Cell Centered Finite Volume Schemes for Multiphase Flow Applications

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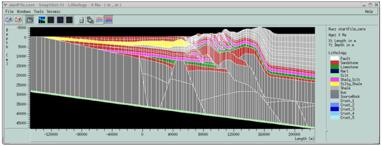
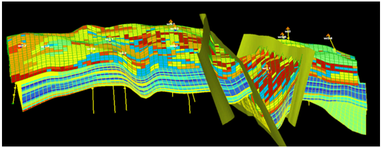
Applications

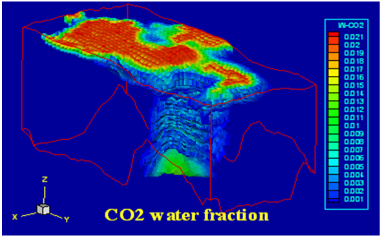
Basin Modeling



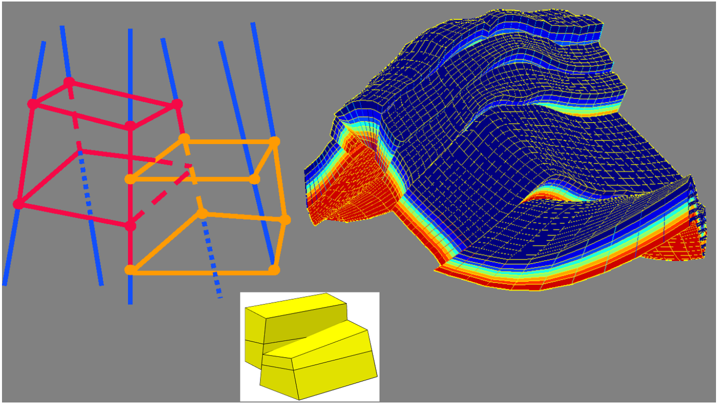
Reservoir simulation



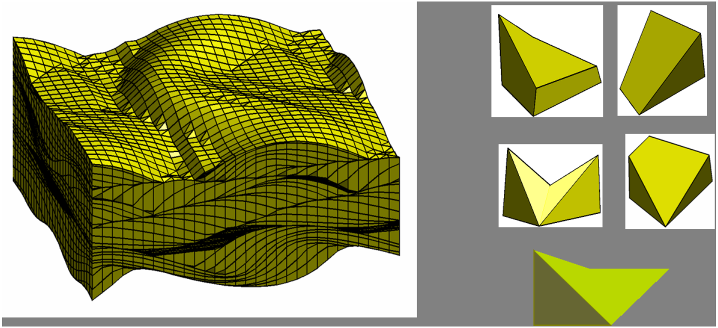
CO2 geological storage



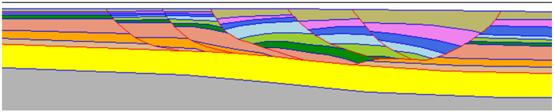
Meshes: corner point geometries with faults

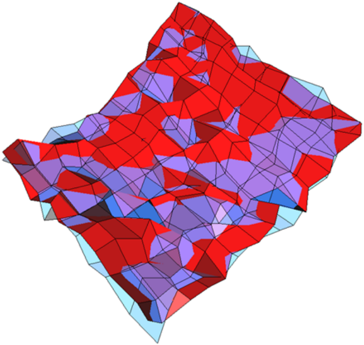


Meshes: corner point geometries with erosions

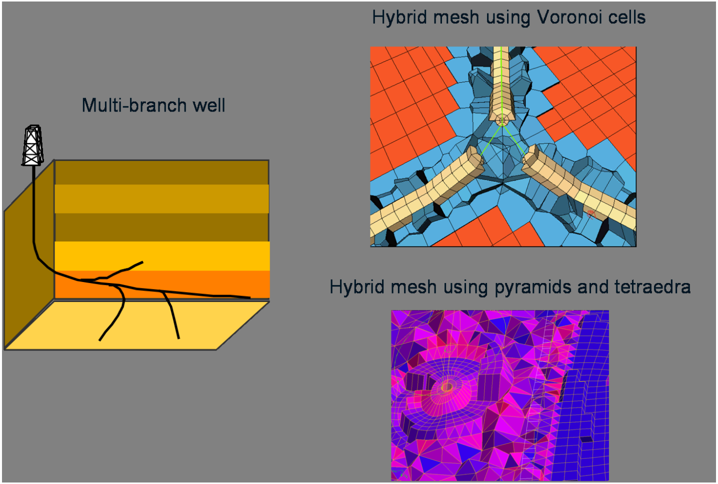


Meshes: basin geometries





Meshes: nearwell meshes



Difficulties

- Degenerated cells due to erosion
- Basin models: dynamic mesh: the scheme must be recomputed at each time step
- Faults in basin models: geometry not always available (overlaps and holes)
- Conductive Faults in basin models
- General polyhedral cells
- Boundary conditions
- Submeshes (dead cells)

Motivations of cell centered schemes for compositional multiphase Darcy flow applications

- One unknown per cell (*N* primary unknowns per cell for multiphase compositional flows)
- Explicit linear fluxes
- Easier to combine TPFA and MPFA
  - Adapted to fully or semi implicit discretization of multiphase compositional Darcy flows

- But “compact” MPFA VF schemes are non symmetric on general meshes
  - Possible lack of robustness due to mesh and diffusion coefficients dependent coercivity

Cell centered schemes currently implemented in ArcGeoSim

- L and G schemes
- O scheme
- GradCell scheme

Model problem

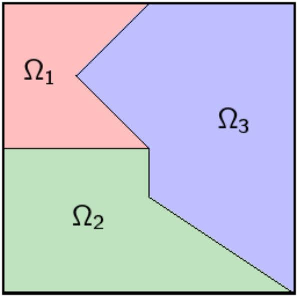
- Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain
- For $f \in L^2(\Omega)$, consider the following problem:

$$\begin{cases} -\mathrm{div}(\nu \nabla u) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

- Let $a(u, v) = \int_\Omega \nu \nabla u \cdot \nabla v$. The weak formulation reads

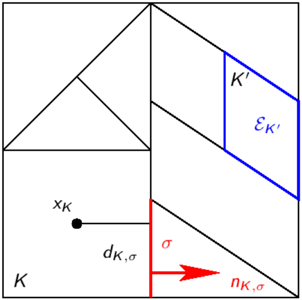
$$\text{Find } u \in H_0^1(\Omega) \text{ such that } a(u, v) = \int_\Omega fv \text{ for all } v \in H_0^1(\Omega) \qquad (\Pi)$$

Model problem



- Let $\{\Omega_i\}_{1 \leq i \leq N_\Omega}$ be a partition of Ω into bounded polygonal sub-domains
- $\nu|_{\Omega_i}$ smooth and $\nu(x)$ is s.p.d. for a.e. $x \in \Omega$

Polyhedral admissible meshes



$\mathcal{T}_h$: set of cells K
$\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b$: set of inner and boundary faces σ
m_σ: surface of the face σ
m_K: volume of the cell K

GradCell scheme: using a discrete variational framework [Agélas et al., 2008]

- Discrete variational formulation

$$a_h(u_h, v_h) = \int_\Omega f v_h \text{ for all } v_h \in V_h.$$

- with a_h based on two cellwise constant gradients and stabilized by residuals

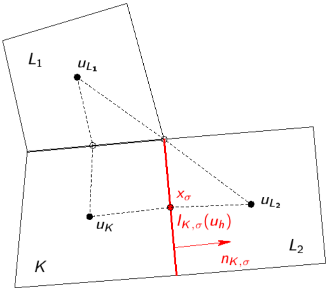
$$a(u_h, v_h) = \sum_{K \in \mathcal{T}_h} m_K \nu_K (\nabla_h u_h)_K \cdot (\widetilde{\nabla}_h v_h)_K + \sum_{K \in \mathcal{T}_h} \eta_K \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} R_{K,\sigma}(u_h) R_{K,\sigma}(v_h)$$

Discrete gradients reconstructions

- The cellwise constant gradients are obtained via the Green formula and trace reconstructions

$$(\nabla_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (I_{K,\sigma}(v_h) - v_K) n_{K,\sigma}$$

$$(\widetilde{\nabla}_h v_h)_K = \frac{1}{m_K} \sum_{\sigma \in \mathcal{E}_K} m_\sigma (\gamma_\sigma(v_h) - v_K) n_{K,\sigma}$$



- Residuals:

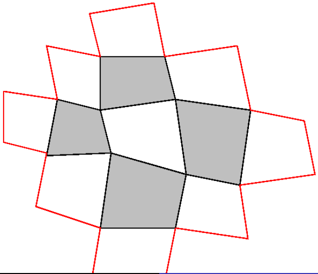
$$R_{K,\sigma}(v_h) = I_{K,\sigma}(v_h) - v_K - (\nabla_h v_h)_K \cdot (x_\sigma - x_K)$$

Fluxes

- Fluxes can be derived from the bilinear form using:

$$a_h(u_h, v_h) = \sum_{\sigma=\mathcal{E}_K\cap\mathcal{E}_L\in\mathcal{E}_h} F_{K,\sigma}(u_h)(v_L - v_K) - \sum_{K\in\mathcal{T}_h}\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_h^b} F_{K,\sigma}(u_h)v_K$$

- Stencil of the scheme: neighbours of the neighbours
- Example for topologicaly cartesian grids
  - 13 cells in 2D
  - 21 cells in 3D



Convergence analysis

- Consistency of $\nabla_h u_h$ for piecewise smooth functions with normal flux continuity: stability of the L-interpolation $\|(A^G)^{-1}\| \geq \beta$
- Coercivity assumption: $a(v_h, v_h) \gtrsim \|v_h\|_{V_h}^2$

Remarks

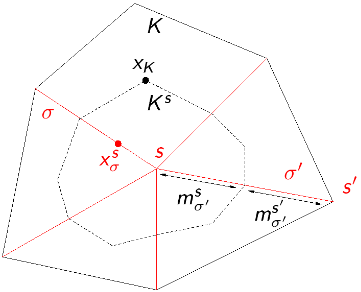
- Remark 1: a linear interpolation $I_{K,\sigma}$ can be used for smooth diffusion tensors ν
- Remark 2: choosing $\widetilde{\nabla}_h u_h = \nabla_h u_h$ yields a symmetric coercive scheme but at the price of a larger stencil (21 in 2D and 81 in 3D) see [Eymard and Herbin, 2007].

O scheme: notations [Agélas and Masson, 2008]

- Choose $x_K \in K \Rightarrow$ unknown u_K
- Choose $x_\sigma^s \in \sigma \Rightarrow$ unknown u_σ^s
- Choose $m_\sigma^s \geq 0$ such that $\sum_{s \in \mathcal{V}_\sigma} m_\sigma^s = m_\sigma \Rightarrow$ subcell K^s

$$\mathrm{m}_K^s = \frac{1}{d} \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} \mathrm{m}_\sigma^s d_{K,\sigma}$$

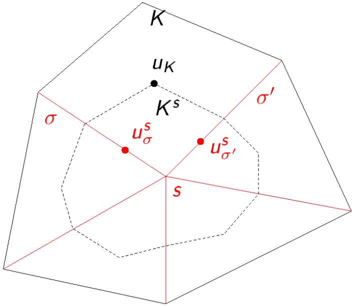
$\mathcal{E}_s$: set of faces connected to s
$\mathcal{E}_K$: set of faces of the cell K
$\mathcal{V}_\sigma$: set of vertices of the edge σ
$\mathcal{V}_K$: set of vertices of the cell K



Discrete gradient

Piecewise constant gradient on each K^s:

$$(\nabla_h u)_K^s = \sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} (u_\sigma^s - u_K)\, g_{K,\sigma}^s$$

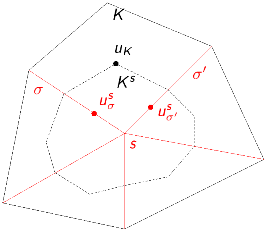


with $g_{K,\sigma}^s \in \mathbb{R}^d$ such that

$$\sum_{\sigma \in \mathcal{E}_K \cap \mathcal{E}_s} v \cdot (x_\sigma^s - x_K)\, g_{K,\sigma}^s = v \quad \text{for all } v \in \mathbb{R}^d.$$

Usual O scheme I

- Assume that $\{x_\sigma^s - x_K\}_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K}$ defines a basis of $\mathbb{R}^d$
- then $(\nabla_h u)_K^s$ is uniquely defined by the gradient of the linear interpolation of (u_K, x_K), $(u_\sigma^s, x_\sigma^s)_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K}$.

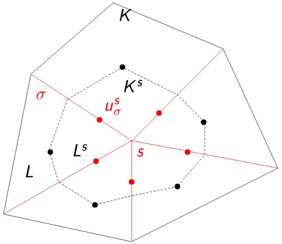


Subfluxes: $F_{K,\sigma}^s(u_h) = m_\sigma^s \nu_K (\nabla_h u)_K^s \cdot n_{K,\sigma}$

Usual O scheme II: Hybrid Finite volume scheme

$$\left\{ \begin{array}{ll} \displaystyle\sum_{\sigma\in\mathcal{E}_K}\Big(\sum_{s\in\mathcal{V}_\sigma} F^s_{K,\sigma}(u_h)\Big) = \int_K f(x)dx & \text{for all } K\in\mathcal{T}_h, \\ F^s_{K,\sigma}(u_h) = -F^s_{L,\sigma}(u_h) & \text{for all } s\in\mathcal{V}_\sigma, \sigma=\mathcal{E}_K\cap\mathcal{E}_L\in\mathcal{E}^i_h. \end{array}\right.$$

$(u^s_\sigma)_{\sigma\in\mathcal{E}_s}$ are eliminated around each vertex s in terms of the cell centered unknowns around s.



Usual O scheme: discrete hybrid variational formulation

Bilinear form on $\mathcal{H}_h$:

$$\mathcal{H}_h = \left\{ (u_K)_{K\in\mathcal{T}_h}, (u_\sigma^s)_{\sigma\in\mathcal{E}_s, s\in\mathcal{V}_h}, \text{ s. t. } u_\sigma^s = 0 \text{ for all } \sigma \in \mathcal{E}_h^b \right\}.$$

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \sum_{s\in\mathcal{V}_K} \mathrm{m}_K^s (\nabla_h u)_K^s \cdot \nu_K (\widetilde{\nabla}_h v)_K^s$$

with

$$(\widetilde{\nabla}_h u)_K^s = \frac{1}{\mathrm{m}_K^s} \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \mathrm{m}_\sigma^s (u_\sigma^s - u_K) n_{K,\sigma}.$$

Finite volume scheme: find $u_h \in \mathcal{H}_h$ such that

$$a_h(u_h, v) = \sum_{K\in\mathcal{T}_h} v_K \int_K f(x)dx \quad \text{for all } v \in \mathcal{H}_h.$$

Generalization: discrete hybrid variational formulation

Bilinear form on $\mathcal{H}_h$:

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} \sum_{s\in\mathcal{V}_K} \Bigg(\mathrm{m}_K^s (\nabla_h u)_K^s \cdot \nu_K (\widetilde{\nabla}_h v)_K^s + \alpha_K^s \sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_s} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} R_{K,\sigma}^s(u) R_{K,\sigma}^s(v) \Bigg)$$

with

$$R_{K,\sigma}^s(u) = u_\sigma^s - u_K - (\nabla_h u)_K^s \cdot (x_\sigma^s - x_K).$$

Finite volume scheme: find $u_h \in \mathcal{H}_h$ such that

$$a_h(u_h, v) = \sum_{K\in\mathcal{T}_h} v_K \int_K f(x)dx \quad \text{for all } v \in \mathcal{H}_h.$$

Hybrid Finite Volume scheme

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h}\sum_{s\in\mathcal{V}_K}\sum_{\sigma\in\mathcal{E}_s\cap\mathcal{E}_K}\sum_{\sigma'\in\mathcal{E}_s\cap\mathcal{E}_K}(T_K^s)_{\sigma,\sigma'}(u_{\sigma'}^s - u_K)(v_\sigma^s - v_K),$$

Let us define the following subfluxes:

$$F_{K,\sigma}^s(u) = \sum_{\sigma'\in\mathcal{E}_s\cap\mathcal{E}_K}(T_K^s)_{\sigma,\sigma'}(u_{\sigma'}^s - u_K),$$

Hybrid Finite volume scheme:

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_K}\Big(\sum_{s\in\mathcal{V}_\sigma} F_{K,\sigma}^s(u_h)\Big) = \int_K f(x)dx & \text{for all } K\in\mathcal{T}_h, \\ F_{K,\sigma}^s(u_h) = -F_{L,\sigma}^s(u_h) & \text{for all } s\in\mathcal{V}_\sigma, \sigma=\mathcal{E}_K\cap\mathcal{E}_L\in\mathcal{E}_h^i. \end{cases}$$

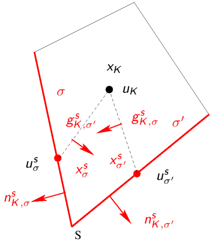
Convergence analysis

- Consistency of $\nabla_h \varphi_h$ for $\varphi \in C_c^\infty(\Omega)$ with $\varphi_h = \left(\varphi(x_K), \varphi(x_\sigma^s)\right)_{K \in \mathcal{T}_h, \sigma \in \mathcal{E}_s, s \in \mathcal{V}_h}$
- Coercivity assumption: $a(v_h, v_h) \gtrsim \|v_h\|_{\mathcal{H}_h}^2$

$$\|v\|_{\mathcal{H}_h} = \left(\sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} \sum_{s \in \mathcal{V}_\sigma} \frac{\mathrm{m}_K^s}{(d_{K,\sigma})^2} (v_\sigma^s - v_K)^2 \right)^{1/2}.$$

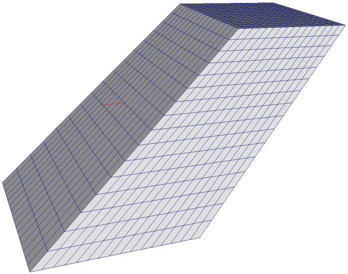
Symmetric unconditionaly coercive cases

- Simplectic cells
- Parallepipedic cells
  - for an ad hoc choice of x^s_σ one has $(\nabla_h u_h)^s_K = (\widetilde{\nabla}_h u_h)^s_K$ i.e. $g^s_{K,\sigma} \parallel n_{K,\sigma}$

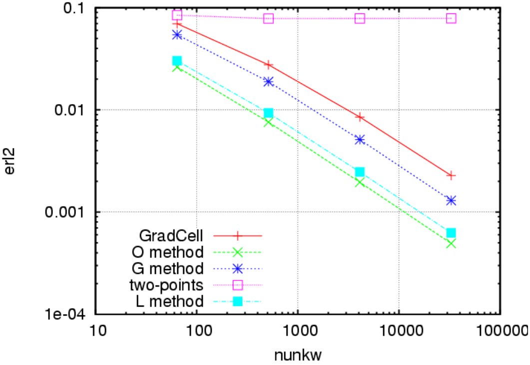


- Symmetrization (C. Lepotier)
  - Choose $(\nabla_h u)^s_K := (\widetilde{\nabla}_h u)^s_K$
  - Fluxes are not consistent except for the above cases

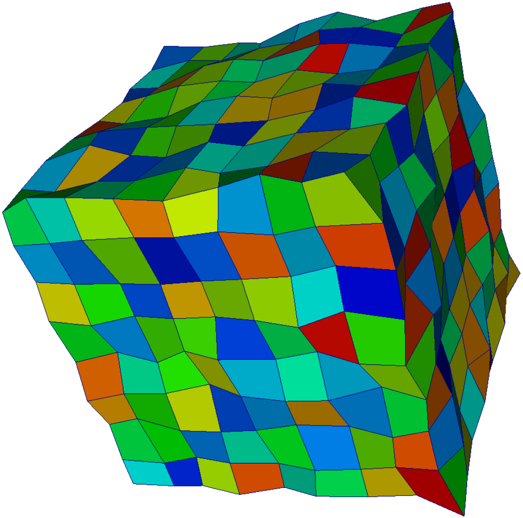
Hexahedral meshes, Identity permeability tensor,
$u(x, y, z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$



The grid

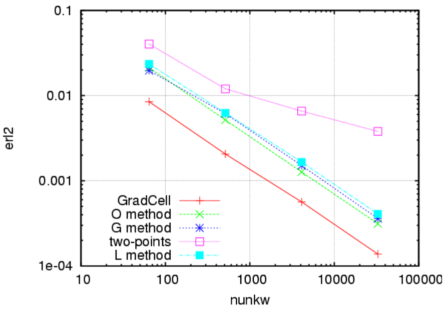


Randomly distorted hexahedral meshes I

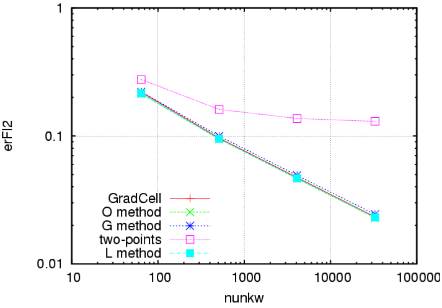


$\nu = \mathrm{I}$

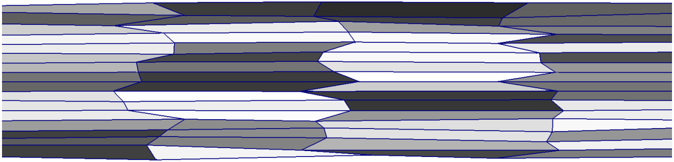
L^2 error on pressure



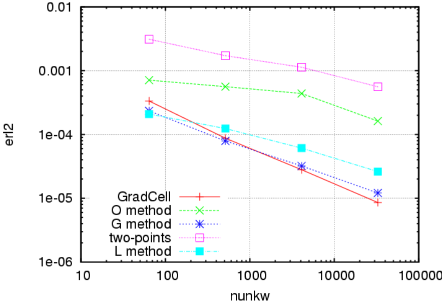
L^2 error on fluxes



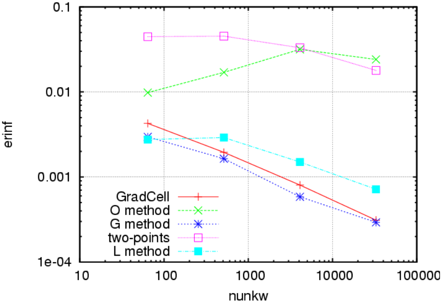
Randomly distorted hexahedral meshes II



$\nu = \mathrm{I}$

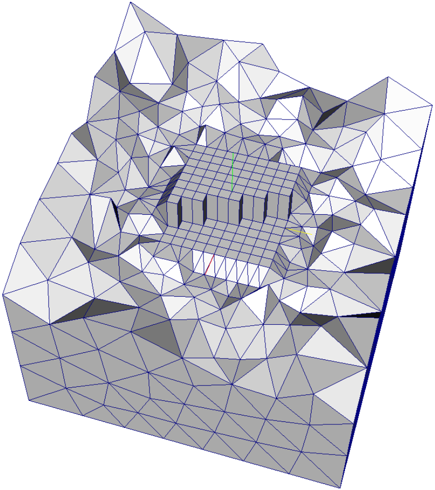


L_2 error for pressure

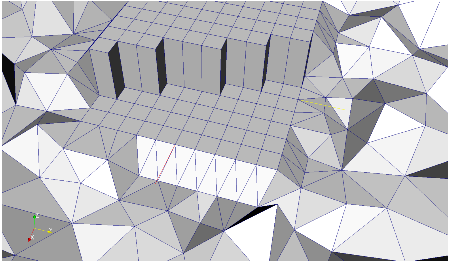


L_∞ error for pressure

Hybrid meshes



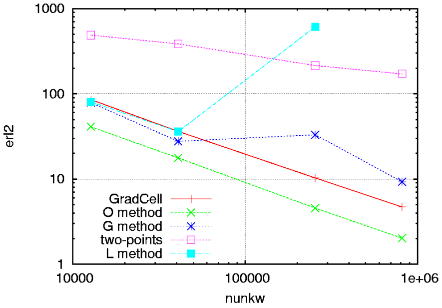
Slice of the mesh



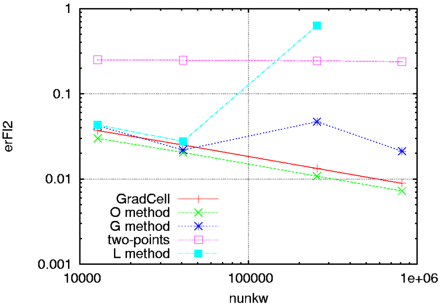
Zoom in

Hybrid meshes, Isotropic tensor ν.

Pressure L_2 error



Flux L_2 error



Hybrid meshes, Isotropic tensor ν, performance with AMG

Number of GMRES iterations vs grid size $\left(\frac{\|r\|}{\|r_0\|} < 10^{-9}\right)$.

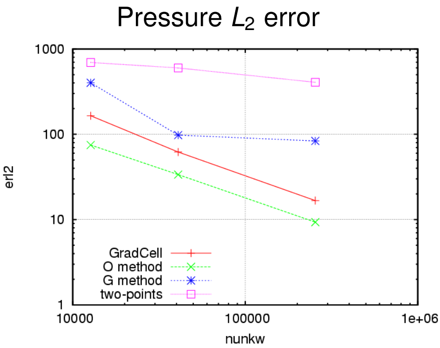
No of cells	O	L	G	GradCell	2-point
12723	7	—	16	8	7
40847	7	—	85	9	7
254645	11	—	—	15	8
813368	8	—	—	11	8

Hybrid meshes, Isotropic tensor ν, coercivity

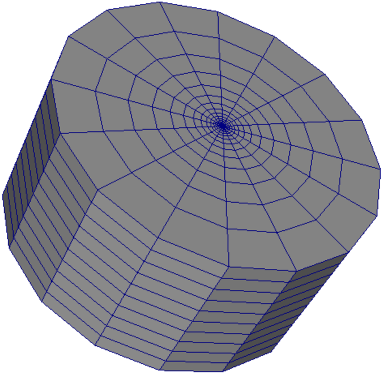
$$\gamma_h = \frac{a_h(u_h - \pi_h(u), u_h - \pi_h(u))}{\|u_h - \pi_h(u)\|_{V_h}^2}$$

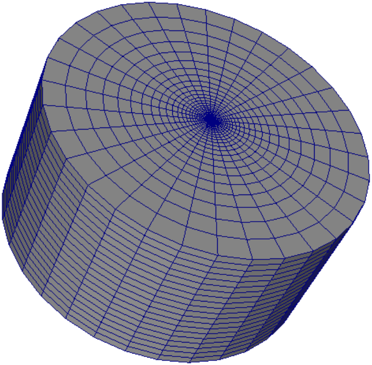
No of cells	O	L	G	GradCell	2-point
12723	$8.07e-01$	$1.35e-01$	$3.63e-01$	$1.45e+00$	$7.94e-01$
40847	$8.29e-01$	$8.52e-02$	$8.65e-01$	$1.51e+00$	$8.02e-01$
254645	$8.10e-01$	$7.20e-06$	$1.77e-02$	$1.47e+00$	$8.09e-01$
813368	$8.10e-01$	–	$3.40e-02$	$1.47e+00$	$8.21e-01$

Anisotropic tensor $\nu = diag\{1, 1, 0.1\}$.



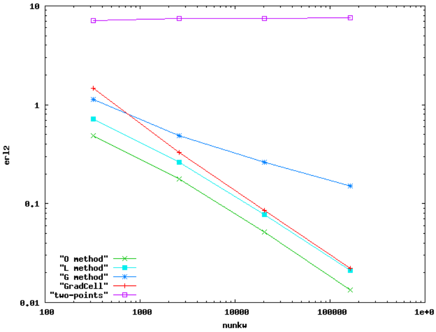
Nearwell radial grids



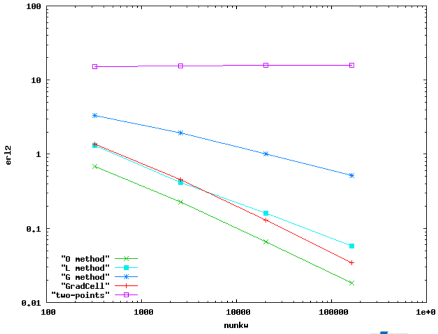


Anisotropic tensor $\nu = diag\{1, 1, 0.05\}$.

L^2 error on pressure



L^2 error on fluxes



Conclusion

- Case dependent results
- O scheme the most accurate but lacks of robustness for meshes with high aspect ratio (or anisotropy) combined with distorsion
- L and G schemes good on hexahedral meshes but may fail for Hybrid meshes
- GradCell exhibits the best robustness but requires two layers of communication in parallel

Perspectives: application dependent choices

- Real test cases on basin models (pressure equation)
- Multiphase compositional Darcy flow on CPG grids and nearwell meshes
- Hybrid Finite Volume scheme (MFD) for faults in basin models

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