EXISTENCE OF A SOLUTION TO A COUPLED ELLIPTIC SYSTEM WITH A SIGNORINI CONDITION

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Abstract. The existence of the solution to an elliptic system arising in electrochemical modelling is proven here. The elliptic system of interest here is composed of two diffusion equations; one of them is posed with a Dirichlet condition which couples it to the other equation on an interface, and a Signorini condition on one of the boundaries. The other one is posed with Neumann conditions, which are also coupled at an interface. The existence of the solution is proven by using Schauder's fixed point theorem, which requires some previous local regularity properties of the solution to the "Signorini problem".

1. Introduction. The modelling of an electrochemical reacting interface [5] gives rise to the following system of equations posed over the physical domain Ω represented in Figure 1. The domain Ω_A represents the electrode and is defined by $\Omega_A =]0,1[\times]0,1[$, the domain Ω_B represents the electrolyte and is defined by $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$. Let $\Omega = \Omega_A \cup I \cup \Omega_B$, where $I = \{(x_1, 0), x_1 \in]0,1[\}$. The boundary $\partial\Omega$ of Ω is composed of seven parts:

$$\Gamma_A^1 = \{(0, x_2), x_2 \in]0,1[\}, \qquad \Gamma_A^2 = \{(x_1, 1), x_1 \in]0,1[\},$$
$$\Gamma_A^3 = \{(1, x_2), x_2 \in]0,1[\}, \qquad \Gamma_B^1 = \{(0, x_2), x_2 \in]x_2^{min}, 0[\},$$
$$\Gamma_B^2 = \{(x_1, x_2^{min}), x_1 \in]0, x_1^{max}[\}, \quad \Gamma_B^3 = \{(x_1^{max}, x_2), x_2 \in], 0[\},$$
$$\Gamma_B^4 = \{(x_1, 0), x_1 \in]1, x_1^{max}[\}.$$

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Figure 1. Domain of study

The electrochemical phenomena which need to be modelled are:

– The diffusion of oxygen in the electrode Ω_A; the oxygen concentration in Ω_A is denoted by u, and the oxygen flux is proportional to ∇u.

– The conduction of electrical (ionic or electronic) current in both electrode Ω_A and electrolyte Ω_B. The electrical current is given by Ohm's law with respect to the electrical potential Φ. The conservation of the oxygen in Ω_A and the conservation of electrical current in Ω_B yield equations (4) and (8) below. Theses equations are coupled by Faraday's law, which states the conservation of electric charges at the interface between Ω_A and Ω_B (conditions (11) and (14) below). They are also coupled through Nernst's law, which expresses the potential jump at the electrolyte with respect to the concentration. A Signorini boundary condition (see (7) below) is written on one of the boundaries of Ω_A to account for the fact that the transmission of oxygen through the electrode wall is limited both for the concentration and concentration flux. For reasons of symmetry, all fluxes are null at $x = 0$ and $x = x_1^{max}$ (conditions (5), (10), (13)).

Let a, c_A, $c_B \in \mathbb{R}_+^\star$ and $b \in H^1(\Gamma_A^3)$ such that $b \geq 0$ and $b(0) = 0$. In order to express Nernst's law, let $\overline{T}$ be a function defined from $L^2(I)$ to $H^2(I)$ such that

$$0 \leq \overline{T}(\varphi)(s) \leq a \qquad \forall \varphi \in L^2(I), \text{ for a.e } s \in I, \tag{1}$$

$$\exists M \in \mathbb{R}^\star, \ \|\overline{T}(\varphi)\|_{H^2(I)} \leq M \qquad \forall \varphi \in L^2(I), \tag{2}$$

$$\overline{T} \text{ is continuous from } L^2(I) \text{ to } H^1(I). \tag{3}$$

With these notations, the conservation equations, interface and boundary conditions lead to seeking $u : \Omega_A \to \mathbb{R}$ and $\Phi : \Omega \to \mathbb{R}$ satisfying

$$-\Delta(u(x)) = 0, \quad x \in \Omega_A \tag{4}$$

$$\nabla u(s) \cdot \mathbf{n} = 0, \quad s \in \Gamma_A^j \quad (j = 1, 2) \tag{5}$$

$$u(s) = \overline{T}(\Phi_{A|_I} - \Phi_{B|_I})(s), \quad s \in I \tag{6}$$

$$\left.\begin{aligned} u(s) &\leq a, \\ \nabla u(s) \cdot \mathbf{n} &\leq b(s), \\ (u(s) - a)(\nabla u(s).\mathbf{n} - b(s)) &= 0, \end{aligned}\right\} \quad s \in \Gamma_A^3, \tag{7}$$

$$-\Delta(\Phi_i(x)) = 0, \quad x \in \Omega_i \quad (i = A, B) \tag{8}$$

$$\Phi_A(s) = 0, \quad s \in \Gamma_A^2 \tag{9}$$

$$\nabla\Phi_A(s) \cdot \mathbf{n} = 0, \quad s \in \Gamma_A^j \quad (j = 1, 3) \tag{10}$$

$$\nabla\Phi_A(s) \cdot \mathbf{n}_I = c_A \nabla u(s) \cdot \mathbf{n}_I, \quad s \in I, \tag{11}$$

$$\Phi_B(s) = 0, \quad s \in \Gamma_B^2 \tag{12}$$

$$\nabla\Phi_B(s) \cdot \mathbf{n} = 0 \quad s \in \Gamma_B^j \quad (j = 1, 3, 4) \tag{13}$$

$$\nabla\Phi_B(s) \cdot \mathbf{n}_I = c_B \nabla u(s) \cdot \mathbf{n}_I, \quad s \in I, \tag{14}$$

where Φ_i denotes the restriction of Φ to the domain Ω_i $(i = A, B)$ and $\mathbf{n}_I$ is the unit vector normal to I external to the domain Ω_A.

Note that in fact, Nernst's law, which is an experimental law, gives the ideal potential jump for an electrochemical reaction with a chemical species of concentration u as:

$$u = T(\Phi_{A|_I} - \Phi_{B|_I}) \quad \text{a.e. on } I,$$

where T is continuous from $\mathbb{R}$ to $[0, a]$. However, with an argument similar to that of [duvaut], we may replace the pointwise dependency of u on Φ through a dependency of u at point x with respect to the values of Φ in the neighborhood of x. Such a model may be obtained by replacing, for instance, T by $\overline{T}$ defined from $L^2(I)$ to $H^2(I)$ such that:

$$\overline{T}(\varphi)(s) = \int_I T(\varphi)(y)\, \rho_h(\psi(s) - y) dy \quad \text{a.e. } s \in I, \tag{15}$$

where ρ_h is the classical mollifier defined by

$$\rho_h(x) = \frac{1}{h}\rho(\frac{x}{h}) \quad \forall x \in \mathbb{R},$$

with $\rho \in C_c^\infty(\mathbb{R}, \mathbb{R}^\star)$ such that supp $\rho \subset [-1,1]$, and $\int_{\mathbb{R}} \rho(x)dx = 1$; thus $\rho_h \in C_c^\infty(\mathbb{R}, \mathbb{R}^\star)$, *supp* $\rho_h \subset [-h,h]$ and $\int_{\mathbb{R}} \rho_h(x)dx = 1$; and ψ is defined by $\psi(x) = x - 2hx + h \quad \forall x \in I$.

The function $\overline{T}$ thus defined clearly satisfies assumptions (1)–(3)). Moreover, we have

$$\overline{T}(\varphi)(0) = \overline{T}(\varphi)(1) = 0 \quad \text{and} \quad \overline{T}(\varphi)'(0) = \overline{T}(\varphi)'(1) = 0. \tag{16}$$

The aim of this paper is to prove the existence of a solution to the system (4)–(14). In order to do so, we need to define in which sense this solution may be obtained. Let us first define the adequate functional spaces and (bi)linear forms for $i = A$ or $i = B$:

$$H^1_{0,2}(\Omega_i) = \{v \in H^1(\Omega_i),\ v_{|_{\Gamma_i^2}} = 0 \text{ a.e}\}, \tag{17}$$

$$\mathcal{A}_i(\Phi, \Psi) = \int_{\Omega_i} \nabla\Phi(x)\nabla\Psi(x)dx, \quad \forall \Phi,\ \Psi \in H^1(\Omega_i), \tag{18}$$

$$L_{u,i}(\Psi) = c_i \int_I \nabla u(s) \cdot \mathbf{n}\, \Psi_{|_I}(s)\, ds, \tag{19}$$

$$\forall \Psi \in H^1(\Omega_i),\ \forall u \in H^1(\Omega_A) \quad \text{s.t. } \nabla u \cdot \mathbf{n} \in L^2(I),$$

$$K_\varphi = \{v \in H^1(\Omega_A),\ v_{|_I} = \overline{T}(\varphi) \text{ a.e},\ v_{|_{\Gamma_A^3}} \leq a \text{ a.e}\}, \quad \forall \varphi \in L^2(I), \tag{20}$$

$$L(v) = \int_{\Gamma_A^3} b(s) v_{|_{\Gamma_A^3}}(s)ds, \quad \forall v \in H^1(\Omega_A). \tag{21}$$

Where $v_{|_I}$ designs the image of $v \in H^1(\Omega_i)$ ($i = A$ or B), by the trace operator defined from $H^1(\Omega_i)$ to $L^2(I)$.

Consider the following variational problem:

$$\begin{cases} \Phi_i \in H^1_{0,2}(\Omega_i)\ (i = A, B), \text{ satisfying :} \\ \qquad \mathcal{A}_i(\Phi_i, \Psi) = L_{u,i}(\Psi), \quad \forall \Psi \in H^1_{0,2}(\Omega_i) \quad (i = A, B); \end{cases} \tag{22}$$

$$\begin{cases} u \in K_\varphi, \text{ satisfying :} \\ \quad \mathcal{A}_A(u, v - u) \geq L(v - u), \quad \forall v \in K_\varphi, \\ \text{with } \varphi = \Phi_{A|_I} - \Phi_{B|_I}. \end{cases} \tag{23}$$

We shall prove in the following section that, under some regularity assumptions, Problem (4)–(14) is equivalent to Problem (22)–(23).

The existence of the solution to the problem (22)-(23) will then be proven by using the Schauder fixed point theorem. Our proof has been inspired by the proof of the existence of the solution to another coupled problem (see [4]). Here, however, the proof of the continuity of the fixed point operator requires some previous regularity results on the solution of the "Signorini problem", i.e., Problem (23) with a given φ; these results are proven in Section 3. Section 4 is devoted to the actual proof of the existence result, which includes the construction of the fixed point operator and the properties which are to be satisfied in order that the assumptions in Schauder's theorem hold.

2. The variational problem.

Proposition 2.1. *Let* $\Omega = \Omega_A \cup I \cup \Omega_B$, *with* $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1, 0), x_1 \in]0,1[\}$. *Let* a, c_A, $c_B \in \mathbb{R}_+^\star$ *and* $b \in H^1(\Gamma_A^3)$ *such that* $b \geq 0$ *and* $b(0) = 0$. *Let* $\overline{T}$ *be a function defined from* $L^2(I)$ *to* $H^2(I)$ *satisfying the assumptions* (1)–(3) *and* (16). *Assume that* $\Phi \in L^2(\Omega)$ *is such that* Φ_i *in* $C^2(\overline{\Omega}_i)$ *for* $i = A, B$; *and* u *in* $C^2(\overline{\Omega}_A)$, *then* (u, Φ) *is solution to variational problem defined by* (17)–(23) *if and only if* (u, Φ) *satisfies equations* (4)–(14).

Proof of Proposition 2.1. Assume u such that $u \in C^2(\overline{\Omega}_A)$, satisfying equations (4)–(7). Thanks to equation (6) and to the first relation in (7), $u \in K_\varphi$ with

$$\varphi = \Phi_{A|_I} - \Phi_{B|_I}.$$

Let $v \in K_\varphi$ with

$$\varphi = \Phi_{A|_I} - \Phi_{B|_I},$$

from equation (4), we obtain

$$\int_{\Omega_A} \nabla u(x)\, \nabla(v-u)(x) dx - \int_{\partial\Omega_A} \nabla u(s) \cdot \mathbf{n}\, (v-u)(s)\, ds = 0$$

Using equation (5) and the fact that u, v are in K_φ, yields

$$\int_{\Omega_A} \nabla u(x)\, \nabla(v-u)(x)\, dx = \int_{\Gamma_A^3} \nabla u(s) \cdot \mathbf{n}\, (v-u)(s)\, ds.$$

Adding and subtracting b and a in the right-hand side, yields

$$\int_{\Omega_A} \nabla u(x)\, \nabla(v-u)(x)\, dx = \int_{\Gamma_A^3} (\nabla u(s)\cdot \mathbf{n} - b(s))\,(v(s)-a)\, ds$$
$$+ \int_{\Gamma_A^3} (\nabla u(s)\cdot \mathbf{n} - b(s))\,(a-u(s))\, ds + \int_{\Gamma_A^3} b(s)\,(v-u)(s)\, ds.$$

Since $v \in K_\varphi$, $v_{|_{\Gamma_A^3}} - a \leq 0$; from conditions (7) on u, it follows that

$$\int_{\Omega_A} \nabla u(x)\, \nabla(v-u)(x) dx \geq \int_{\Gamma_A^3} b(s)\,(v-u)(s)\, ds.$$

Hence, u is solution to the problem (23). The proof that Φ such that $\Phi_i \in C^2(\overline{\Omega}_i)$ $i = A,\ B$ satisfying the equations (8)–(14) also satisfies 922) is straightforward.

Reciprocally, let (u,Φ) satisfying the regularity assumptions $u \in C^2(\overline{\Omega}_A)$ Φ_i in $C^2(\overline{\Omega}_i)$ for $i = A, B$ and the variational problem defined by (17)–(23). Let us show here that u satisfies the third equation in Signorini's conditions (7). Our proof is inspired by Baiocchi and Capelo's one (see [1]), for a similar problem with a Signorini boundary condition on the whole boundary.

Let $\Gamma_a^- = \{s \in \Gamma_A^3$ such that $u(s) < a\}$. Notice that the third equation of (7) is equivalent to $\nabla u \cdot \mathbf{n} = b$, on Γ_a^-. Let g be a function in $C_c^\infty(\Gamma_a^-)$, and denote by $\tilde{g}$ the extension by zero of g on Γ_A^3 and φ_g a function of $H^1(\Omega_A)$ such that $\varphi_{g_{|_{\Gamma_A^3}}} = \tilde{g}$ a.e. and $\varphi_{g_{|_I}} = 0$ a.e.

Let $\mu \geq 0$ be defined by

$$\mu = \frac{\inf_{ess}(a-u(s),\ s \in supp\, g)}{\sup(\,|g(s)|,\ s \in \Gamma_a^-)} \ \text{ if } g \not\equiv 0,\ \mu = 0 \ \text{ if } g \equiv 0.$$

Note that μ is defined for any $u \in H^1(\Omega_A)$.

Let $v_1 = u - \mu\varphi_g$ and $v_2 = u + \mu\varphi_g$, then v_1 and $v_2 \in K_\varphi$ (with $\varphi = \Phi_{A|_I} - \Phi_{B|_I}$); taking v_1 (respectively v_2) in equation (23), yields

$$\int_{\Gamma_a^-} (\nabla u(s)\cdot \mathbf{n} - b(s)) g(s) ds \leq 0$$

(respectively $\int_{\Gamma_a^-} (\nabla u(s)\cdot \mathbf{n} - b(s)) g(s) ds \geq 0$), and therefore,

$$\int_{\Gamma_a^-} (\nabla u(s)\cdot \mathbf{n} - b(s)) g(s)\, ds = 0 \quad \forall g \in C_c^\infty(\Gamma_a^-),$$

which, in turn, yields conditions (7). The other equations (4)–(14) of the strong formulation are easy to obtain. □

In order to use the Schauder fixed point theorem (see, for instance, [7]), we have to study the continuity of a fixed point operator. This requires a previous regularity result for the solution to the Signorini problem (4)–(7) which are studied in the next section.

3. A regularity property of the solution of the "Signorini problem." Let $\Omega_A =]0,1[\times]0,1[$, $a \in \mathbb{R}_+^\star$, $b \in H^1(\Gamma_A^3)$ such that $b(x_2) \geq 0$ $\forall x_2 \in \Gamma_A^3$ and $b(0) = 0$. Let $\psi : I \mapsto \mathbb{R}$ be defined by

$$\begin{aligned} &\psi \in H^2(I), \quad 0 \leq \psi(x_1) \leq a \ \ \forall x_1 \in I, \\ &\psi(0) = \psi(1) = 0 \ \text{ and } \ \psi'(0) = \psi'(1) = 0. \end{aligned} \tag{24}$$

Let u be the unique solution to the following problem:

$$\left\{ \begin{array}{l} u \in K_\psi = \{v \in H^1(\Omega_A),\ v_{|_I} = \psi \text{ a.e},\ v_{|_{\Gamma_A^3}} \leq a \text{ a.e}\}, \quad \text{satisfying :} \\ \int_{\Omega_A} \nabla u(x) \cdot \nabla(v-u)(x)dx \geq \int_{\Gamma_A^3} b(s)(v-u)_{|_{\Gamma_A^3}}(s)ds, \quad \forall v \in K_\psi. \end{array} \right. \tag{25}$$

We shall prove here local H^2 regularity of the solution u to (25). Let us start by proving the following estimate:

Proposition 3.1. *Let $\Omega_A =]0,1[\times]0,1[$, $a \in \mathbb{R}_+^\star$, $b \in H^1(\Gamma_A^3)$ such that $b(x_2) \geq 0$ $\forall x_2 \in \Gamma_A^3$ and $b(0) = 0$. Let ψ be a function defined from I to $\mathbb{R}$ satisfying the assumptions* (24). *Let u be the unique solution to* (25), *there exists $C > 0$ independent of u and of ψ, such that*

$$\|u\|_{H^1(\Omega_A)} \leq C\Big(\|\psi\|_{H^1(I)} + \|b\|_{L^2(\Gamma_A^3)}\Big). \tag{26}$$

Proof of Proposition 3.1. Define, for a.e. $(x_1, x_2) \in \Omega_A$, $u_\psi^0(x_1,x_2) = \psi(x_1)$. Remark that, by assumptions (24), $u_\psi^0 \in K_\psi$. Taking $v = u_\psi^0$ in (25), yields:

$$\begin{aligned} &\int_{\Omega_A} \mid \nabla(u_\psi^0 - u)(x) \mid^2 dx \\ \leq &\int_{\Omega_A} \nabla u_\psi^0 \cdot \nabla(u_\psi^0 - u)(x)dx - \int_{\Gamma_A^3} b(u_\psi^0 - u)_{|_{\Gamma_A^3}}(s)ds; \end{aligned}$$

since $u_\psi^0 - u \in H^1_{0,I}(\Omega_A)$, by Poincaré's inequality and by continuity of the trace operator from $H^1(\Omega_A)$ to $L^2(\Gamma_A^3)$, there exists $C > 0$ such that

$$\|u_\psi^0 - u\|_{H^1(\Omega_A)} \leq C(\|u_\psi^0\|_{H^1(\Omega_A)} + \|b\|_{L^2(\Gamma_A^3)}).$$

which, in turn, yields (26). □

Let us now prove the following maximum principle on the solution u of (25).

Lemma 3.1. *Let $\Omega_A =]0,1[\times]0,1[$, $a \in \mathbb{R}_+^\star$, $b \in H^1(\Gamma_A^3)$ such that $b(x_2) \geq 0$ $\forall x_2 \in \Gamma_A^3$ and $b(0) = 0$. Let ψ be a function defined from I to $\mathbb{R}$ satisfying the assumptions* (24). *Let u be the solution to Problem* (25), *then $u(x_1, x_2) \geq 0$ for a.e. $(x_1, x_2) \in \Omega_A$.*

Proof of Lemma 3.1. Since $-\Delta u = 0$ in $L^2(\Omega_A)$, multiplying by $u^- = -\inf_{\text{ess}}(0, u)$, integrating the product and using the Green formula, yields:

$$\int_{\Omega_A} \nabla u(x) \cdot \nabla u^-(x) dx - \int_{\Gamma_b} b(s) u^-_{|\Gamma_b}(s) ds = 0,$$

where $\Gamma_b = \{(x_1, x_2) \in \Gamma_A^3,\ \text{s.t. } \nabla u(x_1, x_2) \cdot \mathbf{n} = b\}$, (recall that $u^-_{|\Gamma_a} = 0$ on $\Gamma_a = \Gamma/\Gamma_b = \{(x_1, x_2) \in \Gamma_A^3,\ \text{s.t. } u(x_1, x_2) = a\}$, and that $\nabla u \cdot \mathbf{n} = 0$ on $\Gamma_A^1 \cup \Gamma_A^2$).

Since $u^- \in H^1_{0,I}(\Omega_A)$, there exists $C > 0$ such that

$$\|u^-\|_{H^1(\Omega)} \leq -C \int_{\Gamma_b} b(s) u^-_{|\Gamma_b}(s) ds,$$

However, $u^- \geq 0$ and $b \geq 0$ a.e., hence $u^- = 0$ a.e. on Ω_A, which concludes the proof of Lemma 3.1. □

The following H^2 estimate on u holds:

Proposition 3.2. *Let $\Omega_A =]0,1[\times]0,1[$, $\Omega_{A/2} =]0,1[\times]0,\frac{1}{2}[$, $a \in \mathbb{R}_+^\star$ and $b \in H^1(\Gamma_A^3)$ such that $b(x_2) \geq 0$ $\forall x_2 \in \Gamma_A^3$ and $b(0) = 0$. Let ψ be a function defined from I to $\mathbb{R}$ satisfying the assumptions* (24). *Let u be the solution to Problem* (25), *then $u \in H^2(\Omega_{A/2})$, and there exists $C > 0$ independent of u and of ψ such that*

$$\|u\|_{H^2(\Omega_{A/2})} \leq C\Big(\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^2(I)} + \|b\|_{H^1(\Gamma_A^3)}\Big). \tag{27}$$

This estimate and the continuity of the trace operator from $H^2(\Omega_{A/2})$ to $L^2(I)$ yields the following estimate on the trace of the normal derivative of u, which is crucial for the proof of the existence of the solution to the original coupled problem.

Corollary 3.1. *Let* $\Omega_A = (0,1)\times(0,1)$, $\Gamma_A^3 = \{(1,x_2), x_2 \in (0,1)\}$, $I = \{(x_1,0), x_1 \in (0,1)\}$, $a \in \mathbb{R}_+^\star$ *and* $b \in H^1(\Gamma_A^3)$ *such that* $b(x_2) \geq 0\ \forall x_2 \in \Gamma_A^3$ *and* $b(0) = 0$. *Let* ψ *be a function defined from* I *to* $\mathbb{R}$ *satisfying the assumptions* (24). *Let* u *be the solution to problem* (25), *Let* u *be the solution to problem* (25), *then* $\nabla u \cdot \mathbf{n} \in L^2(I)$, *and there exists* $C > 0$ *independent of* u *and of* ψ *such that*

$$\|\nabla u \cdot \mathbf{n}\|_{L^2(I)} \leq C\Big(\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^2(I)} + \|b\|_{H^1(\Gamma_A^3)}\Big).$$

Proof of Proposition 3.2. The proof of Proposition 3.2 consists in two steps. First, a symmetrization of the domain Ω_A is performed, in order to deal with the corners $(x_1 = 0, x_2 = 0)$ and $(x_1 = 1, x_2 = 0)$. A function w is constructed on the symmetrized domain, the restriction to Ω_A of which is equal to u.

The local H^2 regularity of w on the symmetrized domain is then shown by the method of the translations, due to Niremberg, see [2].

Step 1: Symmetrization. Let $\tilde{\Omega}_A =]0,1[\times]-1,0[$ and define $\tilde{u} : \tilde{\Omega}_A \mapsto \mathbb{R}$ by

$$\tilde{u}(x_1,x_2) = -u(x_1,-x_2) + 2\psi(x_1) \quad \text{a.e.} \quad (x_1,x_2) \in \tilde{\Omega}_A;$$

note that, if u is regular enough, one has $\tilde{u}(x_1,0) = u(x_1,0)$ and

$$\frac{\partial u}{\partial x_2}(x_1,0) = \frac{\partial \tilde{u}}{\partial x_2}(x_1,0)$$

for a.e. $x_1 \in]0,1[$.

Let $\Omega = \Omega_A \cup \tilde{\Omega}_A \cup I$ and define $w : \Omega \mapsto \mathbb{R}$ by

$$\begin{aligned} &w(x_1,x_2) = u(x_1,x_2) \quad \text{for a.e. } (x_1,x_2) \in \Omega_A \cup I, \text{ and} \\ &w(x_1,x_2) = \tilde{u}(x_1,x_2) \quad \text{for a.e. } (x_1,x_2) \in \tilde{\Omega}_A. \end{aligned} \tag{28}$$

Let $\tilde{\Gamma}_A^3 = \{(1,x_2),\ x_2 \in]-1,0[\}$ and $\Gamma^3 = \Gamma_A^3 \cup \tilde{\Gamma}_A^3 \cup (1,0)$. Define $\tilde{b} \in L^2(\Gamma^3)$ by

$$\tilde{b}(s) = b(s) \text{ for a.e. } s \in \Gamma_A^3 \text{ and } \tilde{b}(s) = -b(-s) \text{ for a.e. } s \in \tilde{\Gamma}_A^3. \tag{29}$$

Let $f \in L^2(\Omega)$ such that $f(x_1,x_2) = 0$ for a.e. $(x_1,x_2) \in \Omega_A$ and $f(x_1,x_2) = -2\psi''(x_1)$ for a.e. $(x_1,x_2) \in \tilde{\Omega}_A$. We shall now use the following result:

Lemma 3.2. *Let* $\Omega = \Omega_A \cup \tilde{\Omega}_A \cup I$, *with* $\Omega_A =]0,1[\times]0,1[$, $\tilde{\Omega}_A =]0,1[\times]-1,0[$ *and* $I = \{(x_1,0), x_1 \in]0,1[\}$. *Let* $\Gamma^3 = \Gamma^3_A \cup \tilde{\Gamma}^3_A \cup (1,0)$, *with* $\Gamma^3_A = \{(1,x_2), x_2 \in]0,1[\}$ *and* $\tilde{\Gamma}^3_A = \{(1,x_2),\ x_2 \in]-1,0[\}$. *Let* $a \in \mathbb{R}^\star_+$ *and* $\tilde{b} \in L^2(\Gamma^3_A)$ *be defined by* (29). *Let* ψ *be a function defined from* I *to* $\mathbb{R}$ *satisfying the assumptions* (24). *Let* w *be the function defined by* (28), *then* w *satisfies the following problem*

$$\left\{\begin{aligned} & w \in K = \{v \in H^1(\Omega),\ v_{|\Gamma^3_A} \le a,\quad v_{|\tilde{\Gamma}^3_A} \ge -a \ a.e\},\ \textit{satisfying}\ : \\ & \int_\Omega \nabla w(x)\cdot\nabla(v-w)(x)dx \ge \int_{\Gamma^3} \tilde{b}(s)(v-w)_{|\Gamma^3}(s)ds \\ & \qquad\qquad\qquad\qquad + \int_\Omega f(x)(v-w)(x)dx, \quad \forall v \in K \end{aligned}\right. \tag{30}$$

Proof of Lemma 3.2. Let us first show that $w \in K$. Since $\tilde{u}(x_1,0) = u(x_1,0)$ for a.e. $x_1 \in]0,1[$, it is easily seen that $w \in H^1(\Omega)$; moreover, one has, for a.e. $x_2 \in]-1,0[$: $w(1,x_2) = \tilde{u}(1,x_2) = -u(1,-x_2)$. Hence $w_{|\tilde{\Gamma}^3_A} \ge -a$ a.e. and $w \in K$. Let us now show that, for any $v \in K$, w satisfies (30). For $v \in K$, define, for a.e. $(x_1,x_2) \in \Omega_A$, $\tilde{v}(x_1,x_2) = v(x_1,-x_2)$. Integrating by part and thanks to a change of variable, one has

$$\begin{aligned} & \int_{\tilde{\Omega}_A} \nabla\tilde{u}(x)\cdot\nabla(v-\tilde{u})(x)dx \\ &= \int_0^1\int_0^1 \nabla u(x_1,x_2)\cdot\nabla(-\tilde{v}(x_1,x_2) - u(x_1,x_2) + 2\psi(x_1))dx_1dx_2 \\ & -2\int_{-1}^0\int_0^1 \psi''(x_1)(v-\tilde{u})(x_1,x_2)dx_1dx_2. \end{aligned}$$

Therefore, defining $\overline{v}(x_1,x_2) = \frac{v-\tilde{v}}{2}(x_1,x_2) + \psi(x_1)$ for a.e. $(x_1,x_2) \in \Omega_A$, one has

$$\begin{aligned} & \int_\Omega \nabla w(x)\cdot\nabla(v-w)(x)dx \\ &= 2\int_0^1\int_0^1 \nabla u(x_1,x_2)\cdot\nabla(\overline{v}(x_1,x_2) - u(x_1,x_2))dx_1dx_2 \\ & -2\int_{-1}^0\int_0^1 \psi''(x_1)(v-\tilde{u})(x_1,x_2)dx_1dx_2, \end{aligned}$$

since $\overline{v}_{|\Omega_A} \in K_\psi$, inequality (25) with $v = \overline{v}_{|\Omega_A}$ together with the above equality yields

$$\int_\Omega \nabla w(x) \cdot \nabla(v-w)(x)dx$$
$$\geq 2\int_0^1 b(x_2)(\overline{v}-u)(1,x_2)dx_2 - 2\int_{-1}^0\int_0^1 \psi''(x_1)(v-\tilde{u})(x_1,x_2)dx_1dx_2,$$

that is,

$$\int_\Omega \nabla w(x)\cdot\nabla(v-w)(x)dx \geq \int_0^1 \tilde{b}(x_2)(v-u)(1,x_2)dx_2$$
$$+\int_{-1}^0 \tilde{b}(x_2)(v-\tilde{u})(1,x_2)dx_2 - 2\int_{-1}^0\int_0^1 \psi''(x_1)(v-\tilde{u})(x_1,x_2)dx_1dx_2,$$

which concludes the proof of Lemma 3.2.

Remark. Note that, if w is regular enough, then it satisfies the following problem:

$$\left\{\begin{array}{l} -\Delta w = 0, \quad \text{in } \Omega, \\ \nabla w(s)\cdot \mathbf{n} = 0, \quad s \in \Gamma^1,\, \Gamma_A^2,\, \tilde{\Gamma}_A^2, \\ \\ w(s) \leq a, \quad s \in \Gamma_A^3, \\ \nabla w(s)\cdot\mathbf{n} \leq b(s), \quad s\in\Gamma_A^3, \\ (w(s)-a)(\nabla w(s).\mathbf{n} - b(s)) = 0, \quad s \in \Gamma_A^3, \\ \\ w(s) \geq -a, \quad s\in\tilde{\Gamma}_A^3, \\ \nabla w(s)\cdot\mathbf{n} \geq -b(s), \quad s \in \tilde{\Gamma}_A^3, \\ (w(s)+a)(\nabla w(s).\mathbf{n}+b(s)) = 0, \quad s\in\tilde{\Gamma}_A^3, \end{array}\right. \tag{31}$$

with $\tilde{\Gamma}_A^2 = \{(x_1,-1),\ x_1 \in]0,1[\}$ and $\Gamma^1 = \{(0,x_2),\ x_2 \in]-1,1[\}$.

Step 2: Method of translations. Let $\Omega_{1/2} =]0,1[\times]-1/2,1/2[$; we now show that $w \in H^2(\Omega_{1/2})$ and give an estimate of $\|w\|_{H^2(\Omega_{1/2})}$. We use here the method of translations, which was developed by Brezis [2] to show the regularity of the solution of the Laplace operator, and can also be found applied to a problem with a Signorini boundary condition on the whole boundary of the domain in [6].

Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ such that $\varphi \equiv 1$ on $\Omega_{1/2}$, $0 \le \varphi \le 1$ on $\mathbb{R}^2$ and $\mathrm{supp}\varphi \subset \mathbb{R}\times]-\frac{3}{4},\frac{3}{4}[$. Let $h>0$ and $H=(0,h)$. Let us show that, for h small enough, there exists $\varepsilon>0$ depending only on H such that

$$v = w + \varepsilon D_{-H}(\varphi D_H(w)) \in K, \tag{32}$$

where

$$D_{\pm\eta}(x) = \pm \frac{w(x\pm\eta)-w(x)}{|\eta|} \quad \forall \eta \in \mathbb{R}^2, \ \text{ for a.e. } x \in \Omega_{1/2},$$

$|\eta|$ denoting the Euclidean norm of η. For $h<\frac{1}{4}$, v is clearly in $H^1(\Omega)$. Let us show that, for an adequate choice of ε, v is such that $v_{|\Gamma_A^3} \ge -a$ a.e. Let $x_2 \in]-1,0[$, then

$$\begin{aligned} v(1,x_2)+a =& (w(1,x_2)+a)\left(1-\frac{\varepsilon}{h^2}(\varphi(1,x_2)+\varphi(1,x_2-h)\right) \\ &+\frac{\varepsilon}{h^2}\left(\varphi(1,x_2-h)(w(1,x_2-h)+a)+\varphi(1,x_2)(w(1,x_2+h)+a)\right). \end{aligned}$$

Remark that $(w(1,x_2)+a)(1-\frac{\varepsilon}{h^2}(\varphi(1,x_2)+\varphi(1,x_2-h))$ is positive if $\varepsilon \le \frac{h^2}{2}$; hence, let $\varepsilon = \frac{h^2}{2}$.

As $h>0$,

$$\varphi(1,x_2-h)\,(w(1,x_2-h)+a) \ge 0 \quad \text{for a.e. } \ x_2 \in]-1,0[$$

and

$$\varphi(1,x_2)\,(w(1,x_2+h)+a) \ge 0 \quad \text{for a.e. } \ x_2 \in]-1,-h[.$$

By Lemma 3.1, $u(x_1,x_2) \ge 0$ for a.e. $(x_1,x_2) \in \Omega_A$, and therefore,

$$w(1,x_2+h)+a \ge 0 \quad \forall x_2 \in]-h,0[.$$

Hence, $v_{|\Gamma_A^3} \ge -a$ a.e.; in the same way, one may prove that $v_{|\Gamma_A^3} \le a$ a.e.

Then, the function v, defined in (32) with $h<\frac{1}{4}$ and $\varepsilon = \frac{h^2}{2}$, is in K. Hence it may be taken as a test function in (30)

$$\begin{aligned} &\varepsilon \int_\Omega \nabla w(x) \nabla (D_{-H}(\varphi D_H w))(x)dx \\ \ge& \varepsilon \int_{\Gamma^3} \tilde{b}(s) D_{-h}(\varphi D_h w_{|\Gamma^3})(s)ds + \varepsilon \int_\Omega f(x)(D_{-H}(\varphi D_H w))(x)dx, \end{aligned}$$

and therefore

$$\int_\Omega \nabla(D_H w)(x)\cdot\nabla(\varphi D_H w)(x)dx$$
$$\leq \int_{\Gamma^3} \varphi D_h \tilde{b}(s) D_h w_{|_{\Gamma^3}}(s)ds + \int_\Omega \varphi D_H f(x) D_H w(x)dx. \tag{33}$$

Let $\theta \in C_c^\infty(\mathbb{R}^2)$ such that $\theta \geq 0$ and $\theta^2 = \varphi$, (33) becomes

$$\int_\Omega |\theta\nabla(D_H w)(x)|^2 dx + 2\int_\Omega \theta(x) D_H w(x)\nabla\theta(x)\cdot\nabla(D_H w)(x)dx$$
$$\leq \|\theta D_H w\|_{H^1(\Omega)}(C_1\|\theta D_h \tilde{b}\|_{L^2(\Gamma^3)} + \|\theta D_H f\|_{H^{-1}(\Omega)}), \tag{34}$$

where C_1 is the continuity constant of the trace operator from $H^1(\Omega)$ to $L^2(\Gamma^3)$. noticing that

$$\|\theta D_H w\|^2_{H^1(\Omega)} = \int_\Omega |\theta(x) D_H w(x)|^2 dx + \int_\Omega |\nabla\theta(x)\ D_H w(x)|^2 dx \tag{35}$$
$$+ \int_\Omega |\theta(x)\nabla(D_H w)(x)|^2 dx + 2\int_\Omega \theta(x) D_H w(x)\nabla\theta(x)\cdot\nabla(D_H w)(x)dx,$$

from (34) and (35), one has

$$\|\theta D_H w\|^2_{H^1(\Omega)} \leq \int_\Omega |\theta(x) D_H w(x)|^2 dx + \int_\Omega |\nabla\theta(x)\ D_H w(x)|^2 dx \tag{36}$$
$$+ \|\theta D_H w\|_{H^1(\Omega)}\left(C\|\theta D_H \tilde{b}\|_{L^2(\Gamma^3)} + \|\theta D_H f\|_{H^{-1}(\Omega_A)}\right).$$

The right hand side of (36), will now be estimated thanks to the following lemma, [2], p. 153.

Lemma 3.3. *Let Ω be an open set of $\mathbb{R}^N$, $u \in H^1(\Omega)$, ω an open subset of Ω and let $\eta \in \mathbb{R}^N$ such that $\forall x \in \omega$, $x + \eta \in \omega$, then*

$$\|D_\eta u\|_{L^2(\omega)} \leq \|\nabla u\|_{L^2(\Omega)}.$$

Noticing that

$$\int_\Omega |\theta(x) D_H w(x)|^2 dx \leq \int_{\mathrm{supp}\theta\cap\Omega} |D_H w(x)|^2 dx,$$

with

$$\operatorname{supp}\theta \subset \mathbb{R}\times]-\tfrac{3}{4},\tfrac{3}{4}[.$$

and since $h < \frac{1}{4}$, applying Lemma 3.3 yields

$$\int_{\operatorname{supp}\theta\cap\Omega} |D_H w(x)|^2 dx \leq \|\nabla w\|^2_{L^2(\Omega)}. \tag{37}$$

Similarly, one has

$$\int_\Omega |\nabla\theta(x)\, D_H w(x)|^2 dx \leq C_2 \|\nabla w\|^2_{L^2(\Omega)}, \tag{38}$$

with $C_2 > 0$ a constant depending only on θ. Since $\tilde{b} \in H^1(\Gamma^3_A)$, $\tilde{b} \in H^1(\tilde{\Gamma}^3_A)$ and $\tilde{b}$ is continuous at 0, one has $\tilde{b} \in H^1(\Gamma^3)$. Thus, by Lemma 3.3,

$$\int_{\Gamma^3} |\theta_{|\Gamma^3}(t) D_h \tilde{b}(t)|^2 dt \leq \|\nabla\tilde{b}\|_{L^2(\Gamma^3)}. \tag{39}$$

Now, noting that

$$\|\theta D_H f\|_{H^{-1}(\Omega)} \leq \sup_{\psi\in H^1(\Omega)} \frac{|\int_{\operatorname{supp}\theta} f(x) D_H\psi(x) dx\,|}{\|\psi\|_{H^1(\Omega)}},$$

and using Lemma 3.3, one obtains

$$\|\theta D_H f\|_{H^{-1}(\Omega)} \leq \|f\|_{L^2(\Omega)}. \tag{40}$$

Then, inequality (36) becomes, thanks to (37), (38), (39) and (40),

$$\begin{aligned}\|\theta D_H w\|^2_{H^1(\Omega)} \leq{}& (1+C_2)\|\nabla w\|_{L^2(\Omega)} \\ &+ \|\theta D_H w\|_{H^1(\Omega)} \left(C_1 \|\tilde{b}\|_{H^1(\Gamma^3)} + \|f\|_{L^2(\Omega)} \right).\end{aligned}$$

Let

$$C = (1+C_2)\|\nabla w\|_{L^2(\Omega)},\ B = C_1\|\tilde{b}\|_{H^1(\Gamma^3)} + \|f\|_{L^2(\Omega)},\ X = \|\theta D_H w\|^2_{H^1(\Omega)}.$$

Since $X^2 - BX - C \leq 0$ one has

$$X \leq \frac{B + (B^2+4C)^{\frac{1}{2}}}{2} \leq C^{\frac{1}{2}} + B,$$

and therefore,

$$\|\theta D_H w\|_{H^1(\Omega)} \le \big((1+C_2)\|\nabla w\|_{L^2(\Omega)}\big)^{\frac{1}{2}} + C_1\|\tilde{b}\|_{H^1(\Gamma^3)} + \|f\|_{L^2(\Omega)},$$

using the definitions of w, $\tilde{b}$ and f, yields

$$\begin{aligned}\|\theta D_H w\|_{H^1(\Omega)} \le\ & \big((1+C_2)(2\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^1(I)})\big)^{\frac{1}{2}} \\ & + 2C_1\|b\|_{H^1(\Gamma^3_A)} + 2\|\psi\|_{H^2(I)}.\end{aligned}$$

Thus, there exists $C > 0$, independent of ψ and u ($C = 3 + 2C_2 + 2C_1$), such that

$$\|D_H w\|_{H^1(\Omega_{1/2})} \le C\ \Big(\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^2(I)} + \|b\|_{H^1(\Gamma^3_A)}\Big). \tag{41}$$

Let us now show that $w \in H^2(\Omega_{1/2})$, i.e., there exists $g_{i,j} \in L^2(\Omega_{1/2})$ such that

$$\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial x_i \partial x_j}(x)\,dx = \int_{\Omega_{1/2}} g_{i,j}(x)\varphi(x)\,dx \quad \forall\varphi \in C_c^\infty(\Omega_{1/2});\ i,j = 1,2.$$

Let φ in $C_c^\infty(\Omega_{1/2})$, using the Green formula

$$\int_{\Omega_{1/2}} w(x) D_{-H}(\frac{\partial\varphi}{\partial x_i})(x)\,dx = \int_{\Omega_{1/2}} D_H(\frac{\partial w}{\partial x_i})(x)\varphi(x)\,dx, \quad i = 1,2,$$

hence,

$$\Big|\int_{\Omega_{1/2}} w(x) D_{-H}(\frac{\partial\varphi}{\partial x_i})(x)\,dx\Big| \le \|D_H(\frac{\partial w}{\partial x_i})\|_{L^2(\Omega_{1/2})}\|\varphi\|_{L^2(\Omega_{1/2})}, \quad i = 1,2.$$

Using (41) and letting h tend to 0, yields

$$\begin{aligned}&\Big|\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial x_i\partial x_2}(x)dx\Big| \\ &\le C\Big(\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^2(I)} + \|b\|_{H^1(\Gamma^3_A)}\Big)\,\|\varphi\|_{L^2(\Omega_{1/2})}, \\ &\forall\varphi \in C_c^\infty(\Omega_{1/2});\quad i = 1,2.\end{aligned} \tag{42}$$

Since $-\Delta w = f$ in $\mathcal{D}'(\Omega_{1/2})$, one has

$$\begin{aligned}&-\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial^2 x_1}(x)dx\\&=\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial^2 x_2}(x)dx+\int_{\Omega_{1/2}} f(x)\varphi(x)dx,\quad \forall\varphi\in C_c^\infty(\Omega_{1/2}),\end{aligned}$$

hence, there exists $C'>0$, independent of u and ψ ($C'=C+2$), such that

$$\begin{aligned}&\Big|\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x)dx\Big|\\&\leq C'\Big(\|u\|_{H^1(\Omega_A)}+\|\psi\|_{H^2(I)}+\|b\|_{H^1(\Gamma_A^3)}\Big)\|\varphi\|_{L^2(\Omega_{1/2})},\\&\forall\varphi\in C_c^\infty(\Omega_{1/2});\quad i,j=1,2.\end{aligned}\tag{43}$$

Consider the application

$$F_{i,j}:\varphi\mapsto\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x)dx,$$

from $C_c^\infty(\Omega_{1/2})$ to $\mathbb{R}$. Since $F_{i,j}$ is continuous on $C_c^\infty(\Omega_{1/2})$, we can extend by density this application to $G_{i,j}\in(L^2(\Omega_{1/2}))^\star$, such that

$$F_{i,j}(\varphi)=G_{i,j}(\varphi)\quad\forall\varphi\in C_c^\infty(\Omega_{1/2}).$$

By the Riesz Theorem, there exists $g_{i,j}\in L^2(\Omega_{1/2})$ such that

$$G_{i,j}(\varphi)=\int_{\Omega_{1/2}} g_{i,j}(x)\varphi(x)dx$$

for any $\varphi\in \mathrm{L}^2(\Omega_{1/2})$, and therefore

$$\int_{\Omega_{1/2}} w(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x)dx=\int_{\Omega_{1/2}} g_{i,j}(x)\varphi(x)dx\quad\forall\varphi\in C_c^\infty(\Omega_{1/2});\ i,j=1,2,$$

since $g_{i,j}=\frac{\partial^2 w}{\partial x_i\partial x_j}$, $w\in H^2(\Omega_{1/2})$. Moreover, the following estimate holds

$$\|\frac{\partial^2 w}{\partial x_i\partial x_j}\|_{L^2(\Omega_{1/2})}\leq C'\Big(\|u\|_{H^1(\Omega_A)}+\|\psi\|_{H^2(I)}+\|b\|_{H^1(\Gamma_A^3)}\Big),\quad i,j=1,2.\tag{44}$$

To conclude the proof of Proposition 3.2, we recall that $w = u$ in Ω_A, hence $u \in H^2(\Omega_{A/2})$ with $\Omega_{A/2} =]0,1[\times]0,\frac{1}{2}[$, and there exists a constant $C > 0$ independent of u and ψ such that

$$\|u\|_{H^2(\Omega_{A/2})} \leq C\Big(\|u\|_{H^1(\Omega_A)} + \|\psi\|_{H^2(I)} + \|b\|_{H^1(\Gamma_A^3)}\Big).$$

4. Existence of the solution.

Theorem 4.1. *Let*

$$\Omega = \Omega_A \cup I \cup \Omega_B,$$

with $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1,0), x_1 \in]0,1[\}$. *Let* $a, c_A, c_B \in \mathbb{R}_+^\star$ *and* $b \in H^1(\Gamma_A^3)$ *such that* $b \geq 0$ *and* $b(0) = 0$. *Let* $\overline{T}$ *be a function defined from* $L^2(I)$ *to* $H^2(I)$ *satisfying the assumptions* (1)–(3) *and* (16). *There exists* (u, Φ) *satisfying* (17)–(23).

This theorem is proven thanks to Schauder's theorem (see e.g. [7]):

Theorem 4.2. *Let* E *be a Banach space and* F *an application from* E *to* E, *continuous, compact and such that there exists* $R > 0$ *such that* $F(B_E(0,R)) \subset B_E(0,R)$ *(where* $B_E(0,R) = \{u \in E \text{ s.t. } \|u\|_E \leq R\}$*), then* F *admits a fixed point* $u \in B_E(0,R)$, *such that* $u = F(u)$.

In order to use Schauder's theorem, let us first write problem (22)–(23) as a fixed point problem.

4.1. Construction of the fixed point operator. For a given $\varphi \in L^2(I)$, there exists, by a theorem due to Stampacchia (see [2] p. 83) a unique solution $u_\varphi \in H^1(\Omega_A)$ to

$$\begin{cases} u_\varphi \in K_\varphi, \\ \mathcal{A}_A(u_\varphi, v - u_\varphi) \geq L(v - u_\varphi), \quad \forall v \in K_\varphi \end{cases} \tag{45}$$

with

$$K_\varphi = \{v \in H^1(\Omega_A),\ v_{|_I} = \overline{T}(\varphi) \text{ a.e, } v_{|_{\Gamma_A^3}} \leq a \text{ a.e}\},$$

$$\mathcal{A}_A(\Phi, \Psi) = \int_{\Omega_A} \nabla\Phi(x)\nabla\Psi(x)dx \quad \forall \Phi,\ \ \Psi \in H^1(\Omega_i),$$

and

$$L(v) = \int_{\Gamma_A^3} b(s) v_{|_{\Gamma_A^3}}(s) ds \ \ \forall v \in H^1(\Omega_A).$$

Remark. The set K_φ is non empty, thanks to the property (1) of the function $\overline{T}$. Take, for instance, $u^0_\varphi(x_1, x_2) = \overline{T}(\varphi)(x_1)$, $\forall (x_1, x_2) \in \Omega_A$ then $u^0_\varphi \in K_\varphi$.

Consider now the following problem:

$$\begin{cases} \overline{\Phi}_i \in H^1_{0,2}(\Omega_i) \quad (i = A, B), \\ \mathcal{A}_i(\overline{\Phi}_i, \Psi) = L_{u_\varphi, i}(\Psi), \quad \forall \Psi \in H^1_{0,2}(\Omega_i) \quad (i = A, B); \end{cases} \tag{46}$$

with

$$\mathcal{A}_i(\Phi, \Psi) = \int_{\Omega_i} \nabla\Phi(x)\nabla\Psi(x)dx, \quad \forall \Phi,\ \Psi \in H^1(\Omega_A),$$

and

$$L_{u,i}(\Psi) = c_i \int_I \nabla u(s) \cdot \mathbf{n}\Psi_{|_I}(s)\, ds,\ \ \forall \Psi \in H^1(\Omega_i), \quad \forall u \in H^1(\Omega_A)$$

such that $\nabla u \cdot \mathbf{n} \in L^2(I)$.

Lemma 4.1. *Let*

$$\Omega = \Omega_A \cup I \cup \Omega_B,$$

with $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1, 0), x_1 \in]0,1[\}$. *Let* $\varphi \in L^2(I)$ *and* u_φ *be the unique solution to Problem* (45), *then Problem* (46) *has an unique solution.*

Proof of Lemma 4.1. In order to apply Lax-Milgram's Theorem, let us show that u_φ, solution of (45), is such that

$$\nabla u_\varphi \cdot \mathbf{n} \in L^2(I).$$

This result is a consequence of Corollary 3.1. Indeed, thanks to properties (1)–(3) and (16) satisfied by $\overline{T}$, $\psi = \overline{T}(\varphi)$ clearly satisfies the assumptions (24); hence $u_\varphi = u$, where u is the solution to Problem (25) with $\psi = \overline{T}(\varphi)$; therefore, we can apply Corollary 3.1 with $u = u_\varphi$. □

Let $E = L^2(I) \times L^2(\Omega)$, and define the operator F from E into E such that

$$F : (\varphi, \Phi) \longmapsto (\overline{\Phi}_{A|_I} - \overline{\Phi}_{B|_I}, \overline{\Phi}), \tag{47}$$

where $\overline{\Phi}$ is the solution to Problem (46). Note that $F(\varphi, \Phi)$ is independent of Φ. Let $(\varphi, \Phi) \in E$, we define $\|(\varphi, \Phi)\|^2_E = \|\varphi\|^2_{L^2(I)} + \|\Phi\|^2_{L^2(\Omega)}$. Let us prove now that F is continuous and compact from E into a ball of E. Hence, Theorem 4.1 will be proven by Schauder's Theorem.

Proposition 4.1. *Let*

$$\Omega = \Omega_A \cup I \cup \Omega_B,$$

with $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1,0), x_1 \in]0,1[\}$. *Let* $E = L^2(I) \times L^2(\Omega)$ *and* F *be an operator from* E *into* E *defined by* (47). *There exists* $R \in \mathbb{R}_+^\star$, *independent of* φ *and* Φ, *such that*

$$\|F(\varphi,\Phi)\|_E \le R \quad \forall(\varphi,\Phi) \in E.$$

Proof. By definition,

$$\|F(\varphi,\Phi)\|_E^2 = \|\overline{\Phi}_{A|_I} - \overline{\Phi}_{B|_I}\|_{L^2(I)}^2 + \|\overline{\Phi}\|_{L^2(\Omega)}^2,$$

and therefore,

$$\|F(\varphi,\Phi)\|_E^2 \le 2\|\overline{\Phi}_{A|_I}\|_{L^2(I)}^2 + 2\|\overline{\Phi}_{B|_I}\|_{L^2(I)}^2 + \|\overline{\Phi}_A\|_{H^1(\Omega_A)}^2 + \|\overline{\Phi}_B\|_{H^1(\Omega_B)}^2.$$

Therefore, by continuity of the trace operator from $H^1(\Omega_i)$ to $L^2(I)$, $i = A$, B, there exist C_A and $C_B \in \mathbb{R}_+^\star$ such that

$$\|F(\varphi,\Phi)\|_E^2 \le C_A\|\overline{\Phi}_A\|_{H^1(\Omega_A)}^2 + C_B\|\overline{\Phi}_B\|_{H^1(\Omega_B)}^2. \tag{48}$$

Lemma 4.2. *Let*

$$\Omega = \Omega_A \cup I \cup \Omega_B,$$

with $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1,0), x_1 \in]0,1[\}$. *Let* $E = L^2(I) \times L^2(\Omega)$ *and* $(\varphi,\Phi) \in L^2(I) \times L^2(\Omega)$. *Let* u_φ *be the (unique) solution to Problem* (45) *and* $\overline{\Phi}$ *the (unique) solution to Problem* (46), *then there exists* $R_i \in \mathbb{R}_+^\star$, *independent of* φ, u_φ *and* Φ, *such that*

$$\|\overline{\Phi}_i\|_{H^1(\Omega_i)} \le R_i \quad (i = A, B).$$

Proof. Taking $\Psi = \overline{\Phi}_i$ in (46) yields

$$\int_{\Omega_i} \nabla\overline{\Phi}_i \cdot \nabla\overline{\Phi}_i = c_i \int_I \nabla u_\varphi \cdot \mathbf{n}\,\overline{\Phi}_i,$$

and therefore, using Cauchy Schwartz' inequality and the trace operator in the RHS and Poincaré's inequality for the LHS, there exists $C_i > 0$ such that

$$\|\overline{\Phi}_i\|_{H^1(\Omega_i)} \le c_i C_i \|\nabla u_\varphi \cdot \mathbf{n}\|_{L^2(I)}. \tag{49}$$

From Corollary 3.1 with $\psi = \overline{T}(\varphi)$, there exists $C_1 > 0$ independent of u_φ and φ such that

$$\|\nabla u_\varphi \cdot \mathbf{n}\|_{L^2(I)} \leq C_1\Big(\|u_\varphi\|_{H^1(\Omega_A)} + \|\overline{T}(\varphi)\|_{H^2(I)} + \|b\|_{H^1(\Gamma_A^3)}\Big);$$

thanks to Proposition 3.1, there exists $C_2 > 0$ independent of u_φ and φ such that

$$\|\nabla u_\varphi \cdot \mathbf{n}\|_{L^2(I)} \leq C_2\Big(\|\overline{T}(\varphi)\|_{H^2(I)} + \|b\|_{H^1(\Gamma_A^3)}\Big), \tag{50}$$

and since by Assumption (2), $\overline{T}$ is uniformly bounded from $L^2(I)$ to $H^2(I)$, the result of Lemma 4.2 follows from (49) and (50). We deduce from (48) and from Lemma 4.2, the result of Proposition 4.1.

Lemma 4.3. *Let*

$$\Omega = \Omega_A \cup I \cup \Omega_B,$$

with $\Omega_A =]0,1[\times]0,1[$, $\Omega_B =]0, x_1^{max}[\times]x_2^{min}, 0[$, *and* $I = \{(x_1, 0), x_1 \in]0,1[\}$. *Let* $E = L^2(I) \times L^2(\Omega)$ *and* F *be an operator from* E *into* E *defined by* (47). *The operator* F *is compact from* E *to* E.

Proof. Let $(\varphi_n, \Phi_n)_{n\in\mathbb{N}}$ be a sequence of E, let us show that there exists a subsequence $(\varphi_{n_k}, \Phi_{n_k})_{k\in\mathbb{N}}$ and $\overline{w} \in E$ such that $F(\varphi_{n_k}, \Phi_{n_k}) \to \overline{w}$ in E, as $k \to +\infty$.

Let $\overline{\Phi}^n$ be the solution to the following problem

$$\begin{cases} \overline{\Phi}^n \in L^2(\Omega), \ \ \overline{\Phi}_i^n \in H^1_{0,2}(\Omega_i) \ \ (i = A, B), \\ \mathcal{A}_i(\overline{\Phi}_i^n, \Psi) = L_{u_{\varphi_n}, i}(\Psi), \quad \forall \Psi \in H^1_{0,2}(\Omega_i) \quad (i = A, B), \end{cases}$$

where u_{φ_n} is the solution to the following problem

$$\begin{cases} u_{\varphi_n} \in K_{\varphi_n}, \\ \mathcal{A}_A(u_{\varphi_n}, v - u_{\varphi_n}) \geq L(v - u_{\varphi_n}), \quad \forall v \in K_{\varphi_n}, \end{cases}$$

and $\mathcal{A}_i$, $L_{u_{\varphi_n}, i}$, L and K_{φ_n} are defined in (18)–(21).

From Lemma 4.2, one has

$$\|\overline{\Phi}_i^n\|_{H^1(\Omega_i)} \leq R_i \quad \forall n \in \mathbb{N} \quad (i = A, B);$$

hence there exists a subsequence $(\overline{\Phi}_i^{n_k})_{k\in\mathbb{N}}$ and $w_i \in H^1(\Omega_i)$ such that $\overline{\Phi}_i^{n_k} \to w_i$ in $H^1(\Omega_i)$ for the weak topology, as $k \to +\infty$. By compactness of the injection from $H^1(\Omega_i)$ in $L^2(\Omega_i)$ and of the trace operator from

$H^1(\Omega_i)$ in $L^2(I)$, $\overline{\Phi}_i^{n_k} \to w_i$ in $L^2(\Omega_i)$, and $\overline{\Phi}_i^{n_k}{}_{|_I} \to w_{i|_I}$ in $L^2(I)$, as $k \to +\infty$.

Let $w \in L^2(\Omega)$ such that $w_{|_{\Omega_i}} = w_i$ $(i = A, B)$ and $\overline{w} = (w_{A|_I} - w_{B|_I}, w)$, then $F(\varphi_{n_k}, \Phi_{n_k}) \to \overline{w}$ in E, as $k \to +\infty$.

The operator F maps E into a closed ball of E and is compact. We shall show in the next section that it is continuous from E to E. Hence by the Schauder fixed point theorem, there exists $(\varphi, \Phi) \in E$ such that $(\varphi, \Phi) = (\overline{\Phi}_{A|_I} - \overline{\Phi}_{B|_I}, \overline{\Phi})$, where $\overline{\Phi}$ is the solution to Problem (46). This concludes the proof of Theorem 4.1.

4.2. Continuity of the fixed point operator F. In order to prove that F is continuous from E into E, let us first show the following result:

Proposition 4.2. *Let $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(I)$ be such that $\varphi_n \to \varphi$ in $L^2(I)$; let $u_{\varphi_n} \in H^1(\Omega_A)$ be the solution to the following problem*

$$\begin{cases} u_{\varphi_n} \in K_{\varphi_n}, \\ \mathcal{A}_A(u_{\varphi_n}, v - u_{\varphi_n}) \geq L(v - u_{\varphi_n}), \quad \forall v \in K_{\varphi_n}, \end{cases} \tag{51}$$

with $K_{\varphi_n} = \{v \in H^1(\Omega_A),\ v_{|_I} = \overline{T}(\varphi_n)$ a.e, $v_{|_{\Gamma_A^3}} \leq a$ a.e$\}$ and $\mathcal{A}_A$, L defined by (18) *and* (21). *Then $u_{\varphi_n} \to u_\varphi$ in $H^1(\Omega_A)$, for the strong topology* (*recall that u_φ is the solution to problem* (45)).

Proof. From the H^1 estimate obtained for the solution of the Signorini problem (51) (see Proposition 3.1) choosing $\psi = \overline{T}(\varphi_n)$, yields the existence of $C_1 \in \mathbb{R}_+^\star$, independent of φ_n, such that

$$\|u_{\varphi_n}\|_{H^1(\Omega_A)} \leq C_1\Big(\|\overline{T}(\varphi_n)\|_{H^1(I)} + \|b\|_{L^2(\Gamma_A^3)}\Big);$$

and thanks to Assumption (2), there exists $C_2 \in \mathbb{R}_+^\star$ independent of n such that

$$\|u_{\varphi_n}\|_{H^1(\Omega_A)} \leq C_2 \quad \forall n \in \mathbb{N}.$$

Hence, there exists a subsequence $(u_{\varphi_{n_k}})_{k\in\mathbb{N}} \subset H^1(\Omega_A)$ and $\tilde{u} \in H^1(\Omega_A)$ such that

$$u_{\varphi_{n_k}} \to \tilde{u} \ \text{ in } H^1(\Omega_A) \ \text{ for the weak topology, as } \ k \to +\infty. \tag{52}$$

Let us now show that $\tilde{u}$ is the (unique) solution to Problem (45), hence $\tilde{u} = u_\varphi$, and that $\|u_{\varphi_{n_k}}\|_{H^1(\Omega_A)} \to \|u_\varphi\|_{H^1(\Omega_A)}$, which yields the strong convergence of $u_{\varphi_{n_k}}$ towards u_φ in $H^1(\Omega_A)$ as k tends to $+\infty$. The convergence of the whole sequence u_{φ_n} follows by a classical argument.

For the sake of simplicity, let us from now denote by $(u_{\varphi_n})_{n\in\mathbb{N}}$ the subsequence $(u_{\varphi_{n_k}})_{k\in\mathbb{N}}$. For any $n \in \mathbb{N}$, u_{φ_n} satisfies to the following problem

$$\begin{cases} u_{\varphi_n} \in K_{\varphi_n}, \\ \mathcal{A}_A(u_{\varphi_n}, v_n - u_{\varphi_n}) \geq L(v_n - u_{\varphi_n}), \quad \forall v_n \in K_{\varphi_n}. \end{cases} \tag{53}$$

In order to pass to the limit as $n \to +\infty$ in (53) and obtain that $\tilde{u}$ is solution to (45), one possibility is to construct, for any $v \in K_\varphi$, a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in K_{\varphi_n}$ and $v_n \to v$ in $H^1(\Omega_A)$. This, however, does not seem to be straightforward. We shall deal with this problem by showing that Problem (45) is equivalent to the following (easier to deal with) problem

$$\begin{cases} \tilde{u}_\varphi \in \tilde{K}_\varphi = \{v \in K_\varphi,\ v \leq a \text{ a.e in } \Omega_A\}, \\ \mathcal{A}_A(\tilde{u}_\varphi, v - \tilde{u}_\varphi) \geq L(v - \tilde{u}_\varphi), \quad \forall v \in \tilde{K}_\varphi. \end{cases} \tag{54}$$

That is, if u_φ is the solution to (45) and $\tilde{u}_\varphi$ is the (unique) solution to (54), then $u_\varphi = \tilde{u}_\varphi$. Indeed, let u_φ be solution to (45), since $\tilde{K}_\varphi \subset K_\varphi$, one only needs to prove that $u_\varphi \leq a$ a.e. on Ω_A, in order for $\tilde{u}_\varphi$ to be solution to (54). The proof of the maximum principle consists in taking $v = u_\varphi - (u_\varphi - a)^+ \in K_\varphi$ in (45), which yields $\|\nabla(u_\varphi - a)^+\|_{\mathrm{L}^2(\Omega_A)} = 0$; since $(u_\varphi - a)^+ \in H^1_{0,\Gamma^3_A}(\Omega_A)$, this, in turn, yields $u_\varphi \leq a$ a.e..

Hence Problem (45) is equivalent to Problem (54), and we only need to show that $\tilde{u}$ is the solution to Problem (54).

From the equivalence of the Problems (45) and (54), $u_{\varphi_n} \leq a$ a.e. in Ω_A. Therefore,

$$\int_{\Omega_A} (a - u_{\varphi_n})(x)(a - \tilde{u})^-(x)dx \geq 0,$$

and since $u_{\varphi_n} \to \tilde{u}$ in H^1 for the weak topology, $\|(a - \tilde{u})^-\|_{L^2(\Omega_A)} \leq 0$, therefore, $\tilde{u} \leq a$ a.e in Ω_A.

In order to show that $\mathcal{A}_A(\tilde{u}, v - \tilde{u}) \geq L(v - \tilde{u})$, $\forall v \in \tilde{K}_\varphi$, let us state the following lemma, the proof of which will be addressed further.

Lemma 4.4. *For any $\varphi \in L^2(I)$, let*

$$K_\varphi = \{v \in H^1(\Omega_A),\ v_{|_I} = \overline{T}(\varphi) \text{ a.e},\ v_{|_{\Gamma^3_A}} \leq a \text{ a.e}\}$$

and $\tilde{K}_\varphi = \{v \in K_\varphi,\ v \leq a$ a.e in $\Omega_A\}$. Let $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(I)$ such that $\varphi_n \to \varphi$ in $L^2(I)$, then for any $v \in \tilde{K}_\varphi$, there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset K_{\varphi_n}$ such that $v_n \to v$ in $H^1(\Omega_A)$ as $n \to +\infty$.

Let $v \in \tilde{K}_\varphi$, thanks to Lemma 4.4, there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset K_{\varphi_n}$ such that $v_n \to v$ in $H^1(\Omega_A)$; since L is continuous, $L(v_n - u_{\varphi_n}) \to L(v - \tilde{u})$ as $n \to +\infty$, similarly, it is easily seen that $\mathcal{A}_A(u_{\varphi_n}, v_n) \to \mathcal{A}_A(\tilde{u}, v)$; passing to the upper limit on n in (53), one obtains

$$\mathcal{A}_A(\tilde{u}, v) \geq L(v - \tilde{u}) + \limsup_{n\to+\infty} \mathcal{A}_A(u_{\varphi_n}, u_{\varphi_n}), \tag{55}$$

which holds for all $v \in \tilde{K}_\varphi$; taking $v = \tilde{u}$ in (55)

$$\mathcal{A}_A(\tilde{u}, \tilde{u}) \geq \limsup_{n\to+\infty} \mathcal{A}_A(u_{\varphi_n}, u_{\varphi_n}). \tag{56}$$

Since

$$\mathcal{A}_A(u, u) = \int_{\Omega_A} |\nabla u|^2(x)dx,$$

it is easily seen that

$$\mathcal{A}_A(\tilde{u}, \tilde{u}) \leq \liminf_{n\to+\infty} \mathcal{A}_A(u_{\varphi_n}, u_{\varphi_n}). \tag{57}$$

Thus, from (56) and (57), it follows that $\mathcal{A}_A(u_{\varphi_n}, u_{\varphi_n}) \to \mathcal{A}_A(\tilde{u}, \tilde{u})$ as $k \to +\infty$. Hence $\tilde{u}$ is solution to Problem (54) and finally $\tilde{u} = u_\varphi$, which concludes the proof of Proposition 4.2 $\square$

The continuity of the operator F is stated in the following proposition:

Proposition 4.3. *Let $(\varphi_n)_{n\in\mathbb{N}} \subset L^2(I)$ such that $\varphi_n \to \varphi$ in $L^2(I)$. For $n \in \mathbb{N}$, let $\overline{\Phi}^n$ be the unique solution to the following problem*

$$\begin{cases} \overline{\Phi}^n \in L^2(\Omega),\ \overline{\Phi}_i^n \in H^1_{0,2}(\Omega_i)\ (i = A, B), \\ \mathcal{A}_i(\overline{\Phi}_i^n, \Psi) = L_{u_{\varphi_n},i}(\Psi), \quad \forall \Psi \in H^1_{0,2}(\Omega_i) \quad (i = A, B), \end{cases} \tag{58}$$

where u_{φ_n} is the solution to Problem (51). Then, $\overline{\Phi}_i^n \to \overline{\Phi}_i$ in $H^1(\Omega_i)$ for the strong topology, where $\overline{\Phi}_i$ is the solution to

$$\begin{cases} \overline{\Phi} \in L^2(\Omega),\ \overline{\Phi}_i \in H^1_{0,2}(\Omega_i)\ (i = A, B), \\ \mathcal{A}_i(\overline{\Phi}_i, \Psi) = L_{u_\varphi,i}(\Psi), \quad \forall \Psi \in H^1_{0,2}(\Omega_i) \quad (i = A, B); \end{cases}$$

and therefore, $(\overline{\Phi}^n_{A|_I} - \overline{\Phi}^n_{B|_I}, \overline{\Phi}^n) \to (\overline{\Phi}_{A|_I} - \overline{\Phi}_{B|_I}, \overline{\Phi})$ in E $(= L^2(I) \times L^2(\Omega))$.

Proof. Let $\varphi_n \in L^2(I)$, u_{φ_n} be the solution to Problem (51), $L_{u_{\varphi_n},i}$ be defined by (19) and $\overline{\Phi}^n$ be the solution to Problem (58). By continuity, if

$L_{u_{\varphi_n},i} \to L_{u_\varphi,i}$ in $(H^1(\Omega_i))^\star$ as $n \to +\infty$, then $\overline{\Phi}_i^n \to \overline{\Phi}_i$ in $H^1(\Omega_i)$ as $n \to +\infty$. Let us show that $L_{u_{\varphi_n},i} \to L_{u_\varphi,i}$ in $(H^1(\Omega_i))^\star$ as $n \to +\infty$; by definition of $L_{u_\varphi,i}$, this is satisfied if $\nabla u_{\varphi_n} \cdot \mathbf{n} \to \nabla u_\varphi \cdot \mathbf{n}$ in $L^2(I)$. Using Propositions 3.1 and 3.2, taking $\psi = \overline{T}(\varphi_n)$ and thanks to Assumption (2), there exists $C \in \mathbb{R}_+^\star$, independent of n, such that

$$\|u_{\varphi_n}\|_{H^2(\Omega_{A/2})} \le C \quad \forall n \in \mathbb{N},$$

where $\Omega_{A/2} =]0,1[\times]0,\frac{1}{2}[$. Hence there exists a subsequence, still denoted by $(u_{\varphi_n})_{n\in\mathbb{N}} \subset H^2(\Omega_{A/2})$ and $q \in (H^1(\Omega_{A/2}))^2$ such that $\nabla u_{\varphi_n} \to \mathbf{q}$ in $(H^1(\Omega_{A/2}))^2$ for the weak topology. By Proposition 4.2, $\nabla u_{\varphi_n} \to \nabla u_\varphi$ in $(L^2(\Omega_{A/2}))^2$, and therefore $\nabla u_{\varphi_n} \to \nabla u_\varphi$ in $(H^1(\Omega_{A/2}))^2$ for the weak topology, since the trace operator is compact from $H^1(\Omega_{A/2})$ to $L^2(I)$, one has $\nabla u_{\varphi_n} \cdot \mathbf{n} \to \nabla u_\varphi \cdot \mathbf{n}$ in $L^2(I)$, and by a classical argument, the whole sequence converges.

Proof of Lemma 4.4. For $\varphi \in L^2(I)$ define $u^0_\varphi(x_1,x_2) = \overline{T}(\varphi)(x_1)$ $\forall (x_1,x_2) \in \Omega_A$. Let $(\varphi_n)_{n\in\mathbb{N}}$ and $\varphi \in L^2(I)$ such that $\varphi_n \to \varphi$ in $L^2(I)$, and let $v \in \tilde{K}_\varphi$. For $n \in \mathbb{N}$, let $v_n = \min(v - u^0_\varphi + u^0_{\varphi_n}, a)$. Let us show that $v_n \in K_{\varphi_n}$. Clearly $v_n \in H^1(\Omega_A)$ and $v_{n|\Gamma_A^3} \le a$ a.e.; since $v_{n|_I} = \min(u^0_{\varphi_n|_I}, a)$, and $u^0_{\varphi_n|_I} \le a$ a.e. from the property (1) of the function $\overline{T}$, hence $v_{n|_I} = u^0_{\varphi_n|_I} = \overline{T}(\varphi_n)$ a.e..

Let us now show that $v_n \to v$ in $H^1(\Omega_A)$ as $n \to +\infty$; we shall use the following Lemma, the proof of which is given below.

Lemma 4.5. *Let Ω be an open bounded regular domain of $\mathbb{R}^N$, w a function of $H^1(\Omega)$ and $(u_n)_{n\in\mathbb{N}}$ a sequence of $H^1(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$, then* $\min(u_n,w) \to \min(u,w)$ *in $H^1(\Omega)$, as $n \to +\infty$.*

Since, thanks to property (3) of $\overline{T}$, $u^0_{\varphi_n} \to u^0_\varphi$ in $H^1(\Omega_A)$, applying this result with $u_n = v - u^0_\varphi + u^0_{\varphi_n}$, $u = v$ and $w = a$, one deduces that $v_n \to v$ in $H^1(\Omega_A)$, as $n \to +\infty$.

Proof of Lemma 4.5. Let $(u_n)_{n\in\mathbb{N}} \subset H^1(\Omega)$ such that $u_n \to u$ in $H^1(\Omega)$, and $w \in H^1(\Omega)$. First since $\mid \min(u_n,w) - \min(u,w) \mid \le \mid u_n - u \mid$ and $u_n \to u$ in $H^1(\Omega)$, one immediately obtains that $\min(u_n,w) \to \min(u,w)$ in $L^2(\Omega)$. Let us then show that $\min(u_n,w) \to \min(u,w)$ in $H^1(\Omega)$ for the weak topology. We note that

$$\|\nabla(\min(u_n,w))\|^2_{L^2(\Omega)} \le \|\nabla(u_n)\|^2_{L^2(\Omega)} + \|\nabla(w)\|^2_{L^2(\Omega)}.$$

since $u_n \to u$ in $H^1(\Omega)$, there exists $C \in \mathbb{R}_+^\star$ such that $\|\nabla(u_n)\|_{L^2(\Omega)} \leq C$, hence, there exists a subsequence of $(\min(u_n, w))_{n\in\mathbb{N}}$ and g in $H^1(\Omega)$ such that $\min(u_{n_k}, w) \to g$ in $H^1(\Omega)$ for the weak topology, as $k \to +\infty$. By uniqueness of the limit, $\min(u_{n_k}, w) \to \min(u, w)$ in $H^1(\Omega)$ for the weak topology.

Let us now show that

$$\nabla(\min(u_{n_k}, w)) \to \nabla(\min(u, w))$$

in $L^2(\Omega)$ for the strong topology. Assume that there exists a subsequence of $(\min(u_{n_k}, w))_{k\in\mathbb{N}}$, which for the sake of simplicity, will still be denoted by $(\min(u_{n_k}, w))_{k\in\mathbb{N}}$, which is such that $\nabla(\min(u_{n_k}, w))$ does not converge to $\nabla(\min(u, w))$ in $L^2(\Omega)$. Therefore, since $\min(u_n, w) \to \min(u, w)$ in $H^1(\Omega)$ for the weak topology, there exists $\varepsilon > 0$ such that

$$\liminf_{k\to+\infty} \|\nabla(\min(u_{n_k}, w))\|^2_{L^2(\Omega)} = \|\nabla(\min(u, w))\|^2_{L^2(\Omega)} + \varepsilon.$$

Since

$$\mid \nabla(\max(u_{\varphi_{n_k}}, w)) \mid^2 + \mid \nabla(\min(u_{\varphi_{n_k}}, w)) \mid^2 = \mid \nabla u_{\varphi_{n_k}} \mid^2 + \mid \nabla w \mid^2,$$

passing to the limit in k yields

$$\begin{aligned}
&\liminf_{k\to+\infty} \int_\Omega \mid \nabla u_{n_k} \mid^2 d\lambda + \int_\Omega \mid \nabla w \mid^2 d\lambda \\
&= \liminf_{k\to+\infty} \int_\Omega \mid \nabla(\min(u_{n_k}, w)) \mid^2 d\lambda + \liminf_{k\to+\infty} \int_\Omega \mid \nabla(\max(u_{n_k}, w)) \mid^2 d\lambda,
\end{aligned}$$

where λ denotes the $2D$ Lebesgue measure.

From the assumption $\|\nabla(\min(u_{n_k}, w))\|^2_{L^2(\Omega)} \to \|\nabla(\min(u, w))\|^2_{L^2(\Omega)} + \varepsilon$, we obtain

$$\begin{aligned}
&\liminf_{k\to+\infty} \int_\Omega \mid \nabla u_{n_k} \mid^2 d\lambda + \int_\Omega \mid \nabla w \mid^2 d\lambda \\
&= \varepsilon + \int_\Omega \mid \nabla(\min(u, w)) \mid^2 d\lambda + \liminf_{k\to+\infty} \int_\Omega \mid \nabla(\max(u_{n_k}, w)) \mid^2 d\lambda.
\end{aligned}$$

Since $\min(u_{n_k}, w) \to \min(u, w)$ in $H^1(\Omega)$ for the weak topology, (and similarly $\max(u_{n_k}, w) \to \max(u, w)$ in $H^1(\Omega)$ for the weak topology), and since

$u_n \to u$ in $H^1(\Omega)$, one has

$$\int_\Omega \mid \nabla u \mid^2 d\lambda + \int_\Omega \mid \nabla w \mid^2 d\lambda$$
$$\geq \varepsilon + \int_\Omega \mid \nabla(\min(u,w)) \mid^2 d\lambda + \int_\Omega \mid \nabla(\max(u,w)) \mid^2 d\lambda,$$

and therefore,

$$\int_\Omega \mid \nabla u \mid^2 d\lambda + \int_\Omega \mid \nabla w \mid^2 d\lambda \geq \varepsilon + \int_\Omega \mid \nabla(u) \mid^2 d\lambda + \int_\Omega \mid \nabla(w) \mid^2 d\lambda,$$

which is in contradiction with $\varepsilon > 0$. Hence, Lemma 4.5 is proven.

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