Convergence of the MAC scheme for the stationary compressible Stokes equations

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Stationary compressible Stokes equations

$d = 2$ or 3, $\Omega =]0,1[^d$ (or $\Omega = \cup_{i=1}^n R_i$, where R_i's are rectangles if $d = 2$ or parallelipedus rectangulus if $d = 3$).
$\gamma \geq 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\operatorname{div}(\rho u) = 0 \text{ in } \Omega, \; \rho \geq 0 \text{ in } \Omega, \; \int_\Omega \rho(x) dx = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

Aim:

- Discretization by the MAC scheme
- Existence of solution for the discrete problem
- Proof of the convergence (up to subsequence) of the solution of the discrete problem towards a weak solution of the continuous problem (no uniqueness result for this problem) as the mesh size goes to 0

Generalizations

- (Easy) Complete Stokes problem: $-\mu\Delta u - \frac{\mu}{3}\nabla(\mathrm{div}\, u) + \nabla P = f$, with $\mu \in \mathbb{R}_+^\star$ given
- (Ongoing work) Navier-Stokes Equations with $\gamma > 1$ if $d = 2$ and $\gamma > \frac{3}{2}$ if $d = 3$ (probably sharp result with respect to γ without changing the diffusion term or the EOS)
- (Open question) Other boundary condition. Addition of an energy equation
- (Open question) Evolution equation (Stokes and Navier-Stokes)

Weak solution of the stationary compressible Stokes problem

Functional spaces : $u \in H_0^1(\Omega)^d$, $p \in L^2(\Omega)$, $\rho \in L^{2\gamma}(\Omega)$

- Momentum equation:

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega f \cdot v \, dx \text{ for all } v \in H_0^1(\Omega)^d$$

- Mass equation:

$$\int_\Omega \rho u \cdot \nabla \varphi \, dx = 0 \text{ for all } \varphi \in C_c^\infty(\Omega)$$

$$\rho \geq 0 \text{ a.e.}, \quad \int_\Omega \rho dx = M$$

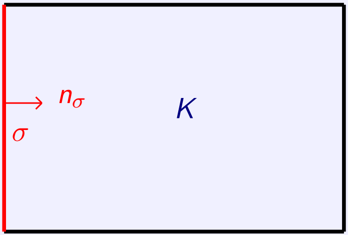
- EOS: $p = \rho^\gamma$

MAC scheme, choice of the discrete unknowns

- $\mathcal{T}$: cartesian mesh of Ω, the mesh size is called h
  $\mathcal{E}$: edges of $\mathcal{T}$
- Discretization of u p and ρ by piecewise constant functions.

n_σ is the normal vector to σ, with $n_\sigma \geq 0$.

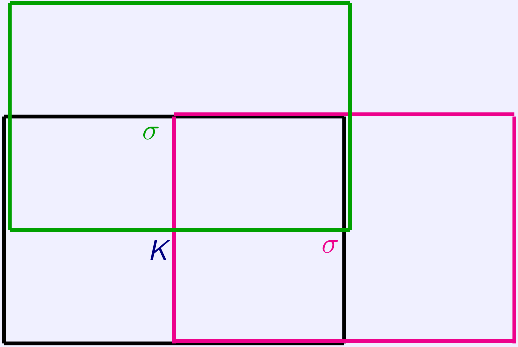
Unknowns for $u_\mathcal{T}$:
u_σ, $\sigma \in \mathcal{E}$. u_σ is an approximate value for $u \cdot n_\sigma$ ($u_\sigma \in \mathbb{R}$)
$u_\sigma = 0$ if $\sigma \subset \partial\Omega$



Unknowns for $p_\mathcal{T}$ and $\rho_\mathcal{T}$:
p_K, ρ_K, $K \in \{\text{rectangles}\}$

MAC scheme, discrete functional spaces, $d = 2$

- $p_{\mathcal{T}}, \rho_{\mathcal{T}} \in X_{\mathcal{T}}$, $p_{\mathcal{T}} = p_K$, $\rho_{\mathcal{T}} = \rho_K$ in K, $K \in \mathcal{T}$ (black cell)
- $u_{\mathcal{T}} = (u_{\mathcal{T}}^{(1)}, u_{\mathcal{T}}^{(2)}) \in H_{\mathcal{T}}$
  $u_{\mathcal{T}}^{(1)} = u_\sigma$ in the magenta cell
  $u_{\mathcal{T}}^{(2)} = u_\sigma$ in the green cell



Discretization of momentum equation (1)

- $v \in H_{\mathcal{T}}$. $\mathrm{div}_{\mathcal{T}} v$ is constant on K, $K \in \mathcal{T}$ and

$$|K| \mathrm{div}_{\mathcal{T}} v = \sum_{\sigma \in \mathcal{E}_K} \varepsilon_{K,\sigma} v_\sigma |\sigma|$$

  $\varepsilon_{K,\sigma} = \mathrm{sign}(n_\sigma \cdot n_{K,\sigma})$, $n_{K,\sigma}$ is the normal vector to σ, outward K

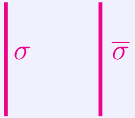
- $u, v \in H_{\mathcal{T}}$, the discretization of $\int_\Omega \nabla u : \nabla v \, dx$ is:

$$\int_\Omega \nabla_{\mathcal{T}} u : \nabla_{\mathcal{T}} v \, dx = \sum_{(\sigma,\overline{\sigma}) \in \mathcal{N}} \frac{h_{\sigma,\overline{\sigma}}}{d_{\sigma,\overline{\sigma}}} (u_\sigma - u_{\overline{\sigma}})(v_\sigma - v_{\overline{\sigma}})$$

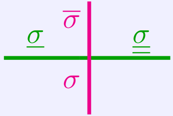
  $d_{\sigma,\overline{\sigma}}$: distance between the centers of σ and $\overline{\sigma}$
  $h_{\sigma,\overline{\sigma}}$ is equal to $|\sigma|$ or to $\frac{1}{2}(|\underline{\sigma}| + |\underline{\underline{\sigma}}|)$, where $\underline{\sigma}$ and $\underline{\underline{\sigma}}$ are "between" σ and $\overline{\sigma}$

Discretization of the momentum equation (2)

Computation of $h_{\sigma,\overline{\sigma}}$ for $(\sigma, \overline{\sigma}) \in \mathcal{N}$

- Case 1: 

$h_{\sigma,\overline{\sigma}} = |\sigma|$

- Case 2: 

$h_{\sigma,\overline{\sigma}} = \frac{1}{2}(|\underline{\sigma}| + |\underline{\underline{\sigma}}|)$

(Slight modification if $\underline{\sigma}$, $\underline{\underline{\sigma}} \subset \partial\Omega$, $u_{\overline{\sigma}} = -u_\sigma$)

Discrete momentum equation

$$u_{\mathcal{T}} \in H_{\mathcal{T}}$$

$$\int_\Omega \nabla_{\mathcal{T}} u_{\mathcal{T}} : \nabla_{\mathcal{T}} v \, dx - \int_\Omega p_{\mathcal{T}} \mathrm{div}_{\mathcal{T}} v \, dx = \int_\Omega f v \, dx, \text{ for all } v \in H_{\mathcal{T}}$$

Discretization of the mass equation

For all $K \in \mathcal{T}$, $\displaystyle\sum_{\sigma \in \mathcal{E}_K} |\sigma| \rho_\sigma \varepsilon_{K,\sigma} u_\sigma + M_K = 0$

with an upstream choice for ρ_σ, that is

$\rho_\sigma = \rho_K \text{ if } u_\sigma \geq 0$

$\rho_\sigma = \rho_L \text{ if } u_\sigma < 0, \ \sigma = K|L$

$$M_K = |K| h^\alpha (\rho_K - \frac{M}{|\Omega|})$$

$\alpha > 0$

The M_K term gives $\int_\Omega \rho_\mathcal{T} dx = M$

Upwinding is enough to ensure (with M) existence (and uniqueness) of a positive solution $\rho_\mathcal{T}$, to the discrete mass equation, for a given $u_\mathcal{T}$.

Discretization of the EOS

Discretization of the EOS:

$$p_K = \rho_K^\gamma$$

for all $K \in \mathcal{T}$

Existence of an approximate solution, convergence result

Existence of a solution $u_{\mathcal{T}}$, $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ of the scheme can be proven using the Brouwer Fixed Point Theorem.

For $\gamma > 1$, convergence of the approximate solution can be proven in the following sense, up to a subsequence:

- $u_{\mathcal{T}} \to u$ in $L^2(\Omega)^d$, $u \in H_0^1(\Omega)^d$
- $p_{\mathcal{T}} \to p$ in $L^q(\Omega)$ for any $1 \leq q < 2$ and weakly in $L^2(\Omega)$
- $\rho_{\mathcal{T}} \to \rho$ in $L^q(\Omega)$ for any $1 \leq q < 2\gamma$ and weakly in $L^{2\gamma}(\Omega)$

where (u, p, ρ) is a weak solution of the compressible Stokes equations

For $\gamma = 1$, the same result holds, at least with only weak convergences of $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$

Proof of convergence, main steps

1. Estimate on the $H_0^1(\Omega)$-discrete norm of the components of $u_{\mathcal{T}}$
2. $L^2(\Omega)$ estimate on $p_{\mathcal{T}}$ and $L^{2\gamma}(\Omega)$ estimate on $\rho_{\mathcal{T}}$

   These two steps give (up to a subsequence), as $h \to 0$,
   - $u_{\mathcal{T}} \to u$ in $L^2(\Omega)$ and $u \in H_0^1(\Omega)^d$
   - $p_{\mathcal{T}} \to p$ weakly in $L^2(\Omega)$
   - $\rho_{\mathcal{T}} \to \rho$ weakly in $L^{2\gamma}(\Omega)$
3. (u, p, ρ) is a weak solution of $-\Delta u + \nabla p = f$, $\mathrm{div}(\rho u) = 0$ $\rho \geq 0$, $\int_\Omega \rho dx = M$
4. Main difficulty, if $\gamma > 1$: $p = \rho^\gamma$ and "strong" convergence of $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$

Preliminary lemma

$\rho \in L^{2\gamma}(\Omega)$, $\gamma > 1$, $\rho \geq 0$ a.e. in Ω, $u \in (H_0^1(\Omega))^d$, $\mathrm{div}(\rho u) = 0$, then:

$$\int_\Omega \rho \mathrm{div}(u) dx = 0$$

$$\int_\Omega \rho^\gamma \mathrm{div}(u) dx = 0$$

The first result (and its discrete counterpart) is used for Step 4 (proof of $p = \rho^\gamma$)

The discrete counterpart (also true for $\gamma = 1$) of the second result is used for Step 1 (estimate for $u_{\mathcal{T}}$)

Preliminary lemma for the approximate solution

Discretization of the mass equation $\mathrm{div}(\rho u) = 0$ and $\int_\Omega \rho \, dx = M$:
For all $K \in \mathcal{T}$, $\sum_{\sigma \in \mathcal{E}_K} |\sigma| \rho_\sigma \varepsilon_{K,\sigma} u_\sigma + M_K = 0$

One proves:

$$\int_\Omega \rho_{\mathcal{T}}^\gamma \mathrm{div}_{\mathcal{T}}\, u_{\mathcal{T}} \, dx \leq C h^\alpha,$$

$$\int_\Omega \rho_{\mathcal{T}} \mathrm{div}_{\mathcal{T}}\, u_{\mathcal{T}} \, dx \leq C h^\alpha.$$

C depends on Ω, M and γ.

Ch^α is due to M_K
$\leq$ is due to upwinding

Estimate on $u_{\mathcal{T}}$

Taking $u_{\mathcal{T}}$ as test function in the discrete momentum equation

$$\int_\Omega \nabla_{\mathcal{T}} u_{\mathcal{T}} : \nabla_{\mathcal{T}} u_{\mathcal{T}}\, dx - \int_\Omega p_{\mathcal{T}} \mathrm{div}_{\mathcal{T}}(u_{\mathcal{T}})\, dx = \int_\Omega f \cdot u_{\mathcal{T}}\, dx.$$

But $p_{\mathcal{T}} = \rho_{\mathcal{T}}^{\gamma}$ a.e., Discrete mass equation and preliminary lemma gives $\int_\Omega p_{\mathcal{T}} \mathrm{div}(u_{\mathcal{T}})\, dx \leq Ch^{\alpha}$.
This gives an estimate on $u_{\mathcal{T}}$:

$$\int_\Omega \nabla_{\mathcal{T}} u_{\mathcal{T}} \cdot \nabla_{\mathcal{T}} u_{\mathcal{T}} dx = \sum_{(\sigma,\overline{\sigma})\in\mathcal{N}} \frac{h_{\sigma,\overline{\sigma}}}{d_{\sigma,\overline{\sigma}}}(u_\sigma - u_{\overline{\sigma}})^2 \leq C_1.$$

Then, up to a subsequence, $u_{\mathcal{T}} \to u$ in $L^2(\Omega)^d$ as $h \to 0$ and $u \in H_0^1(\Omega)^d$

Estimate on $p_{\mathcal{T}}$ (inf-sup condition, Nečas lemma)

Let $m_{\mathcal{T}} = \frac{1}{|\Omega|} \int_\Omega p_{\mathcal{T}} dx$ and $q = p_{\mathcal{T}} - m_{\mathcal{T}}$.

Then, there exists $\overline{v}_{\mathcal{T}} \in (H_0^1(\Omega))^d$ s.t. $\mathrm{div}(\overline{v}_{\mathcal{T}}) = q$ in Ω and $\|\overline{v}_{\mathcal{T}}\|_{(H_0^1(\Omega))^d} \leq C_2 \|q\|_{L^2(\Omega)}$ where C_2 only depends on Ω

One defines $v_{\mathcal{T}} \in H_{\mathcal{T}}$ with $v_\sigma = \frac{1}{|\sigma|} \int_\sigma \overline{v}_{\mathcal{T}} \cdot n_\sigma$ for $\sigma \in \mathcal{E}$.

Then $\mathrm{div}_{\mathcal{T}}(v_{\mathcal{T}}) = p_{\mathcal{T}} - m_{\mathcal{T}}$ and

$$\int_\Omega \nabla_{\mathcal{T}} v_{\mathcal{T}} : \nabla_{\mathcal{T}} v_{\mathcal{T}} \, dx = \sum_{(\sigma,\overline{\sigma}) \in \mathcal{N}} \frac{h_{\sigma,\overline{\sigma}}}{d_{\sigma,\overline{\sigma}}} (v_\sigma - v_{\overline{\sigma}})^2 \leq C_3 \|q\|_{L^2(\Omega)}^2$$

One takes $v_{\mathcal{T}}$ as test function in the discrete momentum equation

Estimate on $p_{\mathcal{T}}$ (2)

$$\int_\Omega \nabla_{\mathcal{T}} u_{\mathcal{T}} : \nabla_{\mathcal{T}} v_{\mathcal{T}} \, dx - \int_\Omega p_{\mathcal{T}} \mathrm{div}_{\mathcal{T}}(v_{\mathcal{T}}) \, dx = \int_\Omega f \cdot v_{\mathcal{T}} \, dx.$$

Using $\int_\Omega \mathrm{div}_{\mathcal{T}}(v_{\mathcal{T}}) dx = 0$:

$$\int_\Omega (p_{\mathcal{T}} - m_{\mathcal{T}})^2 dx = \int_\Omega (f \cdot v_{\mathcal{T}} - \nabla_{\mathcal{T}} u_{\mathcal{T}} : \nabla_{\mathcal{T}} v_{\mathcal{T}}) dx.$$

with the estimate on $u_{\mathcal{T}}$ and the bound on $v_{\mathcal{T}}$ linearly depending on the L^2 norm of $p_{\mathcal{T}} - m_{\mathcal{T}}$, the preceding inequality leads to:

$$\|p_{\mathcal{T}} - m_{\mathcal{T}}\|_{L^2(\Omega)} \leq C_4$$

where C_4 only depends on f and on Ω.

Estimates on $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$

$$\|p_{\mathcal{T}} - m_{\mathcal{T}}\|_{L^2(\Omega)} \leq C_4.$$

$$\int_\Omega p_{\mathcal{T}}^{\frac{1}{\gamma}} dx = \int_\Omega \rho_{\mathcal{T}} dx = M$$

Then:

$$\|p_{\mathcal{T}}\|_{L^2(\Omega)} \leq C_5$$

where C_5 only depends on f, M, γ and Ω.

$p_{\mathcal{T}} = \rho_{\mathcal{T}}^{\gamma}$ a.e. in Ω, then:

$$\|\rho_{\mathcal{T}}\|_{L^{2\gamma}(\Omega)} \leq C_6 = C_5^{\frac{1}{\gamma}}.$$

Convergence of $u_{\mathcal{T}}$, $p_{\mathcal{T}}$, $\rho_{\mathcal{T}}$ (weak for $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$)

Thanks to the estimates on $u_{\mathcal{T}}$, $p_{\mathcal{T}}$, $\rho_{\mathcal{T}}$, it is possible to assume (up to a subsequence) that, as $h \to 0$:

$$u_{\mathcal{T}} \to u \text{ in } L^2(\Omega)^d \text{ and } u \in H_0^1(\Omega)^d,$$

$$p_{\mathcal{T}} \to p \text{ weakly in } L^2(\Omega),$$

$$\rho_{\mathcal{T}} \to \rho \text{ weakly in } L^{2\gamma}(\Omega).$$

Passage to the limit in the momentum equation

Classical proof with FV scheme for elliptic equations

$u \in H_0^1(\Omega)^d$

One proves

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega f \cdot v \, dx \text{ for all } v \in C_c^\infty(\Omega)^d$$

and then, since $u \in H_0^1(\Omega)^d$, one concludes by density

$$\int_\Omega \nabla u : \nabla v \, dx - \int_\Omega p \mathrm{div}(v) \, dx = \int_\Omega f \cdot v \, dx \text{ for all } v \in H_0^1(\Omega)^d$$

Passage to the limit in the mass equation

L^1-weak convergence of $\rho_{\mathcal{T}}$ (and $\rho_{\mathcal{T}} \geq 0$) gives positivity of ρ and convergence of total mass

$$\rho \geq 0 \ \text{ in } \Omega, \ \int_{\Omega} \rho(x) dx = M.$$

Using the fact that $u_{\mathcal{T}}$ converges in L^2 and $\rho_{\mathcal{T}}$ weakly in L^2, one proves

$$\int_{\Omega} \rho u \cdot \nabla \varphi \, dx = 0 \text{ for all } \varphi \in C_c^{\infty}(\Omega)$$

This is quite classical with FV for hyperbolic equations. It uses some weak-BV estimate (to control $\rho_K - \rho_L$ if $\sigma = K|L$) coming from the upwinding of ρ

Quite easy for $\gamma \geq 2$. More difficult for $\gamma < 2$.

Weak-BV estimate, $\gamma \geq 2$

Roughly speaking, upwinding replaces $\mathrm{div}(\rho u) = 0$ by $\mathrm{div}(\rho u) - h\mathrm{div}(|u|\nabla\rho) = 0$ (the term M_K is easy to handle)
Taking ρ as test function leads to

$$-\frac{1}{2}\int_\Omega u \cdot \nabla\rho^2 + h|u||\nabla\rho|^2 = 0$$

which leads to

$$\int_\Omega h|u||\nabla\rho|^2 = -\frac{1}{2}\int_\Omega \mathrm{div}(u)\rho^2 \leq C$$

if ρ is bounded in $L^4(\Omega)$ (since $\mathrm{div}(u)$ is bounded in $L^2(\Omega)$)
This proves the weak-BV estimate on ρ if $\gamma \geq 2$

It allows to pass to the limit in the mass equation using the weak convergence of $\rho_{\mathcal{T}}$ in $L^2(\Omega)$ and the convergence of $u_{\mathcal{T}}$ in $L^2(\Omega)^d$ as $h \to 0$

Weak-BV estimate, $\gamma < 2$

- Method 1: Use ρ-weighted weak-BV estimates
- Method 2: Add another diffusion term in the discrrete mass equation which is a discretization of

$$h^{\xi} \mathrm{div}(\rho^{2-\gamma} \nabla \rho) = 0$$

  ξ is a parameter, $0 < \xi < 2$
  Small diffusion term (ξ close to 2), leading to a weak-BV estimate (taking $\rho^{\gamma-1}$ as test function in the discrete mass equation)

Passage to the limit in EOS

- No problem if $\gamma = 1$, $p = \rho$
- If $\gamma > 1$, question:

$$p = \rho^\gamma \text{ in } \Omega \text{ ?}$$

$p_\mathcal{T}$ and $\rho_\mathcal{T}$ converge only weakly...

Idea : prove $\int_\Omega p_\mathcal{T} \rho_\mathcal{T} \to \int_\Omega p\rho$ and deduce a.e. convergence (of $p_\mathcal{T}$ and $\rho_\mathcal{T}$) and $p = \rho^\gamma$.

$\nabla : \nabla = \mathrm{div}\mathrm{div} + \mathrm{curl} \cdot \mathrm{curl}$

For all $\bar{u}, \bar{v}$ in $H_0^1(\Omega)^d$,

$$\int_\Omega \nabla \bar{u} : \nabla \bar{v} = \int_\Omega \mathrm{div}(\bar{u})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(\bar{u}) \cdot \mathrm{curl}(\bar{v}).$$

Assuming, for simplicity that $u_{\mathcal{T}} \in H_0^1(\Omega)^d$ and $-\Delta u_{\mathcal{T}} + \nabla p_{\mathcal{T}} = f_{\mathcal{T}} \in L^2(\Omega)$, $f_{\mathcal{T}} \to f$ in $L^2(\Omega)^d$ as $h \to 0$ (not true...). Then, for all $\bar{v}$ in $H_0^1(\Omega)^d$

$$\int_\Omega \mathrm{div}(u_{\mathcal{T}})\mathrm{div}(\bar{v}) + \int_\Omega \mathrm{curl}(u_{\mathcal{T}}) \cdot \mathrm{curl}(\bar{v}) - \int_\Omega p_{\mathcal{T}} \mathrm{div}(\bar{v}) = \int_\Omega f_{\mathcal{T}} \cdot \bar{v}.$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_{\mathcal{T}}$ with $\mathrm{curl}(\bar{v}_{\mathcal{T}}) = 0$, $\mathrm{div}(\bar{v}_{\mathcal{T}}) = \rho_{\mathcal{T}}$ and $\bar{v}_{\mathcal{T}}$ bounded in H_0^1 (unfortunately, 0 is impossible).

Then, up to a subsequence,

$\bar{v}_{\mathcal{T}} \to v$ in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$,

$\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof using $\bar{v}_{\mathcal{T}}$ (1)

$$\int_\Omega \mathrm{div}(u_{\mathcal{T}})\mathrm{div}(\bar{v}_{\mathcal{T}}) + \int_\Omega \mathrm{curl}(u_{\mathcal{T}})\cdot\mathrm{curl}(\bar{v}_{\mathcal{T}}) - \int_\Omega p_{\mathcal{T}}\mathrm{div}(\bar{v}_{\mathcal{T}}) = \int_\Omega f_{\mathcal{T}}\cdot\bar{v}_{\mathcal{T}}.$$

But, $\mathrm{div}(\bar{v}_{\mathcal{T}}) = \rho_{\mathcal{T}}$ and $\mathrm{curl}(\bar{v}_{\mathcal{T}}) = 0$. Then:

$$\int_\Omega (\mathrm{div}(u_{\mathcal{T}}) - p_{\mathcal{T}})\rho_{\mathcal{T}} = \int_\Omega f_{\mathcal{T}} \cdot \bar{v}_{\mathcal{T}}.$$

Convergence of $f_{\mathcal{T}}$ in $L^2(\Omega)^d$ to f and convergence of $\bar{v}_{\mathcal{T}}$ in $L^2(\Omega)^d$ to v :

$$\lim_{h\to 0} \int_\Omega (\mathrm{div}(u_{\mathcal{T}}) - p_{\mathcal{T}})\rho_{\mathcal{T}} = \int_\Omega f \cdot v.$$

Proof using $\bar{v}_{\mathcal{T}}$ (2)

But, since $-\Delta u + \nabla p = f$:

$$\int_\Omega \mathrm{div}(u)\mathrm{div}(v) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v) - \int_\Omega p\,\mathrm{div}(v) = \int_\Omega f\cdot v.$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):
$\displaystyle\int_\Omega (\mathrm{div}(u) - p)\rho = \int_\Omega f\cdot v$. Then:

$$\lim_{h\to 0}\int_\Omega (p_{\mathcal{T}} - \mathrm{div}(u_{\mathcal{T}}))\rho_{\mathcal{T}} = \int_\Omega (p - \mathrm{div}(u))\rho.$$

Finally, the preliminary lemma gives, thanks to the mass equations, $\int_\Omega \rho_{\mathcal{T}}\mathrm{div}(u_{\mathcal{T}}) \le Ch^\alpha$ and $\int_\Omega \rho\,\mathrm{div}(u) = 0$. Then, at least for a subsequence

$$\lim_{h\to 0}\int_\Omega p_{\mathcal{T}}\rho_{\mathcal{T}} \le \int_\Omega p\rho.$$

Unfortunately, two difficulties: it is impossible to have $\bar{v}_{\mathcal{T}} \in H^1_0$ and $(u_{\mathcal{T}}, p_{\mathcal{T}})$ is solution of the discrete momentum equation

First difficulty: not 0 at the boundary

Let $w_{\mathcal{T}} \in H_0^1(\Omega)$, $-\Delta w_{\mathcal{T}} = \rho_{\mathcal{T}}$,
One has $w_{\mathcal{T}} \in H^2_{loc}(\Omega)$ since, for $\varphi \in C_c^\infty(\Omega)$, one has $\Delta(w_{\mathcal{T}}\varphi) \in L^2(\Omega)$ and

$$\sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_j(w_{\mathcal{T}}\varphi)\,\partial_i\partial_j(w_{\mathcal{T}}\varphi) = \sum_{i,j=1}^{d} \int_\Omega \partial_i\partial_i(w_{\mathcal{T}}\varphi)\,\partial_j\partial_j(w_{\mathcal{T}}\varphi)$$
$$= \int_\Omega (\Delta(w_{\mathcal{T}}\varphi))^2 < \infty$$

Then, taking $v_{\mathcal{T}} = \nabla w_{\mathcal{T}}$

- $v_{\mathcal{T}} \in (H^1_{loc}(\Omega))^d$,
- $\mathrm{div}(v_{\mathcal{T}}) = \rho_{\mathcal{T}}$ a.e. in Ω,
- $\mathrm{curl}(v_{\mathcal{T}}) = 0$ a.e. in Ω,
- $H^1_{loc}(\Omega)$-estimate on $v_{\mathcal{T}}$ with respect to $\|\rho_{\mathcal{T}}\|_{L^2(\Omega)}$.

Then, up to a subsequence, as $h \to 0$, $v_{\mathcal{T}} \to v$ in $L^2_{loc}(\Omega)$ and weakly in $H^1_{loc}(\Omega)$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof of $\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_\mathcal{T}\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_\mathcal{T}\varphi$:

$$\int_\Omega \mathrm{div}(u_\mathcal{T})\mathrm{div}(v_\mathcal{T}\varphi) + \int_\Omega \mathrm{curl}(u_\mathcal{T}) \cdot \mathrm{curl}(v_\mathcal{T}\varphi) - \int_\Omega p_\mathcal{T}\mathrm{div}(v_\mathcal{T}\varphi)$$
$$= \int_\Omega f_\mathcal{T} \cdot (v_\mathcal{T}\varphi).$$

Using a proof smilar to that given if $\varphi = 1$ (with additionnal terms involving φ), we obtain :

$$\lim_{h\to 0} \int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi,$$

Proving $\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_\mathcal{T}\varphi \in H_0^1(\Omega)^d$)). Taking $\bar{v} = v_\mathcal{T}\varphi$:

$$\int_\Omega \mathrm{div}(u_\mathcal{T})\mathrm{div}(v_\mathcal{T}\varphi) + \int_\Omega \mathrm{curl}(u_\mathcal{T}) \cdot \mathrm{curl}(v_\mathcal{T}\varphi) - \int_\Omega p_\mathcal{T}\mathrm{div}(v_\mathcal{T}\varphi) \\ = \int_\Omega f_\mathcal{T} \cdot (v_\mathcal{T}\varphi).$$

But, $\mathrm{div}(v_\mathcal{T}\varphi) = \rho_\mathcal{T}\varphi + v_\mathcal{T} \cdot \nabla\varphi$ and $\mathrm{curl}(v_\mathcal{T}\varphi) = L(\varphi)v_\mathcal{T}$, where $L(\varphi)$ is a matrix involving the first order derivatives of φ. Then:

$$\int_\Omega (\mathrm{div}(u_\mathcal{T}) - p_\mathcal{T})\rho_\mathcal{T}\varphi = \int_\Omega f_\mathcal{T} \cdot (v_\mathcal{T}\varphi) \\ - \int_\Omega \mathrm{div}(u_\mathcal{T})v_\mathcal{T} \cdot \nabla\varphi - \int \mathrm{curl}(u_\mathcal{T}) \cdot L(\varphi)v_\mathcal{T} + \int_\Omega p_\mathcal{T} v_\mathcal{T} \cdot \nabla\varphi.$$

Weak convergence of $u_\mathcal{T}$ in $H_0^1(\Omega)^d$, weak convergence of $p_\mathcal{T}$ in $L^2(\Omega)$ and convergence of $v_\mathcal{T}$ and $f_\mathcal{T}$ in $L^2_{loc}(\Omega)^d$ and $L^2(\Omega)^d$:

$$\lim_{h\to 0} \int_\Omega (\mathrm{div}(u_\mathcal{T}) - p_\mathcal{T})\rho_\mathcal{T}\varphi = \int_\Omega f \cdot (v\varphi) \\ - \int_\Omega \mathrm{div}(u)v \cdot \nabla\varphi - \int \mathrm{curl}(u) \cdot L(\varphi)v + \int_\Omega pv \cdot \nabla\varphi.$$

Proof of $\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

But, since $-\Delta u + \nabla p = f$:

$$\begin{aligned}\int_\Omega \mathrm{div}(u)\mathrm{div}(v\varphi) + \int_\Omega \mathrm{curl}(u)\cdot\mathrm{curl}(v\varphi) - \int_\Omega p\,\mathrm{div}(v\varphi)\\ = \int_\Omega f\cdot(v\varphi).\end{aligned}$$

which gives (using $\mathrm{div}(v) = \rho$ and $\mathrm{curl}(v) = 0$):

$$\begin{aligned}&\int_\Omega (\mathrm{div}(u) - p)\rho\varphi = \int_\Omega f\cdot(v\varphi)\\ &- \int_\Omega \mathrm{div}(u)v\cdot\nabla\varphi - \int \mathrm{curl}(u)\cdot L(\varphi)v + \int_\Omega pv\cdot\nabla\varphi.\end{aligned}$$

Then:

$$\lim_{h\to 0}\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi.$$

Second difficulty: Discrete momentum equation

Miracle for the MAC scheme: for all $\bar{u}, \bar{v}$ in $H_{\mathcal{T}}$,

$$\int_\Omega \nabla_{\mathcal{T}} \bar{u} : \nabla_{\mathcal{T}} \bar{v} = \int_\Omega \mathrm{div}_{\mathcal{T}}(\bar{u}) \mathrm{div}_{\mathcal{T}}(\bar{v}) + \int_\Omega \mathrm{curl}_{\mathcal{T}}(\bar{u}) \cdot \mathrm{curl}_{\mathcal{T}}(\bar{v}).$$

Then, for all $\bar{v}$ in $H_{\mathcal{T}}$

$$\int_\Omega \mathrm{div}_{\mathcal{T}}(u_{\mathcal{T}}) \mathrm{div}_{\mathcal{T}}(\bar{v}) + \int_\Omega \mathrm{curl}_{\mathcal{T}}(u_{\mathcal{T}}) \cdot \mathrm{curl}_{\mathcal{T}}(\bar{v}) - \int_\Omega p_{\mathcal{T}} \mathrm{div}(\bar{v}) = \int_\Omega f_{\mathcal{T}} \cdot \bar{v}.$$

Choice of $\bar{v}$? $\bar{v} = \bar{v}_{\mathcal{T}}$ with $\mathrm{curl}_{\mathcal{T}}(\bar{v}_{\mathcal{T}}) = 0$, $\mathrm{div}(\bar{v}_{\mathcal{T}}) = \rho_{\mathcal{T}}$ and $\bar{v}_{\mathcal{T}} \in H_{\mathcal{T}}$ and bounded for the natural norm of $H_{\mathcal{T}}$... impossible... (as in the continuous setting)

Choice of the test function in the momentum equation

Let $\{w_K,\ K \in \mathcal{T}\}$ be the FV solution of the $-\Delta w_{\mathcal{T}} = \rho_{\mathcal{T}}$, with the homogeneous Dirichlet boundary condition, that is, for all $K \in \mathcal{T}$,

$$\sum_{\sigma \in \mathcal{E}_K} \frac{|\sigma|}{d_\sigma}(w_K - w_L) = |K|\rho_K$$

In the preceding equality, $\sigma = K|L$, with the usual modification at the boundary

For $\sigma \in \mathcal{E}$, $\sigma = K|L$, $n_{K,\sigma} = n_\sigma \geq 0$, one defines $v_\sigma = u_L - u_K$

A proof similar to the proof for the continous case, gives some discrete-$H^2_{loc}(\Omega)$ estimate on $w_{\mathcal{T}}$ and then some discrete-$H^1_{loc}(\Omega)$ estimate on $v_{\mathcal{T}}$ in term of L^2 norm of $\rho_{\mathcal{T}}$

Furthermore, at least "far" from the boundary, $\mathrm{div}_{\mathcal{T}}(v_{\mathcal{T}}) = \rho_{\mathcal{T}}$ and $\mathrm{curl}_{\mathcal{T}}(v_{\mathcal{T}}) = 0$

Then, up to a subsequence, as $h \to 0$, $v_{\mathcal{T}} \to v$ in $L^2_{loc}(\Omega)$ and $v \in H^1_{loc}(\Omega)^d$, $\mathrm{curl}(v) = 0$, $\mathrm{div}(v) = \rho$.

Proof of $\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi \to \int_\Omega (p - \mathrm{div}(u))\rho\varphi$

Let $\varphi \in C_c^\infty(\Omega)$ (so that $v_\mathcal{T}\varphi_\mathcal{T} \in H_\mathcal{T}$). Taking $\bar{v} = v_\mathcal{T}\varphi_\mathcal{T}$:

$$\int_\Omega \mathrm{div}_\mathcal{T}(u_\mathcal{T})\mathrm{div}_\mathcal{T}(v_\mathcal{T}\varphi) + \int_\Omega \mathrm{curl}_\mathcal{T}(u_\mathcal{T}) \cdot \mathrm{curl}_\mathcal{T}(v_\mathcal{T}\varphi_\mathcal{T}) - \int_\Omega p_\mathcal{T}\mathrm{div}_\mathcal{T}(v_\mathcal{T}\varphi_\mathcal{T}) = \int_\Omega f_\mathcal{T} \cdot (v_\mathcal{T}\varphi_\mathcal{T}).$$

Using a proof smilar to that given in the continuous case we obtain:

$$\lim_{h\to 0} \int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}\varphi = \int_\Omega (p - \mathrm{div}(u))\rho\varphi,$$

Proof of $\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T} \to \int_\Omega (p - \mathrm{div}(u))\rho$

Lemma : $F_\mathcal{T} \to F$ in $D'(\Omega)$, $(F_\mathcal{T})_{n\in\mathbb{N}}$ bounded in L^q for some $q > 1$. Then $F_\mathcal{T} \to F$ weakly in L^1.

With $F_\mathcal{T} = (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T}$, $F = (p - \mathrm{div}(u))\rho$ and since $\gamma > 1$, the lemma gives

$$\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T} \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

Proving $\int_\Omega p_\mathcal{T} \rho_\mathcal{T} \longrightarrow \int_\Omega p\rho$

$$\int_\Omega (p_\mathcal{T} - \mathrm{div}(u_\mathcal{T}))\rho_\mathcal{T} \to \int_\Omega (p - \mathrm{div}(u))\rho.$$

But thanks to the mass equations, the preliminary lemma gives:

$$\int_\Omega \mathrm{div}(u_\mathcal{T})\rho_\mathcal{T} \leq Ch^\alpha,\ \int_\Omega \mathrm{div}(u)\rho = 0;$$

Then:

$$\lim_{h\to 0} \int_\Omega p_\mathcal{T} \rho_\mathcal{T} \leq \int_\Omega p\rho.$$

a.e. convergence of $\rho_{\mathcal{T}}$ and $p_{\mathcal{T}}$

Let $G_{\mathcal{T}} = (\rho_{\mathcal{T}}^{\gamma} - \rho^{\gamma})(\rho_{\mathcal{T}} - \rho) \in L^1(\Omega)$ and $G_{\mathcal{T}} \geq 0$ a.e. in Ω.
Futhermore $G_{\mathcal{T}} = (p_{\mathcal{T}} - \rho^{\gamma})(\rho_{\mathcal{T}} - \rho) = p_{\mathcal{T}}\rho_{\mathcal{T}} - p_{\mathcal{T}}\rho - \rho^{\gamma}\rho_{\mathcal{T}} + \rho^{\gamma}\rho$ and:

$$\int_{\Omega} G_{\mathcal{T}} = \int_{\Omega} p_{\mathcal{T}}\rho_{\mathcal{T}} - \int_{\Omega} p_{\mathcal{T}}\rho - \int_{\Omega} \rho^{\gamma}\rho_{\mathcal{T}} + \int_{\Omega} \rho^{\gamma}\rho.$$

Using the weak convergence in $L^2(\Omega)$ of $p_{\mathcal{T}}$ and $\rho_{\mathcal{T}}$ and $\lim_{h\to 0} \int_{\Omega} p_{\mathcal{T}}\rho_{\mathcal{T}} \leq \int_{\Omega} p\rho$:

$$\lim_{h\to 0} \int_{\Omega} G_{\mathcal{T}} \leq 0,$$

Then (up to a subsequence), $G_{\mathcal{T}} \to 0$ a.e. and then $\rho_{\mathcal{T}} \to \rho$ a.e. (since $y \mapsto y^{\gamma}$ is an increasing function on $\mathbb{R}_+$). Finally:

$\rho_{\mathcal{T}} \to \rho \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2\gamma$,

$p_{\mathcal{T}} = \rho_{\mathcal{T}}^{\gamma} \to \rho^{\gamma} \text{ in } L^q(\Omega) \text{ for all } 1 \leq q < 2$,

and $p = \rho^{\gamma}$.

($\rightsquigarrow$ EOS and EOT ?)

Additional difficulty for stat. comp. NS equations

Ω is a bounded open set of $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz continuous boundary, $\gamma > 1$, $f \in L^2(\Omega)^d$ and $M > 0$

$$-\Delta u + \mathrm{div}(\rho u \otimes u) + \nabla p = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

$$\mathrm{div}(\rho u) = 0 \text{ in } \Omega, \ \rho \geq 0 \text{ in } \Omega, \ \int_\Omega \rho(x) = M,$$

$$p = \rho^\gamma \text{ in } \Omega$$

$d = 2$: no aditional difficulty

$d = 3$: no additional difficulty if $\gamma \geq 3$. But for $\gamma < 3$, no estimate on p in $L^2(\Omega)$.

Estimates in the case of NS equations, $\frac{3}{2} < \gamma < 3$

Estimate on u : Taking u as test function in the momentum leads to an estimate on u in $(H_0^1(\Omega))^d$ since

$$\int_\Omega \rho u \otimes u : \nabla u = 0.$$

Then, we have also an estimate on u in $L^6(\Omega)^d$ (using Sobolev embedding).

Estimate on p in $L^q(\Omega)$, with some $1 < q < 2$ and $q = 1$ when $\gamma = \frac{3}{2}$ (using Nečas Lemma in some L^r instead of L^2).

Estimate on ρ in $L^q(\Omega)$, with some $\frac{3}{2} < q < 6$ and $q = \frac{3}{2}$ when $\gamma = \frac{3}{2}$ (since $p = \rho^\gamma$).

Remark : $\rho u \otimes u \in L^1(\Omega)$, since $u \in L^6(\Omega)^d$ and $\rho \in L^{\frac{3}{2}}(\Omega)$ (and $\frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1$).

NS equations, $\gamma < 3$, how to pass to the limit in the EOS

We prove

$$\lim_{h \to 0} \int_\Omega p_\mathcal{T} \rho_\mathcal{T}^\theta = \int_\Omega p \rho^\theta,$$

with some convenient choice of $\theta > 0$ instead of $\theta = 1$.

This gives, as for $\theta = 1$, the a.e. convergence (up to a subsequence) of $p_\mathcal{T}$ and $\rho_\mathcal{T}$.