Workshop on discretization methods for viscous flows
Porquerolles, June 22 - 26, 2009

Vector penalty-projection methods for incompressible Navier-Stokes equations

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Motivations and objectives

Work focusing on the constraint of free divergence

- How to deal efficiently with the free-divergence constraint with splitting methods (prediction-correction steps) ?
- How to overcome the major drawbacks of the usual projection methods including a scalar correction step of the Lagrange multiplier with a solution of a Poisson-type equation ?

$\Rightarrow$ Key idea : introduce a splitting penalty method for the velocity...

Example of fluid-type models with the pressure field as Lagrange multiplier

$\Rightarrow$ solution of unsteady incompressible Navier-Stokes problems with the primitive variables (velocity and pressure) : $\boldsymbol{\nabla}\cdot \mathbf{v} = \mathbf{0}$
long time simulations, coupling with an advection-dispersion problem, variable density flows...
$\Rightarrow$ very small velocity divergence at each time step..!

Other examples

$\Rightarrow$ solution of magnetohydrodynamics (MHD) problems : $\boldsymbol{\nabla}\cdot \mathbf{B} = \mathbf{0}$

Outlines

1. *Projection methods for incompressible flows*
2. *Scalar penalty-projection methods*
3. *Vector penalty-projection ($VPP_{r,\varepsilon}$) methods*
4. *Conclusion and perspectives*

Outlines

1. ***Projection methods for incompressible flows***
   - Non-homogeneous incompressible flows
   - Fractional-step and projection methods
2. ***Scalar penalty-projection methods***
3. ***Vector penalty-projection ($VPP_{r,\varepsilon}$) methods***
4. ***Conclusion and perspectives***

Navier-Stokes problem for incompressible flows

Incompressible and variable density flows of Newtonian fluids

Navier-Stokes equations in $\boldsymbol{\Omega} \subset \mathbb{R}^{\boldsymbol{d}}$ with mixed boundary conditions : Dirichlet on $\boldsymbol{\partial\Omega_D}$ and open (outflow) B.C. on $\boldsymbol{\partial\Omega_N}$

$$
\begin{cases}
\dfrac{\partial \varrho}{\partial t} + u \cdot \nabla \varrho = 0 & \text{in } \Omega \times]0, T[\\
\varrho \left[\dfrac{\partial u}{\partial t} + (u \cdot \nabla) u \right] - \nabla \cdot \tau(u) + \nabla p = f & \text{in } \Omega \times]0, T[\\
\nabla \cdot u = 0 & \text{in } \Omega \times]0, T[\\
u = u_{\mathbf{D}} & \text{on } \partial\Omega_{\mathbf{D}} \times]0, T[\\
-pn + \tau(u) \cdot n = f_{\mathbf{N}} & \text{on } \partial\Omega_{\mathbf{N}} \times]0, T[\\
\varrho = \varrho_0, \quad \text{and } u = u_0 & \text{in } \Omega \times \{0\}
\end{cases}
$$

$$\nabla \cdot \tau(u) = \nabla \cdot [\mu(\nabla u + \nabla u^{\mathbf{T}})], \text{ or } \mu \Delta u \text{ (for a constant viscosity)}$$

For an homogeneous fluid with constant density, we set $\boldsymbol{\varrho = 1}$.

Semi-implicit method : coupled vs splitted solvers

- Fractional-step methods for NS generally cheaper than fully coupled implicit solvers
- Free-divergence FEM are too much expensive

Theoretical basis of projection methods

$\Rightarrow$ Helmholtz-Hodge decomposition of $\boldsymbol{L^2(\Omega)^d = \mathbf{H} \oplus \mathbf{H}^\perp}$ with :

$$\mathbf{H} = \{\mathbf{u} \in L^2(\Omega)^d,\ \nabla\cdot\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \Gamma\},$$

$$\mathbf{H}^\perp = \{\nabla\phi,\ \phi \in H^1(\Omega)\}$$

$\Rightarrow$ $\boldsymbol{P_{\mathbf{H}}}$: Leray orthogonal projection onto the space of solenoidal fields $\mathbf{H}$

see [Leray, 1934 – Temam, 1986 – Girault and Raviart, 1986]

Projection scheme with scalar pressure-correction
[Chorin, 1968 - Temam, 1969 - Goda, 1979 - Van Kan, 1986...]
Recent review : [Guermond, Minev, Shen, CMAME 2006]

Prediction step – ex. for Euler scheme : $D\tilde{u}^{n+1} = \tilde{u}^{n+1} - u^n$

$$\varrho^{n+1}\left(\frac{D\tilde{u}^{n+1}}{\delta t} + (u^{\star,n+1}\cdot\nabla)\tilde{u}^{n+1}\right) - \nabla\cdot\tau(\tilde{u}^{n+1}) + \nabla p^n = f^{n+1}$$

$\tilde{u}^{n+1} = u_{\mathbf{D}}^{n+1}$ on $\partial\Omega_{\mathbf{D}}$

$-p^n n + \tau(\tilde{u}^{n+1})\cdot n = f_{\mathbf{N}}^{n+1}$ on $\partial\Omega_{\mathbf{N}}$

Projection step – ex. for Euler scheme : $\beta_q = 1$

$$\beta_q\varrho^{n+1}\frac{u^{n+1}-\tilde{u}^{n+1}}{\delta t} + \nabla\phi = 0$$

$\Rightarrow \nabla\phi\cdot n = 0$ on $\partial\Omega_{\mathbf{D}}$ (necessary since here $(u-\tilde{u})\cdot n = 0$)

$\phi = 0$ on $\partial\Omega_{\mathbf{N}}$ (sufficient by orthogonal projection onto H)

$$\nabla\cdot u^{n+1} = 0 \quad\Rightarrow\quad -\nabla\cdot\left(\frac{\delta t}{\varrho^{n+1}}\nabla\phi\right) = -\beta_q\nabla\cdot\tilde{u}^{n+1}$$

Pressure-correction step : ϕ pressure increment

$$p^{n+1} = p^n + \phi$$

Fractional-step and projection-type methods

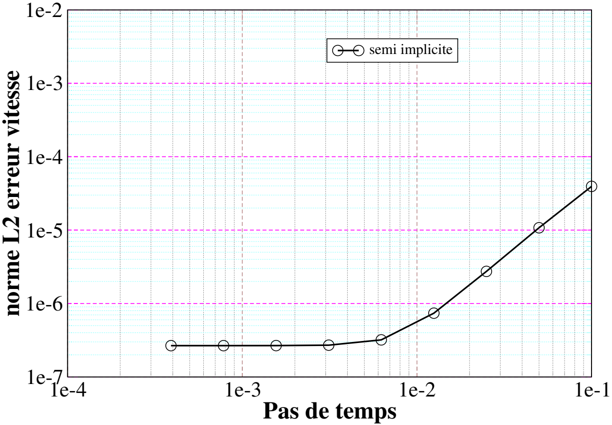
Major drawbacks of the incremental projection methods

- Time order of the splitting error ? generally $\mathcal{O}(\delta t^2)$ or $\mathcal{O}(\delta t^{\frac{3}{2}})$
  i.e. error between the numerical solutions of the implicit (or semi-implicit) method and the fractional-step method
- $\nabla\phi \cdot n = 0$ on $\partial\Omega_{\mathrm{D}}$
  $\Rightarrow$ existence of an artificial pressure boundary layer in space
- $\phi = 0$ on $\partial\Omega_{\mathrm{N}}$
  $\Rightarrow$ convergence in time and space spoiled for outflow boundary conditions : splitting error varying like $\mathcal{O}(\delta t^{\frac{1}{2}})$ (pressure) and no more negligible (for both velocity and pressure) with respect to the time and space discretization error (whatever the time scheme)
  cf the analysis in [Guermond et al., 2005]
- Pressure-correction step strongly dependent on density and viscosity for non-homogeneous flows
  $\Rightarrow$ very poor convergence for large ratios of $\varrho \sim 1000$ to get a small divergence, see [Guermond & Quartapelle, JCP 2000].

Numerical tests (from [Jobelin et al., JCP 2006])

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Velocity error (discrete $l^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt



splitting error in $\mathcal{O}(\delta t^2)$

Time convergence in $\mathcal{O}(\delta t^2)$ with 2nd-order Gear (BDF2) scheme

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

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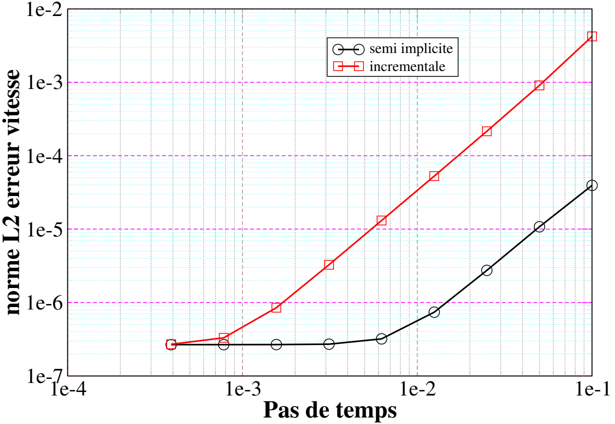

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Outlines

Scalar penalty-projection methods

cf. [Jobelin et al., JCP 2006]

- prediction step with an augmented Lagrangian term for $\boldsymbol{r > 0}$
- consistent projection step by scalar pressure-correction

Previous ideas

- [Shen, 1992] : $\boldsymbol{r = 1/\delta t^2 \gg 1}$ with a different correction of pressure ; only for a theoretical purpose, no numerical experiment
- [Caltagirone and Breil, 1999] : $\boldsymbol{r > 0}$ with a singular projection operator...

Penalty-prediction step : augmentation parameter $r \geq 0$

$$\varrho^{n+1}\left(\frac{D\tilde{u}^{n+1}}{\delta t} + (u^{\star,n+1}\cdot\nabla)\tilde{u}^{n+1}\right) - \nabla\cdot\tau(\tilde{u}^{n+1}) - r_1\nabla\left(\nabla\cdot\tilde{u}^{n+1}\right) + \nabla p^n = f^{n+1}$$

$$\tilde{u}^{n+1} = u_{\mathrm{D}}^{n+1} \text{ on } \partial\Omega_{\mathrm{D}}$$
$$-p^n n + \tau(\tilde{u}^{n+1})\cdot n = f_{\mathrm{N}}^{n+1} \text{ on } \partial\Omega_{\mathrm{N}}$$

Projection step

$$\beta_q\varrho^{n+1}\frac{u^{n+1}-\tilde{u}^{n+1}}{\delta t} + \nabla\phi = 0$$

$\Rightarrow \nabla\phi\cdot n = 0$ on $\partial\Omega_{\mathrm{D}}$ (necessary since here $(u-\tilde{u})\cdot n = 0$)

$\phi = 0$ on $\partial\Omega_{\mathrm{N}}$ (sufficient by orthogonal projection onto H)

$$\nabla\cdot u^{n+1} = 0 \quad \Rightarrow \quad -\nabla\cdot\left(\frac{\delta t}{\varrho^{n+1}}\nabla\phi\right) = -\beta_q\nabla\cdot\tilde{u}^{n+1}$$

Pressure-correction step : ϕ consistent pressure increment

$$p^{n+1} = p^n - r_2\nabla\cdot\tilde{u}^{n+1} + \phi = \tilde{p}^{n+1} + \phi$$

Scalar penalty-projection methods

Algebraic FEM formulation for the Navier-Stokes system

$$\begin{cases} \dfrac{\beta_q}{\delta t}\mathbf{M}_\varrho \tilde{\mathbf{U}}^{n+1} + r_1 \mathbf{B}^{\mathbf{T}} \mathbf{M}_{pl}^{-1} \left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G} \right) + \mathbf{B}^{\mathbf{T}}\mathbf{P}^n \\ \qquad\qquad + \mathbf{A}\tilde{\mathbf{U}}^{n+1} = \mathbf{F} \\ \mathbf{L}_\varrho \mathbf{\Phi} = \dfrac{\beta_q}{\delta t} \left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G} \right) \\ \mathbf{P}^{n+1} = \mathbf{P}^n + r_2 \mathbf{M}_{pl}^{-1} \left(\mathbf{B}\tilde{\mathbf{U}}^{n+1} - \mathbf{G} \right) + \mathbf{\Phi} \\ \mathbf{M}_\varrho \mathbf{U}^{n+1} = \mathbf{M}_\varrho \tilde{\mathbf{U}}^{n+1} + \dfrac{\delta t}{\beta_q} \mathbf{B}^{\mathbf{T}} \mathbf{\Phi} \end{cases}$$

$\Rightarrow$ *Preconditioning the prediction step by one iteration of augmented Lagrangian and consistent scalar projection*

$\Rightarrow$ $r_1 = r_2 = r$ *: penalty-projection method* [Jobelin et al., 2006]

$\Rightarrow$ $r_1 = r,\ r_2 = r + 1/\mathrm{Re}$ *: rotational penalty-projection*

N.B. $r_1 = r_2 = 0$: incremental projection method [Goda, 1979]

$r_1 = 0,\ r_2 = 1/\mathrm{Re}$: rotational projection [Timmermans et al., 1996]

Numerical experiments [Jobelin et al., JCP 2006]

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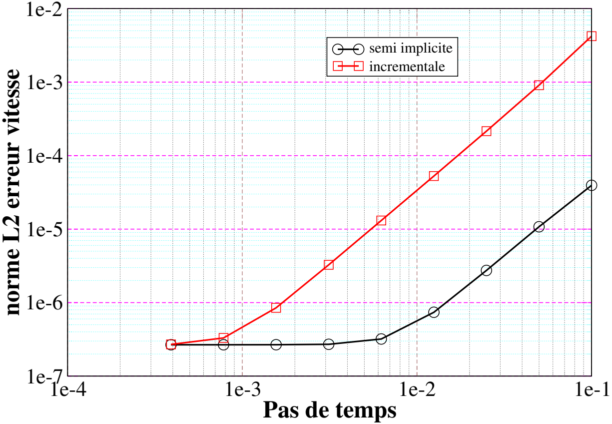
Velocity error (discrete $l^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt



Time convergence in $\mathcal{O}(\delta t^2)$ with 2nd-order Gear (BDF2) scheme

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

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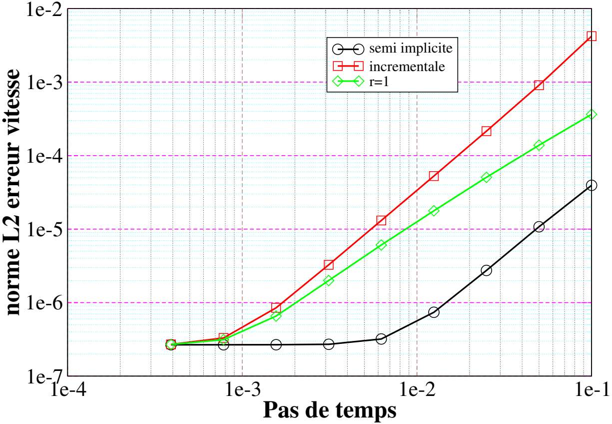

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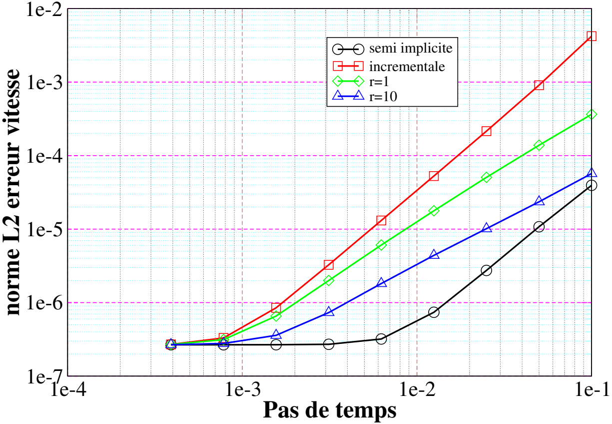

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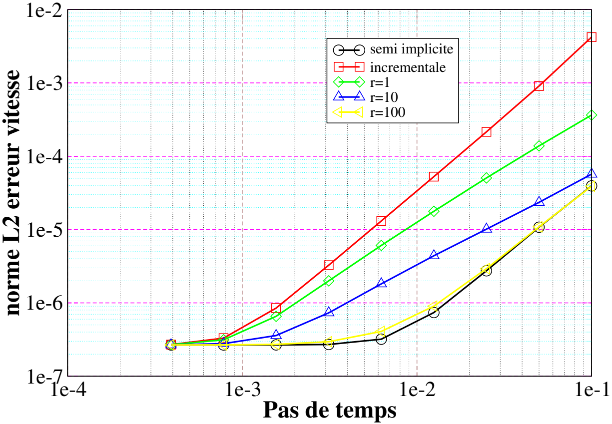

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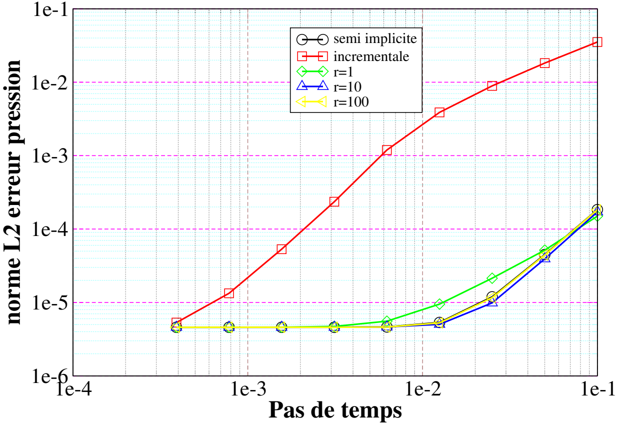
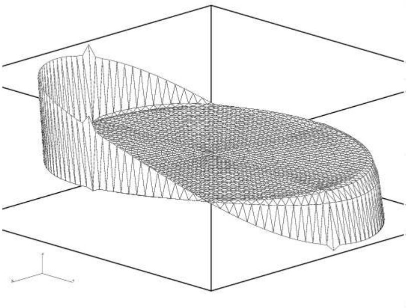
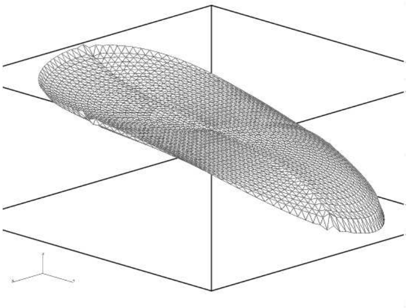


Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Artificial pressure boundary layer : Stokes with Dirichlet B.C. on a disk



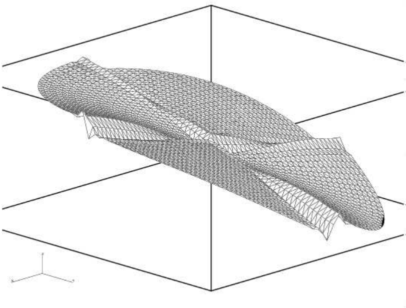
incremental projection
$\|p_h - p\|_{L^\infty(\Omega)} = \mathbf{1.5\ 10^{-2}}$



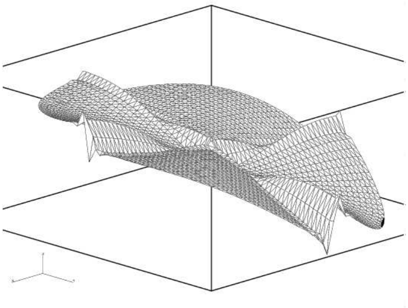
penalty-projection r=1
$\|p_h - p\|_{L^\infty(\Omega)} = \mathbf{2.8\ 10^{-3}}$

Numerical experiments

Artificial pressure boundary layer : Stokes with Dirichlet B.C. on a disk



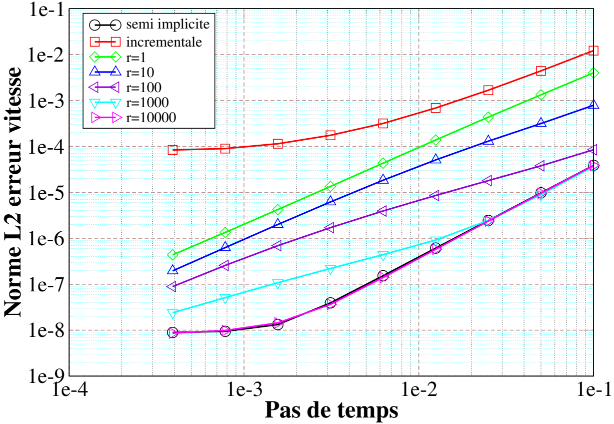
penalty-projection r=100
$\|p_h - p\|_{L^\infty(\Omega)} = \mathbf{2.8\ 10^{-4}}$



implicit scheme
$\|p_h - p\|_{L^\infty(\Omega)} = \mathbf{1.8\ 10^{-4}}$

Stokes with open boundary condition at a channel outflow

Velocity error (discrete $l^2(0,T;L^2(\Omega)^d)$ norm) versus time step δt



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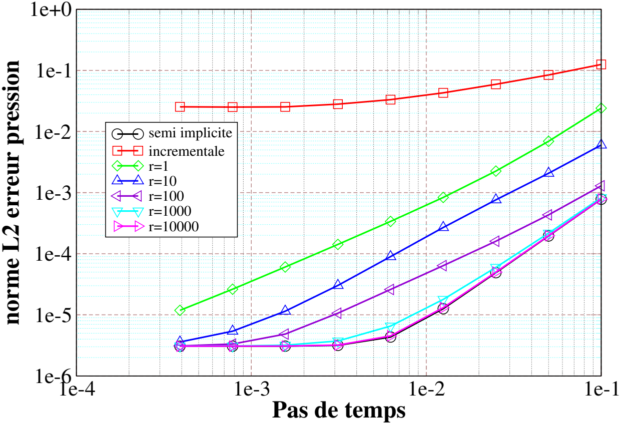

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Some remarks

Interests of penalty-projection methods

- Reduce the splitting error, varying as $\mathcal{O}(\frac{\delta t}{r})$ for $r > 0$, up to make it negligible with respect to the discretization error
- Suppress pressure boundary layers for moderate values of $r \in [1, 10]$
- Recover suitable velocity and pressure optimal convergence with outflow B.C. for $r \simeq 10$
- Require efficient preconditioning of the prediction step for large values of r since Cond $= \mathcal{O}(\frac{r}{h^2})$ for $r > 0$
  see [Février̀e et al., LNCS 2008 - JCAM 2009]
  $\Rightarrow$ multi-level preconditioner for $4th$-order compact FVM on MAC mesh (implicit scheme) : see [Kortas, PhD 1997].

Other works

- Generalization to dilatable and low Mach number flows : $\nabla \cdot \mathbf{u} = \mathbf{G}$ [Jobelin et al., EJCM 2008]
- Theoretical error analysis for fully discrete Stokes problems (1rst-order Euler scheme) [Angot et al., IJFV 2009]
- Theoretical error analysis for Navier-Stokes problems (2nd-order BDF2 scheme) [Févrière et al., JCAM 2009]

Theoretical analysis for Dirichlet-Stokes problem

[Shen, 1992-95 - Guermond, 1996] : standard projection (pressure-correction form)

[Guermond and Shen, 2003] : standard projection (velocity-correction)

[Guermond and Shen, 2004] : rotational variant of [Timmermans et al., 1996]

[Angot, Jobelin, Latché, IJFV 2009] ***: penalty-projection***

Analysis for small values of the augmentation parameter r

Theorem (Splitting error - fully discrete case in time and space)

Energy estimates of splitting errors compared to Euler implicit scheme there exists $\boldsymbol{c = c(\Omega, T, f, u_0, h) > 0}$ *such that : for* $\boldsymbol{1 \leq n \leq N}$,

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c \min(\delta t^2, \frac{\delta t^{3/2}}{r^{1/2}})$$

$$\left[\sum_{k=0}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{\frac{1}{2}} \leq c \max(1, \frac{1}{r^{1/2}}) \delta t^{3/2}.$$

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$$\|e^n\|_0 + \|\tilde{e}^n\|_0 + \left[\sum_{k=0}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c \frac{\sqrt{\delta t}}{r}$$

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{\frac{1}{2}} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{\frac{1}{2}} \leq c \frac{\delta t}{r}$$

$$\|\epsilon^n\|_h \leq c \, \frac{1}{\sqrt{r}}$$

$$\left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{\frac{1}{2}} \leq c \, \frac{1}{r}.$$

Outlines

New approach : a splitting penalty method...

Solve the saddle-point problem (algebraic form)

At each time step $t_n = n\delta t$, $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}$

$\Rightarrow$ Solve the Uzawa augmented Lagrangian problem for $\mathbf{v}$

$$(A + r\, B^T B)\mathbf{v} = \mathbf{f} - B^T p = \mathbf{F}$$

but ill-conditionned and expensive to solve for $r = \dfrac{1}{\varepsilon} \gg 1\,!$

$\Rightarrow$ Solve the splitting penalty problem (equivalent) for $\mathbf{v} = \tilde{\mathbf{v}} + \hat{\mathbf{v}}$ with a penalty parameter $0 < \varepsilon \ll 1$

$$\begin{cases} A\tilde{\mathbf{v}} = \mathbf{F} \\ (A + \frac{1}{\varepsilon} B^T B)\hat{\mathbf{v}} = -\frac{1}{\varepsilon} B^T B \tilde{\mathbf{v}} \\ \text{some pressure reconstruction : e.g. Uzawa...} \end{cases}$$

The limit problem for $\varepsilon \longrightarrow 0$ of the velocity correction

$$(\varepsilon A + B^T B)\hat{\mathbf{v}}_\varepsilon = -B^T B \tilde{\mathbf{v}}$$

has non unique solutions since $\mathrm{Ker}(B^T B) \neq \{0\}$

$||B\mathbf{v}_\varepsilon||_{L^2} = \mathcal{O}(\varepsilon)$, $\Rightarrow$ approximate projection method

A new family of vector penalty-projection methods

The two-parameter family of ($VPP_{r,\varepsilon}$) methods

$\mathbf{v}^0 \in \boldsymbol{H}^1(\Omega)^d$, $p^0 \in L^2_0(\Omega)$ given, for all $n \in \mathbb{N}$ s.t. $(n+1)\delta t \leq T$,

Penalty-prediction step with an augmentation parameter $r \geq 0$

$$\frac{\tilde{\mathbf{v}}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla)\tilde{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\Delta \tilde{\mathbf{v}}^{n+1} - r\nabla\left(\nabla\cdot \tilde{\mathbf{v}}^{n+1}\right) + \nabla p^n = \mathbf{f}^{n+1} \quad \text{in } \Omega$$

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1} \quad \text{on } \Gamma = \partial\Omega$$

$$\tilde{p}^{n+1} = p^n - r\nabla\cdot \tilde{\mathbf{v}}^{n+1} \quad \text{in } \Omega$$

Vector penalty-projection step with a penalty parameter $0 < \varepsilon \leq 1$

$$\varepsilon\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t} + (\mathbf{v}^n \cdot \nabla)\hat{\mathbf{v}}^{n+1} - \frac{1}{\mathrm{Re}}\Delta \hat{\mathbf{v}}^{n+1}\right) - \nabla\left(\nabla\cdot \hat{\mathbf{v}}^{n+1}\right) = \nabla\left(\nabla\cdot \tilde{\mathbf{v}}^{n+1}\right) \quad \text{in } \Omega$$

$$\hat{\mathbf{v}}^{n+1} = 0 \quad \text{on } \Gamma = \partial\Omega$$

Correction step for velocity and pressure

$$\mathbf{v}^{n+1} = \tilde{\mathbf{v}}^{n+1} + \hat{\mathbf{v}}^{n+1} \quad \text{and} \quad p^{n+1} = p^n - r\nabla\cdot \tilde{\mathbf{v}}^{n+1} - \frac{1}{\varepsilon}\nabla\cdot \mathbf{v}^{n+1}$$

A new family of vector penalty-projection methods

($VPP_{r,\varepsilon}$) methods for open boundary conditions on a part Γ_N

For a given stress vector on a part $\boldsymbol{\Gamma_N}$ of $\boldsymbol{\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N}$:

$$(\boldsymbol{\sigma}(\mathbf{v}, p)\cdot \mathbf{n})_{|\Gamma_N} \equiv -p\,\mathbf{n} + \mu\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right)\cdot \mathbf{n} = \mathbf{g}$$

we get for the Dirichlet and Neumann velocity boundary conditions :

Penalty-prediction step :

$$\tilde{\mathbf{v}}^{n+1} = \mathbf{v}_D^{n+1} \quad \text{on } \Gamma_D$$

$$-p^n\,\mathbf{n} + \mu^{n+1}\left(\nabla \tilde{\mathbf{v}}^{n+1} + (\nabla \tilde{\mathbf{v}}^{n+1})^T\right)\cdot \mathbf{n} = \mathbf{g}^{n+1} \quad \text{on } \Gamma_N$$

Vector penalty-projection step :

$$\hat{\mathbf{v}}^{n+1} = \mathbf{0} \quad \text{on } \Gamma_D$$

$$-(\tilde{p}^{n+1} - p^n)\,\mathbf{n} + \mu^{n+1}\left(\nabla \hat{\mathbf{v}}^{n+1} + (\nabla \hat{\mathbf{v}}^{n+1})^T\right)\cdot \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_N$$

$\Rightarrow$ Original boundary conditions not spoiled through a scalar projection step with a Poisson-like pressure correction

($VPP_{r,\varepsilon}$) methods for incompressible and variable density flows

Advection step for density :

$$\frac{\varrho^{n+1}-\varrho^{n}}{\delta t}+\nabla\cdot\left(\varrho^{n+1}\mathbf{v}^{n}\right)=0$$

Penalty-prediction step :

$$\varrho^{n+1}\left(\frac{\tilde{\mathbf{v}}^{n+1}-\mathbf{v}^{n}}{\delta t}+(\mathbf{v}^{n}\cdot\nabla)\tilde{\mathbf{v}}^{n+1}\right)-\nabla\cdot\mu^{n+1}\left(\nabla\tilde{\mathbf{v}}^{n+1}+(\nabla\tilde{\mathbf{v}}^{n+1})^{T}\right)$$
$$-r\nabla\left(\nabla\cdot\tilde{\mathbf{v}}^{n+1}\right)+\nabla p^{n}=\mathbf{f}^{n+1}$$

Vector penalty-projection step :

$$\varepsilon\left(\varrho^{n+1}\left(\frac{\hat{\mathbf{v}}^{n+1}}{\delta t}+(\mathbf{v}^{n}\cdot\nabla)\hat{\mathbf{v}}^{n+1}\right)-\nabla\cdot\mu^{n+1}\left(\nabla\hat{\mathbf{v}}^{n+1}+(\nabla\hat{\mathbf{v}}^{n+1})^{T}\right)\right)$$
$$-\nabla\left(\nabla\cdot\hat{\mathbf{v}}^{n+1}\right)=\nabla\left(\nabla\cdot\tilde{\mathbf{v}}^{n+1}\right)$$

Correction step for velocity and pressure :

$$\mathbf{v}^{n+1}=\tilde{\mathbf{v}}^{n+1}+\hat{\mathbf{v}}^{n+1}\quad\text{and}\quad p^{n+1}=p^{n}-r\nabla\cdot\tilde{\mathbf{v}}^{n+1}-\frac{1}{\varepsilon}\nabla\cdot\mathbf{v}^{n+1}$$

$\Rightarrow$ Velocity correction $\hat{\mathbf{v}}$ all the more quasi-independent on the density ϱ or viscosity μ as $\varepsilon\longrightarrow 0$ and terms possibly dropped in practice

Well-posedness of the ($VPP_{r,\varepsilon}$) methods

Theorem (Global solvability of the ($VPP_{r,\varepsilon}$) method.)

With $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega)^d)$, $\mathbf{v}^0 \in H^1(\Omega)^d$ *and* $p^0 \in L^2_0(\Omega)$ *given, both the prediction and correction steps of the* ($VPP_{r,\varepsilon}$) *method are well-posed for all* $\delta t > 0$, $r \geq 0$ *and* $\varepsilon > 0$, *i.e. for all* $n \in \mathbb{N}$ *such that* $(n+1)\delta t \leq T$, *there exists a unique solution* $(\mathbf{v}^{n+1}, p^{n+1}) \in H^1(\Omega)^d \times L^2_0(\Omega)$ *to the* ($VPP_{r,\varepsilon}$) *scheme such that :*

$$\frac{\mathbf{v}^{n+1} - \mathbf{v}^n}{\delta t} + (\mathbf{v}^n \cdot \nabla)\mathbf{v}^{n+1} - \frac{1}{\mathrm{Re}}\Delta \mathbf{v}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1} \qquad \text{in } \Omega$$

$$(\varepsilon \delta t)\frac{p^{n+1} - p^n}{\delta t} + \nabla \cdot \mathbf{v}^{n+1} + r\varepsilon\, \nabla \cdot \tilde{\mathbf{v}}^{n+1} = 0 \qquad \text{in } \Omega$$

which is the discrete problem effectively solved by the splitting scheme.

N.B. Idem for the fully implicit nonlinear scheme with : $(\mathbf{v}^{n+1} \cdot \nabla)\mathbf{v}^{n+1}$ if δt is taken sufficiently small, as in [Lions, 1969].

An artificial compressibility method with two parameters

with $\tilde{\mathbf{v}}_\varepsilon = \mathbf{v}_D$, $\hat{\mathbf{v}}_\varepsilon = \mathbf{0}$ on Γ and $\tilde{\mathbf{v}}_\varepsilon(0) = \mathbf{v}_0$, $\hat{\mathbf{v}}_\varepsilon(0) = \mathbf{0}$, $p_\varepsilon(0)$ given :

$$\partial_t \tilde{\mathbf{v}}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\tilde{\mathbf{v}}_\varepsilon - \frac{1}{\mathrm{Re}}\Delta\tilde{\mathbf{v}}_\varepsilon - r\nabla\left(\nabla\cdot\tilde{\mathbf{v}}_\varepsilon\right) + \nabla p_\varepsilon = \mathbf{f}$$

$$\varepsilon\left(\partial_t \hat{\mathbf{v}}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\hat{\mathbf{v}}_\varepsilon - \frac{1}{\mathrm{Re}}\Delta\hat{\mathbf{v}}_\varepsilon\right) - \nabla\left(\nabla\cdot\hat{\mathbf{v}}_\varepsilon\right) = \nabla\left(\nabla\cdot\tilde{\mathbf{v}}_\varepsilon\right)$$

$$\mathbf{v}_\varepsilon = \tilde{\mathbf{v}}_\varepsilon + \hat{\mathbf{v}}_\varepsilon \quad \text{and} \quad (\varepsilon\delta t)\partial_t p_\varepsilon + \nabla\cdot\mathbf{v}_\varepsilon + r\varepsilon\,\nabla\cdot\tilde{\mathbf{v}}_\varepsilon = 0$$

Convergence in some sense to the Navier-Stokes system when $\varepsilon \longrightarrow 0$ for all $r \geq 0$ and $\delta t > 0$

$\Rightarrow$ Better convergence properties than the one-step artificial compressibility method of Chorin (1967) and Temam (1968) which suffers from a temporal boundary layer of pressure

$$\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon - \frac{1}{\mathrm{Re}}\Delta\mathbf{v}_\varepsilon + \nabla p_\varepsilon = \mathbf{f}$$

$$\varepsilon\,\partial_t p_\varepsilon + \nabla\cdot\mathbf{v}_\varepsilon = 0$$

with $\mathbf{v}_\varepsilon = \mathbf{v}_D$ on Γ, $\mathbf{v}_\varepsilon(0) = \mathbf{v}_0$ and also $p_\varepsilon(0)$ given.

Stability analysis for small values of $r \geq 0$

Theorem (A priori estimates for VPP$_{r,\varepsilon}$ and stability for N.S.)

There exists $K = K\left(||\mathbf{f}||_{L^2(0,T;H^{-1})}, ||\mathbf{v}_0||_1, ||p_0||_0\right) > 0$, $\delta t_0 > 0$ *and* r_0 *small enough satisfying the additionnal assumption :*

$$(\mathcal{H}_{r,\varepsilon}) \quad 4r_0(\mathrm{Re}+\varepsilon) \leq 1, \quad 4c(\Omega)\sqrt{\mathrm{Re}}\, r_0\varepsilon \leq \sqrt{\delta t}, \quad 0 < \delta t \leq \delta t_0$$

where $c(\Omega)$ *is the Poincaré constant, such that for all* $r \leq r_0$ *we have :*

$$(i) \quad ||\mathbf{v}^{n+1}||_0^2 + \varepsilon\delta t\,||p^{n+1}||_0^2 + \sum_{k=0}^{n} \frac{\delta t}{16\mathrm{Re}}||\nabla \mathbf{v}^{k+1}||_0^2 + \sum_{k=0}^{n}\left(\frac{1}{4}||\mathbf{v}^{k+1} - \mathbf{v}^k||_0^2 + \varepsilon\delta t\,||p^{k+1} - p^k||_0^2\right) \leq K$$

$$(ii) \quad \sum_{k=0}^{n} \delta t\,||p^{k+1}||_0^2 \leq C$$

$$(iii) \quad \sum_{k=0}^{n} \delta t\,||\nabla\cdot \mathbf{v}^{k+1}||_0^2 \leq C\,\varepsilon.$$

$\Rightarrow$ Convergence result with compactness arguments when δt tends to 0

Theorem (Basic error estimates for VPP$_{r,\varepsilon}$ (Stokes problem).)

Assume $(\mathbf{v}, \boldsymbol{p})$ *the solution of the Dirichlet-Stokes problem smooth enough in time and space, well-prepared initial conditions* $\mathbf{v}^0 \in \boldsymbol{H}^1(\Omega)^d$, $\boldsymbol{p}^0 \in \boldsymbol{L}_0^2(\Omega)$ *such that :*

$$(\mathcal{H}_0) \quad \left(1 + \frac{1}{r\varepsilon}\right) ||\mathrm{e}^0||_0^2 + \frac{\delta t}{r} ||\pi^0||_0^2 \leq c\, \delta t^2, \quad 0 < \delta t \leq 1$$

then there exists $C = C(\Omega, T, \mathrm{Re}, \mathrm{f}, \mathrm{v}_0, \mathrm{e}^0, \pi^0) > 0$ *such that we have for all* $n \in \mathbb{N}$ *with* $(n+1)\delta t \leq T$,

$$(i) \quad ||\mathrm{e}^{n+1}||_0^2 + \frac{\varepsilon \delta t}{1 + r\varepsilon} ||\pi^{n+1}||_0^2 + \sum_{k=0}^{n} \frac{\delta t}{\mathrm{Re}} ||\nabla \mathrm{e}^{k+1}||_0^2 \leq C \left(\delta t^2 + \varepsilon \frac{\delta t}{r}\right)$$

$$(ii) \quad \sum_{k=0}^{n} \delta t\, ||\pi^{k+1}||_0^2 \leq C \left((1 + r\varepsilon)\delta t + \frac{\varepsilon}{r} + \varepsilon^2\right)$$

$$(iii) \quad \sum_{k=0}^{n} \delta t\, ||\nabla \cdot \mathrm{v}^{k+1}||_0^2 = \sum_{k=0}^{n} \delta t\, ||\nabla \cdot \mathrm{e}^{k+1}||_0^2 \leq C(r, \varepsilon)\, \varepsilon^2 \delta t.$$

$\Rightarrow$ Improved error estimates with bounds on the time translates errors...

⇒ *Analysis of Navier-Stokes problems with practical algorithms*

Theorem (Error estimates for VPP$_{r,\varepsilon}$ with the Stokes problem.)

Assume $(\mathbf{v}, p)$ *the solution of the Dirichlet-Stokes problem smooth enough in time and space, well-prepared initial conditions and small enough parameters such that,* $\mathbf{c}(\Omega)$ *being the Poincaré constant :*

$$(\mathcal{H}_{r,\varepsilon}) \quad 4r(\mathrm{Re} + \varepsilon) \leq 1, \quad 4c(\Omega)\sqrt{\mathrm{Re}}\, r\varepsilon \leq \sqrt{\delta t}, \quad 0 < \delta t \leq 1$$

then there exists $C = C(\Omega, T, \mathrm{Re}, \mathrm{f}, \mathrm{v}_0, \mathrm{e}^0, \pi^0) > 0$ *such that we have for all* $n \in \mathbb{N}$ *with* $(n+1)\delta t \leq T$,

$$(i) \qquad ||\mathrm{e}^{n+1}||_0^2 + \varepsilon\delta t\, ||\pi^{n+1}||_0^2 + \sum_{k=0}^{n} \frac{\delta t}{\mathrm{Re}} ||\nabla \mathrm{e}^{k+1}||_0^2 \leq C\left(\delta t^2 + \varepsilon^2 \delta t^{\frac{3}{2}}\right)$$

$$(ii) \qquad \sum_{k=0}^{n} \delta t\, ||\pi^{k+1}||_0^2 \leq C\left(\delta t^2 + \varepsilon^2 \delta t\right), \ ||\nabla \mathrm{e}^{n+1}||_0^2 \leq C\, \mathrm{Re}^2 \left(\delta t + \varepsilon^2\right)$$

$$(iii) \qquad \sum_{k=0}^{n} \delta t\, ||\nabla \cdot \mathrm{v}^{k+1}||_0^2 = \sum_{k=0}^{n} \delta t\, ||\nabla \cdot \mathrm{e}^{k+1}||_0^2 \leq C\left(\delta t + \varepsilon\right) \varepsilon \delta t^2.$$

Convergence analysis for small values of $r \geq 0$

Proof : *main steps*

1. basic energy estimates of the errors $\mathbf{e}^n = \mathbf{v}^n - \mathbf{v}(t_n)$, $\tilde{\mathbf{e}}^n = \tilde{\mathbf{v}}^n - \mathbf{v}(t_n)$, $\pi^n = p^n - p(t_n)$
   - eliminate the term $(\pi^{n+1}, \nabla\cdot \mathbf{e}^{n+1})$
   - use Nečas lemma to calculate and estimate : $(\pi^{n+1}, \delta\bar{p}^{n+1}) = (\pi^{n+1}, \nabla\cdot \mathbf{u}^{n+1})$ for some $\mathbf{u}^{n+1} \in H_0^1(\Omega)^d$ ...
   - estimate the term $(\pi^{n+1}, \nabla\cdot \tilde{\mathbf{e}}^{n+1})$ by suitable bounds of $|\tilde{\mathbf{e}}^{n+1}|$, $|\nabla \tilde{\mathbf{e}}^{n+1}|$, $|\nabla\cdot \tilde{\mathbf{e}}^{n+1}|$ with an energy inequality for $\hat{\mathbf{v}}^{n+1}$...
   - absorption of some terms on the left-hand side if r, ε small enough
   - well-prepared initial conditions
2. idem for bounds of the time increments $\delta\mathbf{e}^{n+1} = \mathbf{e}^{n+1} - \mathbf{e}^n$, $\delta\pi^{n+1} = \pi^{n+1} - \pi^n$

   $\Rightarrow$ needs :

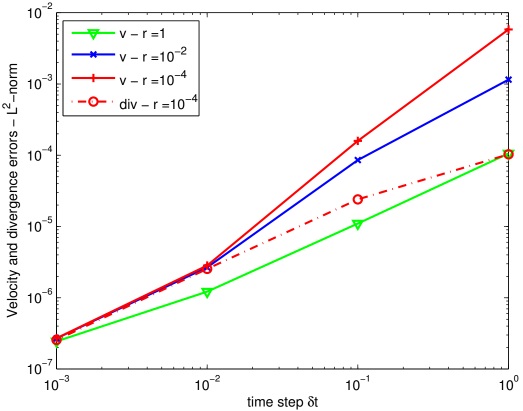
   - additional regularity assumptions

   - stronger assumptions on well-prepared initial conditions
3. establish improved error estimates (quasi-optimal)

It seems also working for regular Navier-Stokes solutions (at least in 2-D)

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Velocity error (discrete $l^\infty(0,T;L^2(\Omega)^d)$ norm) versus time step δt



Velocity convergence in time at $\mathbf{Re}=100$, $t=10$ - $h=1/512$, $\varepsilon=1$

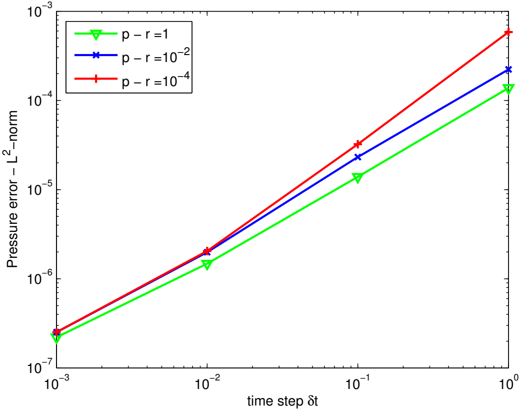
$\Rightarrow$ Time convergence in $\mathcal{O}(\delta t)$ – $||\nabla\cdot \mathbf{v}^n||_{L^2} = \mathcal{O}(\delta t)$ for small r

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical results

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Pressure error (discrete $\boldsymbol{l^\infty(0,T;L^2(\Omega))}$ norm) versus time step $\boldsymbol{\delta t}$



Pressure convergence in time at $\mathbf{Re}=\mathbf{100}$, $\boldsymbol{t=10}$ - $\boldsymbol{h=1/512}$, $\boldsymbol{\varepsilon=1}$

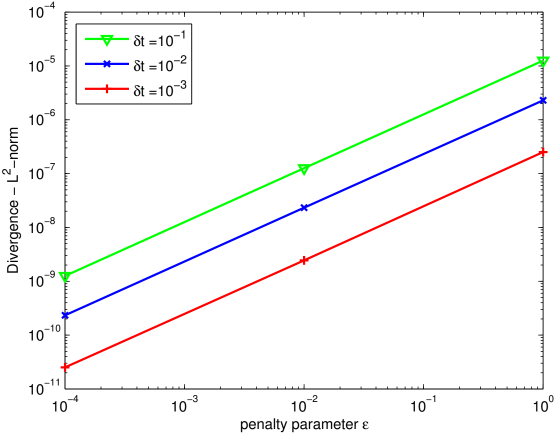
$\Rightarrow$ Time convergence in $\mathcal{O}(\delta t)$

Stagnation threshold = space discretization error in $\mathcal{O}(h^2)$

Numerical results

Green-Taylor vortices : Navier-Stokes with Dirichlet B.C.

Divergence (discrete $l^\infty(0,T;L^2(\Omega))$ norm) versus penalty ε



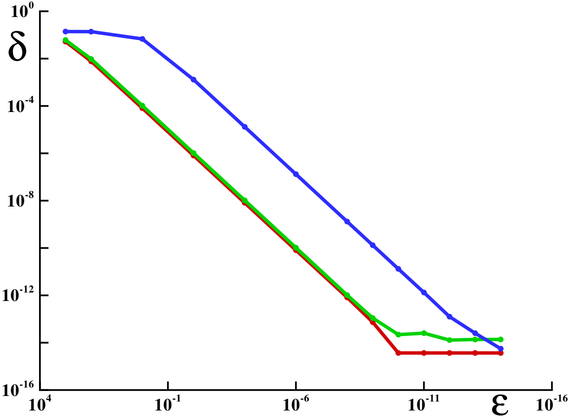
Divergence at $\mathbf{Re} = 100$, $t = 10$ - $h = 1/512$, $r = 1$, $|res|_2 < 10^{-10}$

$\Rightarrow ||\nabla \cdot \mathrm{v}^n||_{L^2} = \mathcal{O}(\varepsilon \delta t)$

Numerical results

Rayleigh-Bénard natural convection in a heated cavity

Convergence of the penalty-correction step : divergence $\boldsymbol{L^2}$-norm $\boldsymbol{\delta}$



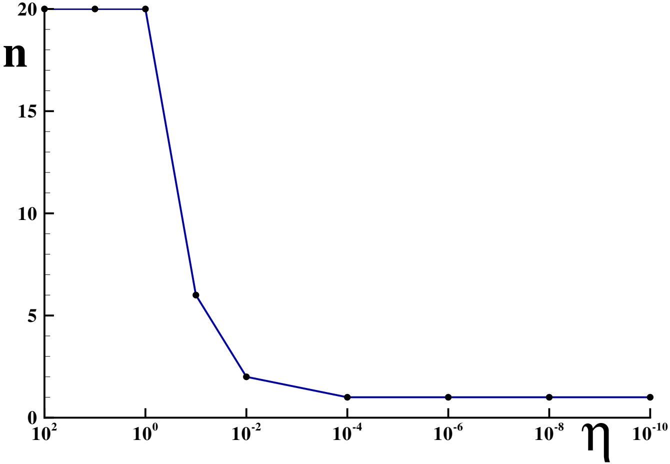
Natural convection at $\boldsymbol{Ra=10^5}$ and $\boldsymbol{t=2\delta t}$ with $\boldsymbol{\delta t=1}$, $\boldsymbol{h=1/256}$ – $\boldsymbol{\mu=0}$ or $\boldsymbol{1.85\,10^{-5}}$ (idem) and $\boldsymbol{\mu=1.85\,10^{-1}}$.

$\Rightarrow ||\nabla\cdot \mathrm{v}||_{L^2} = \mathcal{O}(\varepsilon)$ until $\mathbf{10^{-15}}$ (machine zero)

Numerical results

Rayleigh-Bénard natural convection in a heated cavity

Cost of the penalty-correction step : number of MILU-BiCGStab solver iterations versus $\boldsymbol{\eta = \varepsilon/\delta t}$



$\boldsymbol{Ra = 10^5}$ and $\boldsymbol{t = 2\delta t}$ with $\boldsymbol{\delta t = 1}$, $\boldsymbol{h = 1/256}$, $\boldsymbol{|r_k|_2 \leq 10^{-6}}$.

Outlines

Conclusion

Vector penalty-projection methods for incompressible flows
[Angot, Caltagirone and Fabrie, FVCA5 2008, ...]

- The Lagrangian augmentation with $r > 0$ in the prediction step plays the role of a preconditioner
- Small values of $0 < r \leq 10^{-2}$ sufficient to get a good pressure field
  For $r = 0$, the pressure converges only poorly...
- Approximate projection with a vector penalty-correction step all the cheaper as $\varepsilon\delta t \to 0$
- Same convergence properties as the scalar penality-projection method for $r > 0$
- Vector penalty-correction step all the less dependent on density or viscosity as $\varepsilon\delta t \to 0$
- L^2-norm of velocity divergence as $\mathcal{O}(\varepsilon\delta t)$ until machine precision
  $\Rightarrow$ cheap method for small values of $r \leq 10^{-2}$ and $\varepsilon \leq 10^{-2}$

Some perspectives...

- Other numerical experiments (in progress)
- Other convergence analysis for Navier-Stokes :
  outflow boundary conditions
  variable density flows
- improve the pressure reconstruction by using a consistent pressure correction, as for the scalar penalty-projection method...

$$\Rightarrow \qquad \nabla \cdot \left(\frac{\delta t}{\varrho^{n+1}} \nabla \phi \right) = \nabla \cdot \tilde{u}^{n+1}$$

  only solved to get the pressure
  $\Rightarrow$ precision of $\mathbf{10^{-3}}$ generally sufficient
  but not to get a free-divergence velocity field !

Thank you