A few tools for convergence analysis of Finite Volume schemes

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Workshop on Discretization Methods

Plan of the talk

1. **S.N. Kruzhkov's compactness lemma for evolution problems**
   - The statement of the lemma ("continuous")
   - Three (?) applications
   - The standard method and its variant
2. **The Kruzhkov lemma adapted to FV schemes**
   - Notation and example
   - The statement ("discrete") and comments
   - Proof (sketched)
3. **A gradient reconstruction formula on the plane**
   - The formula and its connection with FV discretizations
   - Applications: consistency + discrete duality

A COMPACTNESS LEMMA

Moduli of continuity

A function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is called modulus of continuity , if

- ω is continuous , non-decreasing, and $\omega(0) = 0$
- ω is sub-additive, i.e. $\omega(r+s) \le \omega(r) + \omega(s)$

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Remarks:

- A modulus of continuity always has a concave envelope, and therefore (the way we use it) it can always be assumed concave and strictly increasing
- Examples:
  – power-like (or Hölder) moduli of continuity $\omega(r) = const\, r^\alpha,\ \alpha \in (0,1]$;
  – "log-Hölder" moduli of continuity $\omega(r) = \frac{const}{\big|\ln|r|\big|}$ (for r small)

Rq. It is not possible to have a modulus of continuity "better" than $\omega(r) = const\, r$

The statement of the lemma ("continuous")

The following lemma due to Kruzhkov ('69) is a sharp tool for proving strong L^1 compactness of families of solutions of evolution PDEs.

It claims that a family of solutions of evolution equations
which possesses uniform space translates in $L^1(Q)$
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– *they satisfy* $\partial_t u^h = \sum_{|\alpha|\leq m} D^\alpha F^h_\alpha$ *in $\mathcal{D}'(Q)$*

– *u^h can be extended outside Q, and one has*

$$\sup_{|\Delta x|\leq\Delta} \int_0^T\int_{\mathbb{R}^d} |u^h(t,x+\Delta x)-u^h(t,x)|\,dxdt \;\leq\; \omega(\Delta),$$

where $\omega(\cdot)$ is a modulus of continuity that does not depend on h.

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where $\omega(\cdot)$ is a modulus of continuity that does not depend on h.

Then $(u^h)_h$ is relatively compact in $L^1(Q)$.

An L^1_{loc} version follows easily.

Relation with the Aubin-Lions-Simon Lemma

Lemma (Aubin-Lions, Simon, the L^1 case)

Let $E \Subset L^1(\Omega) \subset F$ *be a triple of Banach spaces.*

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The Kruzhkov lemma can be compared to this statement.

Typical case:

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(the smallest "classical" Banach spaces compactly embedded in $L^1(\Omega)$).

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Guess : in its full generality, the Aubin-Lions-Simon lemma would be difficult to recast into the discrete framework .

The origin : scalar conservation law in $\mathbb{R}^N$

The lemma was conceived for the passage to the limit in

$$\partial_t u^\varepsilon + \operatorname{div} f(u^\varepsilon) = \varepsilon \Delta u^\varepsilon, \quad u|_{t=0} = u_0 \in L^\infty(\mathbb{R}^d)$$

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– in bounded domains or on non-uniform meshes,
space translation arguments do not apply.

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Our technique in Finite Volumes will be,
discrete *BV* estimates in space + L^1 bounds $\Longrightarrow$ strong L^1 compactness

Parabolic-elliptic problems

General problem (Alt and Luckhaus ('83) :

$$\partial_t b(v) - \operatorname{div} \mathfrak{a}(b(v), \nabla v) = f \quad + \quad BC \; + \; IC : \; b(v)|_{t=0} = b_0.$$

Here b is a continuous non-decreasing function
($\Longrightarrow$ the pb. looks parabolic, but it can degenerate into an elliptic one);
the diffusion a can be of Leray-Lions type (non-Newtonian fluids).

Particular case: the Richards equation (one-phase flow in porous media).

Finite volume studies:
Eymard, Gallouët, Gutnic, Herbin, Hilhorst ,... (quasilinear);
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Technique (think e.g. of Galerkine approximations... but also of FV !):

– $\nabla^h v^h$ is estimated in L^p, $p > 1$
– thus the space translates of $b(v^h)$ are controlled
  (by the modulus of continuity of b and the space translates of v)
– by the Kruzhkov Lemma , $b(v^h)$ have uniform time translates
– passage to the (strong) limit in $b(v^h)$
– some further work (weak compactness of the gradients, Minty argument...)

$\Longrightarrow$ convergence of the approximations.

A cross-diffusion system (squirrels' war)

Consider the quasilinear, strongly coupled reaction-diffusion system

$$\partial_t u - \Delta u - \operatorname{div}\left((u+v)\nabla u + u\nabla v\right) = u(a_1 - b_1 u - c_1 v),$$
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– a uniform bound on u^h, v^h in $L^\infty(0,T;L^2(\Omega))$; on ∇u^h, ∇v^h in $L^2(Q)$;

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The Aubin-Lions-Simon argument can be used : $\partial_t u, \partial_t v$ belong to $L^{4/3}(0,T;W^{-1,4/3}(\Omega)) + L^1(0,T;L^{2^*/2}(\Omega)) \subset L^1(0,T;W^{-1,4/3}+L^{2*/2})$; u, v in $L^2(0,T;H^1(\Omega))$, and $H^1 \Subset L^1 \subset W^{-1,4/3}+L^{2^*/2}(\Omega)$.

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$\Longrightarrow$ by the Kruzhkov Lemma , compactness of u^h, v^h.

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The system is quasilinear + gradients weakly compact $\implies$ convergence Ok.

A system from electrocardiology

Look at the "simplified bidomain model of cardiac electric activity":

$$\partial_t v - \mathrm{div}\, \mathbf{M}_i(x)\nabla u_i + h(v) = I_{\mathrm{app}}, \qquad (t,x) \in Q_T,$$
$$-\partial_t v - \mathrm{div}\, \mathbf{M}_e(x)\nabla u_e - h(v) = -I_{\mathrm{app}}, \qquad (t,x) \in Q_T,$$

with, say, Neumann boundary conditions

$$(\mathbf{M}_{i,e}(x)\nabla u_{i,e}) \cdot n = s_{i,e} \quad \text{on } (0,T) \times \partial\Omega,$$

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- Ω is a time-independent Lipschitz domain in $\mathbb{R}^3$ (the heart does not move...); the space-time domain is $Q_T = (0,T) \times \Omega$;
- u_i, u_e is the intra- (respectively, extra-) cellular electric potential;
- $v := u_i - u_e$ is the transmembrane potential;
- $I_{app}(t,x)$ is a given stimulation current;
- $\mathbf{M}_i(x)$ and $\mathbf{M}_e(x)$ are the intra- and extra-cellular conductivity tensors (assumed symmetric, positive definite, but anisotropic).
- $h(v)$ is an *ad hoc* model for the transmembrane ionic current. We focus on the case where h is close to a cubic polynomial.

A system from electrocardiology (cont[d])

Assume that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function, and there exist $r \in (2, +\infty)$ and constants $C, L, l > 0$ such that

$$\frac{1}{C}|v|^r \leq |h(v)v| \leq C\left(|v|^r + 1\right),$$

$$\tilde{h} : z \mapsto h(z) + Lz + l \quad \text{is strictly increasing on } \mathbb{R}, \text{ with } \tilde{h}(0) = 0.$$

In the known models, the appropriate value is $r = 4$; this means, the nonlinearity h is of cubic growth at infinity. The assumptions are automatically satisfied by any cubic polynomial h with positive leading coefficient.

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In A.,Bendahmane,Karlsen we "make converge" 3D DDFV schemes

— for general r, for the fully implicit scheme ;

— for $r < 4$ (strictly !), for the linearized implicit scheme .

Well, there is some hope for attaining the critical case $r = 4$
(V.V. Zhikov's compensated compactness lemmas...)

A system from electrocardiology (cont[d])

For the fully implicit scheme: standard DDFV convergence proof.

To see the difficulty with the linearized scheme, discretize the "continuous" equations:

$$\frac{v^n - v^{n-1}}{\Delta t} - \operatorname{div} \mathbf{M}_i(x)\nabla u_i^n + \frac{h(v^{n-1})}{v^{n-1}} v^n = I_{\mathrm{app}}^n, \quad \ldots$$

(we've assumed that $h(0) = 0$, $\frac{h(z)}{z} \geq 0$; the general case is similar).

A system from electrocardiology (cont$^{\text{d}}$)

For the fully implicit scheme: standard DDFV convergence proof.

To see the difficulty with the linearized scheme, discretize the "continuous" equations:
$$\frac{v^n - v^{n-1}}{\Delta t} - \operatorname{div} \mathbf{M}_i(x)\nabla u_i^n + \frac{h(v^{n-1})}{v^{n-1}} v^n = I_{\mathrm{app}}^n, \quad \ldots$$
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- Notice that the assumption $r \leq 4$ yields $0 \leq b(z) \leq const\, z^2$.

  This allows to deduce an $L^1(Q)$ integrability estimate on the ionic current term from the decomposition
  $|b(v^h(\cdot - \Delta t))\, v^h(\cdot)| \leq \alpha\, b(v^h(\cdot - \Delta t)) + \frac{1}{\alpha}\, b(v^h(\cdot - \Delta t))\, \left|v^h(\cdot)\right|^2.$

This trick allows to bypass the difficulty and get strong compactness of $(v^h)_h$.

A parabolic-hyperbolic problem

Now, consider degenerate hyperbolic-parabolic problems of the form

$$\begin{cases} \partial_t u + \mathrm{div}\, \mathfrak{f}(u) - \mathrm{div}\, \mathfrak{a}(\nabla A(u)) = s & \text{in } Q := (0,T)\times\Omega, \\ u = 0 \quad \text{on } \Sigma = (0,T)\times\partial\Omega, & u|_{t=0} = u_0 \quad \text{in } \Omega. \end{cases}$$

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– on $\nabla A(u)$ in $L^p(Q)$ $\Longrightarrow$ space translates of $A(u)$
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– integrate the equation between t et $t+\vartriangle$;
– take $A(u)(\cdot+\vartriangle) - A(u)(\cdot)$ for the test function, use Fubini and get

$$\iint_Q |u(t+\vartriangle) - u(t)|\,|A(u(t+\vartriangle)) - A(u(t))| \le \mathrm{Const}\,|\vartriangle|\,.$$

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And if A is not Lipschitz ?

The standard method and its variant

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Much more work (hyperbolic degeneracy) $\Longrightarrow$ convergence of DDFV on orthogonal 2D and 3D meshes (A.,Bendahmane,Karlsen).

One application: a two-phase flow model

Model (Eymard, Ghilani, Marhraoui and Eymard, Henry, Hilhorst):

$$\begin{aligned} u_t &- \operatorname{div}(k_w(u)\nabla p) &&= f_\mu(c)s_+ - f_\mu(u)s_- \\ (1-u)_t &- \operatorname{div}(\mu k_a(u)\nabla(p+p_c(u))) &&= (1-f_\mu(c))s_+ - (1-f_\mu(u))s_- \end{aligned}$$

two-phase flow, no gravity, particular structure of the source term.
Coefficient k_a is degenerate at $u = 1$, k_w is degenerate at $u = 0$.

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There is no particular reason to assume that ζ is Lipschitz;
fortunately, one gets the time translates of $\zeta(u_\mu)$
with the help of the above "moduli of continuity" technique .

A DISCRETE VERSION

Notation and example

We consider a family of finite volume schemes written under the general abstract form

$$(\#) \qquad \text{for } n = 1..N, \quad \frac{u^{\mathfrak{T},n} - u^{\mathfrak{T},(n-1)}}{\Delta t} = \mathrm{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^{\mathfrak{T},n}] + f^{\mathfrak{T},n}.$$

The statement of the Lemma does not depend on the exact nature of the finite volume mesh and the associated discrete divergence operator, but only on a few structural properties shared by many known schemes.

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Let us explain the meaning of notation and assumptions for the standard “admissible schemes” of Eymard, Gallouët, Herbin .

$\mathfrak{T}$ is a mesh on a bounded domain $\Omega \subset \mathbb{R}^d$, Δt is a positive time discretization step. We mean that $\mathfrak{T}$ and Δt are parametrized by a parameter $h > 0$.

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and $\operatorname{div}^{\mathfrak{T}}$ is a discrete divergence operator defined on the mesh $\mathfrak{T}$.

Notation and example (cont^{d})

A mesh $\mathfrak{T}$ consists of volumes supplied with so-called centers; and also of the interfaces between adjacent volumes.
A generic volume is denoted by K, and its center is denoted by x_K.
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A discrete function $v^{\mathfrak{T}}$ is identified with $\sum_K v_K 1\!\!1_K(x) \in L^1(\Omega)$,
where $1\!\!1_K(\cdot)$ stands for the characteristic function of K.

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A discrete function $v^{\mathfrak{T}}$ is identified with $\sum_K v_K 1\!\!1_K(x) \in L^1(\Omega)$, where $1\!\!1_K(\cdot)$ stands for the characteristic function of K.

Therefore, the norm $\|v^{\mathfrak{T}}\|_{L^1(\Omega)}$ of a discrete function $v^{\mathfrak{T}}$ is calculated as $\sum_K m_K |v_K|$.

Notation and example (cont^d)

Further, with each interface $K|L$, one can associate a value $\vec{\mathcal{F}}_{K|L}$
Any set of such values $(\vec{\mathcal{F}}_{K|L})_{K|L}$, denoted $\vec{\mathcal{F}}^{\mathfrak{T}}$, is called a discrete field.

The L^1 norm $\|\vec{\mathcal{F}}^{\mathfrak{T}}\|_{L^1(\Omega)}$ of a discrete field $\vec{\mathcal{F}}^{\mathfrak{T}}$
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In particular, this norm is used for discrete gradients;

for a given discrete function $v^{\mathfrak{T}}$,
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For the case of standard "admissible meshes", $\nabla_{K|L} v^{\mathfrak{T}} = d\,\dfrac{v_L - v_K}{d_{KL}} \nu_{K,L}$,
where $\vec{\nu_{K,L}}$ is the unit normal vector to $K|L$ pointing from K to L: $\vec{\nu_{K,L}} = \frac{x_L - x_K}{d_{KL}}$.

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Further, for a given discrete field $\vec{\mathcal{F}}^{\mathfrak{T}}$, its discrete divergence is usually defined as the discrete function $\operatorname{div}^{\mathfrak{T}} \vec{\mathcal{F}}^{\mathfrak{T}} = (\operatorname{div}_K \vec{\mathcal{F}}^{\mathfrak{T}})_K$ with entries

$$\operatorname{div}_K \vec{\mathcal{F}}^{\mathfrak{T}} := \frac{1}{m_K} \sum_{L \in \mathcal{N}(K)} m_{K|L}\, \vec{\mathcal{F}}_{K|L} \cdot \vec{\nu_{K,L}},$$

where the summation runs over volumes L belonging to the set $\mathcal{N}(K)$ of all the neighbours of K, and K is not a boundary volume.

Notation and example (cont^{d})

For our purposes, the exact nature of $\mathrm{div}^{\mathfrak{T}}$ and $\nabla^{\mathfrak{T}}$ is immaterial; we only require that the following estimate hold:

$$\Big|\sum_K m_K v_K \,\mathrm{div}_K \vec{\mathcal{F}}^{\mathfrak{T}}\Big| \leq C \max_{K|L} |(\nabla^{\mathfrak{T}} v^{\mathfrak{T}})_{K|L}| \times \|\vec{\mathcal{F}}^{\mathfrak{T}}\|_{L^1(\Omega)}$$

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We also need one property which ensures the $W^{1,1}$ discrete Poincaré inequality:

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and a stability bound for discretization of functions in $W_0^{1,\infty}(\Omega)$:

for all $v \in W_0^{1,\infty}(\Omega)$, setting $v_K = \frac{1}{m_K}\int_K v$

one has $\max_{K|L} |(\nabla^{\mathfrak{T}} v^{\mathfrak{T}})_{K|L}| \leq C\, \|\nabla v\|_\infty$

(this imposes a mild regularity assumption on the meshes).

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Such a discrete function is identified with the function $\sum_{n=1}^N\sum_\kappa v^n_\kappa \, 1\!\!1_{Q^n_\kappa}$ on Q, where $Q^n_\kappa := \big((n-1)\Delta t\,,\,n\Delta t\big]\times\kappa$ is the cylinder associated with the space volume κ and the time step n;

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For a discrete field $\vec{\mathcal{F}}^h = (\, \vec{\mathcal{F}}^{\mathfrak{T},n}\,)_{n=1..N}$ on Q,
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The discrete gradient and divergence operators act separately on each time step n, i.e., $\nabla^{\mathfrak{T}}v^h=\big(\,\nabla^{\mathfrak{T}}v^{\mathfrak{T},n}\,\big)_{n=1..N}$ and $\mathrm{div}^{\mathfrak{T}}\vec{\mathcal{F}}^h=\big(\,\mathrm{div}^{\mathfrak{T}}\vec{\mathcal{F}}^{\mathfrak{T},n}\,\big)_{n=1..N}$.

The statement ("discrete")

Let Ω be an open domain in $\mathbb{R}^d$, $T > 0$, $Q = (0, T) \times \Omega$. Let $(\mathfrak{T}^h)_h$ be a family of meshes of Ω and $(\vartriangle t^h)_h$ be the associated time steps.

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Lemma (the discrete statement)

Assume that for some constant C independent of h, the discrete gradient and divergence operators associated with $\mathfrak{T}^h$ verify

- *the "summation-by-parts inequality":*

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For all h, assume that $u^h = \left(u^{\mathfrak{T},n}\right)_{n=1..N}$, $f^h = \left(f^{\mathfrak{T},n}\right)_{n=1..N}$ *and* $\vec{\mathcal{F}}^h = \left(\vec{\mathcal{F}}^{\mathfrak{T},n}\right)_{n=1,\dots,N}$ *satisfy the discrete evolution equations*

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with a family $(u_0^h)_h$ *of initial data,* $u_0^h := u^{0,\mathfrak{T}}$.

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(i) Assume that for all h, u^h *is null on* $\partial\Omega$ *, that the families* $(u^h)_h$, $(f^h)_h$, $(\vec{\mathcal{F}}^h)_h$ *and* $(\nabla^{\mathfrak{T}} u^h)_h$ *are bounded in* $L^1(Q)$ *, and that* $(u_0^h)_h$ *is bounded in* $L^1(\Omega)$ *.*

Then there exists a sequence $(h_i)_{i\in\mathbb{N}}$ *such that* $(u^{h_i})_i$ *is convergent in* $L^1(Q)$.

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$$\sum_{n=1}^{N} \big\| \nabla^{\mathfrak{T}} u^{\mathfrak{T},n} \big\|_{L^1(\Omega')} \le M(\Omega').$$

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Then the claim of (i) holds with $L^1(Q)$ *replaced by* $L^1_{loc}([0,T]\times\Omega)$ *.*

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In some applications, the general local result of the Lemma is not sufficient, because one is interested in compactness of $(u^h)_h$ up to the boundary $(0, T) \times \partial\Omega$. Yet:

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• If only $L^1(Q)$ estimates on $(u^h)_h$ are available, Lemma (i) is one particular case where the $L^1(Q)$ compactness of $(u^h)_h$ holds true. The assumption that u^h is null on $\partial\Omega$ corresponds to the case of the homogeneous Dirichlet boundary condition on the discrete function u^h.

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• For the case of other boundary conditions, different techniques of extension of u^h in a neighbourhood of Q yield compactness results analogous to Lemma (i); one only needs to ensure a uniform $L^1(Neighb(Q))$ bound on $(\nabla^{\mathfrak{T}} u^h)_h$ $(\Longrightarrow$ a uniform space translation estimate on $(u^h)_h)$.

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- it remains to compare $\iint_Q w(t,x)\phi(t,x)$ with $\iint_Q w(t,x)\operatorname{sign} w(t,x)$. Here, the space translates of w enter the stage : the above difference is controlled by the L^1 modulus of continuity $\omega(\delta)$ of $(\tilde{u}^h)_h$.

Proof (sketched)

- from the L^1 bound on $\nabla^{\mathfrak{T}} u^h$, get L^1 translates in space on u^h (trivial), then on the piecewise affine in t interpolation $\tilde{u}^h$ of u^h
- write the discrete equations as evolution equation $\partial_t \tilde{u}^h = \operatorname{div}^{\mathfrak{T}}[\vec{\mathcal{F}}^h] + f^h$
- set $w(t,x) := \tilde{u}^h(t+\tau,x) - \tilde{u}^h(t,x)$
- integrate the equation to make appear $w(t,x)$ in the left-h.side
- try to take $\operatorname{sign} w$ for the test function in the equation obtained. Namely,
  - introduce ϕ the regularization of $\operatorname{sign} w$ by convolution with parameter δ; notice that $\|\nabla\phi\|_\infty \le const\ \delta^{-d}$.
  - consider its space discretization ϕ^h and multiply (pointwise in t) the equation in volume κ by ϕ_K
  - "integrate" on Q, use summation-by-parts , the L^1 bounds on $\vec{\mathcal{F}}^h, f^h$ and Fubini to get
  $$\iint_Q w(t,x)\phi(t,x) \le C\,\tau\,(1 + \|\nabla\phi\|_\infty) = C\,\tau\,(1+\delta^{-d}).$$
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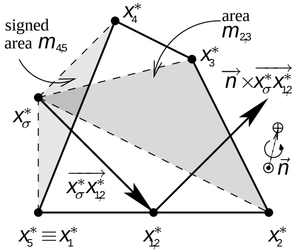
- optimizing in $\delta > 0$ the bound $C\,\tau\,(1+\delta^{-d}) + \omega(\delta)$, we get a modulus of continuity for the L^1 time translates of $\tilde{u}^h$ and u^h.

A RECONSTRUCTION PROPERTY ON THE PLANE

The formula

Polygon $\sigma \subset \Pi$, oriented by $\vec{n} \perp \Pi$



Let Π be a plane in $\mathbb{R}^3$ with a unit normal vector $\vec{n}$, and $\sigma \subset \Pi$ be a polygon. Let $x_\sigma^* \in \Pi$ be a distinguished point .

Introduce the vertices x_i^*, $i = 1, \ldots, l$ (clockwise); and take $x_{i,i+1}^*$ the midpoints of the edges .

Then $m_{i,i+1} = \frac{1}{2} \langle \vec{n}, \overrightarrow{x_\sigma^* x_{i,i+1}^*}, \overrightarrow{x_i^* x_{i+1}^*} \rangle$ is the (signed) area of the triangle $x_i^* x_\sigma^* x_{i+1}^*$. Denote the area of σ by m; we have $m = \sum_{i=1}^{l} m_{i,i+1}$.

Lemma

For all $\vec{r} \parallel \Pi$, $\quad \vec{r} = \dfrac{1}{m} \sum_{i=1}^{l} (\vec{r} \cdot \overrightarrow{x_i^* x_{i+1}^*}) \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right].$

The proof combines two well-known simple formulae (cf. in particular Eymard, Droniou).

The formula (cont^{d})

Corollary (Consistency of the gradient reconstruction)

Take $(w_i^*)_{i=1}^{l} \subset \mathbb{R}$, $w_{l+1}^* := w_1^*$. *Consider the expression*

$$\frac{1}{m} \sum_{i=1}^{l} (w_{i+1}^* - w_i^*) \left[\vec{n} \times \overrightarrow{x_\sigma^* x_{i,i+1}^*}\right].$$

If w_i^* are the values of an affine function w at the vertices x_i^* of the polygon σ, the above expression gives ∇w.

The "complementary volumes" schemes in 2D

The 2D "complementary volumes schemes" were proposed independently in several works in the late 90th and early 00th (Afif and Amaziane ; Handlovičová, Mikula, and Sgallari ; A., Gutnic and Wittbold , ..?).

The idea was to reconstruct the discrete gradient on a given triangulation (affine per triangle) and then write the FV scheme on the dual mesh .

Thus the triangles play the role of "diamonds" of the DDFV schemes. The structural properties of this construction are extremely similar (but simpler !) to those of DDFV schemes.

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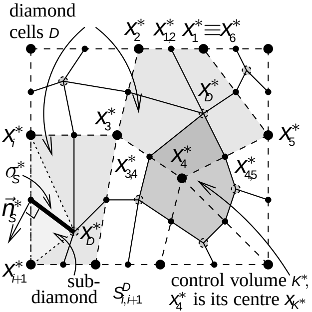
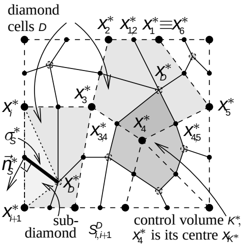


We use the "median dual mesh" (≡"Donald dual mesh").

The "complementary volumes" schemes in 2D (cont^{d})



Here, we do not necessarily need σ's (the "diamonds") to be triangles; any polygon would do (in particular, quadrilaterals are welcome).

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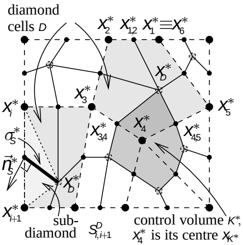

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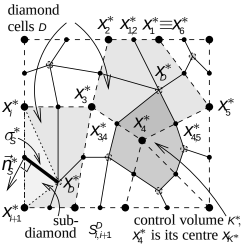
We associate to each "diamond" a value of the discrete gradient (reconstructed from the formula of the Corollary) .

The "complementary volumes" schemes in 2D (cont[d])



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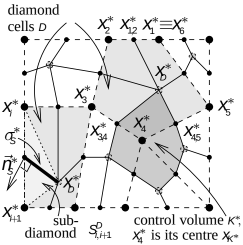
We associate to the mesh the standard FV discrete divergence operator .

The “complementary volumes” schemes in 2D (cont^{d})



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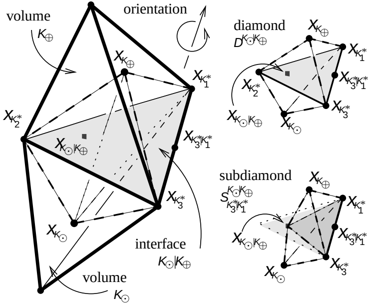
Theorem (for $2D$ complementary volumes schemes)

These discrete gradient and divergence operators are linked by the discrete duality (integration-by-parts) formula.

Rq.: a slight “ideological” difference wrt the Mimetic FD approach.

A “DDFV” scheme in 3D (cf. Hermeline; Pierre; Coudière and Hubert)



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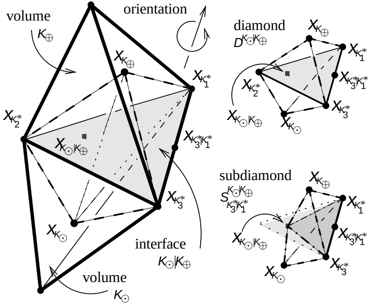

We associate to each "diamond" a value of the discrete gradient:

– its 1D projection on $\overrightarrow{x_{K_\odot} x_{K_\oplus}}$ is reconstructed from the difference $\frac{u_{K_\oplus} - u_{K_\odot}}{d_{KL}}$

– its 2D projection on the interface $K_\odot|K_\oplus$ is reconstructed using the Corollary.

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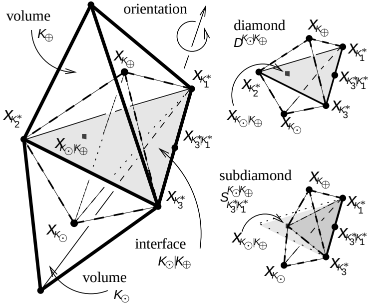

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We associate to the mesh the standard (DD)FV discrete divergence operator.

Theorem (for the 3D DDFV schemes of the above kind)

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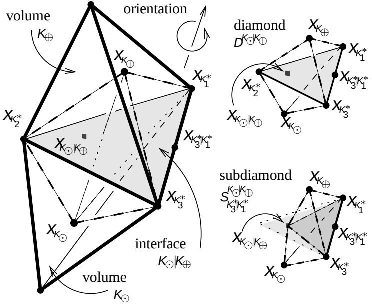

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These discrete gradient and divergence operators are linked by the discrete duality (integration-by-parts) formula.

Rq.: consistency + discrete duality $\Longrightarrow$ convergence proofs !

And that's all...

Thank you !