Stabilized DDFV schemes for Stokes problem with variable viscosity on general 2d meshes

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Workshop on Discretization methods for viscous flows 2009

Outline

1. The DDFV method for the Stokes problem
2. Numerical results
3. The interface problem : Discontinuous viscosity
4. Extension
5. Conclusion

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1. The DDFV method for the Stokes problem
2. Numerical results
3. The interface problem : Discontinuous viscosity
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5. Conclusion

Stokes problem with smooth variable viscosity

- Problem

$$\begin{cases} \operatorname{div}(-2\eta(.)\mathrm{D}\mathbf{u} + p\mathrm{Id}) = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \\ \displaystyle\int_\Omega p(x)\mathrm{d}x = 0. \end{cases} \tag{S}$$

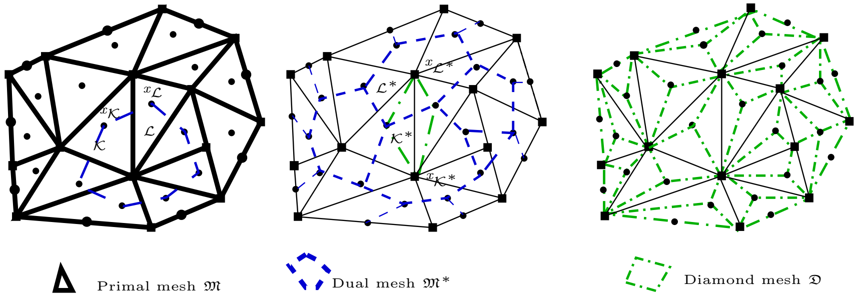
with $\mathrm{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$,

  - Ω a polygonal open bounded connected subset of $\mathbb{R}^2$.
  - $\mathbf{f} \in (L^2(\Omega))^2$,
  - $\eta \in C^2(\Omega)$ with

$$0 < \underline{\mathrm{C}}_\eta \leq \eta(x) \leq \bar{\mathrm{C}}_\eta, \quad \forall\, x \in \Omega.$$

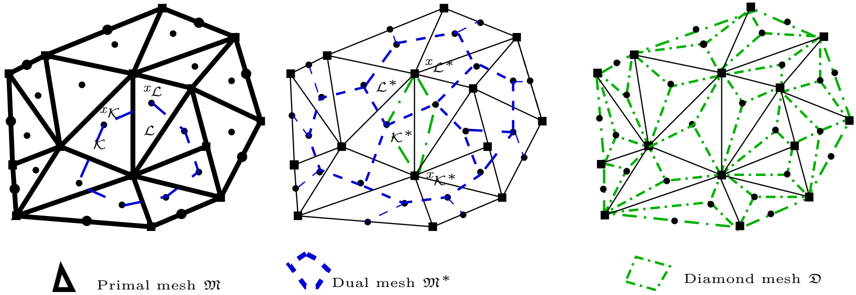
- Goals
  - Write a wellposed DDFV scheme for (S).
  - Prove error estimates for this scheme.

DDFV meshes



$x_{\mathcal{K}}$
$x_{\mathcal{L}}$
$\mathcal{K}$
$\mathcal{L}$
$x_{\mathcal{L}^*}$
$\mathcal{L}^*$
$\mathcal{K}^*$
$x_{\mathcal{K}^*}$
Primal mesh $\mathfrak{M}$
Dual mesh $\mathfrak{M}^*$
Diamond mesh $\mathfrak{D}$

DDFV meshes



Primal cells

$\rightsquigarrow \mathbf{u}^{\mathfrak{M}} = (\mathbf{u}_{\mathcal{K}})_{\mathcal{K}\in\mathfrak{M}}$

Dual cells

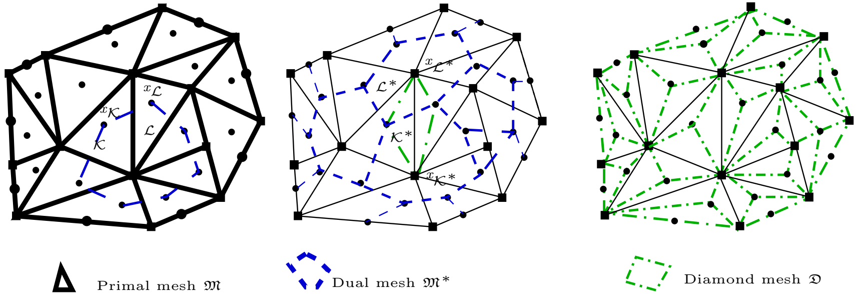
$\rightsquigarrow \mathbf{u}^{\mathfrak{M}^*} = (\mathbf{u}_{\mathcal{K}^*})_{\mathcal{K}^*\in\mathfrak{M}^*}$

Diamond cells

$\rightsquigarrow p^{\mathfrak{D}} = (p^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow \mathbf{u}^{\mathcal{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*})$,

DDFV meshes



Primal cells

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Diamond cells

$\rightsquigarrow p^{\mathfrak{D}} = (p^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}}$

$\rightsquigarrow \mathbf{u}^{\mathcal{T}} = (\mathbf{u}^{\mathfrak{M}}, \mathbf{u}^{\mathfrak{M}^*}),$

$\rightsquigarrow$ Discrete operators : $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}$ and $\mathbf{div}^{\mathcal{T}}(\xi^{\mathfrak{D}})$.

Constant viscosity η

Velocity unknowns : centers and vertices
Pressure unknowns : diamonds cells.

$$\begin{cases} \operatorname{div}(-\nabla \mathbf{u} + p\mathrm{Id}) = \mathbf{f}, \\ \operatorname{div}(\mathbf{u}) = 0. \end{cases}$$

Constant viscosity η

Velocity unknowns : centers and vertices
Pressure unknowns : diamonds cells.

$$\begin{cases} \operatorname{div}(-\nabla \mathbf{u} + p\mathrm{Id}) = \mathbf{f}, \\ \mathrm{Tr}(\nabla \mathbf{u}) = 0. \end{cases}$$

Constant viscosity η

Velocity unknowns : centers and vertices
Pressure unknowns : diamonds cells.

$$\begin{cases} \mathbf{div}^{\mathcal{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathcal{T}}, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) = 0. \end{cases}$$

- We do not know if the discrete problem is wellposed for general meshes.
- The problem is wellposed for $\mathcal{T}$:
  - triangles : conformal meshes with angles $\leq \dfrac{\pi}{2}$
  - rectangles : non-conformal meshes.

Delcourte & Domelevo & Omnès '07

- Existence of uniform discrete inf-sup inequality ?
- Error estimates only for the velocity (when the problem is wellposed).

K. '08

Constant viscosity η

Velocity unknowns : centers and vertices
Pressure unknowns : diamonds cells.

$$\begin{cases} \mathbf{div}^{\mathcal{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathcal{T}}, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) = 0. \end{cases}$$

What can we do ?

▶ Stabilize the mass conservation equation with a term depending on the pressure.

$$\begin{cases} \mathbf{div}^{\mathcal{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathcal{T}}, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + S^{\mathfrak{D}}(p^{\mathfrak{D}}) = 0. \end{cases} \qquad \text{(Stab)}$$

- Existence and uniqueness for general DDFV meshes.
- Error estimates for the velocity and the pressure with a particular stabilization term (inspired of Brezzi-Pitkäranta framework).

K. '09

▶ Approximate the pressure on both centers and vertices of the mesh and the velocity on the diamond cells, using $\Delta = \nabla \mathrm{div} - \mathbf{curl}\ \mathrm{curl}$.

- Existence and uniqueness for general DDFV meshes.

Delcourte & Domelevo & Omnès '07

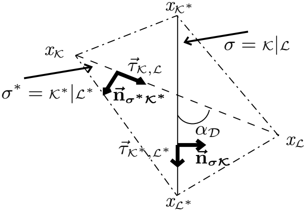
Discrete operators

Discrete gradient of a vector field $\left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \mapsto \left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}}$$

where $\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t\left(\nabla^{\mathcal{D}} u^{\mathcal{T}}\right) \\ {}^t\left(\nabla^{\mathcal{D}} v^{\mathcal{T}}\right) \end{pmatrix}$



Diamond cell

$$\nabla^{\mathcal{D}} v^{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})}\left(\frac{v_{\mathcal{L}} - v_{\mathcal{K}}}{m_{\sigma^*}} \vec{\mathbf{n}}_{\sigma\mathcal{K}} + \frac{v_{\mathcal{L}^*} - v_{\mathcal{K}^*}}{m_{\sigma}} \vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}\right).$$

equivalent definition
$$\begin{cases} \nabla^{\mathcal{D}} v^{\mathcal{T}} \cdot (x_{\mathcal{L}} - x_{\mathcal{K}}) = v_{\mathcal{L}} - v_{\mathcal{K}}, \\ \nabla^{\mathcal{D}} v^{\mathcal{T}} \cdot (x_{\mathcal{L}^*} - x_{\mathcal{K}^*}) = v_{\mathcal{L}^*} - v_{\mathcal{K}^*}. \end{cases}$$

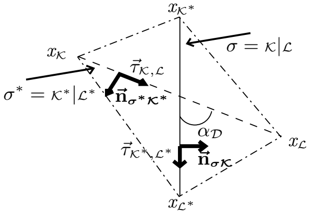
Discrete operators

Discrete gradient of a vector field $\left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \mapsto \left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}}$$

where $\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t(\nabla^{\mathcal{D}} u^{\mathcal{T}}) \\ {}^t(\nabla^{\mathcal{D}} v^{\mathcal{T}}) \end{pmatrix}$



Diamond cell

Discrete divergence of a tensor field $(\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$

$$\mathbf{div}^{\boldsymbol{\mathcal{T}}} : (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \to \left(\mathbb{R}^2\right)^{\mathcal{T}}$$

$$\mathcal{K} \in \mathfrak{M}, \quad \frac{1}{m_{\mathcal{K}}} \int_{\mathcal{K}} \mathrm{div}(\xi(x))\mathrm{d}x = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\subset\partial\mathcal{K}} \int_{\sigma} \xi(s) \vec{\mathbf{n}}_{\sigma\mathcal{K}} \mathrm{d}s$$

$$\mathbf{div}^{\mathcal{K}} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\subset\partial\mathcal{K}} m_{\sigma} \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\sigma\mathcal{K}}$$

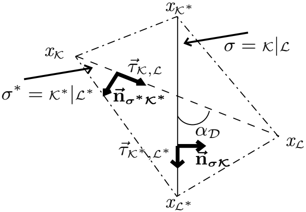
Discrete operators

Discrete gradient of a vector field $\left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \mapsto \left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}}$$

where $\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t\left(\nabla^{\mathcal{D}} u^{\mathcal{T}}\right) \\ {}^t\left(\nabla^{\mathcal{D}} v^{\mathcal{T}}\right) \end{pmatrix}$



Diamond cell

Discrete divergence of a tensor field $\left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$

$$\mathbf{div}^{\boldsymbol{\mathcal{T}}} : \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}} \rightarrow \left(\mathbb{R}^2\right)^{\mathcal{T}}$$

$$\mathcal{K} \in \mathfrak{M}, \quad \mathbf{div}^{\boldsymbol{\mathcal{K}}} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}}} \sum_{\sigma\subset\partial\mathcal{K}} m_\sigma \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma\mathcal{K}}}$$

$$\mathcal{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*, \quad \mathbf{div}^{\boldsymbol{\mathcal{K}^*}} \xi^{\mathfrak{D}} = \frac{1}{m_{\mathcal{K}^*}} \sum_{\sigma^*\subset\partial\mathcal{K}^*} m_{\sigma^*} \xi^{\mathcal{D}} \vec{\mathbf{n}}_{\boldsymbol{\sigma^*\mathcal{K}^*}}$$

$$\mathbf{div}^{\mathfrak{M}} \xi^{\mathfrak{D}} = \left(\left(\mathbf{div}^{\boldsymbol{\mathcal{K}}} \xi^{\mathfrak{D}}\right)_{\mathcal{K}\in\mathfrak{M}}\right) \quad \mathbf{div}^{\mathfrak{M}^*} \xi^{\mathfrak{D}} = \left(\left(\mathbf{div}^{\boldsymbol{\mathcal{K}^*}} \xi^{\mathfrak{D}}\right)_{\mathcal{K}^*\in\mathfrak{M}^*}\right).$$

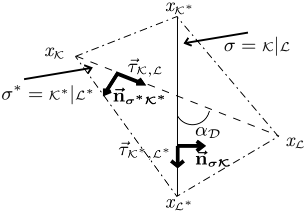
Discrete operators

Discrete gradient of a vector field $(\mathbb{R}^2)^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathcal{T}} \longrightarrow (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \mapsto (\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathcal{D}\in\mathfrak{D}}$$

where $\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t(\nabla^{\mathcal{D}} u^{\mathcal{T}}) \\ {}^t(\nabla^{\mathcal{D}} v^{\mathcal{T}}) \end{pmatrix}$



Diamond cell

Fundamental tool (discrete duality)

$$-\int_{\Omega} \mathbf{div}^{\boldsymbol{\mathcal{T}}}(\xi^{\mathfrak{D}}) \cdot \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \int_{\Omega} (\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}).$$

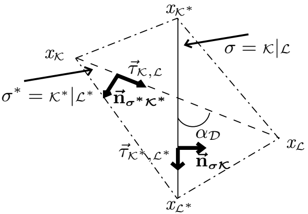
Discrete operators

DISCRETE GRADIENT OF A VECTOR FIELD $\left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\nabla^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \mapsto \left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}}$$

where $\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t\left(\nabla^{\mathcal{D}} u^{\mathcal{T}}\right) \\ {}^t\left(\nabla^{\mathcal{D}} v^{\mathcal{T}}\right) \end{pmatrix}$



Diamond cell

DISCRETE STRAIN RATE TENSOR OF $\left(\mathbb{R}^2\right)^{\mathcal{T}}$

$$\mathrm{D}^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathcal{T}} \longrightarrow \left(\mathcal{M}_2(\mathbb{R})\right)^{\mathfrak{D}}$$

$$\mathbf{u}^{\boldsymbol{\mathcal{T}}} \mapsto \left(\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D}\in\mathfrak{D}}$$

with

$$\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{1}{2}\left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + {}^t\left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)\right)$$

Discrete Korn inequality

Proposition

For all $\mathbf{u}^{\mathcal{T}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$,

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2 \leq \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2.$$

If $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0 : \forall\, \mathcal{K} \in \partial\mathfrak{M},\ \mathbf{u}_{\mathcal{K}} = 0, \quad \forall\, \mathcal{K}^* \in \partial\mathfrak{M}^*,\ \mathbf{u}_{\mathcal{K}^*} = 0,$

$$\mathbf{div}^{\mathcal{T}}\left({}^t\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) = \mathbf{div}^{\mathcal{T}}\left(\mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})\mathrm{Id}\right). \tag{1}$$

Theorem (Discrete Korn inequality, K. 09)

For all $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$,

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2 \leq \sqrt{2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2.$$

▸ Proof

We note

$$\eta_{\mathcal{D}} = \eta(x_{\mathcal{D}}).$$

- On the primal cell $\mathcal{K}$

$$\int_{\mathcal{K}} \mathbf{f} = \int_{\mathcal{K}} \mathrm{div}(-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id}) = \sum_{\sigma \subset \partial\mathcal{K}} \int_{\sigma} (-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathcal{K}}$$

$$\approx m_{\mathcal{K}} \mathrm{div}^{\mathcal{K}}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathfrak{D}}\mathrm{Id}) := \sum_{\sigma \subset \partial\mathcal{K}} m_{\sigma}(-2\eta_{\mathcal{D}} \mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathcal{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathcal{K}}.$$

- On the dual cell $\mathcal{K}^*$

$$\int_{\mathcal{K}^*} \mathbf{f} = \int_{\mathcal{K}^*} \mathrm{div}(-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id}) = \sum_{\sigma^* \subset \partial\mathcal{K}^*} \int_{\sigma^*} (-2\eta \mathrm{D}\mathbf{u} + p\mathrm{Id})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

$$\approx m_{\mathcal{K}^*} \mathrm{div}^{\mathcal{K}^*}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathfrak{D}}\mathrm{Id}) := \sum_{\sigma^* \subset \partial\mathcal{K}^*} m_{\sigma^*}(-2\eta_{\mathcal{D}} \mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + p^{\mathcal{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

- On the diamond cell $\mathcal{D}$

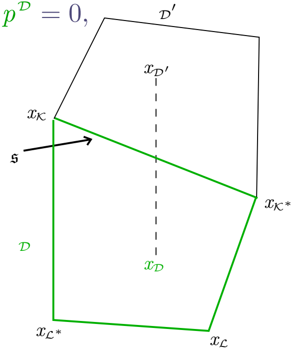
$$\int_{\mathcal{D}} 0 = \int_{\mathcal{D}} \operatorname{div}(\mathbf{u}) = \int_{\mathcal{D}} \operatorname{Tr}(\nabla \mathbf{u}) \approx m_{\mathcal{D}} \operatorname{Tr}(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}).$$

- We stabilize this equation like in **Brezzi & Pitkäranta '84** :

$$\operatorname{Tr}(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) = 0$$

becomes

$$\operatorname{Tr}(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathcal{D}}^2 \Delta^{\mathcal{D}} p^{\mathcal{D}} = 0,$$



with $\lambda > 0$ and

$$\Delta^{\mathcal{D}} p^{\mathfrak{D}} = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \partial \mathcal{D}} \frac{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2}{h_{\mathcal{D}}^2} (p^{\mathcal{D}'} - p^{\mathcal{D}}),$$

$h_{\mathcal{D}}$ is the diameter of the diamond $\mathcal{D}$.

$$\begin{cases} \text{Find } \mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0 \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} \\ \qquad \mathbf{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, \\ \qquad \mathbf{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \qquad \mathrm{Tr}(\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \qquad \displaystyle\sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{cases} \tag{S-DDFV}$$

K. '09

Theorem (Existence and uniqueness)

Let $\mathcal{T}$ be a DDFV mesh.
For any value of the stabilization parameter $\lambda > 0$, the (S-DDFV) *scheme admits an unique solution.*

Proof

Error estimates

Theorem (Error estimates, K. 09)

General and regular DDFV mesh $\mathcal{T}$.

- η *Lipschitz continuous :*

$$|\eta(x)-\eta(x')| \leq \mathrm{C}_\eta |x-x'|, \quad \forall x, x' \in \Omega.$$

- $(\mathbf{u}, p) \in (H^2(\Omega))^2 \times H^1(\Omega)$ *the pair solution of the exact problem* (S),
- $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$ *the pair solution of the scheme* (S-DDFV),

There exists $C > 0$ *:*

$$\|\mathbf{u}-\mathbf{u}^{\mathcal{T}}\|_2 + \|\nabla \mathbf{u} - \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2 \quad \leq \quad C \;\; size(\mathcal{T})$$

and

$$\|p - p^{\mathfrak{D}}\|_2 \quad \leq \quad C \;\; size(\mathcal{T})$$

This convergence rate is optimal.

Main tool : Stability of (S-DDFV) 1/2

- Strategy for stability :

$$B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}) = \int_{\Omega} \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} Id) \cdot \widetilde{\mathbf{u}}^{\mathcal{T}} + \int_{\Omega} (\mathrm{Tr}(\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}) \widetilde{p}^{\mathfrak{D}}.$$

For $\widetilde{\mathbf{u}}^{\mathcal{T}} = \mathbf{u}^{\mathcal{T}}$ and $\widetilde{p}^{\mathfrak{D}} = p^{\mathfrak{D}}$, we want

$$|\!|\!| \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} |\!|\!|_2^2 + \| p^{\mathfrak{D}} \|_2^2 \leq C_2 B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}).$$

But we have

$$|\!|\!| \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} |\!|\!|_2^2 + \| p^{\mathfrak{D}} \|_2^2 \leq C_2 B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}) + C_1 \underbrace{\left(\| p^{\mathfrak{D}} \|_2^2 - | p^{\mathfrak{D}} |_h^2 \right)}_{\text{No uniform control w. r. size}(\mathcal{T})}.$$

where $| p^{\mathfrak{D}} |_h^2 = \sum_{\mathfrak{s} \in \mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2)(p^{\mathcal{D}'} - p^{\mathcal{D}})^2$.

- Idea : construction of $\widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}$ (close to $\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}$).

Eymard & Herbin & Latché '06

Main tool : Stability of (S-DDFV)

Proposition (Stability of (S-DDFV), K. 09)

$\forall$ $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ with $\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$. $\exists$ $(\widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ such that $C_1 > 0$ and $C_2 > 0$:

$$\|\nabla^{\mathfrak{D}}\widetilde{\mathbf{u}}^{\mathcal{T}}\|_2 + \|\widetilde{p}^{\mathfrak{D}}\|_2 \leq C_1 \left(\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2 + \|p^{\mathfrak{D}}\|_2\right),$$

and

$$\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2^2 + \|p^{\mathfrak{D}}\|_2^2 \leq C_2 B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}).$$

Main tool : Stability of (S-DDFV) 2/2

Proposition (Stability of (S-DDFV), K. 09)

$\forall\ (\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ *with* $\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$. $\exists\ (\widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ *such that* $C_1 > 0$ *and* $C_2 > 0$ *:*

$$\|\nabla^{\mathfrak{D}} \widetilde{\mathbf{u}}^{\mathcal{T}}\|_2 + \|\widetilde{p}^{\mathfrak{D}}\|_2 \leq C_1 \left(\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2 + \|p^{\mathfrak{D}}\|_2\right),$$

and

$$\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 + \|p^{\mathfrak{D}}\|_2^2 \leq C_2 B(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathcal{T}}, \widetilde{p}^{\mathfrak{D}}).$$

Corollary

$(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ *the pair solution of the scheme* (S-DDFV), $\exists\ C > 0$

$$\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 + \|p^{\mathfrak{D}}\|_2^2 \leq C \|\mathbf{f}^{\mathcal{T}}\|_2^2.$$

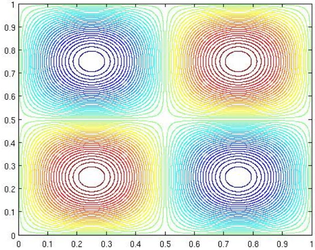
Outline

Case 1 - Green-Taylor vortex - Constant viscosity

$$\mathbf{u}(x,y) = \begin{pmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y) \\ -\frac{1}{2}\cos(2\pi x)\sin(2\pi y) \end{pmatrix},$$

$$p(x,y) = \frac{1}{8}\cos(4\pi x)\sin(4\pi y),$$

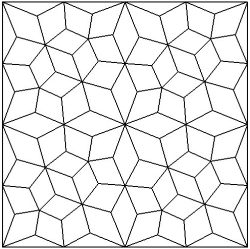
$$\eta(x,y) = 1.$$



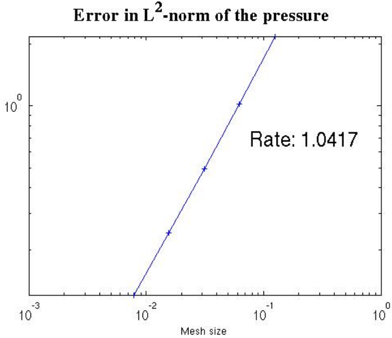
Streamlines

Case 1 - Green-Taylor vortex - Constant viscosity

Mesh



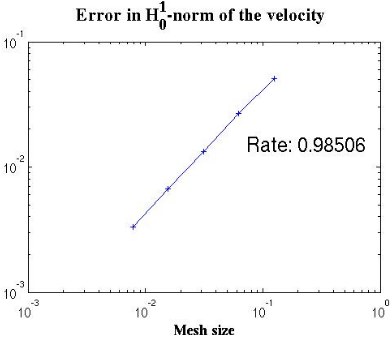
$\|p - p^{\mathfrak{D}}\|_2 / \|p\|_2$



Rate =1.04

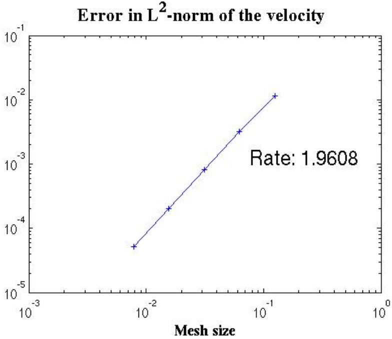
Case 1 - Green-Taylor vortex - Constant viscosity

$$\|\mathbf{u} - \mathbf{u}^{\mathcal{T}}\|_{H_0^1} / \|\mathbf{u}\|_{H_0^1}$$



Rate =0.99

$$\|\mathbf{u} - \mathbf{u}^{\mathcal{T}}\|_2 / \|\mathbf{u}\|_2$$



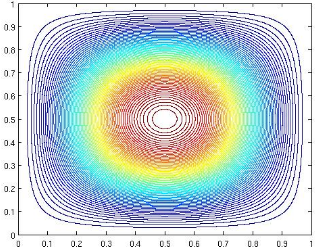
Rate =1.96

Case 2

$$\mathbf{u}(x,y) = \begin{pmatrix} 1000x^2(1-x)^2 2y(1-y)(1-2y) \\ -1000y^2(1-y)^2 2x(1-x)(1-2x) \end{pmatrix},$$

$$p(x,y) = x^2 + y^2 - \frac{2}{3},$$

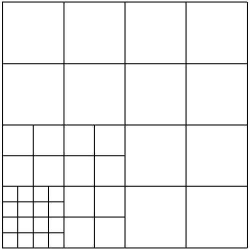
$$\eta(x,y) = 2x + y + 1.$$



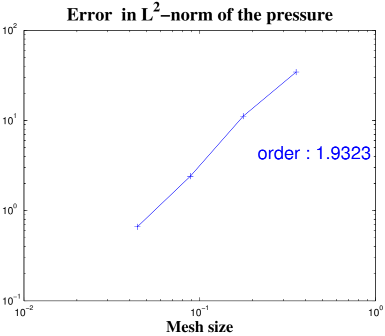
Streamlines

Case 2

Mesh



$\|p - p^{\mathfrak{D}}\|_2 / \|p\|_2$

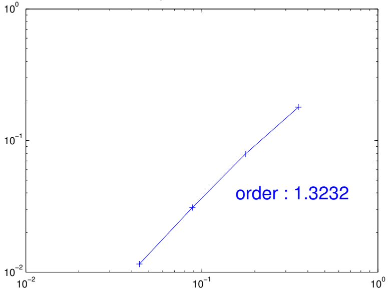


Rate =1.9

Case 2

$$\|\mathbf{u} - \mathbf{u}^{\mathcal{T}}\|_{H_0^1} / \|\mathbf{u}\|_{H_0^1}$$

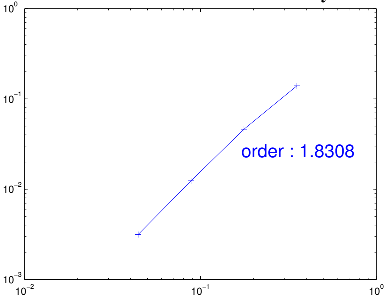
Error in H_0^1–norm of the velocity



Rate =1.3

$$\|\mathbf{u} - \mathbf{u}^{\mathcal{T}}\|_2 / \|\mathbf{u}\|_2$$

Error in L^2–norm of the velocity



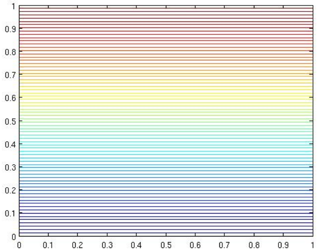
Rate =1.8

Case 3

$$\mathbf{u}(x,y) = \begin{pmatrix} \begin{cases} y^2 - 0.5y & \text{for } y > 0.5 \\ 10^4(y^2 - 0.5y) & \text{else.} \end{cases} \\ 0 \end{pmatrix},$$

$$p(x,y) = 2x - 1,$$

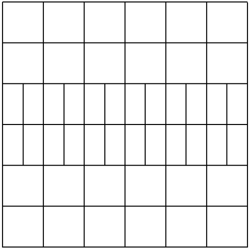
$$\eta(x,y) = \begin{cases} 1 & \text{for } y > 0.5 \\ 10^{-4} & \text{else.} \end{cases}$$



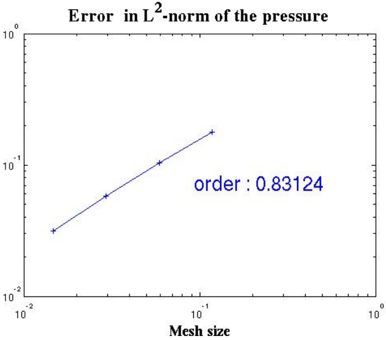
Streamlines

Case 3

Mesh



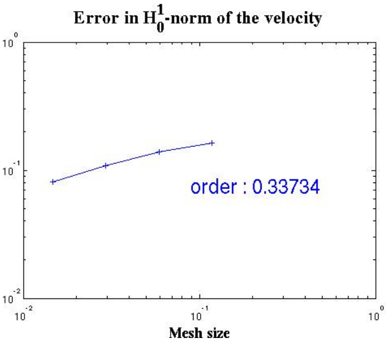
$\|p - p^{\mathfrak{D}}\|_2 / \|p\|_2$



Rate =0.83

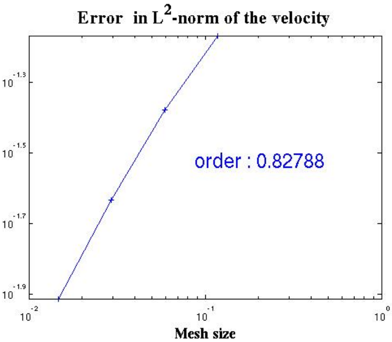
Case 3

$$\|\mathbf{u}-\mathbf{u}^{\mathcal{T}}\|_{H_0^1}/\|\mathbf{u}\|_{H_0^1}$$



Rate =0.34

$$\|\mathbf{u}-\mathbf{u}^{\mathcal{T}}\|_2/\|\mathbf{u}\|_2$$



Rate =0.83

Outline

The interface Stokes problem

- Problem

$$\begin{cases} \operatorname{div}(-2\eta_i \mathrm{D}\mathbf{u} + p\mathrm{Id}) = \mathbf{f}, & \text{in } \Omega_i, \\ \operatorname{div}(\mathbf{u}) = 0, & \text{in } \Omega_i, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \qquad \int_\Omega p(x)\mathrm{d}x = 0, \\ [\mathbf{u}] = 0, & \text{on } \Gamma, \\ [2\eta \mathrm{D}\mathbf{u} - p\mathrm{Id}]\,\vec{\mathbf{n}} = 0, & \text{on } \Gamma, \end{cases} \qquad (S_\Gamma)$$

A piecewise constant viscosity η :

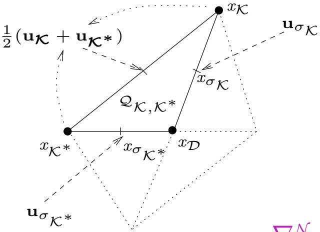
$$\eta = \begin{cases} \eta_1 > 0, & \text{in } \Omega_1, \\ \eta_2 > 0, & \text{in } \Omega_2, \end{cases}$$

satisfying $$0 < \underline{\mathrm{C}}_\eta \leq \eta(x) \leq \bar{\mathrm{C}}_\eta, \quad \forall\, x \in \Omega.$$

- $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$,
- $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$,
- $\vec{\mathbf{n}}$ is an unit normal vector to Γ and $[a]_{|\Gamma} = (a_{|\Omega_1} - a_{|\Omega_2})_{|\Gamma}$.

- $\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}}$ is constant on each quarter diamond cell

$$\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}} = \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}},$$



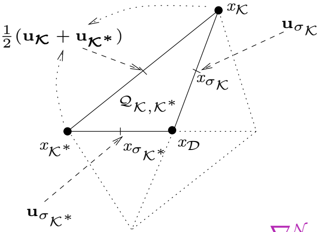
$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} \cdot \frac{1}{2}(x_{\mathcal{D}} - x_{\mathcal{K}}) = u_{\sigma_{\mathcal{K}^*}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*}),$$
$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} \cdot \frac{1}{2}(x_{\mathcal{D}} - x_{\mathcal{K}^*}) = u_{\sigma_{\mathcal{K}}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*}).$$

$$\rightsquigarrow \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}} = \nabla^{\mathcal{D}} u^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}, \ \forall \mathcal{Q} \subset \mathcal{D}.$$

Boyer & Hubert '08

- $\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}}$ is constant on each quarter diamond cell

$$\nabla_{\mathcal{D}}^{\mathcal{N}} u^{\mathcal{T}} = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} 1_{\mathcal{Q}} \nabla_{\mathcal{Q}}^{\mathcal{N}} u^{\mathcal{T}},$$



$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} \cdot \frac{1}{2}(x_{\mathcal{D}} - x_{\mathcal{K}}) = u_{\sigma_{\mathcal{K}^*}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*}),$$

$$\nabla_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}^{\mathcal{N}} u^{\mathcal{T}} \cdot \frac{1}{2}(x_{\mathcal{D}} - x_{\mathcal{K}^*}) = u_{\sigma_{\mathcal{K}}} - \frac{1}{2}(u_{\mathcal{K}} + u_{\mathcal{K}^*}).$$

$$\rightsquigarrow \nabla_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\mathcal{T}} = \nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}, \forall \mathcal{Q} \subset \mathcal{D}.$$

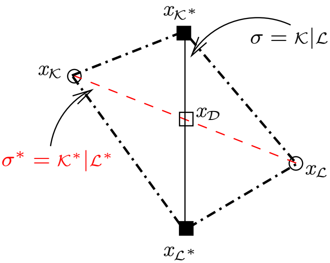
- $B_{\mathcal{Q}}$ is a matrix 2×4 which only depends on the geometry of $\mathcal{Q}$.
- $\delta^{\mathcal{D}} = (\delta_{\mathcal{K}}, \delta_{\mathcal{L}}, \delta_{\mathcal{K}^*}, \delta_{\mathcal{L}^*})^t$ are 8 artificial unknowns to be determined.
- $B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}} = \frac{1}{m_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}} \left(m_{\sigma_{\mathcal{K}}} \vec{\mathbf{n}}_{\mathcal{K}^*\mathcal{L}^*}, 0, m_{\sigma_{\mathcal{K}^*}} \vec{\mathbf{n}}_{\mathcal{K}\mathcal{L}}, 0\right).$

$$\rightsquigarrow \mathrm{D}_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\mathcal{T}} = \frac{1}{2}\left(\nabla_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\mathcal{T}} + {}^t\nabla_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\mathcal{T}}\right), \forall \mathcal{Q} \subset \mathcal{D}.$$

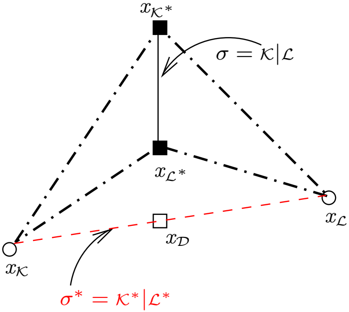
Barycentric dual mesh

Here :

Diamond cells supposed to be convex.



Case of non-convex diamond cells.



Problem in the quarter diamond definition

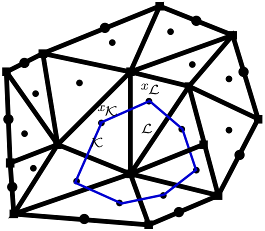
Alternative $\longrightarrow$ Barycentric dual mesh :

Hermeline '00, Delcourte & Domelevo & Omnes '07

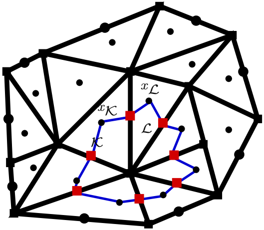
Barycentric dual mesh

Alternative $\longrightarrow$ Barycentric dual mesh :

Hermeline '00, Delcourte & Domelevo & Omnes '07

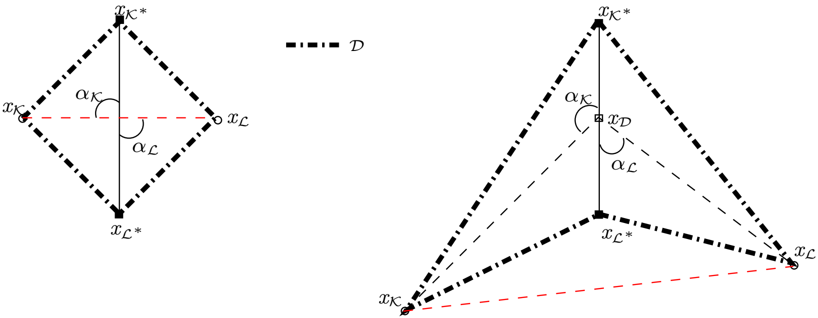


Classic dual mesh



Barycentric dual mesh

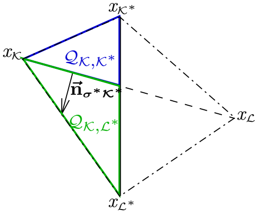
Barycentric dual mesh



$\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$ $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$

▶ Conservativity of the fluxes

Assumption : $p \in H^1(\Omega)$.



The conservativity of the fluxes through $\sigma_{\mathcal{K}}$ is

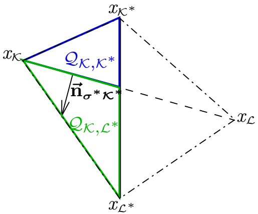
$$\int_{\sigma_{\mathcal{K}}} \eta_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}(s)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}(s)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}\mathrm{d}s = \int_{\sigma_{\mathcal{K}}} \eta_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}(s)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}(s)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}\mathrm{d}s.$$

▶ Conservativity of the numerical fluxes

We note

$$\eta_{\mathcal{Q}} = \eta(x_{\mathcal{Q}}).$$

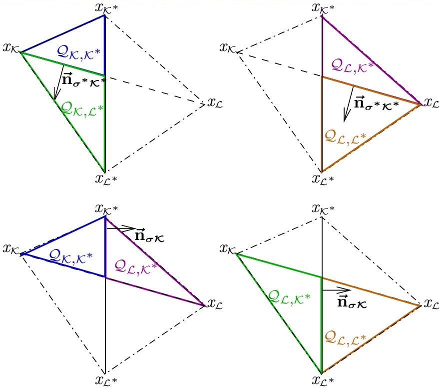
We determine $\delta^{\mathcal{D}}$ matrix 4×2 such that



$$\underbrace{\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(2\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}} + {}^t(B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}))}_{\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\delta^{\mathcal{D}})}\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

$$= \underbrace{\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(2\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}} + {}^t(B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}))}_{\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\delta^{\mathcal{D}})}\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

Derivation of the S-m-DDFV scheme 3/4



$$\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}=\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$
$$\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}=\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$
$$\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma\mathcal{K}}=\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma\mathcal{K}}$$
$$\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma\mathcal{K}}=\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\delta^{\mathcal{D}})\vec{\mathbf{n}}_{\sigma\mathcal{K}}$$

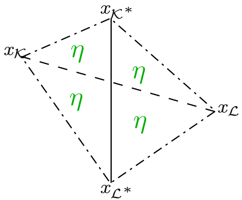
$$\Longleftrightarrow \quad \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\delta^{\mathcal{D}})B_{\mathcal{Q}} = 0.$$

Proposition (K. 09)

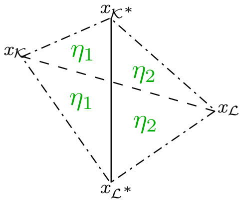
For all $\mathcal{D}\in\mathfrak{D}$ and all $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}\in\mathcal{M}_{2,2}(\mathbb{R})$, there exists a $\delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}})\in\mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R})$ ensuring the fluxes conservativity.

Proof

Examples



$\Longrightarrow \delta^{\mathcal{D}} = 0$ and $\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}$



$\Longrightarrow \delta_{\mathcal{K}} = 0, \quad \delta_{\mathcal{L}} = 0 \quad \text{and} \quad \delta_{\mathcal{K}^*} = \delta_{\mathcal{L}^*}.$

$\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}$ is completely determined by

$$\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + B_{\mathcal{Q}}\delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) + {}^t\delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})^t B_{\mathcal{Q}}.$$

Comparisons between the new and old operators

Proposition (K. 09)

There exists a constant $C > 0$, such that for all $\mathbf{u}^{\mathcal{T}} \in \left(\mathbb{R}^2\right)^{\mathcal{T}}$:

$$\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2 \leq \|\mathrm{D}^{\mathcal{N}}_{\mathfrak{Q}}\mathbf{u}^{\mathcal{T}}\|_2 \leq C\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2.$$

Thanks to the definition of $\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\mathcal{T}}$, we have

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\mathcal{T}}\|^2_{\mathcal{F}} = m_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}\|^2_{\mathcal{F}} + \frac{1}{4}\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}\|^2_{\mathcal{F}}.$$

The second inequality comes from the following estimate

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}\|^2_{\mathcal{F}} \leq Cm_{\mathcal{D}}\|\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}\|^2_{\mathcal{F}}.$$

Find $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that,

$$\begin{cases} \mathbf{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, \\ \mathbf{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{cases} \qquad \text{(S-DDFV)}$$

We will replace, in the S-DDFV scheme, the discrete viscous stress tensor $\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}$ by

$$\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}),$$

We replace, in the S-DDFV scheme, the discrete viscous stress tensor $\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}$ by

$$\varphi_{\mathcal{D}}(\eta, \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}) = \frac{1}{m_{\mathcal{D}}} \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \eta_{\mathcal{Q}} \underbrace{\left(\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + B_{\mathcal{Q}} \delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}) + {}^t(B_{\mathcal{Q}} \delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}))\right)}_{= \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}},$$

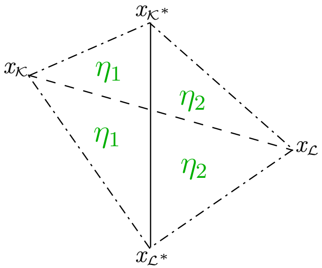
Find $\mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \mathbb{E}_0$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that,

$$\begin{cases} \mathbf{div}^{\mathfrak{M}}(-2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}) + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}}, \\ \mathbf{div}^{\mathfrak{M}^*}(-2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}) + p^{\mathfrak{D}} \mathrm{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{cases} \qquad \text{(S-m-DDFV)}$$

S-m-DDFV scheme : particular case

$$\varphi_{\mathcal{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) = \frac{1}{m_{\mathcal{D}}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}} \underbrace{(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} + B_{\mathcal{Q}}\delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}) + {}^t(B_{\mathcal{Q}}\delta^{\mathcal{D}}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})))}_{=\mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbf{u}^{\boldsymbol{\mathcal{T}}}},$$

In the case where η is constant per primal cells :



$$\Longrightarrow \quad \varphi_{\mathcal{D}}\left(\eta, \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\right) = \begin{pmatrix} 2\frac{\eta_1\eta_2}{\eta_1+\eta_2}\alpha & 2\frac{\eta_1\eta_2}{\eta_1+\eta_2}\gamma \\ 2\frac{\eta_1\eta_2}{\eta_1+\eta_2}\gamma & \frac{\eta_1+\eta_2}{2}\beta \end{pmatrix}.$$

Analysis of the S-m-DDFV scheme

Theorem (K. 09)

For general and regular DDFV mesh $\mathcal{T}$. The S-m-DDFV scheme has an **unique** *solution* $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}})$, *for all* $\lambda > 0$.

▸ Proof

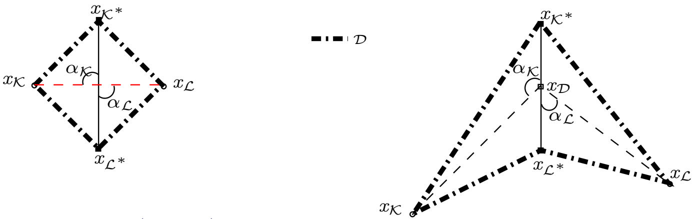
Theorem (Discrete Korn inequality, K. 09)

For all $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$, *there exists a constant* $C > 0$ *such that*

$$\|\nabla^{\mathcal{N}}_{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2 \leq C \|\mathrm{D}^{\mathcal{N}}_{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2.$$

▸ Proof

Technical result



- If $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$, with $(\delta^{\mathcal{D}}, \delta_0) = 0$

$$\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \| B_{\mathcal{Q}} \delta^{\mathcal{D}} \|_{\mathcal{F}}^2 \leq C \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \| B_{\mathcal{Q}} \delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t} B_{\mathcal{Q}} \|_{\mathcal{F}}^2.$$

For each $\mathcal{D} \in \mathfrak{D}$, if $|\alpha_{\mathcal{K}} - \alpha_{\mathcal{L}}| < \epsilon_0$, we choose $x_{\mathcal{D}}$ to be the intersection of the primal edge σ and the dual edge σ^* instead of the middle point of the edge σ.

- If $|\alpha_{\mathcal{K}} - \alpha_{\mathcal{L}}| > \epsilon_0$,

$$\sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \| B_{\mathcal{Q}} \delta^{\mathcal{D}} \|_{\mathcal{F}}^2 \leq C(\sin(\epsilon_0)) \sum_{\mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \| B_{\mathcal{Q}} \delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}\,t} B_{\mathcal{Q}} \|_{\mathcal{F}}^2,$$

  with $C(\sin(\epsilon_0)) \underset{\epsilon_0 \to 0}{\longrightarrow} \infty$.

Analysis of the S-m-DDFV scheme

Theorem (K. 09)

For general and regular DDFV mesh $\mathcal{T}$. We assume that η is Lipschitz continuous per quarter diamond cells : $\forall\ \mathcal{Q} \in \mathfrak{Q}$

$$|\eta(x) - \eta(x')| \leq \mathrm{C}_\eta |x - x'|, \quad \forall x, x' \in \bar{\mathcal{Q}}.$$

If $\mathbf{u}$ *is smooth on each quarter diamond cells* $\mathcal{Q}$ *and* $p \in H^1(\Omega)$*, we have*

$$\|\mathbf{u} - \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 + \|\nabla \mathbf{u} - \nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2 \leq C \ \mathit{size}(\mathcal{T}),$$

$$\|p - p^{\mathfrak{D}}\|_2 \leq C \ \mathit{size}(\mathcal{T}).$$

Ideas of the proof

We need :

- Stability of S-m-DDFV scheme.
- Consistency error. If $\mathbf{u}$ is smooth on each quarter diamond cells $\mathcal{Q}$, the difficulty leads in the proof of

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} |D\mathbf{u}(z) - \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}}\mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}}\mathbf{u}(z)|^2 \mathrm{d}z \leq Ch^2_{\mathcal{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} (|\nabla \mathbf{u}|^2 + |D^2\mathbf{u}|^2)\mathrm{d}z,$$

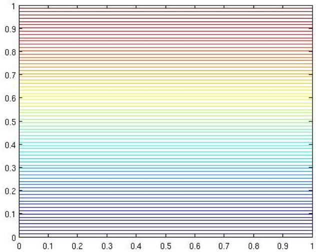
  use the continuity of the normal part of the viscous stress tensor accross of egdes.

Recall Case 3

$$\mathbf{u}(x,y) = \begin{pmatrix} \begin{cases} y^2 - 0.5y & \text{for } y > 0.5 \\ 10^4(y^2 - 0.5y) & \text{else.} \end{cases} \\ 0 \end{pmatrix},$$

$$p(x,y) = 2x - 1,$$

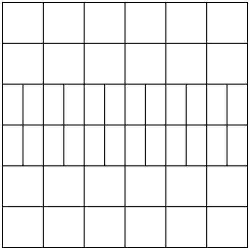
$$\eta(x,y) = \begin{cases} 1 & \text{for } y > 0.5 \\ 10^{-4} & \text{else.} \end{cases}$$



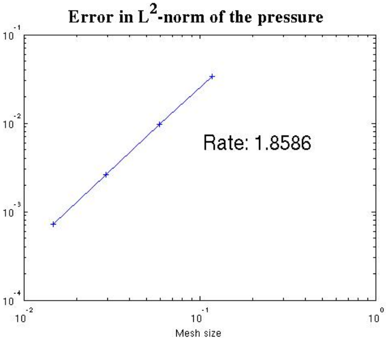
Streamlines

Case 3

Mesh



$\|p - p^{\mathfrak{D}}\|_2 / \|p\|_2$

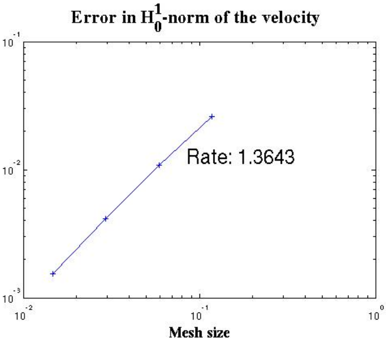


Rate for S-m-DDFV =1.85

Rate for S-DDFV =0.83

Case 3

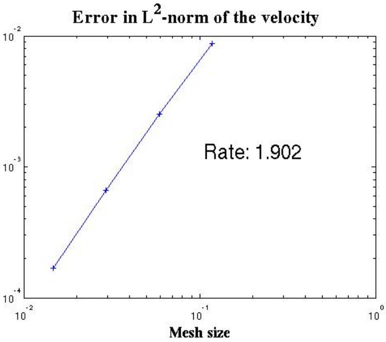
$$\|\mathbf{u}-\mathbf{u}^{\mathcal{T}}\|_{H^1_0}/\|\mathbf{u}\|_{H^1_0}$$



Rate for S-m-DDFV =1.36

Rate for S-DDFV =0.34

$$\|\mathbf{u}-\mathbf{u}^{\mathcal{T}}\|_2/\|\mathbf{u}\|_2$$

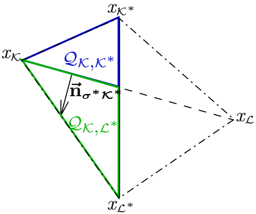


Rate for S-m-DDFV =1.9

Rate for S-DDFV =0.83

Outline

▶ **Conservativity of the fluxes**



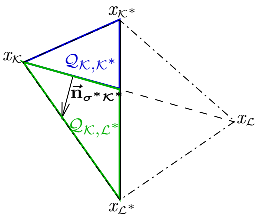
$\rightsquigarrow$ 4 new pressure unknowns $p^{\mathcal{Q}}$ on the quarter diamond cells

The conservativity of the fluxes through $\sigma_{\mathcal{K}}$ is

$$\int_{\sigma_{\mathcal{K}}} (2\eta_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}(s)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}(s) - p_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}}(s)\mathrm{Id})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}\mathrm{d}s$$

$$= \int_{\sigma_{\mathcal{K}}} (2\eta_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}(s)\mathrm{D}\mathbf{u}_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}(s) - p_{|\overline{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}\mathrm{d}s.$$

▶ Conservativity of the fluxes



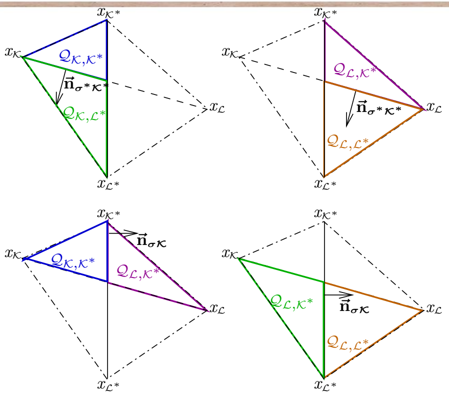
$\rightsquigarrow$ 4 new pressure unknowns $p^{\mathcal{Q}}$ on the quarter diamond cells

We determine $\delta = (\delta^{\mathcal{D}}, p_{\mathcal{D}}^{\mathfrak{Q}})$ such that

$$\underbrace{(\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(2\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\tau}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}} + {}^t(B_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\delta^{\mathcal{D}}) - p_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}\mathrm{Id})}_{\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\delta)})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

$$= \underbrace{(\eta_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(2\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\tau}} + B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}} + {}^t(B_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\delta^{\mathcal{D}}) - p_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}\mathrm{Id})}_{\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\delta)})\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$

Case of discontinuous pressure 2/3



$$\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{K}^*}}(\delta)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}=\varphi_{\mathcal{Q}_{\mathcal{K},\mathcal{L}^*}}(\delta)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$
$$\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{K}^*}}(\delta)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}=\varphi_{\mathcal{Q}_{\mathcal{L},\mathcal{L}^*}}(\delta)\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}$$
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$$\mathrm{Tr}(B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0, \forall \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$$
$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} p^{\mathcal{Q}} = m_{\mathcal{D}} p^{\mathcal{D}}.$$

Proposition (K. 09)

For all $\mathcal{D} \in \mathfrak{D}$ and all $(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}, p^{\mathcal{D}}) \in \mathcal{M}_{2,2}(\mathbb{R}) \times \mathbb{R}$, there exists a $\delta = (\delta^{\mathcal{D}}, p_{\mathcal{D}}^{\mathfrak{Q}}) \in \mathcal{M}_{n_{\mathcal{D}},2}(\mathbb{R}) \times \mathbb{R}^{n_{\mathcal{D}}}$ ensuring the fluxes conservativity.

Case of discontinuous pressure 3/3

Theorem (K. 09)

General and regular $\mathcal{T}$ DDFV mesh. The S-m-DDFV scheme has an **unique** *solution $(\mathbf{u}^{\boldsymbol{\mathcal{T}}}, p^{\mathfrak{D}})$, for all $\lambda > 0$.*

Error estimates in progress.

Consistency error. If $(\mathbf{u}, p)$ are smooth on each quarter diamond cells $\mathcal{Q}$, the difficulty leads in the proof

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} |D\mathbf{u}(z) - \mathrm{D}^{\mathcal{N}}_{\mathcal{Q}} \mathbb{P}^{\boldsymbol{\mathcal{T}}}_{\boldsymbol{c}} \mathbf{u}(z)|^2 \mathrm{d}z \leq C h_{\mathcal{D}}^2 \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} \int_{\mathcal{Q}} (|\nabla \mathbf{u}|^2 + |D^2\mathbf{u}|^2 + |\nabla p|^2) \mathrm{d}z.$$

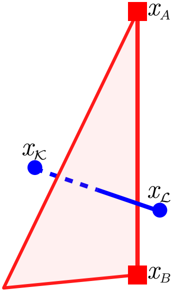
DDFV schemes in 3D

Several approaches

- Unknowns at centers of control volumes, at vertices **Coudière & Pierre '07**
  $\rightsquigarrow$ Restrictions on the meshes. **Andreianov and *al* '08**
- Unknowns at centers of control volumes, at vertices and at the faces **Hermeline 08'** $\rightsquigarrow$ Restrictions on the meshes.
- Unknowns at centers of control volumes, vertices, faces and edges. **Coudière & Hubert '09** $\rightsquigarrow$ Works for general meshes.

Construction of the diamond cells

- We need three complementary directions to reconstruct the discrete gradient
- A natural choice, for any face $F = \partial\mathcal{K} \cap \partial\mathcal{L}$, any edge $e \in \partial F$, whose vertices $A, B \in \partial e$.



- The direction $x_{\mathcal{K}}x_{\mathcal{L}}$
- The direction $x_A x_B$
- The direction $x_F x_e$

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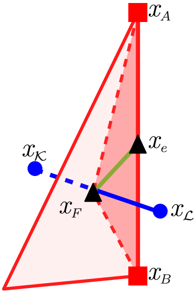

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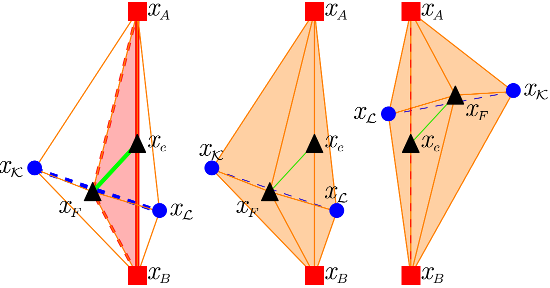
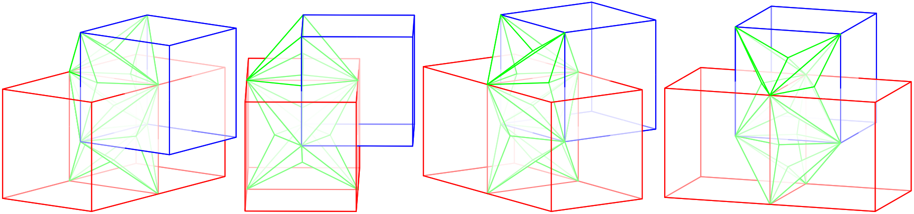


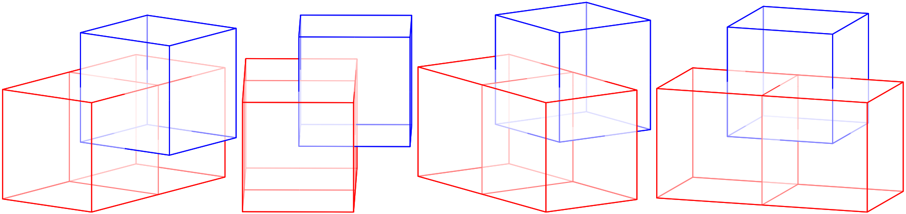
- The direction $x_\mathcal{K} x_\mathcal{L}$
- The direction $x_A x_B$
- The direction $x_F x_e$

Example of regular hexahedrical mesh

The three meshes



The primal mesh and a node cell



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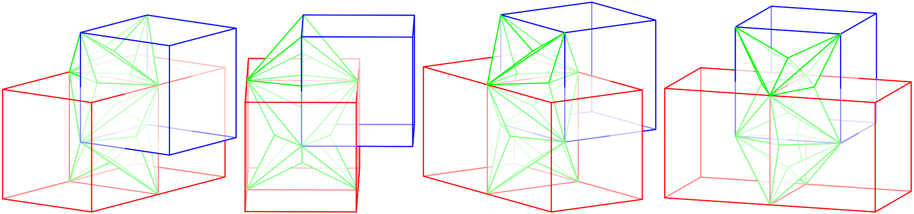
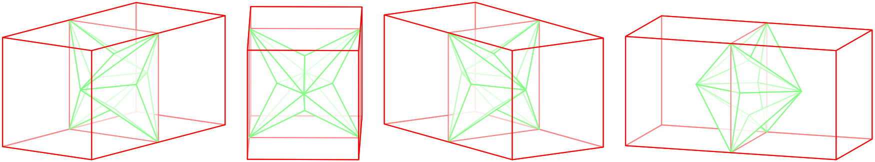
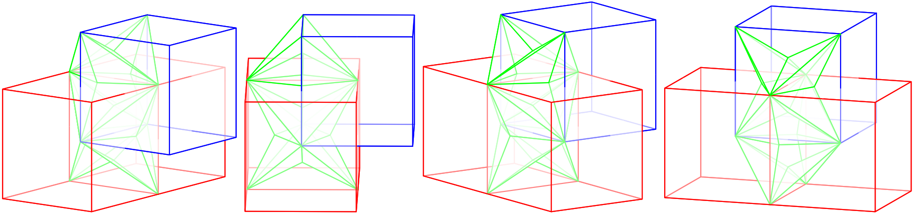


The primal mesh and a face cell

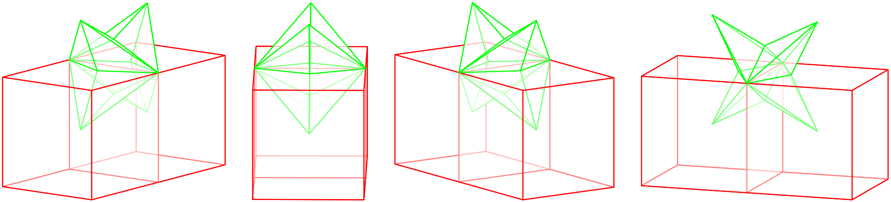


Example of regular hexahedrical mesh

The three meshes



The primal mesh an an edge cell



The discrete operators for scalar-value functions

The discrete operators with

$$\nabla^{\mathfrak{D}} : \mathbb{R}^{\mathcal{T}} \to (\mathbb{R}^3)^{\mathfrak{D}}, \ \mathrm{div}^{\mathcal{T}} : (\mathbb{R}^3)^{\mathfrak{D}} \to \mathbb{R}^{\mathfrak{M} \cup \mathfrak{M}^* \cup \mathfrak{N}}.$$

The discrete gradient

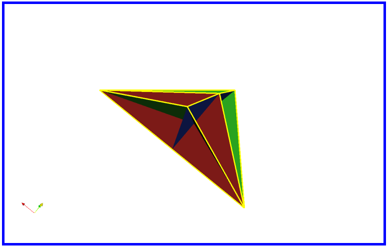
$$\forall \mathcal{D} \in \mathfrak{D}, \quad \nabla^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{3 m_{\mathcal{D}}} \Big((u_{\mathcal{L}} - u_{\mathcal{K}}) \vec{\mathbf{N}}_{\mathcal{K}\mathcal{L}} + (u_{B} - u_{A}) \vec{\mathbf{N}}_{AB} + (u_{F} - u_{e}) \vec{\mathbf{N}}_{eF} \Big).$$

with

$$\begin{aligned}
\vec{\mathbf{N}}_{\mathcal{K}\mathcal{L}} &= \frac{1}{2}(x_B - x_A) \times (x_F - x_e) = \int_{\bar{\mathcal{K}} \cap \bar{\mathcal{L}} \cap \mathcal{D}} n_{\mathcal{K}\mathcal{L}} \, ds \\
\vec{\mathbf{N}}_{AB} &= \frac{1}{2}(x_F - x_e) \times (x_{\mathcal{L}} - x_{\mathcal{K}}) = \int_{\bar{A} \cap \bar{B} \cap \mathcal{D}} n_{AB} \, ds \\
\vec{\mathbf{N}}_{eF} &= \frac{1}{2}(x_{\mathcal{L}} - x_{\mathcal{K}}) \times (x_B - x_A) = \int_{\bar{e} \cap \bar{F} \cap \mathcal{D}} n_{eF} \, ds
\end{aligned}$$

with the orientation choosen in such a way that

$$\det(x_B - x_A, x_F - x_e, x_{\mathcal{L}} - x_{\mathcal{K}}) > 0$$



The discrete operators for scalar-value functions

The discrete gradient

$$\forall \mathcal{D} \in \mathfrak{D}, \quad \nabla^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{3m_{\mathcal{D}}} \left((u_{\mathcal{L}} - u_{\mathcal{K}}) \vec{\mathbf{N}}_{\boldsymbol{\mathcal{KL}}} + (u_{B} - u_{A}) \vec{\mathbf{N}}_{\boldsymbol{AB}} + (u_{F} - u_{e}) \vec{\mathbf{N}}_{\boldsymbol{eF}} \right).$$

The discrete divergence

$$m_{\mathcal{K}} \mathrm{div}^{\mathcal{K}} \boldsymbol{\phi}^{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}_{\mathcal{K}}} \boldsymbol{\phi}^{\mathcal{D}} \cdot \vec{\mathbf{N}}_{\boldsymbol{\mathcal{KL}}}, \quad m_{A} \mathrm{div}^{A} \boldsymbol{\phi}^{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}_{A}} \boldsymbol{\phi}^{\mathcal{D}} \cdot \vec{\mathbf{N}}_{\boldsymbol{AB}},$$

$$m_{e} \mathrm{div}^{e} \boldsymbol{\phi}^{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}_{e}} \boldsymbol{\phi}^{\mathcal{D}} \cdot \vec{\mathbf{N}}_{\boldsymbol{eF}}, \quad m_{F} \mathrm{div}^{F} \boldsymbol{\phi}^{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}_{F}} \boldsymbol{\phi}^{\mathcal{D}} \cdot \left(-\vec{\mathbf{N}}_{\boldsymbol{eF}} \right).$$

Remark that for all $\mathbb{C} \in \mathcal{T}$, if $n_{\mathbb{C}}$ is the unit normal to $\partial\mathbb{C}$ outward of $\mathbb{C}$,

$$|\mathbb{C}| \mathrm{div}_{\mathbb{C}} \xi^{\mathfrak{D}} = \int_{\partial\mathbb{C}} \xi^{\mathfrak{D}}(x) \cdot n_{\mathbb{C}}(x) d\sigma(x).$$

The discrete operators for vector-value functions

THE DISCRETE GRADIENT

$\nabla^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^3\right)^{\mathcal{T}} \mapsto (\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}})_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_3(\mathbb{R}))^{\mathfrak{D}}$, as follows :

$$\nabla^{\mathcal{D}}\mathbf{u}^{\boldsymbol{\mathcal{T}}} = \begin{pmatrix} {}^t(\nabla^{\mathcal{D}} u_1^{\mathcal{T}}) \\ {}^t(\nabla^{\mathcal{D}} u_2^{\mathcal{T}}) \\ {}^t(\nabla^{\mathcal{D}} u_3^{\mathcal{T}}) \end{pmatrix}, \quad \forall \mathcal{D} \in \mathfrak{D},$$

where $\nabla^{\mathcal{D}} u_i^{\mathcal{T}}$ is defined below, for $i = 1, 2, 3$.

THE DISCRETE DIVERGENCE

$\mathbf{div}^{\boldsymbol{\mathcal{T}}} : \xi^{\mathfrak{D}} = (\xi^{\mathcal{D}})_{\mathcal{D}\in\mathfrak{D}} \in (\mathcal{M}_3(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\boldsymbol{\mathcal{T}}}\xi^{\mathfrak{D}} \in (\mathbb{R}^3)^{\mathfrak{M}\cup\mathfrak{M}^*\cup\mathfrak{N}}$, as follows :

$$m_{\mathcal{K}}\mathbf{div}^{\boldsymbol{\mathcal{K}}}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}_{\mathcal{K}}} \xi^{\mathcal{D}}\vec{\mathbf{N}}_{\boldsymbol{\mathcal{K}\mathcal{L}}}, \quad m_{A}\mathbf{div}^{A}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}_A} \xi^{\mathcal{D}}\vec{\mathbf{N}}_{AB},$$

$$m_{e}\mathbf{div}^{e}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}_e} \xi^{\mathcal{D}}\vec{\mathbf{N}}_{eF}, \quad m_{F}\mathbf{div}^{F}\xi^{\mathfrak{D}} = \sum_{\mathcal{D}\in\mathfrak{D}_F} \xi^{\mathcal{D}}\left(-\vec{\mathbf{N}}_{eF}\right),$$

The discrete operators for vector-value functions

The discrete strain rate tensor

$\mathrm{D}^{\mathfrak{D}} : \mathbf{u}^{\boldsymbol{\mathcal{T}}} \in \left(\mathbb{R}^3\right)^{\mathcal{T}} \mapsto \left(\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)_{\mathcal{D} \in \mathfrak{D}} \in \left(\mathcal{M}_3(\mathbb{R})\right)^{\mathfrak{D}}$, such that

$$\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} = \frac{\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}} + {}^{t}\left(\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\right)}{2}.$$

The stabilization term

$\Delta^{\mathfrak{D}} : p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \Delta^{\mathfrak{D}} p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, and defined as follows :

$$\Delta^{\mathcal{D}} p^{\mathfrak{D}} = \frac{1}{m_{\mathcal{D}}} \sum_{\mathfrak{s} = \mathcal{D} | \mathcal{D}' \in \mathcal{E}_{\mathcal{D}}} \frac{h_{\mathcal{D}}^3 + h_{\mathcal{D}'}^3}{h_{\mathcal{D}}^3} \left(p^{\mathcal{D}'} - p^{\mathcal{D}}\right), \qquad \forall\, \mathcal{D} \in \mathfrak{D}.$$

The DDFV scheme

$$\begin{cases} \text{Find } \mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0 \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} \\ \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = \mathbf{f}^{\mathcal{T}}, \\ \mathrm{Tr}\nabla^{\mathfrak{D}}(\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^3 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0, \end{cases} \qquad \text{(3D-S-DDFV)}$$

with $\lambda > 0$ given.

Theorem (Existence and uniqueness, K. & Manzini 09)

Let $\mathcal{T}$ be a DDFV mesh.
For any value of the stabilization parameter $\lambda > 0$, the (3D-S-DDFV) *scheme admits an unique solution.*

Error estimates is under study.

Outline

Conclusion

- Successful extension for more general flows

$$\begin{cases} \mathrm{div}(-2\eta(.)D(\mathbf{u}) + p\mathrm{Id}) = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega. \end{cases}$$

even for discontinuous η viscosity .

- Perspectives
  - Further numerical tests in process.
  - Error estimates for pressures that are only smooth per quarter diamonds.
  - Error estimates in 3D.
  - Handle other boundary conditions.
  - Take into account the dependency of η on $D\mathbf{u}$ (non-newtonian flows / LES models).
  - Add the non-linear term $\mathbf{u} \cdot \nabla \mathbf{u}$ of the Navier-Stokes equations.

Proof of discrete Korn inequality

- Proof of $\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2 \leq \sqrt{2}\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2$:

$$2\|\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2^2 = \|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2^2 + \int_\Omega ({}^t(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}).$$

Using the Stokes formula Theorem and (1), we have

$$\int_\Omega \left({}^t(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) = -\int_\Omega \mathbf{div}^{\mathcal{T}}\left({}^t(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})\right)\cdot\mathbf{u}^{\mathcal{T}} = -\int_\Omega \mathbf{div}^{\mathcal{T}}(\mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})\mathrm{Id})\cdot\mathbf{u}^{\mathcal{T}}$$

Using the Stokes formula Theorem and $\mathrm{Tr}\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} = (\mathrm{Id} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})$:

$$\int_\Omega \left({}^t(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) = \int_\Omega (\mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})\mathrm{Id} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) = \|\mathrm{Tr}\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\|_2^2 \geq 0.$$

Return

Let $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that :

$$\begin{cases} \mathbf{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = 0, \\ \mathbf{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) = 0, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{cases}$$

$$\int_\Omega \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}}\mathrm{Id}) \cdot \mathbf{u}^{\mathcal{T}} = \int_\Omega \left(2\eta^{\mathfrak{D}}\mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}} : \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) - \int_\Omega \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})p^{\mathfrak{D}}$$

Furthermore, the mass conservation equation gives :

$$-\int_\Omega \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}})p^{\mathfrak{D}} = -\int_\Omega \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} p^{\mathfrak{D}} = \lambda |p^{\mathfrak{D}}|_h^2,$$

where $|p^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s}\in\mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2)(p^{\mathcal{D}'} - p^{\mathcal{D}})^2$.

Using the discrete Korn inequality :

$$0 = \int_\Omega \mathbf{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}} \mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \mathrm{Id}) \cdot \mathbf{u}^{\mathcal{T}} \geq \underline{\mathrm{C}}_\eta \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 + \lambda |p^{\mathfrak{D}}|_h^2.$$

We finally get

$$\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 = 0 \quad \text{and} \quad |p^{\mathfrak{D}}|_h^2 = 0.$$

We deduce $\mathbf{u}^{\mathcal{T}} = \mathbf{0}$ and $p^{\mathfrak{D}} = c$. And we have $\sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$ so $p^{\mathfrak{D}} = 0$.

Return

Proof of existence of $\delta^{\mathcal{D}}$

We have

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\varphi_{\mathcal{Q}}(\delta^{\mathcal{D}})B_{\mathcal{Q}} = 0 \Longleftrightarrow \mathcal{A}\delta = \mathcal{B}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}),$$

with $\mathcal{B}(\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}) = 0$ if $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}} = 0$.

$$\text{Existence} \Longleftrightarrow \mathrm{Ker}\mathcal{A} = \{0\}$$

Multiplying by $\delta^{\mathcal{D}}$

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}(\underbrace{2\eta_{\mathcal{Q}}\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}} + \eta_{\mathcal{Q}}(B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}})}_{\varphi_{\mathcal{Q}}(\delta^{\mathcal{D}})} : B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0.$$

Since $\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}$ is zero, we obtain

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}({}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} : B_{\mathcal{Q}}\delta^{\mathcal{D}}) = \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}\|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}\|_{\mathcal{F}}^2 = 0.$$

Therefore, it implies

$${}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = 0, \quad \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}.$$

Proof of existence of $\delta^{\mathcal{D}}$ 2/2

- If $\alpha_{\mathcal{K}} \neq \alpha_{\mathcal{L}}$,
  ${}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = 0, \quad \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$ implies $\delta^{\mathcal{D}} = 0$.
- If $\alpha_{\mathcal{K}} = \alpha_{\mathcal{L}}$,
  ${}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}} + B_{\mathcal{Q}}\delta^{\mathcal{D}} = 0, \quad \forall\, \mathcal{Q} \in \mathfrak{Q}_{\mathcal{D}}$ implies

$$\mathrm{Ker}\mathcal{A} = \mathrm{Span}\begin{pmatrix} -\dfrac{{}^t\vec{\mathbf{n}}_{\sigma\mathcal{K}}}{m_{\sigma_{\mathcal{K}}}} \\ \dfrac{{}^t\vec{\mathbf{n}}_{\sigma\mathcal{K}}}{m_{\sigma_{\mathcal{L}}}} \\ \dfrac{{}^t\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}}{m_{\sigma_{\mathcal{K}^*}}} \\ -\dfrac{{}^t\vec{\mathbf{n}}_{\sigma^*\mathcal{K}^*}}{m_{\sigma_{\mathcal{L}^*}}} \end{pmatrix} := \mathrm{Span}(\delta_0).$$

Need to impose $(\delta^{\mathcal{D}}, \delta_0) = 0$ for uniqueness and verify that the second member belongs to the range of $\mathcal{A}$.

Return

Proof of wellposedness of the scheme 1/2

Let $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that :

$$\begin{cases} \mathbf{div}^{\mathfrak{M}}(-2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + p^{\mathfrak{D}}\mathrm{Id}) = 0, \\ \mathbf{div}^{\mathfrak{M}^*}(-2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + p^{\mathfrak{D}}\mathrm{Id}) = 0, \\ \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \displaystyle\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0. \end{cases}$$

$$\int_\Omega \mathbf{div}^{\mathcal{T}}(-2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) + p^{\mathfrak{D}}\mathrm{Id}) \cdot \mathbf{u}^{\mathcal{T}} = \int_\Omega \left(2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) + \lambda |p^{\mathfrak{D}}|_h^2.$$

$$\begin{aligned} \int_\Omega 2(\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) &= \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}} : 2\mathrm{D}^{\mathcal{D}}\mathbf{u}^{\mathcal{T}}) \\ &= \sum_{\mathcal{D}\in\mathfrak{D}} \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}} : 2\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}} - B_{\mathcal{Q}}\delta^{\mathcal{D}} - {}^t\delta^{\mathcal{D}\,t}B_{\mathcal{Q}}) \\ &= \int_\Omega 2\left(\eta^{\mathfrak{Q}}\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}} : \mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}}\right). \end{aligned}$$

Thanks to $\displaystyle\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}}\eta_{\mathcal{Q}}(\mathrm{D}_{\mathcal{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}} : B_{\mathcal{Q}}\delta^{\mathcal{D}}) = 0.$

Proof of wellposedness of the scheme 2/2

Using the new discrete Korn inequality :

$$0 = \int_\Omega \left(2\varphi^{\mathfrak{D}}(\eta, \mathrm{D}^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}) : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathcal{T}}\right) + \lambda |p^{\mathfrak{D}}|_h^2 \geq C \|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}}\|_2^2 + \lambda |p^{\mathfrak{D}}|_h^2.$$

We finally get

$$\|\nabla_{\mathfrak{Q}}^{\mathcal{N}}\mathbf{u}^{\mathcal{T}}\|_2^2 = 0 \quad \text{and} \quad |p^{\mathfrak{D}}|_h^2 = 0.$$

We deduce $\mathbf{u}^{\mathcal{T}} = \mathbf{0}$ and $p^{\mathfrak{D}} = c$. And we have $\sum_{\mathcal{D}\in\mathfrak{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$ so $p^{\mathfrak{D}} = 0$.

Return

Proof of new discrete Korn inequality

We have

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|\nabla_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2 = m_{\mathcal{D}} \|\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2 + \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|_{\mathcal{F}}^2.$$

Combining the two estimates

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}}\|_{\mathcal{F}}^2 \leq C \sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}\|_{\mathcal{F}}^2,$$

and

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|B_{\mathcal{Q}}\delta^{\mathcal{D}} + {}^t\delta^{\mathcal{D}}\,{}^tB_{\mathcal{Q}}\|_{\mathcal{F}}^2 \leq C m_{\mathcal{D}} \|\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2,$$

we get

$$\sum_{\mathcal{Q}\in\mathfrak{Q}_{\mathcal{D}}} m_{\mathcal{Q}} \|\nabla_{\mathcal{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2 \leq m_{\mathcal{D}} \|\nabla^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2 + C m_{\mathcal{D}} \|\mathrm{D}^{\mathcal{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_{\mathcal{F}}^2$$

Using the discrete Korn inequality Theorem 1 and Proposition 4 , we conclude

$$\|\nabla_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2 \leq C \|\mathrm{D}^{\mathfrak{D}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2 \leq C \|\mathrm{D}_{\mathfrak{Q}}^{\mathcal{N}} \mathbf{u}^{\boldsymbol{\mathcal{T}}}\|_2^2.$$

Return