solution (1) writes: find u such that

$$\begin{aligned} & u \in E_{\mathcal{D}}(\Omega), \\ & \eta \int_{\Omega} u(x) \cdot v(x) dx + \nu [u, v]_{\mathcal{D}} + b_{\mathcal{D}}(u, u, v) = \int_{\Omega} f(x) \cdot v(x) dx, \ \forall v \in E_{\mathcal{D}}(\Omega), \end{aligned} \tag{13}$$

where, for u, v and $w \in H_{\mathcal{D}}(\Omega)$, we define the following centered approximation $b_{\mathcal{D}}$ of the trilinear form b (6):

$$\begin{aligned} & b_{\mathcal{D}}(u, v, w) = \sum_{K \in \mathcal{M}} \sum_{k=1,2} w_K^{(k)} \sum_{S \in \mathcal{V}_K} v_S^{(k)} \sum_{i=1,2} A_{K,S}^{(i)} \, u_K^{(i)} \\ & v_S^{(k)} = \frac{1}{\mathrm{m}(S)} \sum_{K \in \mathcal{M}_S} \mathrm{m}(K \cap S) \, v_K^{(k)}, \ \forall S \in \mathcal{V}, \ k = 1, 2. \end{aligned} \tag{14}$$

It may be shown that the trilinear form $b_{\mathcal{D}}(u, v, w)$ satisfies some continuity properties in $(H_{\mathcal{D}}(\Omega))^3$ (see [5] for the proof).

Lemma 1. [Continuity of the trilinear form in discrete H^1 space] *Under Hypothesis (2), let* $\mathcal{D}$ *be an admissible discretization in the sense of Definition 2, let* $\alpha > 0$ *be such that* $\mathrm{angle}(\mathcal{D}) \geq \alpha$, *let* $H_{\mathcal{D}}(\Omega)$ *be the space of piecewise constant functions defined in 3 and let* $b_{\mathcal{D}}$ *be the trilinear form defined by (14).*

Then there exists $C_1 > 0$, *only depending on* α, *such that:*

$$|b_{\mathcal{D}}(u, v, w)| \leq C_1 \, |u|_{\mathcal{D}} |v|_{\mathcal{D}} |w|_{\mathcal{D}}. \tag{15}$$

As in the case of the linear problem (see [4]), we use the following penalized approximation of (13):

$$\begin{aligned} & (u, p) \in (H_{\mathcal{D}}(\Omega))^2 \times L_{\mathcal{D}}(\Omega), \\ & \nu \left([u, v]_{\mathcal{D}} \right) - \int_{\Omega} p(x) \mathrm{div}_{\mathcal{D}}(v)(x) dx + b_{\mathcal{D}}(u, u, v) = \int_{\Omega} f(x) \cdot v(x) dx, \quad \forall v \in (H_{\mathcal{D}}(\Omega))^2, \\ & \mathrm{div}_{\mathcal{D}}(u) = -\lambda \, \mathrm{size}(\mathcal{D}) \, p, \end{aligned} \tag{16}$$

3 Convergence of the scheme

The following proposition gives a sufficient condition for the existence and uniqueness of a solution to the scheme (with or without penalization), under the classical assumption that the data are small, or the viscosity is large enough (see [14] Theorem 1.3 page 167 for the continuous case). Note that in the continuous case, the "small data" assumption is only required to prove uniqueness, not existence. Here, however, this assumption is also required for the existence of a discrete solution. Moreover, uniqueness is only proven for "small enough" solutions.

Proposition 1 [Existence and uniqueness of small discrete solutions in the small data case, with or without a penalization] *Under hypotheses (2)-(4), let $\mathcal{D}$ be an admissible discretization of Ω in the sense of Definition 2 and let $\alpha > 0$ with* $\mathrm{angle}(\mathcal{D}) \geq \alpha$. *Let C_1 be the real value which only depends on α, given by (15) of Lemma 1. Assume that the condition*

$$\frac{1}{\nu^2}\left(\sum_{i=1,2} \|f^{(i)}\|_{L^2(\Omega)}\right) < C_2 := \frac{1}{4\mathrm{diam}(\Omega)C_1} \tag{17}$$

is fulfilled. Then there exists one and only one function $u \in (H_{\mathcal{D}}(\Omega))^2$ such that

$$|u|_{\mathcal{D}} \leq C_3 := \frac{1}{2C_1}\left[\nu - \left(\nu^2 - 4\left(\sum_{i=1,2} \|f^{(i)}\|_{L^2(\Omega)}\right)\mathrm{diam}(\Omega)C_1\right)^{1/2}\right], \tag{18}$$

and u is solution to (13) and (14) (no penalization), or u is such that there exists a function p with (u,p) solution to (16) and (14) for a given $\lambda \in (0,+\infty)$. Furthermore, in the latter case, the following inequality holds:

$$\lambda\ \mathrm{size}(\mathcal{D})\ \|p\|^2_{L^2(\Omega)} \leq \mathrm{diam}(\Omega)\left(\sum_{i=1,2} \|f^{(i)}\|_{L^2(\Omega)}\right) + C_1\, {C_3}^2. \tag{19}$$

and the function p is unique too.

As in the case of the linear problem, we may prove the convergence of the scheme to the continuous solution for the scheme (16), (14). This is stated in the next proposition.

Proposition 2 [Convergence of the centered penalized scheme in the nonlinear case] *Under Hypotheses (2)-(4), let $\alpha > 0$ be given and let $C_2 > 0$ be given by Proposition 1. We assume that the property (17) holds. Let $\lambda \in (0,+\infty)$ be given and let $(\mathcal{D}^{(n)})_{n\in\mathbb{N}}$ be a sequence of admissible discretization of Ω in the sense of Definition 2, such that* $\lim_{n\to\infty} \mathrm{size}(\mathcal{D}^{(n)}) = 0$ *and* $\mathrm{angle}(\mathcal{D}^{(n)}) \geq \alpha$, *for all $n \in \mathbb{N}$. Let $(u^{(n)}, p^{(n)}) \in (H_{\mathcal{D}^{(n)}}(\Omega))^2 \times L_{\mathcal{D}^{(n)}}(\Omega)$ be a solution to (16), (14),(18). Then there exists a subsequence of the sequence $(u^{(n)})_{n\in\mathbb{N}}$ which converges in $L^2(\Omega)^2$ to u, weak solution of the Navier-Stokes problem in the sense of (5). If C_2 is taken small enough, the uniqueness property of the solution entails the convergence of the whole sequence.*

4 Numerical results

The implementation of the scheme was performed using the F90 language on a Unix system. The linear systems are solved using a direct method and the nonlinearities are treated with a Newton iteration. The grid generator proceeds from a given number of initial grid blocks, which can be triangular or rectangular, describing the geometry, which are then uniformly refined as desired.

Experiments with an analytical solution were performed. For the centered scheme, the results indicate a rate of convergence of h^2 for the velocities, and better than $h^{0.5}$ for the pressures in the case of unstructured triangular meshes. In the case of rectangular meshes or structured triangular meshes, we obtain an order h^2 for the velocities and better than h for the pressures. For the upstream weighting scheme on structured meshes, we obtain an order $h^{0.8}$ for the velocities.

Some experiments were also carried out for the classical example of the lid driven cavity. We refer to [5] for these, and shall only give here some results on the backward facing step, for a Reynolds number equal to 800. This is a well documented case in the literature (see e.g. [1], and allows to test the performance of methods with respect to the precision on the zones of recirculating flow. The geometrical data of the backward step is taken from [1]. We computed the streamlines using a reconstruction of a discrete potential Φ_σ, located at the edges $\sigma \in \mathcal{E}$ of the mesh (see [5]). We present in Figure 2 the streamlines in three different cases: starting form the top, the first figure is obtained with the centered scheme, using a 25200 rectangular grid blocks mesh, the second one with the centered scheme using a 2800 rectangular grid blocks mesh, the third one with the upstream scheme using a 2800 rectangular grid blocks mesh, and the two last ones with respectively the centered and the upstream scheme for 847 cells. It is clear from these figures that the centered scheme is, as one could expect, more precise, but that it becomes unstable for coarser meshes. In fact, for a mesh of 700 cells, the Newton iterations do not converge, even when using an under-relaxation procedure.

The numerical solution obtained with the centered scheme, using a 25200 rectangular grid blocks mesh seems to be precise enough (comparing the separation and reattachment lengths with those of the literature, see [5]) to be used as a reference solution for experiments carried out on coarser meshes. This allows to compute a rate of convergence of h^2.

We conclude from these numerical tests that the upstream scheme is too diffusive and cannot be used for accurate results, although it has the advantage of remaining stable even on coarse meshes. The centered scheme yields accurate results for a reasonable number of Newton iterations (typically between 5 and 15).

Future developments will concentrate on the extension to three-dimensional meshes and to the time-dependent case.

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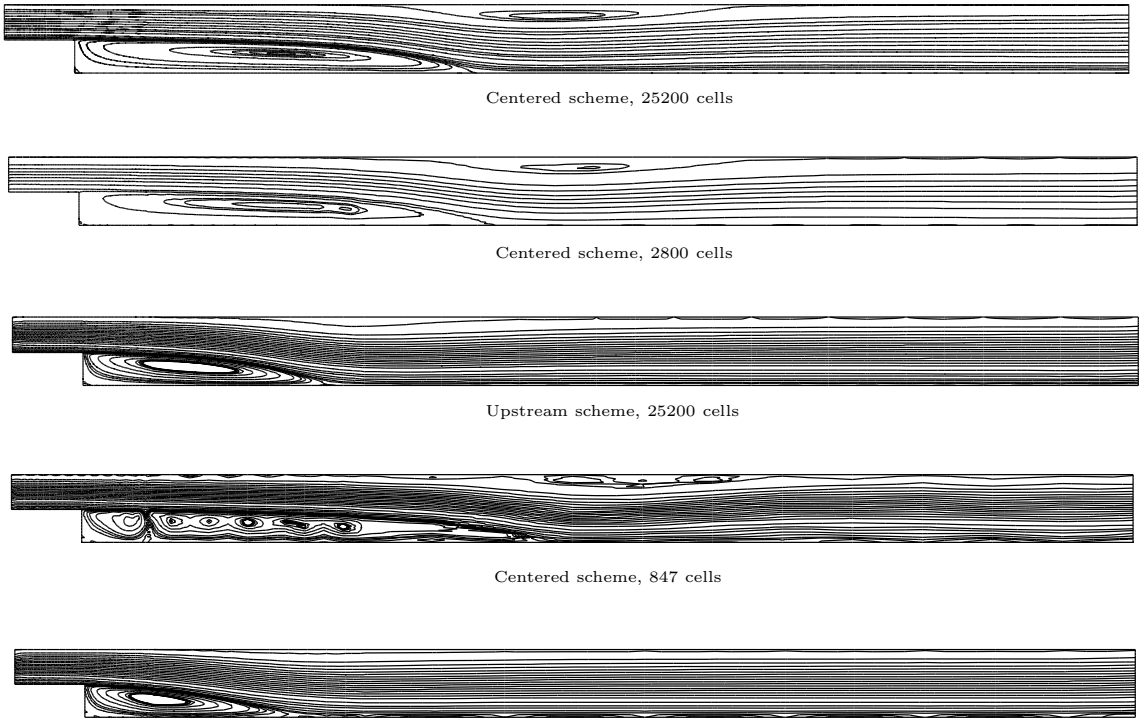

Fig. 2. Streamlines for the backward step

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