The G-scheme

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joint work with L. Agelas (IFP) and D. Di Pietro (IFP)

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Contents

1. Introduction and generic notions
2. The G-scheme
3. A few words on the theoretical results
4. Some numerical results

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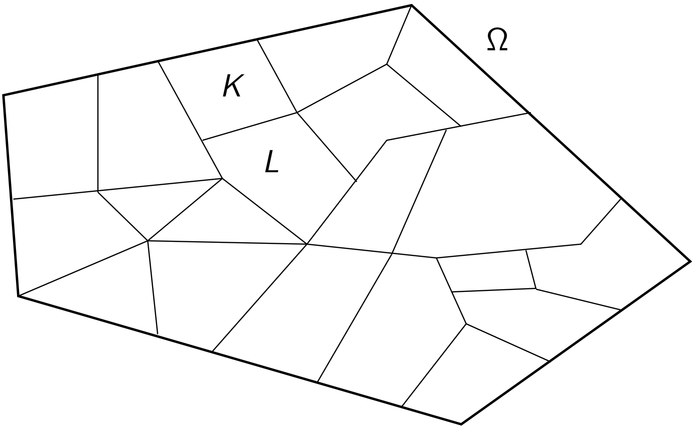
Equation

$$\begin{cases} -\operatorname{div}(\Lambda \nabla \bar{u}) = f & \text{in } \Omega, \\ \bar{u} = g & \text{on } \partial\Omega \end{cases}$$

with:

- Ω open subset of $\mathbb{R}^d$ ($d \geq 2$),
- $\Lambda : \Omega \to M_d(\mathbb{R})$ a (symmetric) uniformly elliptic diffusion tensor,
- $f \in L^2(\Omega)$,
- $g \in H^{1/2}(\partial\Omega)$.

Grids and unknowns



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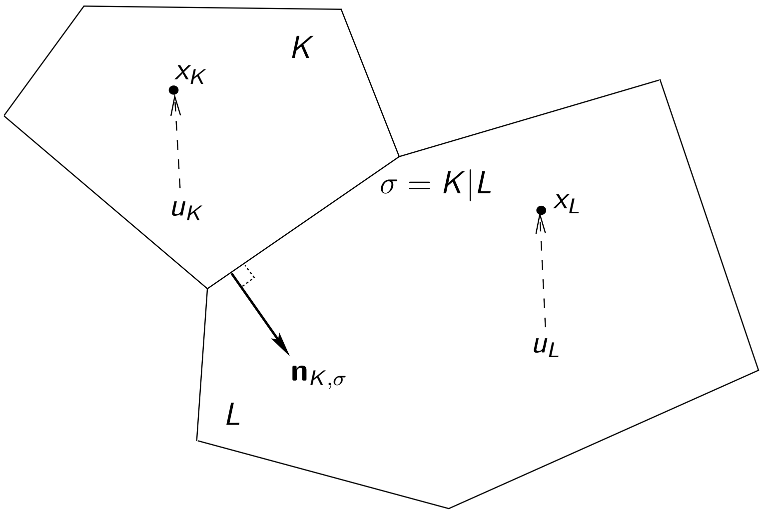

Basic principle

Define consistent flux approximations $F_{K,\sigma}(u)$ using $u = (u_K)_K$

$$F_{K,\sigma}(u) \approx \int_\sigma -\Lambda\nabla\bar{u}\cdot\mathbf{n}_{K,\sigma}.$$

Once this is done, the scheme is given by:

Flux balance

$$\forall K \ : \ \sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma} = \int_K f$$

with $\mathcal{E}_K$= edges (faces in 3d) of the cell K.

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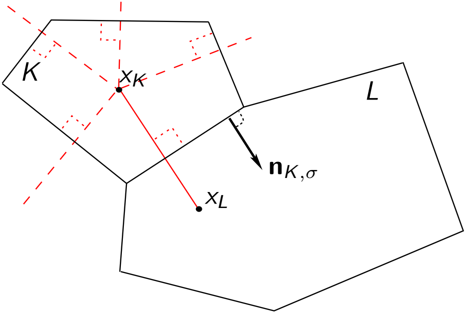
It might also be reasonnable to have:

Flux conservativity:

$$\forall \sigma \text{ between } K \text{ and L} \ : \ F_{K,\sigma} + F_{L,\sigma} = 0.$$

A simple case: 2-point flux Finite Volume scheme

$\Lambda = \mathrm{Id}$ (model problem: $-\Delta \bar{u} = f$) and



With m_σ= measure of σ, then $F_{K,\sigma} \approx \dfrac{\mathrm{m}_\sigma}{\mathrm{dist}(x_K, x_L)}(u_K - u_L)$.

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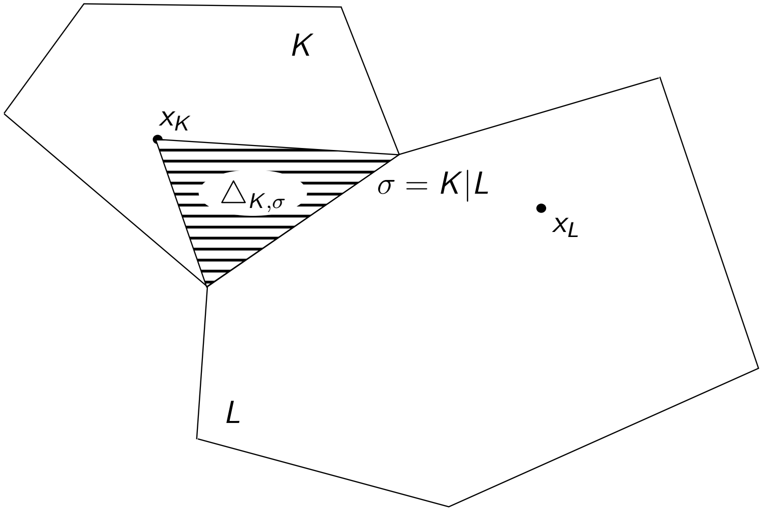
First idea: half-diamond gradients

Half-diamond:



First idea: half-diamond gradients

Local gradients $\nabla_{K,\sigma} u \approx (\nabla u)_{|\triangle_{K,\sigma}}$ defined by:

Coherent values on the edge

$$\forall \sigma \text{ between } K \text{ and } L, \forall x \in \sigma \ :$$
$$u_K + (\nabla_{K,\sigma} u) \cdot (x - x_K) = u_L + (\nabla_{L,\sigma} u) \cdot (x - x_L)$$

Conservativity of the fluxes

$$\forall \sigma \text{ between } K \text{ and } L \ : \ \Lambda_K (\nabla_{K,\sigma} u) \cdot \mathbf{n}_{K,\sigma} + \Lambda_L (\nabla_{L,\sigma} u) \cdot \mathbf{n}_{L,\sigma} = 0.$$

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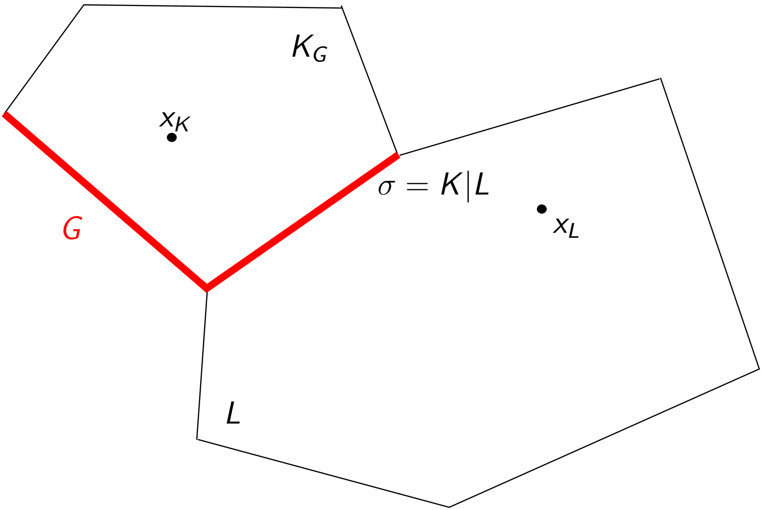
- Not enough relations...

Group gradients

Groups of edges = sets of d edges of a same cell and with a common vertex.

K_G: cell containing the group G.



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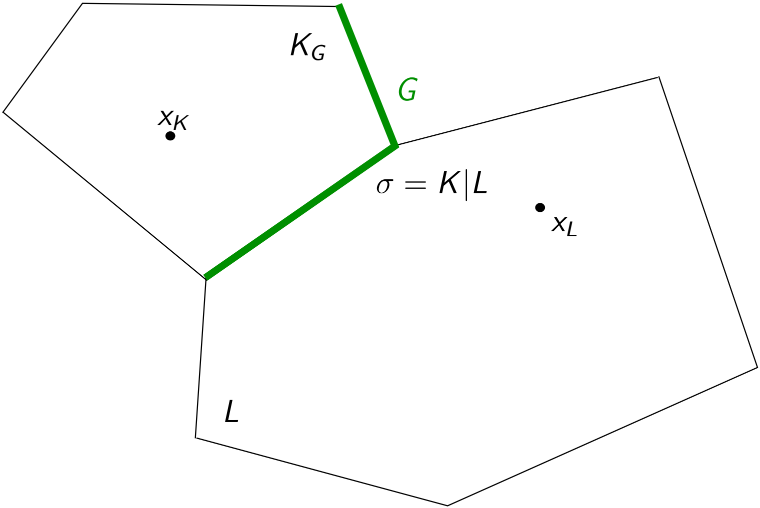

Group gradients

For all $\sigma \in G$ and K containing σ, $\nabla^G_{K,\sigma} u \approx (\nabla u)_{|\triangle_{K,\sigma}}$ defined by:

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Specific role of K_G: **we impose** that $\nabla^G_{K_G,\sigma} u$ does not depend on $\sigma \in G$, i.e.

$$\forall \sigma, \sigma' \in G \ : \ \nabla^G_{K_G,\sigma} u = \nabla^G_{K_G,\sigma'} u \quad (=: \nabla^G_{K_G} u)$$

- These group gradients are (nearly always) uniquely defined from $(u_K)_K$.

Definition of the fluxes

Group fluxes:

$$F_{K,\sigma}^{G}(u) = -\mathrm{m}_{\sigma}\Lambda_K(\nabla_{K,\sigma}^{G}u)\cdot\mathbf{n}_{K,\sigma}.$$

Full fluxes: convex combination of all possible group fluxes for each edge σ.

- $\mathcal{G}_{\sigma} = \{G \text{ group s.t. } \sigma \in G\}$,
- $(\theta_{\sigma}^{G})_{G\in\mathcal{G}_{\sigma}}$ s.t. $\theta_{\sigma}^{G} \geq 0$ and $\sum_{G\in\mathcal{G}_{\sigma}} \theta_{\sigma}^{G} = 1$,
- Fluxes:

$$F_{K,\sigma}(u) = \sum_{G\in\mathcal{G}_{\sigma}} \theta_{\sigma}^{G} F_{K,\sigma}^{G}(u).$$

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Convergence tools

Coercivity of the scheme: the G-scheme is only *conditionnally* coercive.

- Local sufficient coercivity condition (can be numerically tested).

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Consistency: let's see...

Consistency of the fluxes: not on regular functions

- In general, the gradients $\nabla^G_{K,\sigma} u$ (and thus the fluxes $F_{K,\sigma}(u)$) are **not consistent** on C^∞ functions:

$$\text{if } \varphi \in C^\infty \text{ and } \varphi_{\mathcal{T}} = (\varphi(x_K))_K,$$

$$F_{K,\sigma}(\varphi_{\mathcal{T}}) = \int_\sigma -\Lambda \nabla \varphi \cdot \mathbf{n}_{K,\sigma} + \mathrm{m}_\sigma \mathcal{O}(1)$$

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(we would expect $\mathrm{m}_\sigma \mathcal{O}(h)$).

Cause: Conservativity of the fluxes.

$$\Lambda_K(\nabla^G_{K,\sigma} u) \cdot \mathbf{n}_{K,\sigma} + \Lambda_L(\nabla^G_{L,\sigma} u) \cdot \mathbf{n}_{L,\sigma} = 0$$

with possibly $\Lambda_K \neq \Lambda_L$.

Consistency of the fluxes: for non-standard test functions

We assume that Λ is piecewise regular on a partition $(\Omega_i)_{i=1,k}$ of Ω.

Space of test functions: the fluxes **are consistent** on the space $\mathcal{S}$ of functions φ such that

$$\varphi : \overline{\Omega} \to \mathbb{R} \text{ is continuous and vanishes on } \partial\Omega,$$
$$\forall i = 1, \ldots, k\,,\ \varphi_{|\Omega_i} \in C^2(\overline{\Omega_i}),$$
$$\forall i, j = 1, \ldots, k\,,\ (\Lambda\nabla\varphi)_{|\Omega_i} \cdot \mathbf{n}_{\Omega_i} + (\Lambda\nabla\varphi)_{|\Omega_j} \cdot \mathbf{n}_{\Omega_j} = 0 \text{ on } \overline{\Omega_i} \cap \overline{\Omega_j}.$$

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Lemma

$\mathcal{S}$ is dense in $H_0^1(\Omega)$.

Convergence result

Using the space $\mathcal{S}$ and classical discrete compactness results:

Theorem

Under a coercivity assumption and if the grids do not degenerate as $h \longrightarrow 0$, the solution u to the G-scheme converges strongly in L^2 to the weak solution $\bar{u}$ of the elliptic problem.

The discrete gradients of u also converge in L^2 to $\nabla \bar{u}$.

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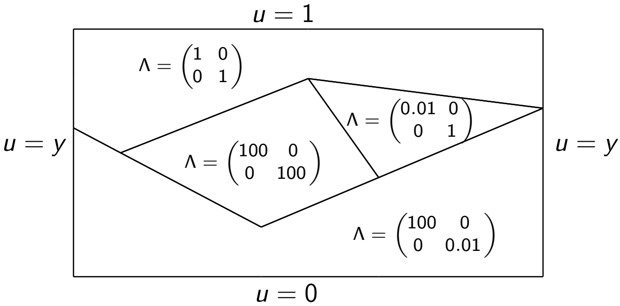
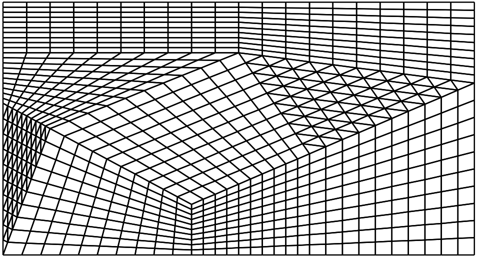
General behavior

- Exact on affine solutions.
- As many other FV method, general order 2 convergence for the solution and order 1 convergence for the gradient.

A strong anisotropy and heterogeneity test





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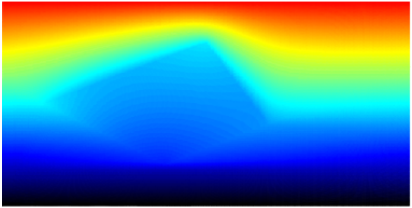
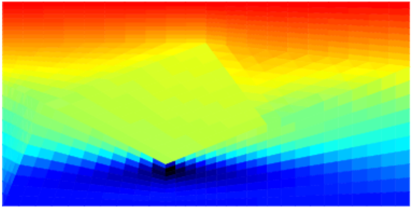
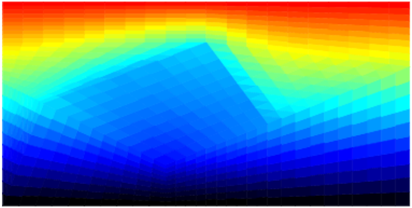


Exact solution



HMM solution



G-scheme solution

A strong anisotropy and heterogeneity test

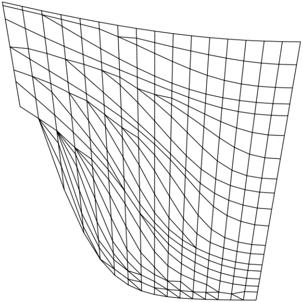
- L^2 errors:
  - HMM: 22%
  - G-scheme: 4%.

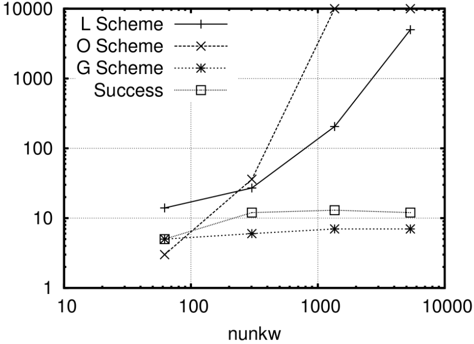
- Minima and maxima of the solutions:

	min	max
theoretical	0	1
HMM	-0.36	0.99
G-scheme	0.00967	0.99

An element of comparison with MPFA L

$\Lambda = \mathrm{diag}(0.1, 1)$, $\bar{u}(x, y) = \sin(\pi x)\sin(\pi y)$.





… and recall also some results presented in the benchmark session…

Thanks.

Thanks.

- to lunch!