3D Benchmark; some results with the hybrid FV scheme

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Linear diffusion equation

Ω : open bounded connected polygonal subset of $\mathbb{R}^d$, $d \geq 1$.

$$-\text{div}(\Lambda(x)\nabla u)) = f \text{ in } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

$\partial\Omega = \overline{\Omega} \setminus \Omega$.

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$$u = 0 \text{ on } \partial\Omega,$$

$\partial\Omega = \overline{\Omega} \setminus \Omega$.

Weak formulation:

$u \in H_0^1(\Omega)$,

$\int_\Omega \Lambda(x)\nabla u(x)) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx, \ \forall v \in H_0^1(\Omega).$

Objective

Discretization of this linear diffusion problem with two constraints:

- "General" meshes (not adapted to the Finite Element method)
- One discrete unknown per cell (and one unknown by edge)

First approximate problem (1)

$\mathcal{T}$: mesh of Ω.

$H_{\mathcal{T}}$: $\begin{cases} \text{unknowns } (u_K)_{K\in\mathcal{T}} \text{ and } (u_\sigma)_{\sigma\in\mathcal{E}} \\ \text{function } x \in K \mapsto u_K \end{cases}$

First idea:

For $u \in H_{\mathcal{T}}$, define $\nabla_{\mathcal{T}} u$ (convenient approximation of the gradient on u)

Approximate problem :

$$\int_\Omega \Lambda(x)\nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

Definition of the approximate gradient

For $K \in \mathcal{T}$,

$$\frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma n_{K,\sigma} (x_\sigma - x_K)^t = Id.$$

For $u \in H_{\mathcal{T}}$, the value of the approximate gradient, $\nabla_{\mathcal{T}} u$ is, on K :

$$\nabla_K u = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (u_\sigma - u_K) n_{K,\sigma}.$$

First approximate problem (2)

First approximate problem :

$$\int_{\Omega} \Lambda(x) \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx = \int f v \, dx, \; \forall v \in H_{\mathcal{T}}.$$

But, no existence, no uniqueness, no coercivity on $H_{\mathcal{T}}$...

$\nabla_{\mathcal{T}} u = 0 \not\Rightarrow u = 0$

Stabilization using consistency estimate,

$u \in H_{\mathcal{T}}$, $K \in \mathcal{T}$, $\sigma \in \mathcal{E}_K$, the value of $R_{\mathcal{T}} u$ is on the cone $D_{K,\sigma}$:

$$R_{K,\sigma} u = \frac{1}{d_{K,\sigma}} (u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)).$$

If u is a regular function $u_K = u(x_K)$, $R_{K,\sigma} u \to 0$ as the mesh size goes to 0.

Approximate problem

$$u \in H_{\mathcal{T}},$$

$$\int_\Omega \Lambda(x) \nabla_{\mathcal{T}} u \cdot \nabla_{\mathcal{T}} v \, dx + b(u,v) = \int f v \, dx, \ \forall v \in H_{\mathcal{T}}.$$

$$b(u,v) = \int_\Omega R_{\mathcal{T}} u \, R_{\mathcal{T}} v \, dx$$

main properties:

- Coercivity of the discrete operator, existence of a solution and estimate
- Monotony of b, uniqueness of the solution
- $b(u,u) \geq 0$
- Convenient bound on u, and v such that $v_K = \psi(x_K)$ with ψ regular ($v = P_{\mathcal{T}} \psi$) gives $b(u,v) \to 0$ as the $\mathrm{size}(\mathcal{T}) \to 0$

Getting rid of the hybrid edges: SUSHI with 0 edge unknowns or "SUCCES"

For $K \in \mathcal{T}$, choose $x_K \in \mathcal{T}$ (K is "x_K star shaped").
For σ interior edge (interface) of $\mathcal{T}$, x_σ is the center of σ.
Decomposition of x_σ : $x_\sigma = \sum_{M \in \mathcal{T}} a_{M,\sigma} x_M$.

Then, if $u \in H_\mathcal{T}$, one sets:

$$u_\sigma = \sum_{M \in \mathcal{T}} a_{M,\sigma} u_M.$$

If σ is an edge on the boundary, one sets $u_\sigma = 0$.

Tests 1 Loc. Ref.

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	81	855	1.86E+00	2.38E+00	5.86E+00	5.70E+00
2	564	6180	2.38E-01	3.37E-01	3.18E+00	3.02E+00
3	4176	46800	7.20E-02	7.92E-02	1.79E+00	2.17E+00
4	32064	363840	2.04E-02	2.12E-02	1.86E+00	1.94E+00
5	251136	2868480	5.39E-03	6.56E-03	1.94E+00	1.71E+00

i	umin	umax
1	-2.87E+00	2.44E+00
2	-4.34E-01	8.23E-01
3	-8.29E-01	9.77E-01
4	-9.55E-01	9.98E-01
5	-9.89E-01	1.00E+00

Tests 1 Voronoi

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	984	21828	3.77E-01	3.91E-01	4.43E+00	4.42E+00
2	6493	171131	1.32E-01	1.65E-01	1.67E+00	1.37E+00

i	umin	umax
1	-1.20E+00	1.48E+00
2	-1.15E+00	1.11E+00

TEST2 hexa

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	44	380	1.47E+00	3.84E-01	-	-
2	304	2992	4.19E-01	2.55E-01	1.95E+00	6.38E-01
3	2240	23744	1.07E-01	6.81E-02	2.05E+00	1.98E+00
4	17152	189184	2.72E-02	1.78E-02	2.02E+00	1.98E+00
5	134144	1510400	6.83E-03	4.50E-03	2.01E+00	2.00E+00

i	umin	umax
1	-2.47E+00	2.47E+00
2	-6.30E-01	6.30E-01
3	-8.56E-01	8.56E-01
4	-9.60E-01	9.60E-01
5	-9.90E-01	9.90E-01

TEST2 tetra

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	6311	46371	8.98E-02	3.78E-01	3.98E+00	3.49E+00
2	12146	90106	5.81E-02	3.07E-01	2.00E+00	9.58E-01
3	23859	178079	3.67E-02	2.46E-01	2.04E+00	9.89E-01
4	46957	352277	2.34E-02	1.99E-01	1.99E+00	9.42E-01
5	93267	702867	1.47E-02	1.56E-01	2.03E+00	1.04E+00

i	umin	umax
1	-1.02E+00	1.03E+00
2	-1.05E+00	1.02E+00
3	-1.02E+00	1.03E+00
4	-1.02E+00	1.01E+00
5	-1.01E+00	1.01E+00

TEST2 tetra with SUCCES

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	6311	46371	8.98E-02	3.78E-01	-	-
1	2609	72673	1.52E-01	3.46E-01	-	-
2	12146	90106	5.81E-02	3.07E-01	2.00E+00	9.58E-01
2	4803	141827	1.17E-01	2.83E-01	1.30E+00	9.85E-01
3	23859	178079	3.67E-02	2.46E-01	2.04E+00	9.89E-01
3	9167	282153	8.08E-02	2.29E-01	1.84E+00	1.07E+0
4	46957	352277	2.34E-02	1.99E-01	1.99E+00	9.42E-01
5	93267	702867	1.47E-02	1.56E-01	2.03E+00	1.04E+0

i	umin	umax
1	-1.02E+00	1.03E+00
1	-9.01E-01	8.79E-01
2	-1.05E+00	1.02E+00
2	-9.52E-01	9.27E-01
3	-1.02E+00	1.03E+00
3	-9.49E-01	9.80E-01
4	-1.02E+00	1.01E+00
5	-1.01E+00	1.01E+00

TEST2 Loc. ref.

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiog
1	81	855	1.80E+00	2.18E+00	-	-
1	48	710	1.66E+00	1.61E+00	-	-
2	564	6180	4.50E-01	3.95E-01	2.14E+00	2.64E-
2	252	5036	4.18E-01	3.78E-01	2.50E+00	2.62E
3	4176	46800	1.18E-01	1.04E-01	2.00E+00	2.00E-
3	1488	32768	1.22E-01	1.48E-01	2.08E+00	1.58E-
4	32064	363840	3.11E-02	2.92E-02	1.97E+00	1.86E-
5	251136	2868480	7.97E-03	8.98E-03	1.99E+00	1.72E-

i	umin	umax
1	-2.82E+00	2.47E+00
1	-2.65E+00	2.40E+00
2	-6.19E-01	8.28E-01
2	-5.94E-01	7.94E-01
3	-8.50E-01	9.68E-01
3	-8.25E-01	9.51E-01
4	-9.58E-01	9.95E-01
5	-9.89E-01	9.99E-01

TEST2 Voronoi

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	984	21828	4.48E-01	4.72E-01	-	-
2	6493	171131	1.47E-01	2.14E-01	1.77E+00	1.26E+00

i	umin	umax
1	-1.29E+00	1.41E+00
2	-1.15E+00	1.12E+00

TEST2 Voronoi with SUCCES

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	984	21828	4.48E-01	4.72E-01	-	-
1	304	13618	2.36E-01	7.12E-01	-	-
2	6493	171131	1.47E-01	2.14E-01	1.77E+00	1.26E+00
2	1372	105560	1.43E-01	3.70E-01	9.99E-01	1.30E+00

i	umin	umax
1	-1.29E+00	1.41E+00
1	-8.42E-01	1.02E+00
2	-1.15E+00	1.12E+00
2	-9.01E-01	9.10E-01

TEST2 Loc. ref.

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	7400	67400	6.20E-02	2.12E-01	-	-
2	57600	537600	1.85E-02	1.19E-01	1.77E+00	8.48E-01
3	192600	1812600	9.39E-03	8.37E-02	1.69E+00	8.72E-01
4	454400	4294400	5.68E-03	6.46E-02	1.76E+00	9.07E-01

i	umin	umax
1	-1.05E+00	1.05E+00
2	-9.96E-01	9.96E-01
3	-1.00E+00	1.00E+00
4	-9.98E-01	9.98E-01

i	nunkw	nnmat	erl2	ergrad	ratiol2	ratiograd
1	6541	92741	7.78E-02	7.69E-02	-	-
2	46081	679281	2.19E-02	2.79E-02	1.94E+00	1.56E+00
3	148621	2221621	1.02E-02	1.50E-02	1.96E+00	1.59E+00

i	umin	umax
1	-1.04E+00	1.04E+00
2	-9.85E-01	9.85E-01
3	-1.01E+00	1.01E+00