# A finite volume method for the Stokes equations

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# Plan

1. Introduction : the MAC and covolume schemes
2. Generalization to non-orthogonal meshes
3. Definition of discrete differential operators
4. Properties of the discrete operators
5. Application to the Stokes problem
6. Numerical results
7. Extension to Navier-Stokes
8. Conclusion and perspectives

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# Introduction

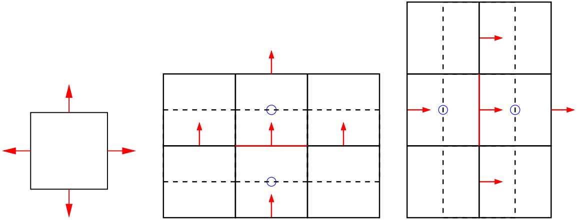
"Marker and Cell" scheme for the Stokes problem

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (2)$$

Unknowns : pressure at the centers of the cells and normal components of the velocity at the edges

Equations : discretization of (2) on the cells and discretization of the normal component of (1) on the edges.



Associated equations

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \frac{v_{i,j+1} - v_{i,j}}{\Delta y} = 0$$

$$\frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{\Delta x^2} + \frac{-u_{i,j+1} + 2u_{i,j} - u_{i,j-1}}{\Delta y^2} + \frac{p_{i,j} - p_{i-1,j}}{\Delta x} = (\mathbf{f}_x)_{i,j}$$

$$\frac{-v_{i+1,j} + 2v_{i,j} - v_{i-1,j}}{\Delta x^2} + \frac{-v_{i,j+1} + 2v_{i,j} - v_{i,j-1}}{\Delta y^2} + \frac{p_{i,j} - p_{i,j-1}}{\Delta y} = (\mathbf{f}_y)_{i,j}$$

Finite element interpretation (Girault - Raviart 79)

Finite volume interpretation and generalization to triangular (Delaunay) meshes by Nicolaides and coworkers (90 - 92 - 96).

Remark : $-\Delta \mathbf{u} = -\nabla\nabla\cdot\mathbf{u} + \nabla\times\nabla\times\mathbf{u}$.

So that the Stokes may be written

$$\nabla\times\nabla\times\mathbf{u} - \nabla\nabla\cdot\mathbf{u} + \nabla p = \mathbf{f}, \quad (3)$$

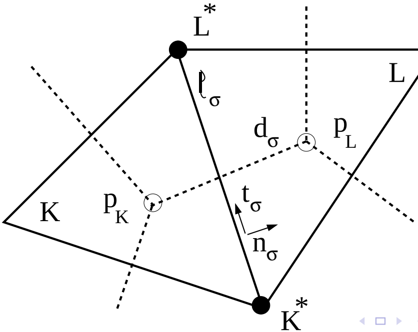
$$\nabla\cdot\mathbf{u} = 0. \quad (4)$$

Like in the MAC scheme, unknowns : $p$ at the circumcenters of the cells and $\mathbf{u}\cdot\mathbf{n}$ on the edges.
Like in the MAC scheme, equations : integration of (4) over the cells and finite differences on the normal component of (3) on the edges.

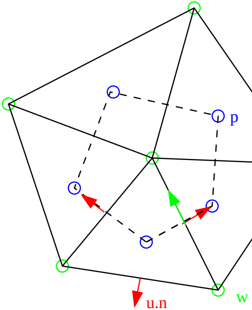
$$(\nabla \cdot \mathbf{u})_K = \frac{1}{|K|}\int_K \nabla \cdot \mathbf{u} = \frac{1}{|K|}\int_{\partial K} \mathbf{u}\cdot\mathbf{n} \approx \frac{1}{|K|}\sum_{\sigma\subset\partial K} \ell_\sigma \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K}$$

$$\nabla p \cdot \mathbf{n}_{\sigma,KL} \approx \frac{p_L - p_K}{d_\sigma} \quad ; \quad \nabla(\nabla\cdot\mathbf{u})\cdot\mathbf{n}_{\sigma,KL} \approx \frac{(\nabla\cdot\mathbf{u})_L - (\nabla\cdot\mathbf{u})_K}{d_\sigma}$$

$$\nabla\times(\nabla\times\mathbf{u})\cdot\mathbf{n}_{\sigma,KL} = \nabla(\nabla\times\mathbf{u})\cdot\mathbf{t}_{\sigma,KL} \approx \frac{(\nabla\times\mathbf{u})_{L^*} - (\nabla\times\mathbf{u})_{K^*}}{\ell_\sigma}$$



$$(\nabla \times \mathbf{u})_{K^*} = \frac{1}{|K^*|}\int_{K^*} \nabla \times \mathbf{u} = \frac{1}{|K^*|}\int_{\partial K^*} \mathbf{u}\cdot\mathbf{t}$$
$$\approx \frac{1}{|K^*|}\sum_{\sigma=K|L\perp\sigma^*\subset\partial K^*} d_\sigma \mathbf{u}_\sigma \cdot \mathbf{n}_{\sigma,K}$$



## Properties

- Duality of the operators $\nabla_K \cdot$ and $-\nabla_\sigma$ :
$(\nabla_K \cdot \mathbf{u}, p)_K = -(\mathbf{u}, \nabla_\sigma p)_\sigma +$ B.T.
- $\nabla_K \cdot (\nabla_\sigma \times) = 0$ and $\nabla_{K^*} \times \nabla_\sigma = 0$
- Discrete Hodge decomposition (here in a simply connected domain)

For a given set of edge normal velocities $v_\sigma$, there exists a unique set of cell and vertex values $(\phi_K), (\psi_{K^*})$ such that

$$v_\sigma = \nabla_\sigma(\phi_K) + \nabla_\sigma \times (\psi_{K^*}) = \frac{\phi_L - \phi_K}{d_\sigma} + \frac{\psi_{L^*} - \psi_{K^*}}{\ell_\sigma}$$

with $\phi_K = 0$ at the midpoints of boundary edges and $\sum_{K^*} |P_{K^*}| \psi_{K^*} = 0$

Proof : Euler's formula ($E = T + V - 1$) then $\nabla_K \cdot (\nabla_\sigma \times) = 0$, duality between $\nabla_K \cdot$ and $-\nabla_\sigma$ and injectivity of $\nabla_\sigma$.

# Generalization to non-orthogonal meshes

| Orthogonal meshes | General meshes |
|---|---|
| $p$ at the centers<br>$\omega$ at the vertices<br>$\mathbf{u} \cdot \mathbf{n}$ at the edges | $p$ at the centers and at the vertices<br>$\omega$ at the centers and at the vertices<br>both components of $\mathbf{u}$ at the edges |
| $\nabla \cdot \mathbf{u} = 0$ primal mesh<br>$\omega = \nabla \times \mathbf{u}$ dual mesh<br>Normal components at the edges<br>of the momentum equation | $\nabla \cdot \mathbf{u} = 0$ primal and dual meshes<br>$\omega = \nabla \times \mathbf{u}$ primal and dual meshes<br>both components<br>of the momentum equation |

## Primal and dual meshes

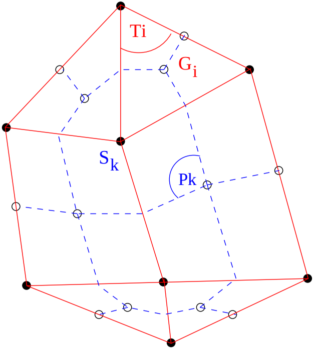



FIG.: Primal and associated dual meshes

## Diamond mesh

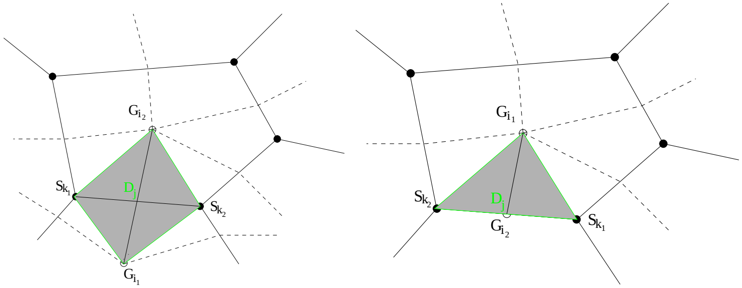
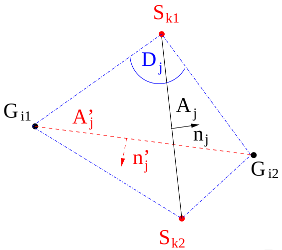



Fig.: Interior and boundary diamond cells

# Discrete differential operators

Discrete gradient and curl

$$\nabla\phi \cdot \overrightarrow{G_{i_1}G_{i_2}} \approx \phi^T_{i_2} - \phi^T_{i_1}$$
$$\nabla\phi \cdot \overrightarrow{S_{k_1}S_{k_2}} \approx \phi^P_{k_2} - \phi^P_{k_1}$$




We obtain *the definition* of the discrete gradient $\nabla_h^D$ on $D_j$

$$(\nabla_h^D \phi)_j := \frac{1}{2\,|D_j|} \left\{ \left[\phi^P_{k_2} - \phi^P_{k_1}\right] |A'_j|\, \mathbf{n}'_j + \left[\phi^T_{i_2} - \phi^T_{i_1}\right] |A_j|\, \mathbf{n}_j \right\}$$

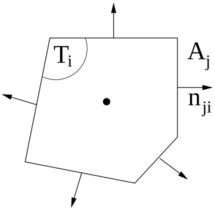
and that of the curl $\nabla_h^D \times$ on $D_j$

$$(\nabla_h^D \times \phi)_j := -\frac{1}{2\,|D_j|} \left\{ \left[\phi^P_{k_2} - \phi^P_{k_1}\right] |A'_j|\, \mathbf{n'}_j^{\perp} + \left[\phi^T_{i_2} - \phi^T_{i_1}\right] |A_j|\, \mathbf{n}_j^{\perp} \right\}$$

## Discrete divergence operator

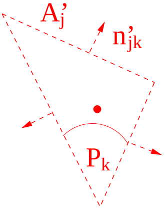
Definition of the discrete divergence on the primal cells :

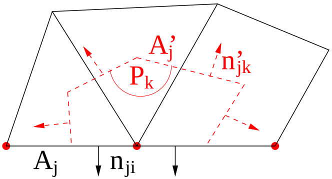
$$(\nabla_h^T \cdot \mathbf{u})_i := \frac{1}{|T_i|} \sum_{j \in \mathcal{V}(i)} |A_j| \mathbf{u}_j \cdot \mathbf{n}_{ji}$$




Definition of the discrete divergence on the inner dual cells :

$$(\nabla_h^P \cdot \mathbf{u})_k := \frac{1}{|P_k|} \sum_{j \in \mathcal{E}(k)} |A'_j| \mathbf{u}_j \cdot \mathbf{n}'_{jk}$$






Definition of the discrete divergence on the boundary dual cells :

$$(\nabla_h^P \cdot \mathbf{u})_k := \frac{1}{|P_k|} \left( \sum_{j \in \mathcal{E}(k)} |A'_j| \mathbf{u}_j \cdot \mathbf{n}'_{jk} + \sum_{j \in \mathcal{E}(k) \cap \partial\Omega} \frac{1}{2} |A_j| \mathbf{u}_j \cdot \mathbf{n}_j \right)$$

We have the following discrete Green formula

$$-(\mathbf{u}, \nabla_h^D \phi)_D + (\mathbf{u} \cdot \mathbf{n}, \phi)_{\partial\Omega} = \frac{1}{2} \left[ (\nabla_h^T \cdot \mathbf{u}, \phi^T)_T + (\nabla_h^P \cdot \mathbf{u}, \phi^P)_P \right]$$

## Discrete curls

We define in the same way a discrete scalar curl operator on primal and dual cells :

$$(\nabla_h^T \times \mathbf{u})_i := \frac{1}{|T_i|} \sum_{j \in \mathcal{V}(i)} |A_j| \mathbf{u}_j \cdot \mathbf{t}_{ji}$$

$$(\nabla_h^P \times \mathbf{u})_k := \frac{1}{|P_k|} \sum_{j \in \mathcal{E}(k)} |A_j'| \mathbf{u}_j \cdot \mathbf{t}_{jk}'$$

$$(\nabla_h^P \times \mathbf{u})_k := \frac{1}{|P_k|} \left( \sum_{j \in \mathcal{E}(k)} |A_j'| \mathbf{u}_j \cdot \mathbf{t}_{jk}' + \sum_{j \in \mathcal{E}(k) \cap \partial\Omega} \frac{1}{2} |A_j| \mathbf{u}_j \cdot \mathbf{t}_j \right)$$

We have the following discrete Green formula

$$(\mathbf{u}, \nabla_h^D \times \phi)_D + (\mathbf{u} \cdot \mathbf{t}, \phi)_{\partial\Omega} = \frac{1}{2} \left[ (\nabla_h^T \times \mathbf{u}, \phi^T)_T + (\nabla_h^P \times \mathbf{u}, \phi^P)_P \right]$$

# Properties of the operators

For all $\phi = (\phi_i^T, \phi_k^P)$,

$$(\nabla_h^T \cdot (\nabla_h^D \times \phi))_i = 0 \ \ ; \ \ (\nabla_h^T \times (\nabla_h^D \, \phi))_i = 0 \ \ \forall T_i$$

$$(\nabla_h^P \cdot (\nabla_h^D \times \phi))_k = 0 \ \ ; \ \ (\nabla_h^P \times (\nabla_h^D \, \phi))_k = 0 \ \ \forall P_k \notin \Gamma$$

Hodge decomposition (on simply connected domains) : for all $\mathbf{u} \in (\mathbb{R}^J)^2$, there exist $\phi$ and $\psi$ such that $\psi = 0$ on $\Gamma$ and $\phi$ with vanishing mean value on $\Omega$ and, for all diamond cells

$$\mathbf{u}_j = (\nabla_h^D \, \phi)_j + (\nabla_h^D \times \psi)_j \ .$$

The decomposition is orthogonal

# Application to Stokes

Vorticity velocity pressure formulation

$$\nabla \times \nabla \times \mathbf{u} - \nabla \nabla \cdot \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega$$
$$\text{with} \quad \mathbf{u} \cdot \mathbf{n} = 0, \ \nabla \times \mathbf{u} = 0 \ \text{ on } \ \Gamma \text{ and } \int_\Omega p = 0$$

(also works for $\mathbf{u} \cdot \mathbf{t}$ and $p$ given on the boundary.)

**1st step :**Hodge decomposition of $\mathbf{f}$ :

$$\left\{ \begin{array}{rcl} (\nabla_h^D \times \omega)_j + (\nabla_h^D p)_j & = & \mathbf{f}_j, \quad \forall j \in [1, J], \\ \omega^T_{i \in [I+1, I+J^\Gamma]} & = & \omega^P_{k \in [K-J^\Gamma+1, K]} = 0, \\ \sum_{i \in [1,I]} |T_i| \, p_i^T & = & \sum_{k \in [1,K]} |P_k| \, p_k^P = 0 \end{array} \right.$$

## Pressure/Vorticity : 2 Laplacians

**Vorticity :** $(\omega_i^T, \omega_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ such that

$$\left\{\begin{array}{rcl} -(\nabla_h^T \cdot \nabla_h^D \omega)_i & = & (\nabla_h^T \times \mathbf{f})_i, \quad \forall i \in [1,I] \\ -(\nabla_h^P \cdot \nabla_h^D \omega)_k & = & (\nabla_h^P \times \mathbf{f})_k, \quad \forall k \in [1,K-J^\Gamma] \\ \omega^T_{i\in[I+1,I+J^\Gamma]} & = & \omega^P_{k\in[K-J^\Gamma+1,K]} = 0 \end{array}\right.$$

**Pressure :** $(p_i^T, p_k^P)_{i\in[1,I+J^\Gamma],k\in[1,K]}$ such that

$$\left\{\begin{array}{rcl} (\nabla_h^T \cdot \nabla_h^D p)_i & = & (\nabla_h^T \cdot \mathbf{f})_i, \quad \forall i \in [1,I] \\ (\nabla_h^P \cdot \nabla_h^D p)_k & = & (\nabla_h^P \cdot \mathbf{f})_k, \quad \forall k \in [1,K] \\ (\nabla_h^D p)_j \cdot \mathbf{n}_j & = & \mathbf{f}_j \cdot \mathbf{n}_j, \quad \forall j \in [J-J^\Gamma+1, J] \\ \sum_{i\in[1,I]} |T_i|\; p_i^T & = & \sum_{k\in[1,K]} |P_k|\; p_k^P = 0 \end{array}\right.$$

**2nd step :** Velocity : Solution of a "div-curl" problem

$$\nabla \cdot \mathbf{u} = 0 \ , \ \ \nabla \times \mathbf{u} = \omega \ \text{ in } \Omega \text{ and } \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma.$$

Unknowns : $(\mathbf{u}_j)$

$$\begin{aligned} (\nabla_h^T \cdot \mathbf{u})_i &= 0, \ \ \forall T_i \\ (\nabla_h^P \cdot \mathbf{u})_k &= 0, \ \ \forall P_k \\ (\nabla_h^T \times \mathbf{u})_i &= \omega_i^T, \ \ \forall T_i \\ (\nabla_h^P \times \mathbf{u})_k &= \omega_k^P, \ \ \forall P_k \notin \Gamma \\ \mathbf{u}_j \cdot \mathbf{n}_j &= 0, \ \ \forall D_j \in \Gamma \end{aligned}$$

Hodge decomposition : $\mathbf{u}_j = (\nabla_h^D \phi)_j + (\nabla_h^D \times \psi)_j$
where $\phi$ has vanishing mean value and $\psi$ vanishes on $\Gamma$

The unknowns are $(\phi^T_{[1,I+J^\Gamma]}, \phi^P_{[1,K]})$ and $(\psi^T_{[1,I+J^\Gamma]}, \psi^P_{[1,K]})$.

$$\left\{ \begin{array}{rcl} (\nabla_h^T \cdot \nabla_h^D \phi)_i & = & 0, \quad \forall i \\ (\nabla_h^P \cdot \nabla_h^D \phi)_k & = & 0, \quad \forall k \\ (\nabla_h^D \phi)_j \cdot \mathbf{n}_j & = & 0, \quad \forall j \in \Gamma \end{array} \right.$$

$$\sum_{i \in [1,I]} |T_i| \phi_i^T = \sum_{k \in [1,K]} |P_k| \phi_k^P = 0,$$

(and so $\phi$ identically zero) and

$$\left\{ \begin{array}{rcl} (\nabla_h^T \times \nabla_h^D \times \psi)_i & = & \omega_i^T, \quad \forall i \\ (\nabla_h^P \times \nabla_h^D \times \psi)_k & = & \omega_k^P, \quad \forall k \notin \Gamma \\ \psi_i^T = \psi_k^P & = & 0, \quad \forall i \in \Gamma, \forall k \in \Gamma \end{array} \right.$$

## Numerical analysis of the scheme

Reformulation into three discrete Laplacians ($p$, $\omega$ and $\psi$). The numerical analysis has been conducted and we get, on arbitrary meshes (with a condition on the angles of the diamond cells) and under regularity conditions of $p$, $\omega$ and $\psi$. (Domelevo Omnes M$^2$AN 2005, Delcourte Domelevo Omnes SIAM J. Num. Anal. 2007)

| | theoretical | numerical |
|---|---|---|
| $p$ ($\nabla p$) | 1 (1) | 2 (1) |
| $\omega$ ($\nabla \times \omega$) | 1 (1) | 2 (1) |
| $\psi$ ($\nabla \times \psi$) | 1 (1) | 2 (1) |
| $\mathbf{u}$ | 1 | 1 |

Superconvergence at the order 1.5 of $\nabla p$, $\nabla \times \omega$, $\nabla \times \psi$ and $\mathbf{u}$ and at the order 2 of $p$, $\omega$ and $\psi$ on regularly refined meshes.

Pressure velocity formulation for Dirichlet B.C. on the velocity
$\mathbf{u} = 0$ on $\Gamma$, $\int_\Omega p = 0$

Hypothesis on the primal mesh : boundary primal cells have only one edge on the boundary.
Unknowns : $(\mathbf{u}, p) = (\mathbf{u}_j, p_i^T, p_k^P)$

$$\begin{aligned}
(\nabla_h^D \times \nabla_h^{T,P} \times \mathbf{u})_j - (\nabla_h^D \nabla_h^{T,P} \cdot \mathbf{u})_j + (\nabla_h^D p)_j &= \mathbf{f}_j^D, \quad \forall D_j \notin \Gamma \\
(\nabla_h^T \cdot \mathbf{u})_i &= 0, \quad \forall T_i \\
(\nabla_h^P \cdot \mathbf{u})_k &= 0, \quad \forall P_k \\
\mathbf{u}_j &= 0, \quad \forall D_j \in \Gamma \\
\sum_{i \in [1,I]} |T_i| \, p_i^T = \sum_{k \in [1,K]} |P_k| \, p_k^P &= 0
\end{aligned}$$

Existence and uniqueness : thanks to the div-curl problem for the velocity and to the hypothesis on the mesh for the pressure uniqueness. Equivalence with a discrete bilaplacian on $\psi$. No numerical analysis up to now.

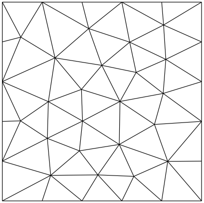
# Numerical results for Stokes with Dirichlet boundary conditions

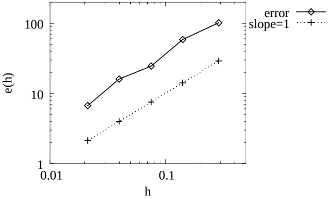
$$\Omega = [-0.5; 0.5]^2$$

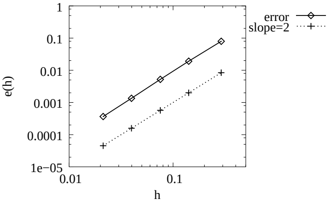
$$\mathbf{u} = \begin{pmatrix} \exp(x)\cos(\pi y) \\ x\sin(\pi y) + \cos(\pi x) \end{pmatrix} \qquad p = xy\exp(x)\cos(\pi y)$$

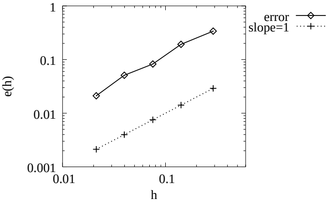
$$\nabla \cdot \mathbf{u} \neq 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{u} \neq 0 \text{ on } \Gamma$$

# Unstructured meshes



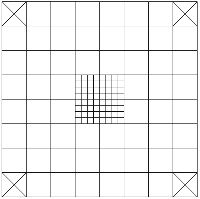


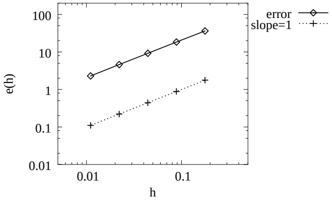



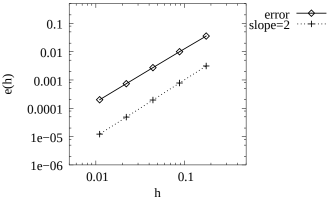



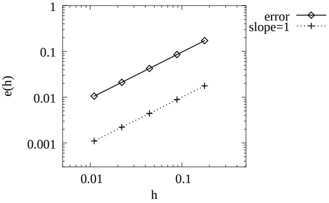

Errors on $||p||$, $||\mathbf{u}||$ and $||\nabla \cdot \mathbf{u}|| + ||\nabla \times \mathbf{u}||$ in the discrete $L^2$ norm

# Non conforming meshes at the center



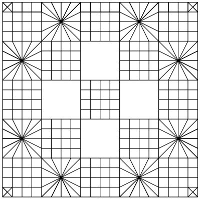


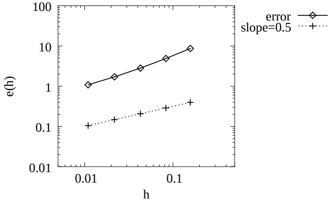



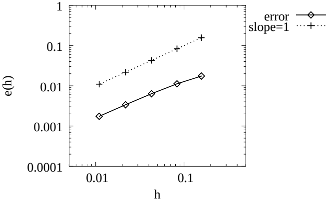



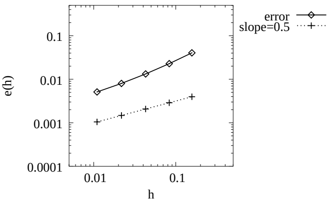

Errors on $||p||$, $||\mathbf{u}||$ and $||\nabla \cdot \mathbf{u}|| + ||\nabla \times \mathbf{u}||$ in the discrete $L^2$ norm

# Non conforming meshes






Errors on $||p||$, $||\mathbf{u}||$ and $||\nabla \cdot \mathbf{u}|| + ||\nabla \times \mathbf{u}||$ in the discrete $L^2$ norm

# Extension to stationary Navier-Stokes

$$-\Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \;\; \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

and $\mathbf{u} = 0$ on $\Gamma$, $\int_\Omega p = 0$.

Since $\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(\frac{\mathbf{u}^2}{2}) + (\nabla \times \mathbf{u})\; \mathbf{u} \times \mathbf{e}_z$, and using the "Bernoulli pressure" $\pi = p + \frac{\mathbf{u}^2}{2}$, we obtain :

$$-\nabla\nabla \cdot \mathbf{u} + \nabla \times \nabla \times \mathbf{u} + (\nabla \times \mathbf{u})\; \mathbf{u} \times \mathbf{e}_z + \nabla \pi = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega$$

and $\mathbf{u} = 0$ on $\Gamma$, $\int_\Omega \pi = 0$.

Unknowns : $(\mathbf{u}, \pi) = (\mathbf{u}_j, \pi_i^T, \pi_k^P)$ ; an iterative procedure is used

$$
\begin{aligned}
(\nabla_h^D \times \nabla_h^{T,P} \times \mathbf{u}^n)_j - (\nabla_h^D \nabla_h^{T,P} \cdot \mathbf{u}^n)_j & & \\
+(\nabla \times \mathbf{u}^{n-1})_{|D_j} \mathbf{u}_j^n \times \mathbf{e}_z + (\nabla_h^D \pi^n)_j &= \mathbf{f}_j^D, & \forall D_j \notin \Gamma \\
(\nabla_h^T \cdot \mathbf{u}^n)_i &= 0, & \forall T_i \\
(\nabla_h^P \cdot \mathbf{u}^n)_k &= 0, & \forall P_k \\
\mathbf{u}_j^n &= 0, & \forall D_j \in \Gamma \\
\sum_{i \in [1,I]} |T_i| \; \pi_i^T = \sum_{k \in [1,K]} |P_k| \; \pi_k^P &= 0 &
\end{aligned}
$$

Existence and uniqueness of the solution $(\mathbf{u}_j^n, \pi_i^T, \pi_k^P)$ : similarly to the Stokes problem using the fact that $\mathbf{u}_j \times \mathbf{e}_z \cdot \mathbf{u}_j = 0$.

We deduce $(p_i^T, p_k^P)$ by computing : $p = \pi - \frac{|\widetilde{\mathbf{u}}|^2}{2}$, where $\widetilde{\mathbf{u}}$ is a quadrature formula defined on the primal and dual cells, using the $\mathbf{u}_j$ defined on the diamond cells. Then, the sets $(p_i^T, p_k^P)$ are projected so that they have a vanishing mean value.

# Conclusion and perspectives

- Discrete differential operators on arbitrary meshes
- Properties analogous to continuous operators
- Derivation of a priori error estimations for the Stokes problem with non-standard B.C.

- A priori and a posteriori error estimations for Stokes - Navier-Stokes
- Extension to the non stationary Navier-Stokes problem
- Extension to 3D