Yet another nine-point scheme

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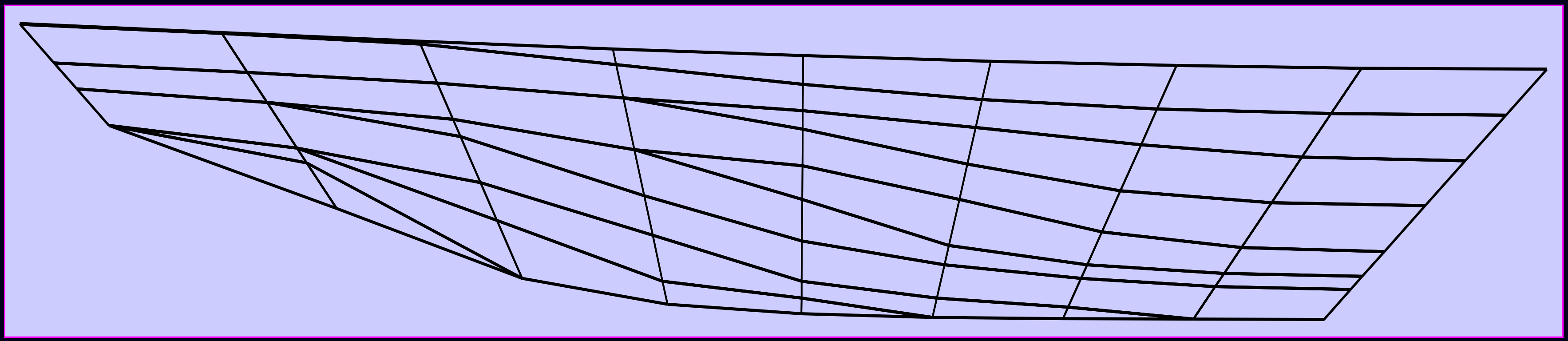
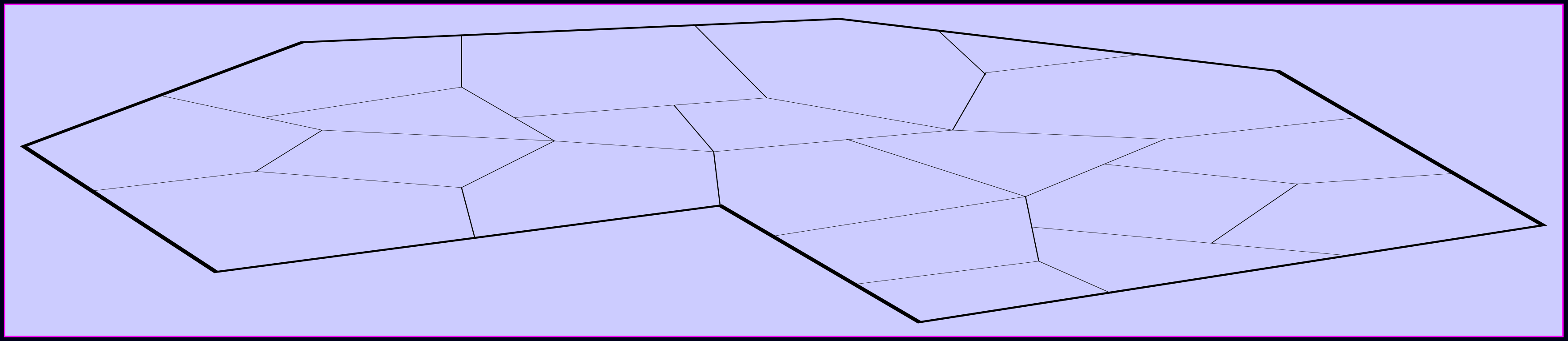
Specifications for the approximation of diffusion terms

be valid on 3D generalizations of such grids see 3D benchmark: R. Herbin, F. Hubert





Specifications for the approximation of diffusion terms

apply to anisotropic heterogeneous diffusion operators

$$-\mathrm{div}(\Lambda(x)\nabla u)$$

examples

geological data $\Lambda(x) = \lambda_h(x)\ e_h(x) \otimes e_h(x) + \lambda_v(x)\ e_v(x) \otimes e_v(x)$

diffusion “dispersion” $\Lambda(x) = (\lambda_f + \mu_t |v(x)|)\mathrm{Id} + \dfrac{\mu_l - \mu_t}{|v(x)|}\ v(x) \otimes v(x)$

matricial relative permeabilities

Specifications for the approximation of diffusion terms

provide conservative fluxes between control volumes

$$F_{KL}(u) + F_{LK}(u) = 0$$

used for transport of species $\sum_L w_{KL} F_{KL}(u)$

examples:

oil engineering

water resources management

nuclear waste storage simulation

Specifications for the approximation of diffusion terms

be exact on coarse mesh with heterogeneous anisotropic diffusion when affine solution

(cf. 2 point flux in 1D problems)

be precise on 2D and 3D meshes with one layer per rock type

with homogeneous control volumes

$$-\Delta u = 0$$
$$\partial_t w - \operatorname{div}(w \nabla u) = 0$$

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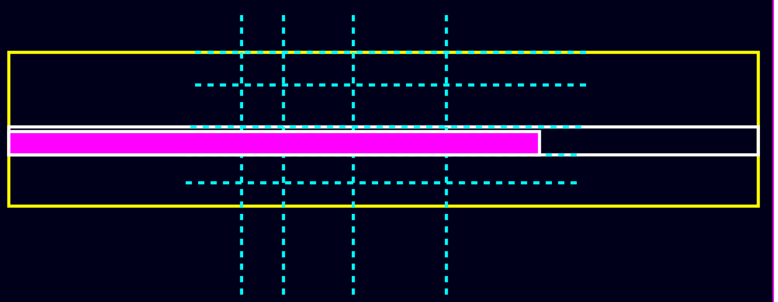

$$\sum_L F_{KL}(u) = 0$$

$$|K|\frac{w_K^{n+1} - w_K^n}{\delta t} + \sum_L (w_K^n F_{KL}(u)^+ - w_L^n F_{KL}(u)^-) = 0$$

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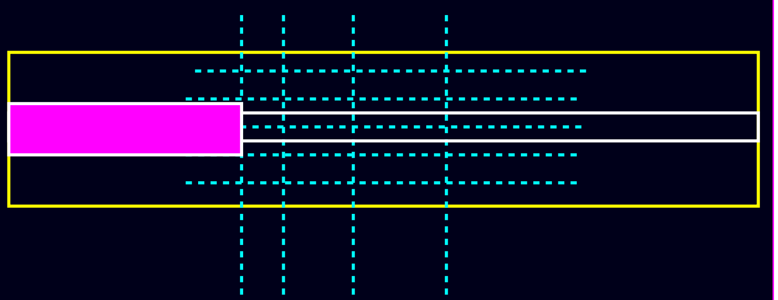

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Specifications for the approximation of diffusion terms

use values $(u_K)_{K\in\mathcal{M}}$ as primary variables

elimination of any $(u_\sigma)_{\sigma\in\mathcal{E}}$

stencil: $\mathcal{M}_K = \{L \in \mathcal{M}, \exists u,\ F_{KL}(u) \neq 0\}$

for parallel architectures, 9-point in 2D, 27-point in 3D

Specifications for the approximation of diffusion terms

convergence properties

$$\sum_L F_{KL}(u) = \int_K f$$ implies u converges to continuous solution

resulting from coercivity properties

$$\|u\|_D^2 \leq \alpha \sum_K \sum_L F_{KL}(u)(u_K - u_L)$$

consistency properties

$$\sum_K \sum_L F_{KL}(u)(\varphi_K - \varphi_L) \text{ converges to } \int_\Omega \nabla u \cdot \Lambda \nabla \varphi$$

symmetry properties (necessary for "convective nine-point scheme")

$$\sum_K \sum_L F_{KL}(u)(v_K - v_L) = \sum_K \sum_L F_{KL}(v)(u_K - u_L)$$

Specifications for the approximation of diffusion terms

satisfy a local maximum principle for solution of $$\sum_L F_{KL}(u) = 0$$

$$\exists \mathcal{M}_K, \min_{L \in \mathcal{M}_K} u_L \le u_K \le \max_{L \in \mathcal{M}_K} u_L$$

or less strong properties...

ensured by (non)linear schemes such that

$$F_{KL}(u) = \sum_M T_{KL}^M(u)(u_K - u_M) \text{ with } T_{KL}^M(u) \ge 0$$

ex: Le Potier's works

Schemes and specifications

restricted on linear schemes

schemes	gen. 2D mesh	gen. 3D mesh	gen. diff. matrix	exact aff. sol.	stencil	coerc. consist. symm.	max.
2-point	N	N	N	Y	Y	Y Y Y	Y
P1 FE	Y/N	Y/N	Y	N	Y	Y Y Y	Y/N
Mix.FE	Y/N	Y/N	Y	Y	N	Y Y Y	N
HMM	Y	Y	Y	Y	N	Y Y Y	N
DDFV	Y	Y	Y	Y	N	Y Y Y	Y/N
MPFA	Y	Y	Y	Y	Y	N Y N	N
G-Scheme	Y	Y	Y	Y	Y	Y Y N	N
SUSHI	Y	Y	Y	N/Y	N	Y Y Y	N
YA9PS	Y	N	Y	Y	Y	Y Y Y	N

no scheme with only “Y”...

YA9PS: new scheme inspired from MPFA, HMM and SUSHI

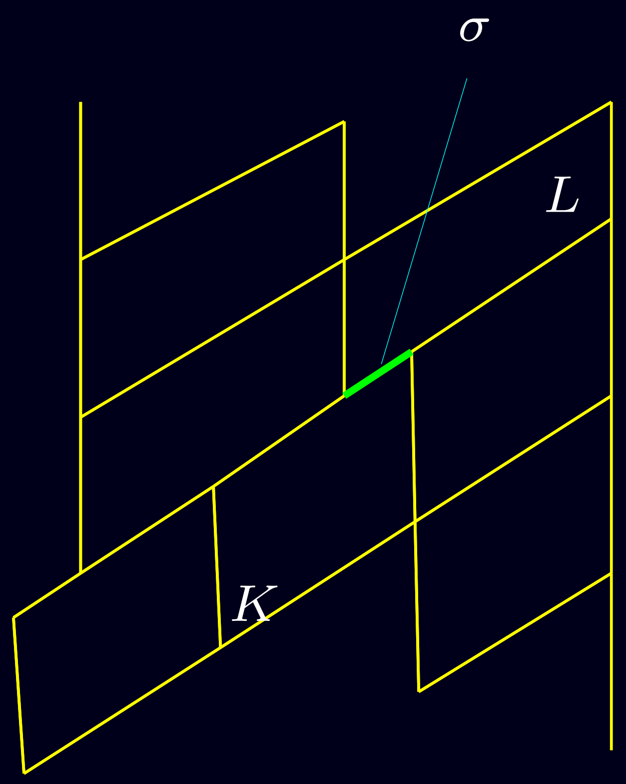
first step: recall HMM

second step: use HMM and MPFA for YA9PS

Notations for HMM methods

continuous problem: $-\mathrm{div}\Lambda\nabla\bar{u} = f$ **in** Ω **and** $\bar{u} = 0$ **on** $\partial\Omega$

$$\sum_{\sigma\in\mathcal{E}_K} \overline{F}_{K,\sigma}(\bar{u}) = \int_K f(x)\mathrm{d}x \text{ with } \overline{F}_{K,\sigma}(\bar{u}) = -\int_\sigma \Lambda(x)\nabla\bar{u}(x)\cdot n_{K,\sigma}\mathrm{d}s(x)$$



find $u \in X_{\mathcal{D},0}$ **s.t.** $\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u) = \int_K f(x)\mathrm{d}x$

and $F_{K,\sigma}(u) + F_{K,\sigma}(u) = 0$ **if** $\mathcal{M}_\sigma = \{K, L\}$

with

$X_{\mathcal{D},0} = \{(u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}},\ u_\sigma = 0 \text{ for } \sigma \in \mathcal{E}_{\mathrm{ext}}\}$

$\sigma \in \mathcal{E}_{\mathrm{int}} :\ \mathcal{M}_\sigma = \{K, L\}$, $\sigma \in \mathcal{E}_{\mathrm{ext}} :\ \mathcal{M}_\sigma = \{K\}$

Construction of HMM methods

first step: write FV scheme under weak form

$$X_{\mathcal{D},0} = \{(u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}},\ u_\sigma = 0 \text{ for } \sigma \in \mathcal{E}_{\text{ext}}\}$$

find $u \in X_{\mathcal{D},0}$ s.t. $\displaystyle\sum_{\sigma\in\mathcal{E}_K} F_{K,\sigma}(u) = \int_K f(x)\mathrm{d}x$

and $F_{K,\sigma}(u) + F_{K,\sigma}(u) = 0$ if $\mathcal{M}_\sigma = \{K, L\}$

$$\langle u, v\rangle_{\mathcal{D}} = \sum_{K\in\mathcal{M}}\sum_{\sigma\in\mathcal{E}_K}(v_K - v_\sigma)F_{K,\sigma}(u)$$

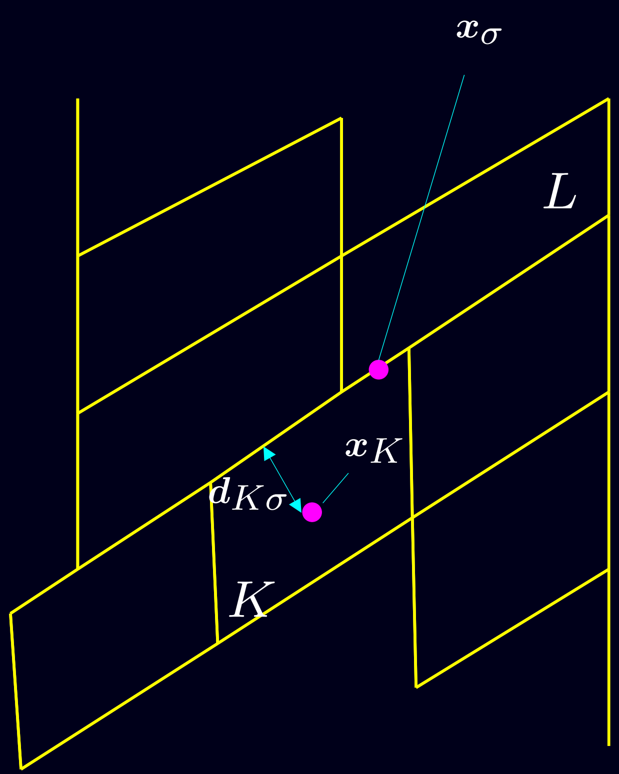
$\Pi_{\mathcal{M}} v(x) = v_K$ if $x \in K$

find $u \in X_{\mathcal{D},0}$ s.t.

$$\langle u, v\rangle_{\mathcal{D}} = \int_\Omega f(x)\Pi_{\mathcal{M}} v(x)\mathrm{d}x,\ \forall v \in X_{\mathcal{D},0}$$

Construction of HMM methods

second step: use discrete gradient, defined thanks to magical formula



$$\forall G \in \mathbb{R}^d,\ \forall K \in \mathcal{M}, |K| G = \sum_{\sigma \in \mathcal{E}_K} |\sigma| (x_\sigma - x_K) \cdot G\, n_{K,\sigma}$$

$$\nabla_K u = \sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{|K|} (u_\sigma - u_K) n_{K,\sigma}$$

$$(R_K u)_\sigma = \frac{1}{d_{K,\sigma}} \left(u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)\right)$$

$$\langle u, v \rangle_{\mathcal{D}} = \sum_{K \in \mathcal{M}} |K| \Big(\nabla_K u \cdot \Lambda_K \nabla_K u + (R_K u)^T B_K R_K u \Big)$$

where B_K symmetric positive definite matrix with side $\#\mathcal{E}_K$ and bounded eigenvalues

thm (DEGH2009): HMM coercive consistent symmetric

SUSHI scheme

elimination of selected interfaces unknowns by barycentric averaging

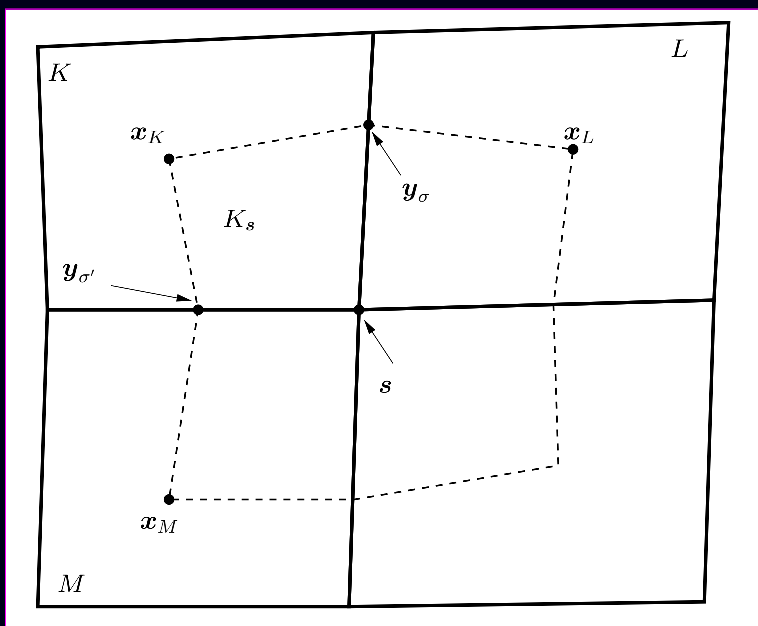
find $$u \in X_{\mathcal{B}} \text{ s.t. } \langle u, v \rangle_{\mathcal{D}} = \int_{\Omega} f(x) \Pi_{\mathcal{M}} v(x) \mathrm{d}x, \ \forall v \in X_{\mathcal{B}}$$

where $$X_{\mathcal{B}} = \{u \in X_{\mathcal{D},0}, \ u_\sigma = \sum_K a_\sigma^K u_K \text{ for } \sigma \in \mathcal{B}\}$$

setting $$x_\sigma = \sum_K a_\sigma^K x_K \text{ for all } \sigma \in \mathcal{B}$$

symmetric scheme

Build 9-point scheme using MPFA and SUSHI ideas



Build gradient in K_s

using magical formula

with values $u^{\epsilon}_{K,s}$ at centers of edges

$\epsilon = [x_K, y_{\sigma'}][y_{\sigma'}, s][s, y_\sigma][x_K, y_\sigma]$

values $u^{\epsilon}_{K,s}$ for $\epsilon = [y_{\sigma'}, s][s, y_\sigma]$

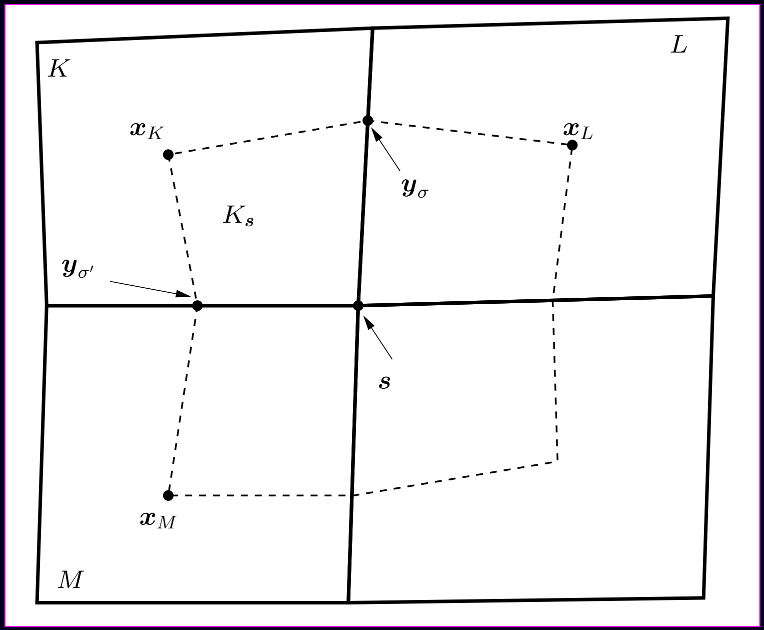
provided by flux conservation
(as in MPFA O-scheme)

values on $u^{\epsilon}_{K,s}$ for $\epsilon = [x_K, y_{\sigma'}][x_K, y_\sigma]$

provided by averaging expressions

$$\langle u, v\rangle_{\mathcal{D}} = \sum_K \sum_s \sum_{\epsilon \in \mathcal{E}_{K,s}} \sum_{\epsilon' \in \mathcal{E}_{K,s}} A^{\epsilon'\epsilon}_{K,s}(u^{\epsilon'}_{K,s} - u_K)(v^{\epsilon}_{K,s} - v_K)$$

Build 9-point scheme using MPFA and SUSHI ideas



elimination of $u^{\epsilon}_{K,s}$ for $\epsilon = [y_{\sigma'}, s][s, y_\sigma]$ provides

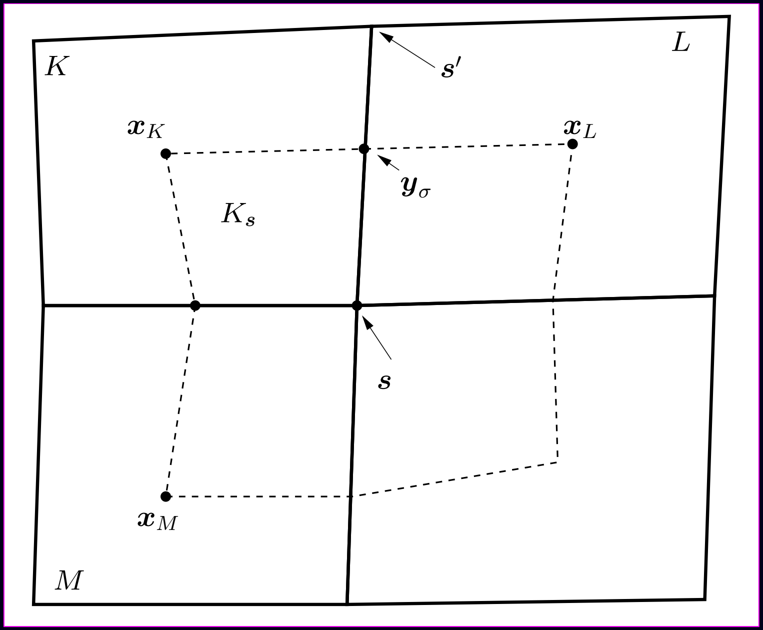
$u^{\epsilon}_{K,s}$ lin.comb. of four u_L

and of eight $u^{\epsilon'}_{K,s}$ for $\epsilon' = [x_K, y_\sigma] \ldots$

$$u^{\epsilon'}_{K,s} = \frac{1}{2}(u_K + u_\sigma)$$

9-point stencil needs $u_\sigma = \alpha u_K + (1 - \alpha) u_L \ldots$

Averaging on edges



first idea:

$$y_\sigma = [x_K, x_L] \cap [s, s']$$

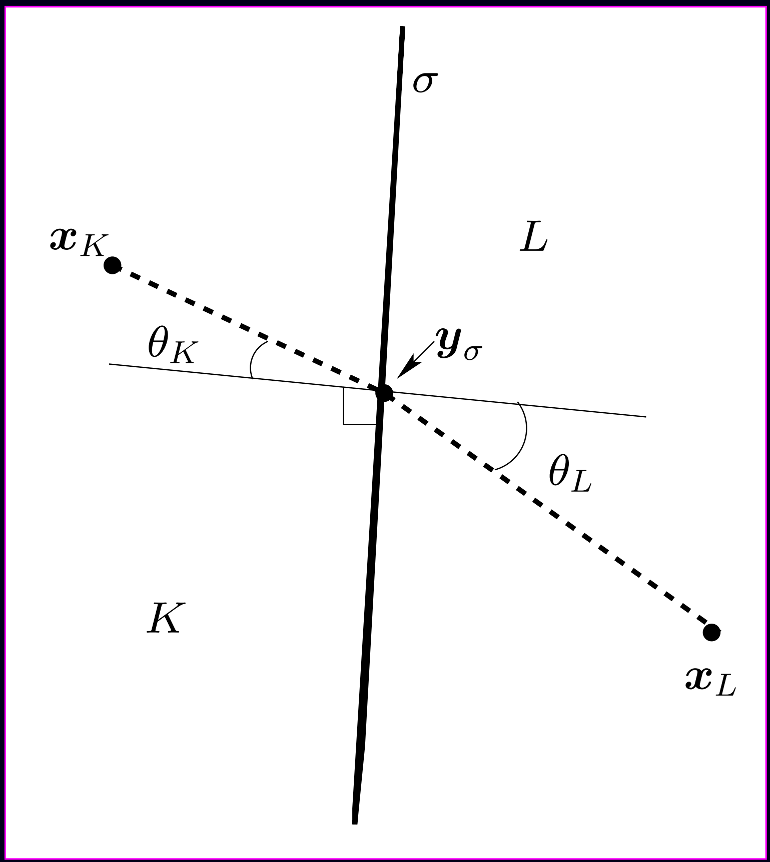
then barycentric expression

does not respect specification !

second idea:
use “harmonic averaging” instead
of barycentric value...

Need two-point expression in y_σ

A problem which looks like optics...



Problem: find point y_σ

s.t., for all u affine in K and L

continuous at the interface

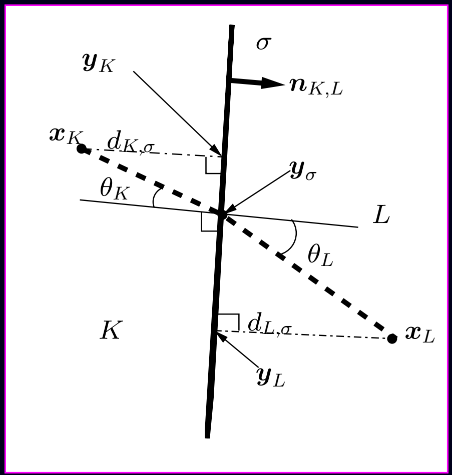
with $\boldsymbol{\Lambda_K \nabla u_{|K} \cdot n_{KL} = \Lambda_L \nabla u_{|L} \cdot n_{KL}}$

then $u(\boldsymbol{y}_\sigma)$ **given by lin. comb. of** $u(\boldsymbol{x}_K)$ **and** $u(\boldsymbol{x}_L)$

Recall: Snell-Descartes law

$$\boldsymbol{n_K \sin \theta_K = n_L \sin \theta_L}$$

Harmonic averaging points....



$$y_\sigma = \frac{\lambda_L d_{K,\sigma} y_L + \lambda_K d_{L,\sigma} y_K}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} + \frac{d_{K,\sigma} d_{L,\sigma}}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}} (\lambda_K^\sigma - \lambda_L^\sigma)$$

with

$$\lambda_K = n_{KL} \cdot \Lambda_K n_{KL} \quad \lambda_K^\sigma = (\Lambda_K - \lambda_K \mathrm{Id}) n_{KL}$$

$$\lambda_L = n_{KL} \cdot \Lambda_L n_{KL} \quad \lambda_L^\sigma = (\Lambda_L - \lambda_L \mathrm{Id}) n_{KL}$$

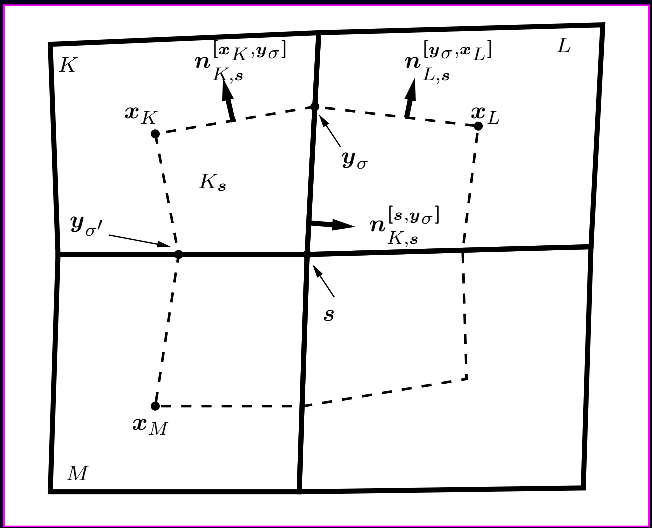
for all continuous function u affine in K and L s.t. $\Lambda_K \nabla u_{|K} \cdot n_{KL} = \Lambda_L \nabla u_{|L} \cdot n_{KL}$

$$u(y_\sigma) = \frac{\lambda_L d_{K,\sigma} u(x_L) + \lambda_K d_{L,\sigma} u(x_K)}{\lambda_L d_{K,\sigma} + \lambda_K d_{L,\sigma}}$$

Remark: if $\lambda_K^\sigma = \lambda_L^\sigma = 0$ then $\lambda_K \frac{y_K - y_\sigma}{d_{K,\sigma}} = \lambda_L \frac{y_\sigma - y_L}{d_{L,\sigma}}$ i.e. $\lambda_K \tan\theta_K = \lambda_L \tan\theta_L$

... no optics

Resulting construction of $\langle u, v\rangle_{\mathcal{D}}$



$$X_{\mathcal{B}} = \{(u_K)_{K\in\mathcal{M}}, (u_\sigma)_{\sigma\in\mathcal{E}}, (u_{\sigma,s})_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}}$$
u_σ given by harmonic point averaging$\}$

$u_{K,s}^{\epsilon} = \dfrac{u_K + u_\tau}{2}$ if $\epsilon = [x_K, y_\tau]$ and
$u_{K,s}^{\epsilon} = u_{\tau,s}$ if $\epsilon = [s, y_\tau]$ for $\tau = \sigma$ and σ'

$$|K_s|\nabla_{K,s}u = \sum_{\epsilon\in\mathcal{E}_{K,s}} |\epsilon|(u_{K,s}^{\epsilon} - u_K)n_{K,s}^{\epsilon}$$

$$\langle u, v\rangle_{\mathcal{D}} = \sum_{K\in\mathcal{M}}\sum_{s\in\mathcal{V}_K} |K_s| \left(\Lambda_K\nabla_{K,s}u\cdot\nabla_{K,s}v + \sum_{\tau=\sigma,\sigma'} \alpha_{K\tau}R_{K,s}^{\tau}uR_{K,s}^{\tau}v\right)$$

with $\alpha_{K\tau} > 0$, $R_{K,s}^{\tau}u = \dfrac{1}{d_{K\tau}}(u_\tau - u_K - \nabla_{K,s}u\cdot(y_\tau - x_K))$, for $\tau = \sigma$ et σ'

properties of scheme: coercive, symmetric, convergent

YA9PS: 9-point finite volume scheme

scheme can be expressed by

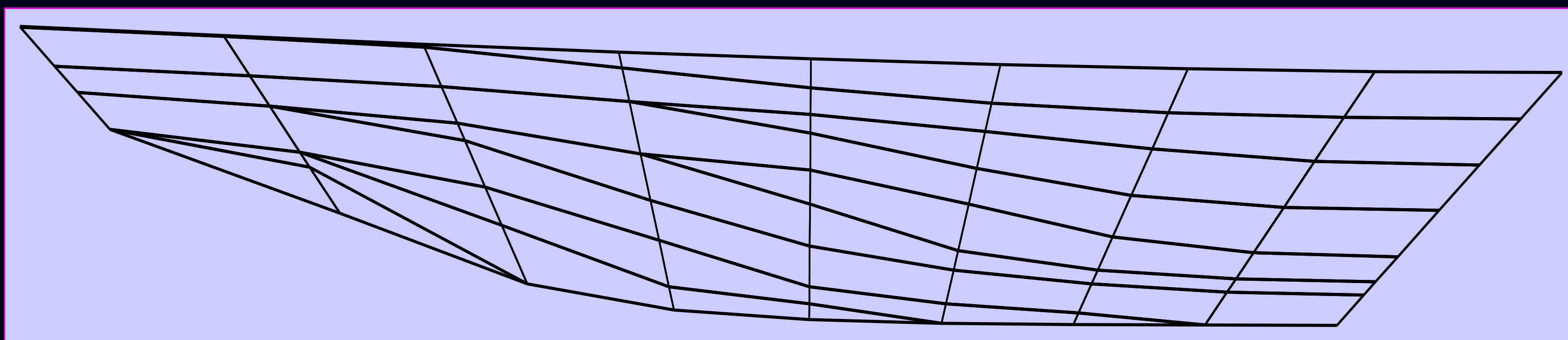
$$\forall K \in \mathcal{M}, \sum_{\substack{\sigma \in \mathcal{E}_K \\ \sigma = K|L}} F_{K,L}(u) + \sum_{\substack{\sigma \in \mathcal{E}_K \\ \sigma \subset \partial\Omega}} F_{K,\sigma}(u) = \int_K f(x) \mathrm{d}x$$

where expression of $F_{K,L}(u)$ can be deduced from

$$\langle u, v \rangle_{K,s} = \sum_{\epsilon \in \mathcal{E}_{K,s}} \sum_{\epsilon' \in \mathcal{E}_{K,s}} A_{K,s}^{\epsilon'\epsilon} (u_{K,s}^{\epsilon'} - u_K)(v_{K,s}^{\epsilon} - v_K) = \sum_{\epsilon \in \mathcal{E}_{K,s}} F_{K,s}^{\epsilon}(u) \; (v_{K,s}^{\epsilon} - v_K)$$

using conservation of fluxes and averaging expressions

Numerical results



$$u(x,y) = \sin(\pi x)\sin(\pi y)$$

	mesh 1	mesh 2	mesh 3	mesh 4	mesh 5
$\#\mathcal{M}$	62	302	1357	5363	21031
# hybrid edges	1	3	6	10	17
L^2-error	$9.15\ 10^{-3}$	$3.07\ 10^{-3}$	$9.30\ 10^{-4}$	$2.66\ 10^{-4}$	$6.89\ 10^{-5}$

Conclusions

YA9PS satisfies a few properties but not 3D extension

ideal scheme remains an open problem