A MUSCL finite volume scheme for axisymmetric compressible Euler equations

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22–26, June 2009

Main motivation

- Numerical simulation of inductive plasma torches (ICP: *Inductively Coupled Plasma Torch*)
- This study is part of a TRP (Technology Research Program) with ESA (European Space Agency)

Principle:

- A plasma torch is a quantitative chemical analysis technique (*e.g.* for detection of trace metals in environmental samples).
- It consists in ionizing a sample by injecting it in a plasma (generally in Argon): Atoms are ionized by a hot flame (6 000 to 8 000 K).
- The sample experiences fusion (solid), vaporization, and then ionization.
- High temperature is maintained by magnetic induction (thanks to a HF generator).
- Ions are detected either by mass or emission spectrometry.

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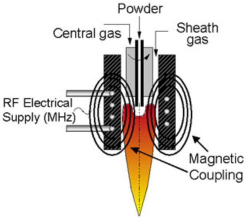
Powder
Central gas
Sheath gas
RF Electrical Supply (MHz)
Magnetic Coupling

Summary

- A mathematical model for ICP
- Axisymmetric Euler equations
- A finite volume method
- MUSCL schemes
- Application to Euler equations
- Stationary radial solutions
- Numerical experiments

A mathematical model for ICP

Mathematical modelling of ICP takes into account various phenomena;

- Electromagnetic Induction: We use a quasi-static eddy current model (we neglect displacement currents). The major difficulty lies in the fact that an unknown part of the gas is transformed into plasma and becomes electrically conducting.
- Gas Dynamics: We are dealing with a compressible flow that we assume stationary and laminar.
- We use an axisymmetric description thanks to the device geometry.

1. Electromagnetism

Eddy current equations read in quasi-static regime (time harmonic):

$$\begin{cases} \mathbf{curl}\,\mathbf{H} = \mathbf{J} \\ i\omega\mu_0\mathbf{H} + \mathbf{curl}\,\mathbf{E} = 0 \\ \mathbf{J} = \sigma\,\mathbf{E} + \mathbf{J}_0 \end{cases}$$

- $\mathbf{J}$: Current density
- $\mathbf{J}_0$: Source current
- $\mathbf{E}$: Electric field
- $\mathbf{H}$: Magnetic field
- ω : Angular frequency
- σ : Electric conductivity
- μ_0 : Magnetic permeability of the free space

We neglect here current transport by the fluid flow (In fact, we have $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \mathbf{J}_0$).

In this model, we choose a formulation in terms of the electric field.
We have

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + i\omega\mu_0\sigma\mathbf{E} = -i\omega\mu_0\mathbf{J}_0 & \text{in } \mathbb{R}^3 \\ |\mathbf{E}(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-1}) & |\mathbf{x}| \to \infty \end{cases}$$

where $\sigma = \sigma(e)$ with

$$\sigma(e) = \begin{cases} 0 & \text{if } e \le e_0, \\ > 0 & \text{otherwise} \end{cases}$$

where e is the internal energy and e_0 is the energy required for ionization.

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2. Flow of gas–plasma

We use compressible Euler equations (we neglect viscosity and thermal diffusion) with the following features:

- The gas motion is driven by the Lorentz force that we average on a time period.
- The energy source is given by Joule heating (also time averaged).

$$\nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \rho \mathbf{g} + \frac{\mu_0}{2} \mathrm{Re}\,(\mathbf{J} \times \overline{\mathbf{H}})$$

$$\nabla \cdot (\rho \mathbf{u}) = 0$$

$$\nabla \cdot ((E + p)\, \mathbf{u}) = \frac{1}{2} \mathrm{Re}\,(\mathbf{J} \cdot \overline{\mathbf{E}}) - R$$

$$p = p(\rho, e)$$

where $\mathbf{u}$ is the velocity, p is the pressure, ρ is the density, $\mathbf{g}$ is the gravity vector, e is the internal specific energy and E is the total energy defined by $E = \rho\,(e + \frac{1}{2}\,|\mathbf{u}|^2)$, R is a radiation source.

In the following, we restrict the model to an ideal gas

$$p = (\gamma - 1)\,\rho e \qquad \gamma : \text{ratio of specific heats}$$

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Axisymmetric Euler equations

- We consider compressible time dependent Euler equations.
- The geometry of the domain suggests using an axisymmetric model.
- We don't take into account, in the sequel, sources (Joule and Lorentz)

Let (r, θ, z) stand for cylindrical coordinates and let (u_r, u_θ, u_z) stand for the components of a vector in this system, we obtain the system of equations (taking into account θ–invariance):

$$\frac{\partial}{\partial t}(r\rho) + \frac{\partial}{\partial r}(r\rho u_r) + \frac{\partial}{\partial z}(r\rho u_z) = 0$$

$$\frac{\partial}{\partial t}(r\rho u_r) + \frac{\partial}{\partial r}(r\rho u_r^2 + rp) + \frac{\partial}{\partial z}(r\rho u_r u_z) = \rho u_\theta^2 + p$$

$$\frac{\partial}{\partial t}(r\rho u_z) + \frac{\partial}{\partial r}(r\rho u_r u_z) + \frac{\partial}{\partial z}(r\rho u_z^2 + rp) = 0$$

$$\frac{\partial}{\partial t}(r\rho u_\theta) + \frac{\partial}{\partial r}(r\rho u_\theta u_r) + \frac{\partial}{\partial z}(r\rho u_\theta u_z) = -\rho u_\theta u_r$$

$$\frac{\partial}{\partial t}(rE) + \frac{\partial}{\partial r}(ru_r(E + p)) + \frac{\partial}{\partial z}(ru_z(E + p)) = 0$$

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We can write this system under the conservative formulation:

$$\frac{\partial}{\partial t}(rU) + \frac{\partial}{\partial r}(rF_r(U)) + \frac{\partial}{\partial z}(rF_z(U)) = G(U)$$

where

$$U = \begin{pmatrix} \rho \\ \rho u_r \\ \rho u_z \\ \rho u_\theta \\ E \end{pmatrix}, \quad F_r(U) = \begin{pmatrix} \rho u_r \\ \rho u_r^2 + p \\ \rho u_z u_r \\ \rho u_\theta u_r \\ u_r(E+p) \end{pmatrix}, \quad F_z(U) = \begin{pmatrix} \rho u_z \\ \rho u_r u_z \\ \rho u_z^2 + p \\ \rho u_\theta u_z \\ u_z(E+p) \end{pmatrix}, \quad G(U) = \begin{pmatrix} 0 \\ \rho u_\theta^2 + p \\ 0 \\ -\rho u_\theta u_r \\ 0 \end{pmatrix}$$

This formulation involves a divergence form that can be discretized by finite volumes, source terms being treated separately.

A finite volume method

Consider a triangulation of the domain Ω of parameters (r, z). Let:

- T_i : Triangle, $1 \leq i \leq n_T$
- e_{ij} : Common edge to triangles T_i and T_j
- $\mathbf{n}_{ij} = (n_{ij,r}, n_{ij,z})$: Unit normal to the triangle T_i oriented toward T_j
- $\nu(i)$: Index set of the (3) neighbour triangles of T_i

Let us integrate the system of equations over a triangle T_i and use the divergence theorem. We get

$$\frac{d}{dt}\int_{T_i} U(r,z,t)\, r\, dr\, dz + \int_{\partial T_i} (F_r(U)n_{ij,r} + F_z(U)n_{ij,z})\, r\, d\sigma = \int_{T_i} G(U)\, dr\, dz$$

Let $(t^n = n\,\delta t)_{n\in\mathbb{N}}$ denote a uniform subdivision of the interval $[0, \infty)$. We have

$$\begin{aligned}\int_{T_i} U(r,z,t^{n+1})\, r\, dr\, dz = &\int_{T_i} U(r,z,t^n)\, r\, dr\, dz \\ &- \int_{t^n}^{t^{n+1}}\int_{\partial T_i} (F_r(U)n_{ij,r} + F_z(U)n_{ij,z})\, r\, d\sigma\, dt \\ &+ \int_{t^n}^{t^{n+1}}\int_{T_i} G(U)\, dr\, dz\, dt\end{aligned}$$

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$$|T_i| = \int_{T_i} dr\, dz,\ |T_i|_r = \int_{T_i} r\, dr\, dz,\ |e_{ij}| = \int_{e_{ij}} d\sigma,\ |e_{ij}|_r = \int_{e_{ij}} r\, d\sigma,$$

and define the approximation

$$U_i^n \approx \frac{1}{|T_i|_r} \int_{T_i} U(r, z, t^n)\, r\, dr\, dz.$$

We consider the approximate flux:

$$F_{ij}^n \approx \frac{1}{\delta t\, |e_{ij}|_r} \int_{t^n}^{t^{n+1}} \int_{e_{ij}} (F_r(U) n_{ij,r} + F_z(U) n_{ij,z})\, r\, d\sigma\, dt$$

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$$G_i^n \approx \frac{1}{\delta t\, |T_i|} \int_{t^n}^{t^{n+1}} \int_{T_i} G(U)\, dr\, dz\, dt.$$

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$$|T_i|_r U_i^{n+1} = |T_i|_r U_i^n - \delta t \sum_{j \in \nu(i)} |e_{ij}|_r F_{ij}^n + \delta t\, |T_i| G(U_i^n) \qquad 1 \le i \le n_T.$$

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The finite volume scheme is entirely determined by the choice of F_{ij}^n and G_i^n.
For instance, the Rusanov scheme consists in defining the flux:

$$F_{ij}^n = \frac{1}{2}(F_r(U_i) + F_r(U_j))n_{ij,r} + \frac{1}{2}(F_z(U_i) + F_z(U_j))n_{ij,z} - \lambda_{ij}(U_j - U_i)$$

where λ_{ij} is large enough to ensure stability.

Other alternative schemes:

- Godunov: It consists in solving exactly the obtained Riemann problems.
- HLL (Harten, Lax, Van Leer): Approximate solution of Riemann problems.
- HLLC (+ Contact): Adaptation of HLL to contact discontinuities.

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MUSCL schemes

Consider the conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \qquad x \in \mathbb{R},\ t > 0$$

The basic finite volume scheme uses a piecewise constant approximation. For instance, the first-order upwind scheme reads:

$$\frac{du_i}{dt} + \frac{f(u_i) - f(u_{i-1})}{\delta x} = 0$$

This scheme is known to be very diffusive *i.e.* it smooths Shocks and discontinuities.

To obtain less numerical diffusion, we can consider a piecewise linear approximation like:

$$\frac{du_i}{dt} + \frac{f(u_{i+\frac{1}{2}}) - f(u_{i-\frac{1}{2}})}{\delta x} = 0$$

where

$$u_{i+\frac{1}{2}} := \frac{1}{2}(u_i + u_{i+1}), \quad u_{i-\frac{1}{2}} := \frac{1}{2}(u_{i-1} + u_i).$$

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SKIP: TVD Schemes

The total variation is defined by:

$$TV(u) = \sum_i |u_{i+1} - u_i|.$$

A scheme is said to be TVD (Total Variation Diminishing) if

$$\frac{d}{dt} TV(u) \leq 0$$

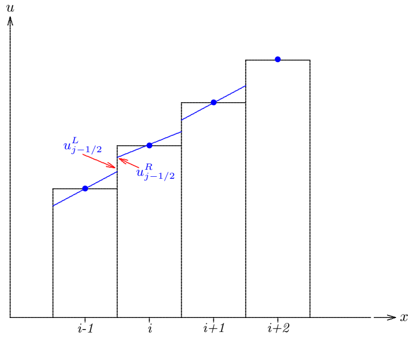
or, after time discretization:

$$TV(u^{n+1}) \leq TV(u^n).$$

We can then use a MUSCL scheme:

$$\frac{du_i}{dt} + \frac{f^*_{i+\frac{1}{2}} - f^*_{i-\frac{1}{2}}}{\delta x} = 0$$

Numerical fluxes $f^*_{i\pm\frac{1}{2}}$ correspond to a nonlinear combination of first and second order approximations of $f(u)$.



We define:

$$u^*_{i\pm\frac{1}{2}} = u^*_{i\pm\frac{1}{2}}(u^L_{i\pm\frac{1}{2}}, u^R_{i\pm\frac{1}{2}})$$
$$u^L_{i+\frac{1}{2}} = u_i + \frac{1}{2}\phi(r_i)(u_{i+1} - u_i)$$
$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}\phi(r_{i+1})(u_{i+2} - u_{i+1})$$
$$r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$$

The function ϕ is a slope limiter ensuring that the obtained solution is TVD, with

$$\phi(r) = 0 \quad \text{if } r \leq 0, \quad \phi(1) = 1.$$

The literature contains a large variety of slope limiters.
For instance the limiter minmod is defined by

$$\phi(r) = \max(0, \min(1, r)), \quad \lim_{r\to\infty} \phi(r) = 1.$$

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$$u^*_{i\pm\frac{1}{2}} = u^*_{i\pm\frac{1}{2}}(u^L_{i\pm\frac{1}{2}}, u^R_{i\pm\frac{1}{2}})$$
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$$u^R_{i+\frac{1}{2}} = u_{i+1} - \frac{1}{2}\phi(r_{i+1})(u_{i+2} - u_{i+1})$$
$$r_i = \frac{u_i - u_{i-1}}{u_{i+1} - u_i}$$

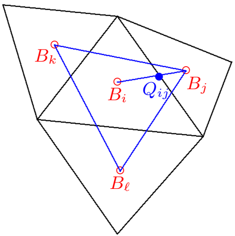
The function ϕ is a slope limiter ensuring that the obtained solution is TVD, with

$$\phi(r) = 0 \quad \text{if } r \leq 0, \quad \phi(1) = 1.$$

The literature contains a large variety of slope limiters.
For instance the limiter minmod is defined by

$$\phi(r) = \max(0, \min(1, r)), \quad \lim_{r\to\infty} \phi(r) = 1.$$

For a triangle T_i, we denote by B_i its centroid and by Q_{ij} the intersection of the segment $[B_i, B_j]$ with the edge e_{ij} for all $j \in \nu(i)$.



We introduce the barycentric coordinates $(\rho_{ij})_{j\in\nu(i)}$ by

$$\sum_{j\in\nu(i)} \rho_{ij} B_j = B_i, \quad \sum_{j\in\nu(i)} \rho_{ij} = 1.$$

We assume that B_i is strictly in the interior of the triangle formed by the centroids of neighbour triangles. Thus $\rho_{ij} > 0$.
We define the direction

$$t_{ij} = \frac{B_i B_j}{|B_i B_j|}$$

MUSCL schemes for Euler equations

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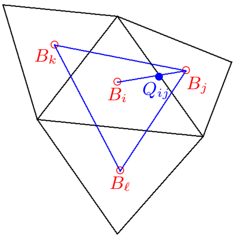

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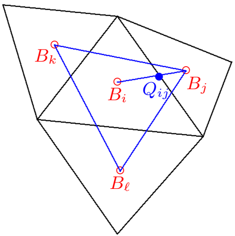

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We obtain the decomposition

$$t_{ij} = \sum_{\substack{j \in \nu(i) \\ k \neq i}} \beta_{ijk} t_{ik}, \qquad \beta_{ijk} = -\frac{\rho_{ik}}{\rho_{ij}} \frac{|B_i B_k|}{|B_i B_j|}$$

We want now to reconstruct the values U_{ij} on the edge e_{ij}.
Let v denote any component of U (piecewise constant).
We define a first downwind slope by

$$p_{ij}^{+} = \frac{v_j - v_i}{|B_i B_j|} \qquad \forall \ j \in \nu(i), \ 1 \leq i \leq n_T.$$

Therefore p_{ij}^{+} is an approximation of the derivative of v in the direction t_{ij}.
The upwind slope is defined by:

$$p_{ij}^{-} = -\sum_{\substack{k \in \nu(i) \\ k \neq j}} \beta_{ijk} p_{ik}^{+} \qquad \forall \ j \in \nu(i), \ 1 \leq i \leq n_T.$$

The slopes p_{ij} are then obtained by a limiter. For instance

$$p_{ij} := \text{minmod}\left(p_{ij}^{+}, p_{ij}^{-}\right)$$

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The slopes p_{ij} are then obtained by a limiter. For instance

$$p_{ij} := \text{minmod}\,(p_{ij}^{+}, p_{ij}^{-})$$

and the reconstruction of v on e_{ij} is given by

$$v_{ij} := v_i + p_{ij}\,|B_i Q_{ij}|$$

Remarks

- This reconstruction is exact for affine functions: $v(Q_{ij}) = v_{ij}$ if v is piecewise linear.
- The main advantage is that the reconstruction is typically 1-D. This enables using classical 1-D slope limiters.
- The property $\rho_{ij} > 0$ implies $\beta_{ijk} < 0$. Therefore, if v_i is a local extremum, we have $p_{ij}^{+} p_{ij}^{-} \leq 0$. Hence $p_{ij} = 0$. This means that extrema do not increase.
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Stationary radial solutions

In order to test the numerical scheme, we construct a radial steady state solution of the equations: We look for a solution $(u_r, u_\theta, u_z, p, e)$ that depends on r only and such that $u_z = u_\theta = 0$. We obtain the system:

$$
\begin{aligned}
&\frac{d}{dr}(r\rho u_r) = 0\\
&\frac{d}{dr}(r(\rho u_r^2 + p)) = p\\
&\frac{d}{dr}(r u_r(e + p)) = 0\\
&p = (\gamma - 1)\rho e
\end{aligned}
$$

We deduce for $\alpha, \beta \in \mathbb{R}$

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\frac{d\rho}{dr} = \frac{\rho}{\left(\alpha\rho^2 r^2 - \frac{\gamma+1}{2(\gamma-1)}\right)(\gamma - 1)r}, \qquad u_r = \frac{\beta}{\rho r}
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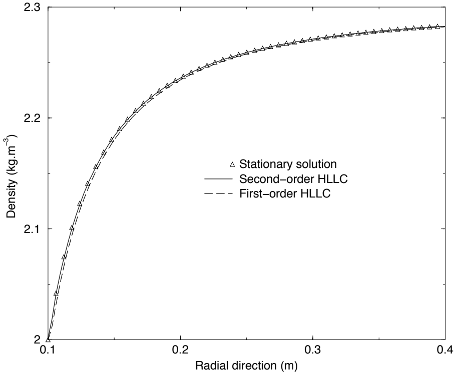
Numerical tests

1. A stationary radial solution
2. Shock tube (SOD)
3. A supersonic flow in a channel

A stationary radial solution



Shock tube

Let Ω denote the domain of parameters

$$\Omega = \{(r, z);\ r \in [0, 1),\ z \in (0, 1)\}.$$

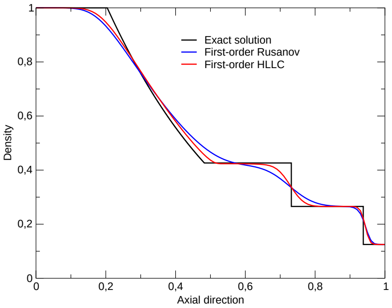
We define $\Omega_L = (0, 1) \times (0, \frac{1}{2})$, $\Omega_R = (0, 1) \times (\frac{1}{2}, 1)$ and the initial conditions:

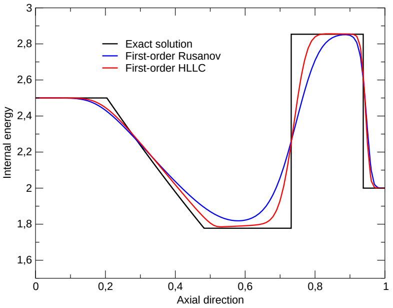
$$U(t = 0) = \begin{cases} U_L & \text{in } \Omega_L \\ U_R & \text{in } \Omega_R \end{cases}$$

Shock tube: Test 1

We test a configuration with a left rarefaction wave, a contact discontinuity and a right shock wave. For this we prescribe:

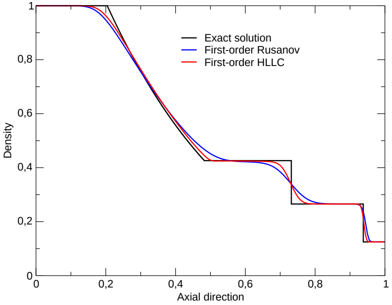
$$\rho_L = 1,\ \ \rho_R = 0.125, \quad u_L = u_R = 0, \quad p_L = 1,\ \ p_R = 0.1$$

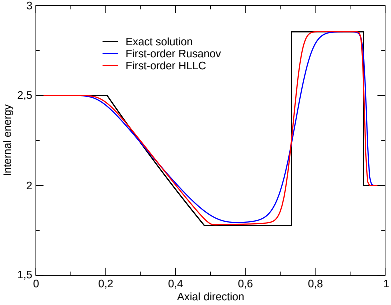




First Order: Rusanov and HLLC schemes: Mesh size 1/100

Shock tube: Test 1





First Order: Rusanov and HLLC schemes: Mesh size 1/200

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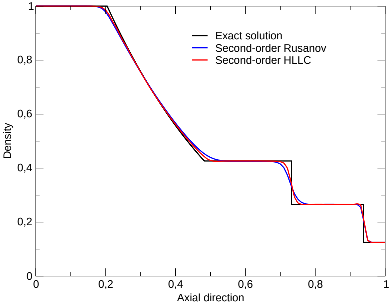
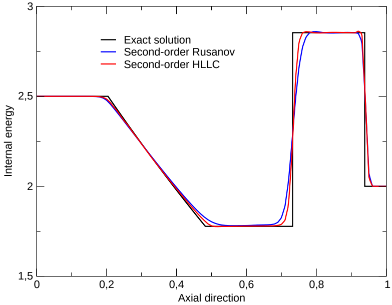
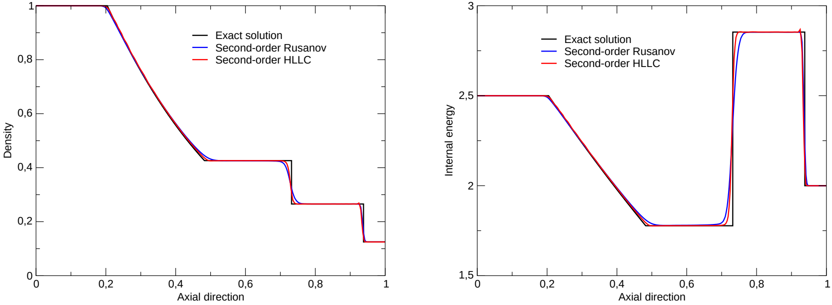




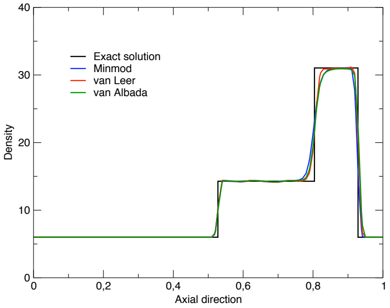
Second Order: Rusanov and HLLC schemes: Mesh size 1/100

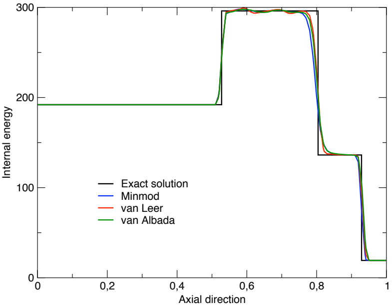


Second order: Rusanov and HLLC schemes: Mesh size 1/200

We consider a configuration with a double shock and a contact discontinuity. This is obtained from the conditions:

$$\rho_L = \rho_R = 6, \quad u_L = 19.6, \ u_R = -6.2, \quad p_L = 460, \ p_R = 46$$



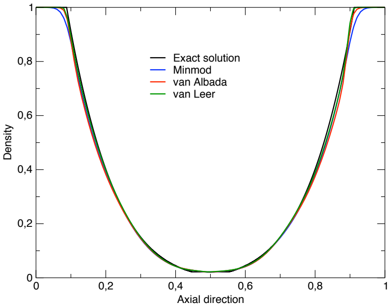


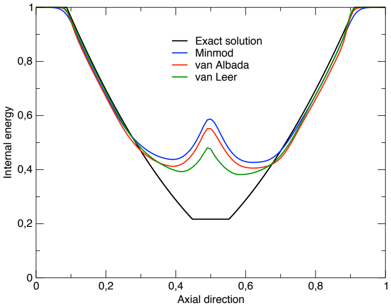
Second order: Rusanov and HLLC schemes: Mesh size 1/200

Shock tube: Test 3

We now test a configuration with 2 rarefactions and a contact discontinuity and where the solution presents a vacuum like situation. This is obtained with the conditions:

$$\rho_L = \rho_R = 1, \quad u_L = -2, \; u_R = 2, \quad p_L = 1, \; p_R = 0.4$$





Second Order: Rusanov and HLLC: Mesh size 1/200

Supersonic flow in a channel

We consider a compressible Euler flow in a channel with an oblique obstacle (10 degrees) forming a cone.
Problem data:

$$P_\infty = 10^5 Pa,\ \ \rho_\infty = 1.16 Kg/m^3, M_\infty = 2$$

Mesh: 5176 triangles.

Cone: Density contours

Animation

Cone: Mach contours

Animation