# An error estimate for a finite volume scheme for a diffusion convection problem on a triangular mesh

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**Abstract** We study here a finite volume scheme for a diffusion-convection equation on an open bounded set $\Omega$ of $\mathbb{R}^2$, using a triangular mesh for the discretization of $\Omega$. The 4-point numerical scheme is presented along with the geometrical assumptions on the mesh. An error estimate of order $h$ on the discrete $L^2$ norm is obtained, where $h$ denotes the "size" of the mesh. The proof uses an estimate of order $h$ of the consistency error on the fluxes and an estimate of the number of edges of the mesh between one given triangle and the boundary $\Omega$.

## 1 Introduction

Let $\Omega$ be an open bounded polygonal set of $\mathbb{R}^2$, $k, b$ functions from $\Omega$ into $\mathbb{R}_+$ of class $C^1$, such that there exists $\kappa > 0$ such that $k(\mathbf{x}) \geq \kappa$, $\forall \mathbf{x} \in \Omega$, $\mathbf{v}$ a function from $\Omega$ to $\mathbb{R}^2$ of class $C^1$, such that $\mathrm{div}\mathbf{v} \geq 0$, and $f \in C(\Omega), g \in C(\partial\Omega)$; our aim here is the discretization of the following diffusion-convection problem on a triangular mesh:

$$-\mathrm{div}(k(\mathbf{x})\nabla u(\mathbf{x}) + \mathrm{div}(u(\mathbf{x})\mathbf{v}(\mathbf{x})) + b(\mathbf{x})u(\mathbf{x}) - f(\mathbf{x}) = 0,\ \mathbf{x} \in \Omega, \tag{1}$$

$$u(\mathbf{x}) = g(\mathbf{x}),\ \mathbf{x} \in \partial\Omega, \tag{2}$$

Let $\mathcal{T}$ be a triangular mesh of $\Omega$, satisfying the following assumptions :

$$\text{For any angle } \theta \text{ of a triangle of } \mathcal{T}, \text{ one has: } 0 < \theta < \frac{\pi}{2}. \tag{3}$$

There exist $\alpha_1 > 0, \alpha_2 > 0$ and $h > 0$ such that $\forall T \in \mathcal{T}$, and for any edge $a$ of the mesh,

$$\alpha_1 h^2 \leq S(T) \leq \alpha_2 h^2, \text{ and } \alpha_1 h \leq \ell(a) \leq \alpha_2 h, \tag{4}$$

where $S(T)$ denotes the area of triangle $T$ and $\ell(a)$ denotes the length of edge $a$.

Using this triangular mesh, a four point finite volume scheme will be defined in section 2 for the numerical solution of problem (1)-(2), for which an error estimate of order $h$ will be proven in the next sections. Note that other conditions than the Dirichlet conditions (2) which are considered here can be treated with this finite volume scheme (e.g. mixed type boundary conditions), yielding the same error estimate.

Finite volume methods have ben extensively used for industrial problems, in the case of hyperbolic equations ([4], [23],[16]), elliptic equations ([9], [11]) or coupled systems of equations ([12], [13], [9], [24], [19]). The advantage of finite volume schemes using triangular meshes is clear for convection diffusion equations: on one hand, the stability and convergence properties of the finite volume scheme (using an upwind choice for the convective flux) ensure a robust scheme for any mesh satisfying the above assumptions, without any need of refinement in the areas of a large convection flux. On the other hand, the use of a triangular mesh allows the computation of a solution for any shape of the physical domain.

Error estimates for finite volume schemes can be found in [14] in the case of a regular rectangular mesh for a 2D diffusion equation. The technique used in [14] is based on a special decomposition of the discrete operator and does not seem to extend to other grid structures, except in special cases [9]. Special finite volume methods (with the unknowns located at the vertices) defined on triangles may also be viewed as a finite element method and then the error estimate follows from the finite element framework [2]. In the same idea, recent work [1] (which was in fact mentionned to the author after completion of this work), compares the finite volume which is considered here, with a mixed finite element formulation, using a numerical integration technique; the authors then use the framework of the mixed finite element method to obtain an error estimate in the case of a pure diffusion operator. The originality of the present work lies in the fact that a convection-diffusion operator is considered and that the proof of the error estimate is obtained directly, without any reference to finite element techniques. Let us also mention that error estimates for finite volume schemes on triangular meshes have been obtained by several authors for hyperbolic equations [18], [8], [5], [25].

Numerical experiments using this method have been performed for a diffusion operator and show a good numerical convergence rate compared to the piecewise linear finite element method on triangles. These may be found in [20] where a comparison with other finite volume type methods and for different equations (convection-diffusion, hyperbolic) is also performed.

# 2 The four point finite volume scheme

The principle of the finite volume scheme (see e.g. [15] [8]) is to integrate the conservation equation (1) on a control volume $T$, which yields:

$$-\int_{\partial T} k(\mathbf{x})\nabla u(\mathbf{x}).\mathbf{n}(\mathbf{x})d\sigma(\mathbf{x}) + \int_{\partial T} u(\mathbf{x})\mathbf{v}(\mathbf{x}).\mathbf{n}(\mathbf{x})d\sigma(\mathbf{x}) +$$

$$\int_T b(\mathbf{x})u(\mathbf{x})d\mathbf{x} = \int_T f(\mathbf{x})d\mathbf{x}, \forall T \in \mathcal{T}, \tag{5}$$

where $\mathbf{n}(\mathbf{x})$ is the outward normal unit vector to the boundary $\partial T$, and $\sigma$ is the usual one-dimensional measure on $\partial T$; next the fluxes need to be approximated on the boundary $\partial T$ of the control volume. Hence we shall approximate the integral of $-k\nabla u.\mathbf{n} + u\mathbf{v}.\mathbf{n}$ over each edge of the mesh. Before we define the numerical scheme, some notations need to be introduced. For $T \in \mathcal{T}$, denote by :

- $S(T)$ the area of triangle $T$,
- $\mathbf{x}_T$ the intersection of the orthogonal bisectors of the three sides of triangle $T$,
- $c_i(T), i = 1, 2, 3$ the three sides of triangle $T$,
- $d(x_T, c_i(T))$ the distance between $\mathbf{x}_T$ and the side $c_i(T)$, for $i = 1, 2, 3$,
- $f_T = \frac{1}{S(T)} \int_T f(\mathbf{x})d\mathbf{x}$ and $b_T = \frac{1}{S(T)} \int_T b(\mathbf{x})d\mathbf{x}$,

and define one discrete unknown per triangle, which will be denoted $u_T$. Let $\mathcal{A}$ be the set of the edges of the mesh; $\mathcal{A} = \mathcal{A}_{int} \cup \mathcal{A}_{ext}$, where $\mathcal{A}_{ext} = \mathcal{A} \cap \partial\Omega$ and $\mathcal{A}_{int} = \mathcal{A} \setminus \partial\Omega$; for $a \in \mathcal{A}$, we define:

- $\{u\}_a$ an approximation of $u$ on the edge $a$ ;
- $\mathbf{x}_a$, the center point of edge $a$.
- $\mathbf{n}_a$, a unit normal vector to the edge $a$ such that $\int_a \mathbf{v}(\mathbf{x}).\mathbf{n}_a d\sigma(\mathbf{x}) \geq 0$;
- $v_a = \frac{1}{\ell(a)} \int_a \mathbf{v}(\mathbf{x}).\mathbf{n}_a d\sigma(\mathbf{x})$;
- if $a \in \mathcal{A}_{int}$, define:
  * $T_a^+$ and $T_a^-$ the two triangles for which $a$ is an edge; $T_a^+$ (resp. $T_a^-$) is the upstream (resp. downstream) with respect to the convection velocity $\mathbf{v}$, i.e.: such that $\mathbf{n}_a$ is exterior (resp. interior) to $T_a^+$ (resp. $T_a^-$).
  * $d_a = d(\mathbf{x}_{T_a^+}, a) + d(\mathbf{x}_{T_a^-}, a)$
  * the "exchange term" at interface $a$, $E(T_a^+, T_a^-)$, by:

$$E(T_a^+, T_a^-) = (-k_a \frac{u_{T_a^-} - u_{T_a^+}}{d_a} + v_a\{u\}_a)\ell(a), \tag{6}$$

  and $E(T_a^-, T_a^+) = -E(T_a^+, T_a^-)$, where $k_a$ is the value of $k$ at the center of interface $a$ and $\{u\}_a = u_{T_a^+}$ (upstream choice).

- If $a \in \mathcal{A}_{ext}$, let
  - $*$ $g_a$ be the value of $g$ at the center of interface $a$,
  - $*$ $T_a$ the triangle for which $a$ is an edge. We then set $\{u\}_a$ as the upstream choice: $\{u\}_a = u_{T_a^+}$ if $T_a = T_a^+$, that is if $\mathbf{n}_a$ is exterior to $T_a^+$, and $\{u\}_a = g_a$ otherwise.

Let $T \in \mathcal{T}$ and $i \in \{1,2,3\}$, we define the following terms:

- If $c_i(T) \not\subset \partial\Omega$,

$$F_{T,i} = \begin{cases} \frac{E(T^+_{c_i(T)}, T^-_{c_i(T)})}{\ell(c_i(T))} \text{ if } T = T^+_{c_i(T)} \\ \frac{E(T^-_{c_i(T)}, T^+_{c_i(T)})}{\ell(c_i(T))} \text{ if } T = T^-_{c_i(T)} \end{cases} \tag{7}$$

- If $c_i(T) \subset \partial\Omega$, then

$$F_{T,i} = \begin{cases} -k_{c_i(T)} \frac{g_{c_i(T)} - u_T}{d(x_T, c_i(T))} + v_{c_i(T)} u_{T^+_{c_i(T)}} \text{ if } T = T^+_{c_i(T)}, \\ -k_{c_i(T)} \frac{g_{c_i(T)} - u_T}{d(x_T, c_i(T))} - v_{c_i(T)} g_{c_i(T)} \text{ if } T = T^-_{c_i(T)}, \end{cases} \tag{8}$$

The numerical scheme for the discretization of equations (1)-(2) is then:

$$\sum_{i=1}^{3} F_{T,i}\ell(c_i(T)) + S(T) b_T u_T = S(T) f_T, \forall T \in \mathcal{T}. \tag{9}$$

Relations (6) (7), (8) and (9) uniquely define the discrete unknowns $u_T, T \in \mathcal{T}$ (see remark 3). The numerical scheme thus defined is convergent, and the following error estimate will be proven in section 3.

**Theorem 1** *Let $(u_T)_{T\in\mathcal{T}}$ be defined by the numerical scheme (6)-(9), $\overline{u}_T = u(\mathbf{x}_T)$ for $T \in \mathcal{T}$, where $u$ denotes the exact solution to problem (1)-(2) and $\mathbf{x}_T$ the intersection of the perpendicular bisectors of triangle $T$; define the error by: $e_T = u_T - \overline{u}_T$. There exists $C \geq 0$ depending only on $u$, $\kappa$, $\alpha_1$ and $\alpha_2$ such that $\left(\sum_{T\in\mathcal{T}} h^2 |e_T|^2\right)^{\frac{1}{2}} \leq Ch$.*

**Remark 1** *This estimate shows that the $L^2$ norm of the error is of order $h$, since the area of each triangle is of order $h^2$.*

Furthermore, the numerical scheme satisfies a discrete maximum principle, that is:

**Proposition 1** *if $f_T \geq 0$ $\forall T \in \mathcal{T}$, and $g_a \geq 0$, $\forall a \in \mathcal{A}_{ext}$, then the solution $(u_T)_{T\in\mathcal{T}}$ of (6)-(9) satisfies $u_T \geq 0$.*

# 3 Proof of the results

## 3.1 Maximum principle

Let us start with the proof of Proposition 1. Assume that $f_T \geq 0$ $\forall T \in \mathcal{T}$, and $g_a \geq 0$, $\forall a \in \mathcal{A}_{ext}$. Let $T_0$ be a triangle such that $u_{T_0} = \min\{u_T, T \in \mathcal{T}\}$. Assume first that $T_0$ is an interior triangle and that $u_{T_0} \leq 0$. Then, from (9),

$$\sum_{i=1}^{3} F_{T_0,i}\ell(c_i(T_0)) \geq 0; \tag{10}$$

since for any neighbour $T_i, i = 1,2,3$ of $T_0$, one has $u_{T_i} \geq u_{T_0}$, then, noting that $\text{div}\mathbf{v} \geq 0$, one must have: $u_{T_i} = u_{T_0}$ for any neighbour $T_i, i = 1,2,3$. Hence $u_T = u_{T_0}$ for all $T \in \mathcal{T}$. Therefore, the minimum is attained on a triangle neighbouring the boundary.

Assume then that $T_0$ is a triangle neighbouring the boundary, and that $u_{T_0} < 0$. Then, for the edge $a$ of $T$ which is part the boundary, relation (8) $g_a < 0$, which is in contradiction with the assumption. Hence Proposition 1 is proven.

**Remark 2** *This result immediately yields the existence and uniqueness of the solution of the numerical scheme (6)-(9).*

Before the proof of the error estimate is given, a few geometrical results are needed in order to "count the edges and the triangles".

## 3.2 Some geometrical lemmas

Technical lemmas are given here, which will be needed in the sequel.

**Lemma 1** *For any triangle $T \in \mathcal{T}$, one has $\mathbf{x}_T \in T \setminus \partial T$.*

The proof of this lemma is easy by contradiction, using the geometrical assumptions on the mesh.

The number of edges of the mesh $\mathcal{T}$ is finite, and therefore there exists a direction $\mathcal{D}$ which may be assumed, without loss of generality, to be parallel to the $x$ axis, such that $\mathcal{D}$ is parallel to none of the edges of the mesh.

The following result gives an estimate of the number of edges of the mesh which lay "in the shadow" of one given edge $a$ along the direction $\mathcal{D}$.

**Lemma 2** *For $a \in \mathcal{A}$, let $\mathbf{x}_a = (x_a, y_a)$ be the center of the edge $a$ and $\mathcal{D}_a^- = \{(x, y_a); x \leq x_a\}$ and $\mathcal{D}_a^+ = \{(x, y_a); x \geq x_a\}$. Let $\mathcal{A}_a^-$ (resp. $\mathcal{A}_a^+$) be the set of edges which intersect $\mathcal{D}_a^-$ (resp. $\mathcal{D}_a^+$): $\mathcal{A}_a^- = \{b \in \mathcal{A}; b \cap \mathcal{D}_a^- \neq \emptyset\}$ (resp. $\mathcal{A}_a^+ = \{b \in \mathcal{A}; b \cap \mathcal{D}_a^+ \neq \emptyset\}$. Define $\mathcal{S}_a^- = \{T \in \mathcal{T}; \exists i \in \{1,2,3\}; a \in \mathcal{A}_{c_i(T)}^-$ (resp. $\mathcal{S}_a^+ = \{T \in \mathcal{T}; \exists i \in \{1,2,3\}; a \in \mathcal{A}_{c_i(T)}^+$. There exists $C > 0$ such that* *for any $a \in \mathcal{A}$,* $\mathrm{card}\mathcal{A}_a^- \leq \frac{C}{h}$ *(resp.* $\mathrm{card}\mathcal{A}_a^+ \leq \frac{C}{h}$*) and* $\mathrm{card}\mathcal{S}_a^- \leq \frac{C}{h}$ *(resp.* $\mathrm{card}\mathcal{S}_a^+ \leq \frac{C}{h}$*)*

Proof Let $a \in \mathcal{A}$, then there exists $\eta > 0$ depending on $\alpha_1$ and $\alpha_2$ such that, for any $b \in \mathcal{A}_a$, and any triangle $T$ for which $b$ is an edge, $T \subset \mathcal{R}_a^m = \{(x,y); x \leq x_a, y \in [y_a - \eta h, y_a + \eta h]\}$. It is easily checked that $\sum_{T \in \mathcal{R}_a^m} \mathrm{meas}(T) \leq \mathrm{meas}(\mathcal{R}_a^m \cap \Omega) \leq d_\Omega \eta h$, where $\mathrm{meas}(T)$ denotes the 2D Lebesgue measure of $T$, and $d_\Omega$ denote the diameter of $\Omega$. Therefore, with assumption $(H2)$, card $\mathcal{R}_a^m \leq \frac{d_\Omega \eta}{\alpha_1 h}$ and therefore card $\mathcal{S}_a^- \leq \frac{d_\Omega \eta}{\alpha_1 h}$ and card $\mathcal{A}_a^- \leq \frac{3 d_\Omega \eta}{\alpha_1 h}$. The same technique clearly yields card $\mathcal{S}_a^+ \leq \frac{d_\Omega \eta}{\alpha_1 h}$ and card $\mathcal{A}_a^+ \leq \frac{3 d_\Omega \eta}{\alpha_1 h}$.

$\square$

### 3.3 Error estimate

Let us first define the "consistency error on the fluxes" (see e.g. [10], [15], [8]). Denote by:

$$\overline{F}_a = \frac{1}{\ell(a)} \int_a \Big( - k(\mathbf{x}) \nabla u(\mathbf{x}).\mathbf{n}_a + \mathbf{v}(\mathbf{x}).\mathbf{n}_a u(\mathbf{x}) \Big) d\sigma(\mathbf{x}) \tag{11}$$

the "exact" flux on edge $a$, and

$$F_a = -k(\mathbf{x}_a) \frac{u_{T_a^-} - u_{T_a^+}}{d_a} + v_a u_{T_a^+} \tag{12}$$

the "approximate flux" on edge $a$. From the definition of $\overline{F}_a$, since $u \in C^1(\Omega)$, a first order Taylor expansion yields that:

$$\sum_{i=1,2,3} \overline{F}_{T,i} \ell(c_i(T)) + S(T) b_T u_T = S(T) f_T + S(T) b_T \varepsilon_T, \forall T \in \mathcal{T} \tag{13}$$

where $\varepsilon_T \leq c_u h, \forall T \in \mathcal{T}$, and $c_u \geq 0$ depends only on $u$ and $\alpha_2$. The consistency error over the flux is denoted by $R_a$ and defined by:

$$R_a = \overline{F}_a + k(\mathbf{x}_a) \frac{\overline{u}_{T_a^-} - \overline{u}_{T_a^+}}{d_a} - v_a \overline{u}_{T_a^+}. \tag{14}$$

The following result holds:

**Lemma 3** *There exists $C \geq 0$ depending on $u$, $\alpha_1, \alpha_2$, $\kappa$ and $\mathbf{v}$, such that $|R_a| \leq Ch$.*

PROOF Using a first order Taylor expansion, it is easily shown that there exists $c_1 \geq 0$ depending only on $\alpha_1, \alpha_2$ the second order derivative of $u$ and the first order derivatives of $k, \mathbf{v}$, such that,for any $a \in \mathcal{A}$:

$$\begin{aligned}|(-k(\mathbf{x})\nabla u(\mathbf{x}).\mathbf{n}_a + \mathbf{v}(\mathbf{x}).\mathbf{n}_a u(\mathbf{x}) - (-k(\mathbf{x}_a)\nabla u(\mathbf{x}_a).\mathbf{n}_a \quad &+ \\ \mathbf{v}(\mathbf{x}_a).\mathbf{n}_a u(\mathbf{x}_a)| \quad &\leq \quad c_1 h, \forall \mathbf{x} \in a (15)\end{aligned}$$

Similarly, there exists $c_2 \geq 0$ and $c_3 \geq 0$ depending only on $\alpha_1, \alpha_2$ such that :

$$| - k(\mathbf{x}_a)\nabla u(\mathbf{x}_a).\mathbf{n}_a - k(\mathbf{x}_a)\frac{\overline{u}_{T_a^-} - \overline{u}_{T_a^+}}{d_a}| \leq c_2 h, \tag{16}$$

and

$$|\mathbf{v}(\mathbf{x}_a).\mathbf{n}u(\mathbf{x}_a) - v_a \overline{u}_{T_a^+}| \leq c_3 h. \tag{17}$$

Hence the result.

□

Let $e_T = u_T - \overline{u}_T$, for $T \in \mathcal{T}$, then the following lemma holds:

**Lemma 4** *There exists $C > 0$ such that*

$$\sum_{a \in \mathcal{A}_{int}} (e_{T_a^+} - e_{T_a^-})^2 + \sum_{a \in \mathcal{A}_{ext}} e_{T_a}^2 \leq C h^2, \tag{18}$$

PROOF For $a \in \partial\Omega$, define $\{e\}_a = 0$ and define $G_{T,i} = F_{T,i} - \overline{F}_{T,i}$; substracting (9) from (13) yields:

$$\sum_{i=1,2,3} G_{T,i}\ell(c_i(T)) + S(T) b_T e_T \leq S(T) b_T \varepsilon_T, \forall T \in \mathcal{T}. \tag{19}$$

Multiplying equation (19) by $e_T$, summing over $T \in \mathcal{T}$ yields:

$$\sum_{T \in \mathcal{T}} \sum_{i=1,2,3} G_{T,i}\ell(c_i(T)) e_T + \sum_{T \in \mathcal{T}} S(T) b_T e_T^2 \leq c_u h \sum_{T \in \mathcal{T}} S(T) b_T e_T. \tag{20}$$

and therefore, using Young's inequality:

$$\sum_{T \in \mathcal{T}} \sum_{i=1,2,3} G_{T,i}\ell(c_i(T)) e_T \leq K h^2. \tag{21}$$

where $K = \frac{c_u^2}{2} \sup_{\mathbf{x} \in \Omega} |b(\mathbf{x})| \text{meas}(\Omega)$.

For $a \in \mathcal{A}_{ext}$, set $e_{T_a^+} = 0$ and $e_{T_a^-} = e_{T_a}$, (resp. $e_{T_a^-} = 0$ and $e_{T_a^+} = e_{T_a}$) if $\mathbf{n}_a$ is inward (resp. outward) to $\Omega$. Replacing the sum over the triangles in (21) by a sum over the edges and using the definition (14) of $R_a$ yields:

$$\sum_{a\in\mathcal{A}}\Big(k_a\frac{(e_{T_a^+}-e_{T_a^-})^2}{d_a} \quad + \quad v_a e_{T_a^+}(e_{T_a^+}-e_{T_a^-})\Big)\ell(a) \leq$$

$$- \quad \sum_{a\in\mathcal{A}} R_a(e_{T_a^+}-e_{T_a^-})\ell(a) + Kh^2 \qquad (22)$$

Next, remark that

$$\sum_{a\in\mathcal{A}} v_a e_{T_a^+}(e_{T_a^+}-e_{T_a^-})\ell(a) = \frac{1}{2}\sum_{a\in\mathcal{A}} v_a\Big((e_{T_a^+}-e_{T_a^-})^2+(e_{T_a^+}^2-e_{T_a^-}^2)\Big)\ell(a) \quad (23)$$

and, thanks to the assumption div$\mathbf{v}\geq 0$,

$$\sum_{a\in\mathcal{A}} v_a(e_{T_a^+}^2-e_{T_a^-}^2)\ell(a) = \sum_{T\in\mathcal{T}}\int_{\partial T}\mathbf{v}(\mathbf{x}).\mathbf{n}d\sigma(\mathbf{x})\, e_T^2 \geq 0. \qquad (24)$$

Hence:

$$\sum_{a\in\mathcal{A}} k_a\frac{(e_{T_a^+}-e_{T_a^-})^2}{d_a}\ell(a) \leq -\sum_{a\in\mathcal{A}} R_a(e_{T_a^+}-e_{T_a^-})\ell(a) + Kh^2 \qquad (25)$$

Thanks to the assumptions $k(\mathbf{x}) \geq \kappa\ \forall\mathbf{x}\in\Omega$, and $\alpha_1 h \leq \ell(a) \leq \alpha_2 h$, remarking that $\frac{1}{d_a}\geq\frac{1}{2\alpha_2 h}$ and using lemma 3, (25) yields:

$$\kappa\frac{\alpha_1}{2\alpha_2}\sum_{a\in\mathcal{A}}(e_{T_a^+}-e_{T_a^-})^2 \leq Ch^2\sum_{a\in\mathcal{A}}|e_{T_a^+}-e_{T_a^-}| + Kh^2, \qquad (26)$$

where $C\geq 0$ depends only on $u$, $\alpha_1$, $\alpha_2$, $k$ and $\mathbf{v}$. Note that:

$$|e_{T_a^+}-e_{T_a^-}| \leq \kappa\frac{\alpha_1}{4\alpha_2}\frac{1}{Ch^2}\sum_{a\in\mathcal{A}}(e_{T_a^+}-e_{T_a^-})^2 + \frac{4\alpha_2}{\kappa\alpha_1}Ch^2, \qquad (27)$$

Next remark that there exists $\gamma\geq 0$ depending only on $\alpha_1$, $a_2$ and meas$(\Omega)$ such that card$\mathcal{A}\leq\frac{\gamma}{h^2}$. Hence, summing (27) over $a\in\mathcal{A}$ and using (26):

$$\sum_{a\in\mathcal{A}}(e_{T_a^+}-e_{T_a^-})^2 \leq \frac{4\alpha_2}{\kappa\alpha_1}(K+\frac{4\alpha_2}{\kappa\alpha_1}C^2\gamma)h^2; \qquad (28)$$

hence the result.

□

In order to complete the proof of Theorem 1, let us define, for any $T\in\mathcal{T}$, $\mathcal{D}_{\mathbf{x}_T}=\{(x,y_T)\in\Omega; x\leq x_T\}$ (recall that $\mathbf{x}_T=(x_T,y_T)$ is the intersection of the perpendicular bisectors of the three edges of triangle $T$). Let $\mathcal{A}_T=$

$\{a \in \mathcal{A}_{int}; a \cap \mathcal{D}_{\mathbf{x}_T} \neq \emptyset\}$, and $a_0$ be the edge of the boundary containing the intersection of $\mathcal{D}_{\mathbf{x}_T}$ with $\partial\Omega$. Then:

$$|e_T| \leq \sum_{a \in \mathcal{A}_T} |e_{T_a^+} - e_{T_a^-}| + |e_{T_{a_0}}| \tag{29}$$

Therefore, by Cauchy-Schwarz' inequality,

$$|e_T|^2 \leq \Big( \sum_{a \in \mathcal{A}_T} |e_{T_a^+} - e_{T_a^-}|^2 + |e_{T_{a_0}}|^2 \Big) (1 + \mathrm{card}\mathcal{A}_T) \tag{30}$$

By lemma 2, card $A_T \leq \frac{C}{h}$. Hence, summing over $T \in \mathcal{T}$:

$$\sum_{T \in \mathcal{T}} |e_T|^2 \leq \frac{C}{h} \sum_{T \in \mathcal{T}} \sum_{a \in \mathcal{A}_T} |e_{T_a^+} - e_{T_a^-}|^2 + |e_{T_{a_0}}|^2. \tag{31}$$

Swapping the summations over $T$ and $a$ yields:

$$\sum_{T \in \mathcal{T}} |e_T|^2 \leq \frac{C}{h} \Big( \sum_{a \in \mathcal{A}_{int}} |e_{T_a^+} - e_{T_a^-}|^2 \sum_{T \in \mathcal{S}_a} 1 + \sum_{a \in \mathcal{A}_{ext}} |e_{T_a}|^2 \sum_{T \in \mathcal{S}_a} 1 \Big). \tag{32}$$

Since, by lemma 2, card $\mathcal{S}_a \leq \frac{C}{h}$, one obtains:

$$\sum_{T \in \mathcal{T}} |e_T|^2 \leq \frac{C}{h^2} \Big( \sum_{a \in \mathcal{A}_{int}} |e_{T_a^+} - e_{T_a^-}|^2 + \sum_{a \in \mathcal{A}_{ext}} |e_{T_a}|^2 \Big). \tag{33}$$

By lemma 4, this last inequality completes the proof of Theorem 1.

**Remark 3** *If $f_T = 0$ $\forall T \in \mathcal{T}$ and $g_a = 0$ $\forall a \in \mathcal{A}_{ext}$ the technique of the above proof yields that the numerical scheme (6)-(9) has one unique solution, which is $u_T = 0, \forall T \in \mathcal{T}$. Hence the existence and uniqueness of the solution to (6)-(9) for any data $(f_T)_{T \in \mathcal{T}}$ and $(g_a)_{a \in \mathcal{A}_{ext}}$.*

# References


[1] Baranger J., Maître J.F. and F. Oudin: Condensation de la matrice de masse pour certaines méthodes éléments finis mixtes, liens avec les volumes finis, Congrès National d'Analyse Numérique, Les Karellis, 1994.

[2] Cai Z., Mandel J. and Mc Cormick S. : The finite volume Element Method for Diffusion Equations on General Triangulations, SIAM J. Numer. Anal., 28,2, 392-402, 1991.

[3] Champier S., Gallouët T. , Herbin R., Convergence of an Upstream finite volume Scheme on a Triangular Mesh for a Nonlinear Hyperbolic Equation, Numer. Math. 66, 139-157 (1993).

[4] Chevrier P. (1990): Simulation Numérique de l'Interaction Arc Electrique - Ecoulement Gazeux dans les Disjoncteurs Moyenne et Haute Tension, PhD Thesis, INPG, Grenoble, France. See also : Chevrier P., Galley H.: A Van Leer finite volume Scheme for the Euler Equations on Unstructured Meshes, $M^2AN$, 27,2, 183 (1993).

[5] Cockburn B., Coquel F., Le Floch P., An error estimate for high order accurate finite volume methods for scalar conservation laws, to appear in Math. Comput.

[6] Eymard R., Gallouët T.: Convergence d'un Schéma de type Eléments Finis - Volumes Finis pour un Système Couplé Elliptique - Hyperbolique, $M^2AN$, 27, 7, 843 (1993).

[7] Eymard R., Gallouët T. (1991): Traitement des changements de phase dans la modélisation de gisements pétroliers, Journées numériques de Besançon.

[8] Eymard R., Gallouët T. and Herbin R., The finite volume method, in preparation for the "Handbook of Numerical Analysis", Ph. Ciarlet et J.L. Lions eds.

[9] Faille I. A control volume method to solve an elliptic equation on a 2D irregular meshing,Comp. Meth. Appl. Mech. Engrg., 100, 275 (1992) ; See also : Modélisation Bidimensionnelle de la Genèse et la Migration des Hydrocarbures dans un Bassin Sédimentaire, PhD thesis, University of Grenoble, France (1992).

[10] I. Faille, T. Gallouët and R. Herbin, Des mathématiciens découvrent les volumes finis, Matapli, Avril1991.

[11] Fiard J.M., Herbin R., Comparison between finite volume finite element methods for the numerical simulation of an elliptic problem arising in electrochemical engineering, Comput. Meth. Appl. Mech. Engin. 115, 315 (1994).

[12] Forsyth P.A., A control volume finite element method for local mesh refinement, SPE 18415, 85-96 (1989).

[13] Forsyth P.A., A control volume finite element approach to NAPL groundwater contamination, SIAM J. Sci. Stat. Comput., 12, 5, 1029-1057 (1991).

[14] Forsyth P.A., Sammon P.H., Quadratic Convergence for Cell-Centered Grids, Appl. Num. Math. 4, 377-394 (1988).

[15] T. Gallouët, An Introduction to Finite Volume Methods, *Cours CEA/EDF/INRIA, Octobre 1992.*

[16] Fezoui L., Lanteri S., Larrouturou B., Olivier C. : Résolution Numérique des Equations de Navier-Stokes pour un Fluide Compressible en Maillage Triangulaire, INRIA report 1033 (1989).

[17] Gallouët T. Herbin R.: A Uniqueness Result for Measure Valued Solutions of a Nonlinear Hyperbolic Equations, accepted for publication in Journal of Diff. Int. Equations.

[18] Gallouët T. Herbin R.: Error estimate of a finite volume scheme for a nonlinear hyperbolic equation, in preparation.

[19] Labergerie O.: Traitement numérique de la migration des hydrocarbures et de la compaction dans un bassin sédimentaire. Application d'un schéma de type élements finis- volume finis pondérés. Rapport de DESS, Université de Savoie, 1993.

[20] Labergerie O.: Comparaison de diverses méthodes numériques pour la résolution de certains systèmes d'équations aux dérivées partielles. Rapport de DEA, Université de Grenoble 1, 1994.

[21] Manteufel T.A., White A.B. (1986): The numerical solution of second order boundary value problem on non uniform meshes, Math. Comp., 47, 511-536.

[22] Patankar S.V. : Numerical Heat Transfer and Fluid Flow, Series in Computational Methods in Mechanics and Thermal Sciences, Minkowycz and Sparrow Eds., Mc Graw Hill, New York, 1980.

[23] Selmin V. (1993): The node-centred finite volume approach: Bridge between finite differences and finite elements, Comp. Meth. in Appl. Mech. Engin., 102, 107-138.

[24] Sonier F., Eymard R., Mathematical and numerical properties of control-volume finite-element scheme for reservois simulation, paper SPE 25267 presented at the 12th symposium on reservoir simulation, New Orleans, 1993.

[25] Vila J.P., Convergence and error estimate in finite volume schemes for general multidimensional conservation laws, I. explicit monotone schemes, M2AN, 28, 3, 267-285 (1994).