SAW, Restriction & CFT
« question/problem/etc »

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Aims/Questions…

— Characterising (at least constraining) the CFT for SAW in 2D & 3D by mixing random geometry arguments — the restriction property — and bootstrap arguments — operator algebra & OPE.

— in 2D… This should be possible since, as proved by Lawler-Schramm-Werner, the restriction property + conformal invariance fixes uniquely a measure on simple curve in a simple connected domain

—> restriction property yield information on the behavior of the measure on domain deformations…

« rules » : use the restriction property, global conformal symmetry, if necessary local conformal invariance (but no null vector), and OPE/bootstrap constraints on CFT .

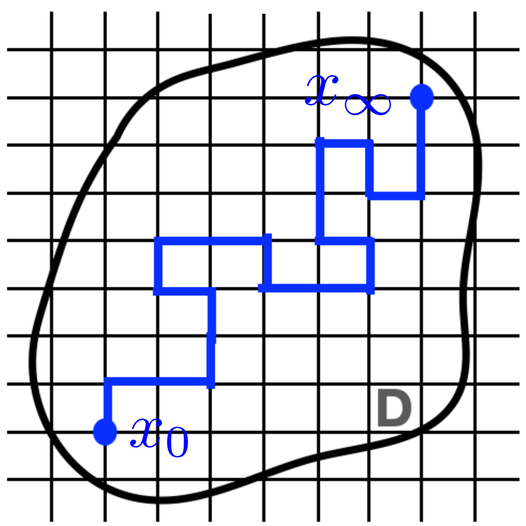
— in 3D… Restriction - global conformal invariance - bootstrap still valid…

SAW : Self avoiding walks

— Self-avoiding walks :

- square (say) lattice in domain $\mathbb{D} \subset \mathbb{R}^D$
- two marked points (bulk or boundary) : x_0 and x_∞
- weights assign to SAW from x_0 to x_∞ in $\mathbb{D}$

weight : $w_\gamma = \mu^{|\gamma|}$

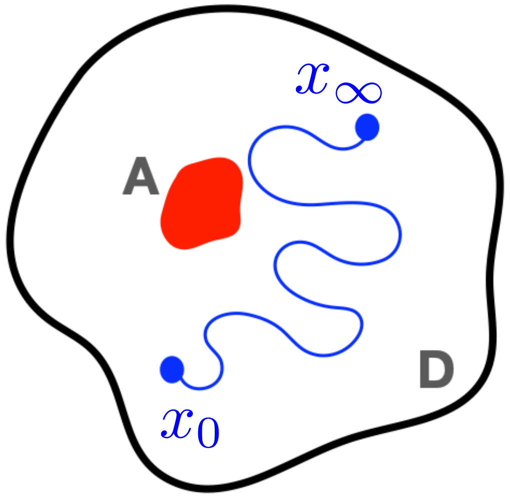


Partition function $Z_{x_0,x_\infty;\mathbb{D}} := \sum_{\gamma(x_0 \leftrightarrow x_\infty) \subset \mathbb{D}} \mu^{|\gamma|}$

— Visiting probabilities and partition functions

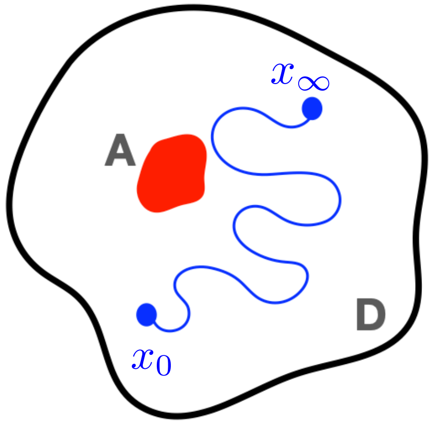
—> the probabilities not to visit a sub-domain

$$\mathbb{P}_{x_0,x_\infty;\mathbb{D}}[\gamma \cap \mathbb{A} = \emptyset] = \frac{1}{Z_{x_0,x_\infty;\mathbb{D}}} \cdot \sum_{\substack{\gamma \subset \mathbb{D} \\ \gamma \cap \mathbb{A} = \emptyset}} \mu^{|\gamma|} = \frac{Z_{x_0,x_\infty;\mathbb{D}\setminus\mathbb{A}}}{Z_{x_0,x_\infty;\mathbb{D}}}$$



The restriction property

— The restriction property [Lawler, Schramm, Werner]



$$\mathbb{P}_{x_0,x_\infty;\mathbb{D}}\big[\mathcal{E}\big|\gamma\cap\mathbb{A}=\emptyset\big]=\mathbb{P}_{x_0,x_\infty;\mathbb{D}\setminus\mathbb{A}}\big[\mathcal{E}\big]$$

that is: conditioning = cutting

- Valid in any dimension, at criticality or not
- For SLE, this fixes SLE(8/3)
- Restriction mesure for *conformally invariant* hulls in 2D : one parameter family

— Proof from partition functions for SAW

Consider the event not to visit a sub-domain B, not intersecting A

$$\mathbb{P}_{\mathbb{D}}\big[\gamma\cap\mathbb{B}=\emptyset\big|\gamma\cap\mathbb{A}=\emptyset\big] = \frac{Z_{\mathbb{D}\setminus\mathbb{A}\cup\mathbb{B}}}{Z_{\mathbb{D}}}\,\Big(\frac{Z_{\mathbb{D}\setminus\mathbb{A}}}{Z_{\mathbb{D}}}\Big)^{-1}=\frac{Z_{\mathbb{D}\setminus\mathbb{A}\cup\mathbb{B}}}{Z_{\mathbb{D}\setminus\mathbb{A}}}$$

$$= \mathbb{P}_{\mathbb{D}\setminus\mathbb{A}}\big[\gamma\cap\mathbb{B}=\emptyset\big]$$

Field theory re-writing

— Partition functions as correlation functions

In the scaling limit, partition function is identified as the expectation of fields *creating* a curve.

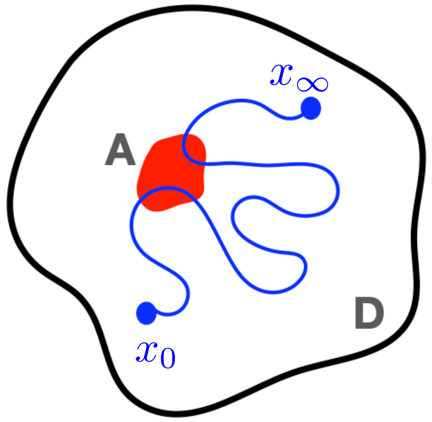
- bdry-to-bdry: $Z_{x_0,x_\infty;\mathbb{D}} \simeq a^{2h_\perp}\ \langle \psi_\perp(x_0)\psi_\perp(x_\infty)\rangle_{\mathbb{D}}$
- bulk-to-bulk: $Z_{x_0,x_\infty;\mathbb{D}} \simeq a^{2\Delta_|}\ \langle \Phi_|(x_0)\Phi_|(x_\infty)\rangle_{\mathbb{D}}$

In 2D: $h_\perp = 5/8;\ \Delta_| = 5/48$ $D_F = 4/3$

In 3D: $h_\perp = 1.3303(3);\ \Delta_| = ...$ [Kennedy] $D_F = 1,701...$

— Conditioning and pinching fields

Conditioning the curve to visit a sub-domain defines a (non-local) field: $\Phi_{\mathbb{A}} := \mathbf{1}_{\gamma\cap\mathbb{A}\neq\emptyset}$



Conditioning the curve to visit a small ball defines a local field: $\Phi_\bullet(x) := \lim_{\epsilon\to 0} \epsilon^{D_F-D}\ \Phi_{\mathbb{B}_\epsilon(x)}.$

Fractal dimension <=> Scaling dimension of the pinching field : $\Delta_\bullet = D - D_F$

The restriction property in field theory

– Since the field $\Phi_{\mathbb{A}} := \mathbf{1}_{\gamma\cap\mathbb{A}\neq\emptyset}$ conditions to (not) visit a sub-domain, the visiting probabilities can be written as ratio of correlation functions (with insertions of that field).

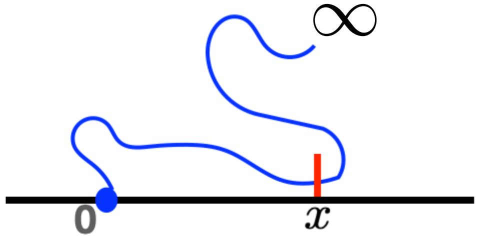
– The restriction property (tested against some observable/event) is a series of non-linear (Ward) identities :

$$\langle\mathcal{O}\rangle_{\mathbb{D}} - \langle\mathcal{O}\rangle_{\mathbb{D}\setminus\mathbb{A}} = \langle\mathcal{O}\,\Phi_{\mathbb{A}}\rangle_{\mathbb{D}} - \langle\Phi_{\mathbb{A}}\rangle_{\mathbb{D}}\,\langle\mathcal{O}\rangle_{\mathbb{D}\setminus\mathbb{A}}$$

—> How do they constrain the operator algebra ?
—> How is the restriction prop. related to some (hidden) field theory symmetry ?

– A simple application... [Friedrich-Werner]

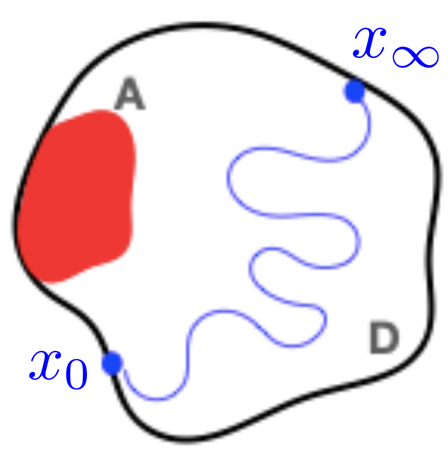
$$\mathbb{P}_{0,\infty;\mathbb{H}}[\gamma\cap[x,x+i\epsilon]\neq\emptyset] = \langle\psi_{\perp}(0)\psi_{\vee}(x)\psi_{\perp}(\infty)\rangle_{\mathbb{H}} \simeq \frac{1}{x^2}$$



— Another application of restriction in CFT...

Probability not to visit a sub-domain attached to the boundary

[Lawler, Schramm, Werner]



$$\mathbb{P}_{x_0,x_\infty;\mathbb{D}}\left[\gamma \cap \mathbb{A} = \emptyset\right] = \frac{Z_{x_0,x_\infty;\mathbb{D}\setminus\mathbb{A}}}{Z_{x_0,x_\infty;\mathbb{D}}}$$

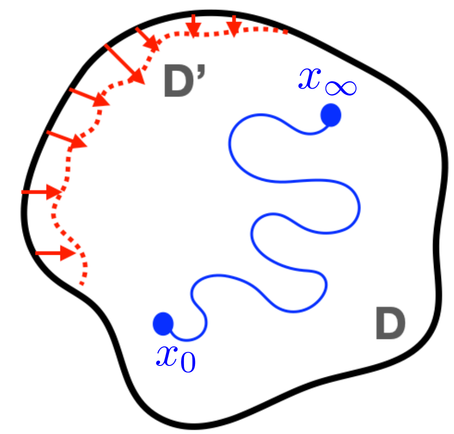
$$= \frac{\langle \psi_\perp(x_0)\psi_\perp(x_\infty)\rangle_{\mathbb{D}\setminus\mathbb{A}}}{\langle \psi_\perp(x_0)\psi_\perp(x_\infty)\rangle_{\mathbb{D}}}$$

$$= \left[\varphi'_{\mathbb{A}}(x_0)\varphi'_{\mathbb{A}}(x_\infty)\right]^{h_\perp}$$

with $\varphi_{\mathbb{A}}(z)$ the uniformizing map from D-A to D fixing x_0 and x_∞

The boundary OPE of the pinching field

—> In field theory, deformation of the geometry
in implemented by insertion of the stress-tensor $T_{\mu\nu}$



$$\delta_\xi \langle \mathcal{O} \rangle_{\mathbb{D}} = -\int_{\partial\mathbb{D}} ds\, \xi(x) \left[\langle \mathcal{O}\, T_{nn}(x) \rangle_{\mathbb{D}} - \langle T_{nn}(x) \rangle_{\mathbb{D}} \langle \mathcal{O} \rangle_{\mathbb{D}} \right]$$

deformation parameter

normal component of $T_{\mu\nu}$

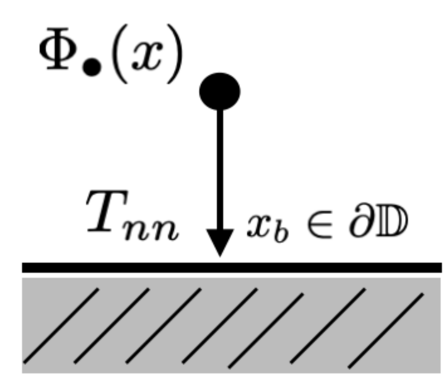
— By the restriction property, pinching SAW = deforming the geometry

—> conditioning field at the boundary = the stress-tensor

—> boundary OPE of the pinching field => the stress-tensor

$$\Phi_\bullet(x) \simeq_{x \to x_b \in \partial\mathbb{D}} \text{const.}\ |n \cdot (x - x_b)|^{D_F}\, T_{nn}(x_b) + \cdots$$

[T]=D



—> Encoding the restriction property
in the OPE / bootstrap...

Does the OPE fix the fractal dimension?

—> If yes, we would have completed (largely) our programme...

— In 2D, in the upper half plane, OPE is : $\Phi_\bullet(z,\bar z) \simeq_{y\to}\, y^{D_F}\, \big(T(x)+\cdots\big)$ with higher order...

$$\Phi_\bullet \simeq y^{D_F}\, \big(T(x) + {}_{\#}\, yL_{-1}T + {}_{\#}\, y^2 L_{-2}T + {}_{\#}\, y^2 L_{-1}^2 T + \cdots\big)$$

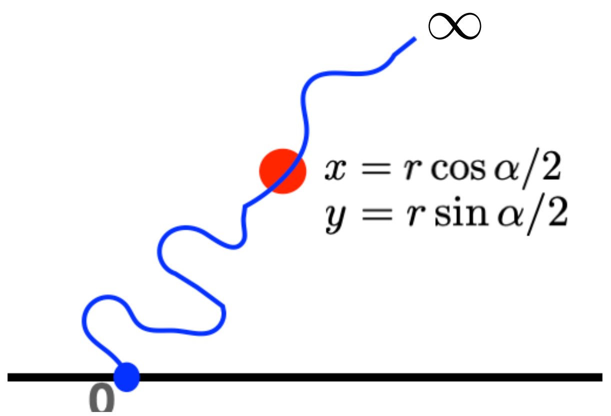
—> Easy constraints : central charge $c=0$ and $h_\vee = 2 \quad (\Psi_\vee = T)$

(no use of the null-vector for $\psi_\perp$)

— Constraints from the OPE/bootstrap *plus* an ansatz...

Assume the simple ansatz for the one-point bulk visiting probability:

$$\langle \psi_\perp(\infty)\Phi_{\mathbb{B}_\epsilon(z)}\psi_\perp(0)\rangle = (\frac{\epsilon}{y})^{\Delta_\bullet}\, (\sin\alpha/2)^2$$



Then

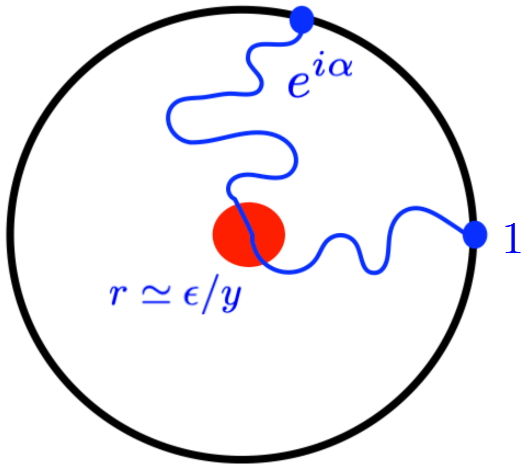
- OPE at order 2 => : $2h_\perp + \Delta_\bullet(5h_\perp + 1) = 4$
- OPE at order 4 => : $h_\perp = 5/8\ ,\ \Delta_\bullet = 2/3 \implies D_F = 4/3$!!

What is missing ?

— How restriction fixes the pinching one-point function (Green function) ?

Why restriction implies this simple structure ?

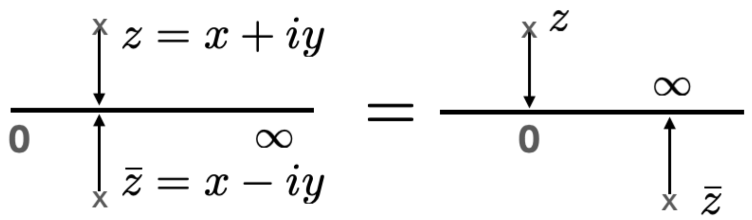
$$\langle \psi_\perp(e^{i\alpha}) \Phi_{\mathbb{B}_r(0)} \psi_\perp(1) \rangle_{\mathbb{U}} \simeq_{r\to 0} r^{\Delta_\bullet} (\sin\alpha/2)^2$$



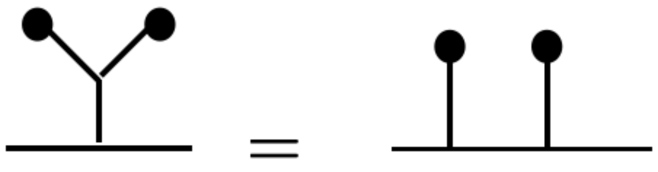
- its form (known from SLE/null vector)
- its geometrical interpretation
- naturalness (in 2D / in 3D)

— Other geometrical event or correlation functions ?

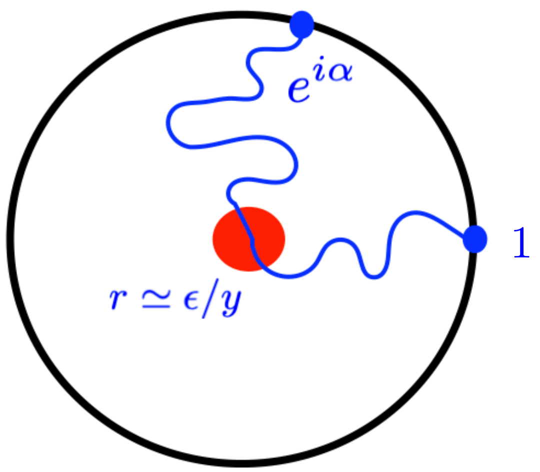
- other bootstrap constraints on $\langle \psi_\perp \Phi_\bullet(z,\bar z) \psi_\perp \rangle$? But ....



- other use of bootstrap constraints on other visiting probability for bdry-to-bdry SAW or bulk-to-bulk SAW ?... e.g. $\langle \Phi_|(z_0) \Phi_|(z_\infty) \rangle$



Any suggestion on how to explain
why the restriction property
implies this simple form for
the one-point bulk visiting probability ?



$$\mathbb{P} \simeq r^{\Delta_\bullet} (\sin \alpha/2)^2$$

Thank you !!!