Non-compact boundary conditions for the XXZ spin chain and conformal boundary loop models

Dmitry Chernyak

ENS

Based on arXiv:2207.12772 with A. Gainutdinov and H. Saleur and upcoming paper(s) with A. Gainutdinov, J. Jacobsen and H. Saleur

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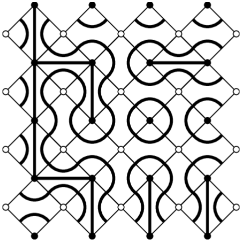
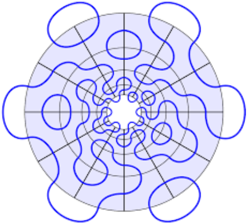
Motivation

Why loop models ?





A priori complicated geometrical models... But actually :

- They admit a **local** lattice formulation,
- A finite-dimensional spin chain presentation,
- A (non-unitary) CFT description in the continuum,
- They have nice algebraic properties even on the lattice.
- They are integrable.

Outline

The Hamiltonian of the open XXZ spin chain of length N is given by

$$H^{\text{open}}_{\mathrm{XXZ}} := \frac{1}{2}\sum_{i=1}^{N-1}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}\sigma_i^z\sigma_{i+1}^z\right)$$

for some complex parameter $\mathfrak{q}$ and acts on the Hilbert space $(\mathbb{C}^2)^{\otimes N}$ with

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\mathfrak{q} \neq 1$, the global SU(2) symmetry of the XXX spin chain breaks down to U(1).

We can recover the larger symmetry by deforming SU(2) into the $U_{\mathfrak{q}}\mathfrak{sl}_2$ **quantum group** and changing the boundary conditions as

$$H_{\mathrm{XXZ}} := H^{\text{open}}_{\mathrm{XXZ}} + \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma_N^z - \sigma_1^z).$$

Definition

$U_\mathfrak{q}\mathfrak{sl}_2$ is generated by E, F, K and K^{-1} with relations

$$\mathsf{KEK}^{-1} = \mathfrak{q}^2\mathsf{E}\,, \quad \mathsf{KFK}^{-1} = \mathfrak{q}^{-2}\mathsf{F}\,, \quad [\mathsf{E},\mathsf{F}] = \frac{\mathsf{K}-\mathsf{K}^{-1}}{\mathfrak{q}-\mathfrak{q}^{-1}}\,.$$

It is a $\mathfrak{q}$-deformation of the Lie algebra $\mathfrak{sl}_2$: in the limit $\mathfrak{q}\to 1$ we recover the commutation relations of the $\mathfrak{sl}_2$ triple $(\mathsf{E},\mathsf{F},\mathsf{H})$ with $\mathsf{K}^{\pm 1} = \mathfrak{q}^{\pm\mathsf{H}}$.

Representations

Very similar to $\mathfrak{sl}_2$. For example, the spin-$\frac{1}{2}$ representation in the basis $\{|\uparrow\rangle\,, |\downarrow\rangle\}$ is given by

$$\mathsf{E}_{\mathbb{C}^2} = \sigma^+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,, \qquad \mathsf{F}_{\mathbb{C}^2} = \sigma^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\,,$$

$$\mathsf{K}^{\pm 1}_{\mathbb{C}^2} = \mathfrak{q}^{\pm\sigma^z} = \begin{pmatrix} \mathfrak{q}^{\pm 1} & 0 \\ 0 & \mathfrak{q}^{\mp 1} \end{pmatrix}\,.$$

The main difference with $\mathfrak{sl}_2$ is the **coproduct structure**, i.e. the way we define tensor products of representations.

Coproduct

The coproduct map $\Delta : U_{\mathfrak{q}}\mathfrak{sl}_2 \to U_{\mathfrak{q}}\mathfrak{sl}_2 \otimes U_{\mathfrak{q}}\mathfrak{sl}_2$ is given by

$$\Delta(\mathsf{E}) = 1 \otimes \mathsf{E} + \mathsf{E} \otimes \mathsf{K}\,, \;\; \Delta(\mathsf{F}) = \mathsf{K}^{-1} \otimes \mathsf{F} + \mathsf{F} \otimes 1\,, \;\; \Delta(\mathsf{K}^{\pm 1}) = \mathsf{K}^{\pm 1} \otimes \mathsf{K}^{\pm 1}$$

and is non-symmetric under permutation of the two tensor factors.

Applying Δ $N-1$ times, we obtain the $U_{\mathfrak{q}}\mathfrak{sl}_2$-action on $(\mathbb{C}^2)^{\otimes N}$

$$\mathsf{E} = \sum_{j=1}^{N} \sigma_j^+ \otimes \mathfrak{q}^{\sigma_{j+1}^z + \ldots + \sigma_N^z}\,, \qquad \mathsf{F} = \sum_{j=1}^{N} \mathfrak{q}^{-\sigma_1^z - \ldots - \sigma_{j-1}^z} \otimes \sigma_j^-$$

$$\mathsf{K}^{\pm 1} = \mathfrak{q}^{\pm \sum_{j=1}^{N} \sigma_j^z}$$

One checks that H_{XXZ} commutes with the generators E, F and $\mathsf{K}^{\pm 1}$ so with the whole algebra $U_{\mathfrak{q}}\mathfrak{sl}_2$.

We want to define new boundary conditions while keeping the $U_q\mathfrak{sl}_2$ symmetry.

Strategy

- Take an irrep $\mathcal{V}$ of $U_q\mathfrak{sl}_2$ and consider the bigger Hilbert space $\mathcal{V} \otimes (\mathbb{C}^2)^{\otimes N}$.
- Look for the most general $U_q\mathfrak{sl}_2$-invariant operator b acting only on the two leftmost sites $\mathcal{V} \otimes \mathbb{C}^2$.
- Define the new $U_q\mathfrak{sl}_2$-invariant Hamiltonian $H_b := -\mu b + H_{\mathrm{XXZ}}$ with some coupling constant μ.

For $\mathcal{V}$ we will take infinite-dimensional **Verma modules** of $U_q\mathfrak{sl}_2$.

Definition

Take $\alpha \in \mathbb{C}$ and set $\mathcal{V}_\alpha := \bigoplus_{0\leq n} \mathbb{C}\,|n\rangle$. Then $U_{\mathfrak{q}}\mathfrak{sl}_2$ acts on $\mathcal{V}_\alpha$ as

$$\begin{aligned} \mathsf{E}_{\mathcal{V}_\alpha}\,|n\rangle &= [n]_{\mathfrak{q}}[\alpha-n]_{\mathfrak{q}}\,|n-1\rangle\ , \\ \mathsf{F}_{\mathcal{V}_\alpha}\,|n\rangle &= |n+1\rangle\ , \\ \mathsf{K}^{\pm 1}_{\mathcal{V}_\alpha}\,|n\rangle &= \mathfrak{q}^{\pm(\alpha-1-2n)}\,|n\rangle\ , \end{aligned}$$

for all $n \geq 0$ with $[x]_{\mathfrak{q}} := \frac{\mathfrak{q}^x-\mathfrak{q}^{-x}}{\mathfrak{q}-\mathfrak{q}^{-1}}$.

Remarks

- These are the same as $\mathfrak{sl}_2$ Verma modules except that we use $\mathfrak{q}$-deformed coefficient $[n]_{\mathfrak{q}}[\alpha-n]_{\mathfrak{q}}$ instead of the usual one $n(\alpha-n)$.
- The parameter α can be thought of as a generalised continuous "spin".
- $\mathcal{V}_\alpha$ is "unique" and (generically) irreducible.

What is the most general $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant operator b acting on $\mathcal{V}_\alpha \otimes \mathbb{C}^2$?

One can show that we have the $U_{\mathfrak{q}}\mathfrak{sl}_2$ irrep decomposition

$$\mathcal{V}_\alpha \otimes \mathbb{C}^2 = \mathcal{V}_{\alpha+1} \oplus \mathcal{V}_{\alpha-1}\,.$$

Introducing the projectors $b_\pm$ on $\mathcal{V}_{\alpha\pm1}$ therefore we must have

$$b = \mu_+ b_+ + \mu_- b_-$$

But $b_+ + b_- = 1$ so up to a constant

$$H_b = -\mu b + H_{\mathrm{XXZ}}$$

with $b := b_+$, $\mu := \mu_+$, $\mu_- = 0$.

Same strategy for the other boundary :

- We consider the Hilbert space $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta$
- We take the $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant projector b' on the $\mathcal{V}_{\beta+1}$ summand of $\mathbb{C}^2 \otimes \mathcal{V}_\beta = \mathcal{V}_{\beta+1} \oplus \mathcal{V}_{\beta-1}$
- We define the new two-boundary Hamiltonian

$$H_{2b} := -\mu b - \mu' b' + H_{\mathrm{XXZ}}$$

Explicitly :

$$b = \frac{1}{[\alpha]_{\mathfrak{q}}} \begin{pmatrix} \frac{\mathfrak{q}^{\alpha} - \mathfrak{q}^{-1}\mathsf{K}^{-1}}{\mathfrak{q}-\mathfrak{q}^{-1}} & \mathsf{F} \\ \mathfrak{q}\mathsf{K}^{-1}\mathsf{E} & \frac{\mathfrak{q}\mathsf{K}^{-1} - \mathfrak{q}^{-\alpha}}{\mathfrak{q}-\mathfrak{q}^{-1}} \end{pmatrix}, \quad b' = \frac{1}{[\beta]_{\mathfrak{q}}} \begin{pmatrix} \frac{\mathfrak{q}\mathsf{K} - \mathfrak{q}^{-\beta}}{\mathfrak{q}-\mathfrak{q}^{-1}} & \mathfrak{q}\mathsf{K}\mathsf{F} \\ \mathsf{E} & \frac{\mathfrak{q}^{\beta} - \mathfrak{q}^{-1}\mathsf{K}}{\mathfrak{q}-\mathfrak{q}^{-1}} \end{pmatrix}$$

written as 2×2 matrices with elements in $\mathrm{End}(\mathcal{V}_\alpha)$ or $\mathrm{End}(\mathcal{V}_\beta)$.

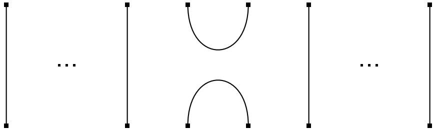
1. XXZ spin chains
   - The open $U_q\mathfrak{sl}_2$-invariant XXZ spin chain
   - New $U_q\mathfrak{sl}_2$-invariant boundary conditions

2. Loop Models and their lattice algebras

3. Conformal scaling limit
   - Free fermions at $\mathfrak{q} = i$
   - General case for $|\mathfrak{q}| = 1$

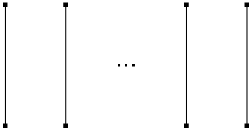
4. Summary and Outlook

Let N be an integer. For all $1 \leq i \leq N-1$ consider the diagrams

$e_i =$ 

$i \quad i+1$

and

$1 =$ 

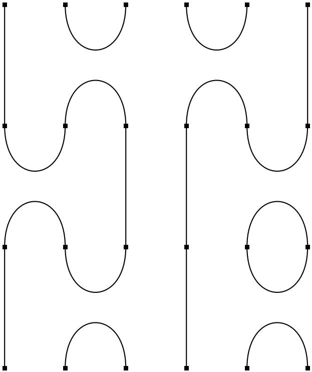
Configurations are built by stacking these diagrams on top of each other.

For example, a configuration on $N = 6$ sites :

$e_4 e_2 =$

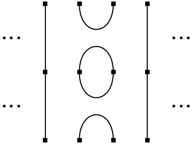
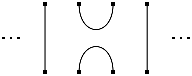
$e_5 e_1 =$

$e_5 e_2 =$

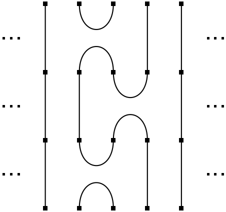
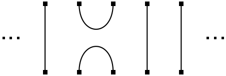


$= e_4 e_2 e_5 e_1 e_5 e_2$

Graphical rules :

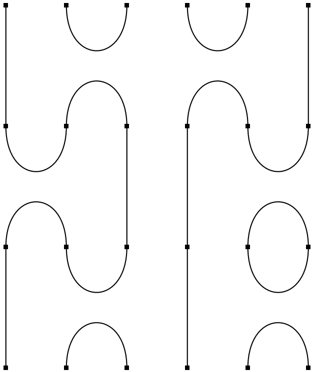
$e_i^2 =$  $= \delta$  $= \delta e_i$

δ : weight of a closed loop.

$e_i e_{i+1} e_i =$  $=$  $= e_i$

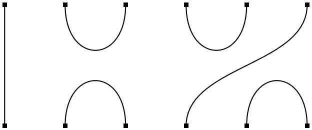
The resulting algebra is called the **Temperley-Lieb (TL) algebra** and denoted $\mathrm{TL}_{\delta,N}$.

Example :



$=$

δ

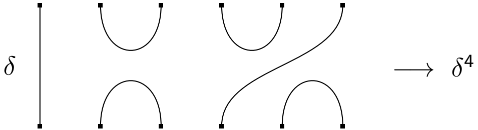


It remains to impose boundary conditions.

We take periodic boundary conditions by gluing the top and the bottom of the diagram.

We need to associate the statistical weight δ to the new loops we form.

For example :



This procedure defines a **quantum trace** on the TL algebra, denoted $\mathrm{qtr} : \mathsf{TL}_{\delta,N} \to \mathbb{C}$.

We now have all the ingredients to build our loop model !

Consider the transfer matrix

$$T = \prod_{i \text{ odd}} (1 + e_i) \prod_{i \text{ even}} (1 + e_i) \in \mathsf{TL}_{\delta,N} \,.$$

T^M generates all possible loop configurations on an $N \times M$ lattice.

Taking the quantum trace we obtain the partition function on the $N \times M$ cylinder

$$Z_{N\times M}(\delta) = \sum_{\text{loop configs.}} \delta^{\#\text{loops}} = \text{qtr } T^M \,.$$

How is this related to our XXZ spin chain ?

Introduce the local Hamiltonian densities

$$e_i = -\frac{1}{2}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}(\sigma_i^z\sigma_{i+1}^z - 1)\right) - \frac{\mathfrak{q}-\mathfrak{q}^{-1}}{4}(\sigma_{i+1}^z - \sigma_i^z)$$

such that

$$H_{\mathrm{XXZ}} = \frac{\mathfrak{q}+\mathfrak{q}^{-1}}{2}(N-1) - \sum_{i=1}^{N-1} e_i \,.$$

The e_i satisfy the defining relations of the TL algebra

$$e_i^2 = (\mathfrak{q}+\mathfrak{q}^{-1})e_i\,, \qquad e_i e_{i\pm1} e_i = e_i\,, \qquad [e_i, e_j] = 0 \quad |i-j| \geq 2\,,$$

with $\delta = \mathfrak{q}+\mathfrak{q}^{-1}$. The $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant XXZ spin chain generates a representation of the loop model !

But does this spin chain representation contain all the information about the abstract loop model ?

Yes ! $\mathrm{TL}_{\delta,N}$ has exactly $N/2+1$ irreps $\mathcal{T}_j$, $0 \leq j \leq N/2$ and all of them are contained in the spin chain. Actually we have an even stronger result : an instance of (quantum) **Schur-Weyl duality**.

Theorem (Jimbo, Jones, Wenzl, Martin...)

i) $U_{\mathfrak{q}}\mathfrak{sl}_2$ *and* $\mathrm{TL}_{\delta,N}$ *are mutual centralisers on* $(\mathbb{C}^2)^{\otimes N}$.

ii) The Hilbert space decomposes as a $(U_{\mathfrak{q}}\mathfrak{sl}_2, \mathrm{TL}_{\delta,N})$*-bimodule*

$$(\mathbb{C}^2)^{\otimes N} = \bigoplus_{j=0}^{N/2} \mathbb{C}^{2j+1} \otimes \mathcal{T}_j$$

where $\mathbb{C}^{2j+1}$ *are spin-*j *representations of* $U_{\mathfrak{q}}\mathfrak{sl}_2$.

iii) The partition function decomposes as

$$Z_{N\times M}(\delta) = \mathrm{qtr}\ T^M = \sum_{j=0}^{N/2} [2j+1]_{\mathfrak{q}}\, \mathrm{tr}_{\mathcal{T}_j}\, T^M .$$

How do our new boundary conditions fit into this picture ?

If we modify only the left boundary we have an additional generator b which :

- Commutes with the $U_{\mathfrak{q}}\mathfrak{sl}_2$ action.
- Commutes with the e_i for all $2 \leq i \leq N-1$.
- Is a projector so $b^2 = b$.

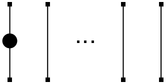
Moreover, by direct computation

$$e_1 b e_1 = y e_1 \qquad \text{with} \qquad y := \frac{[\alpha+1]_{\mathfrak{q}}}{[\alpha]_{\mathfrak{q}}} .$$

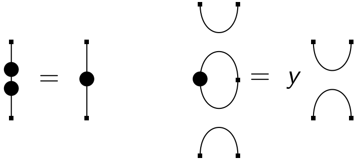
The boundary coupling b and the e_i define a representation of the **Blob algebra** $\mathrm{B}_{\delta,y,N}$ on $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N}$ which commutes with the $U_{\mathfrak{q}}\mathfrak{sl}_2$ action !

What kind of loop model corresponds to the Blob algebra ?

Let us represent the new generator b, also known as **blob**, by

$b =$ 

Then the rules $b^2 = b$ and $e_1 b e_1 = y e_1$ simply mean that



A loop touching the left boundary and carrying a blob has weight y instead of δ.

The new transfer matrix is

$$T_b = (1 + \mu b) \prod_{i \text{ odd}} (1 + e_i) \prod_{i \text{ even}} (1 + e_i) \in \mathsf{B}_{\delta,y,N}$$

and T^M generates all possible loop configurations on the $N \times M$ lattice with some probability for a loop touching the left boundary to carry a blob.

For example, in the limit $\mu \to \infty$ all loops touching the left boundary carry a blob.

There is also a notion of quantum trace for the blob algebra.

The partition function on the $N \times M$ cylinder is then

$$Z_{N\times M}(\delta, y, \mu) = \sum_{\text{loop configs.}} y^{\#\text{loops}\bullet} \delta^{\#\text{loops}\emptyset} = \text{qtr}\ T_b^M \,.$$

Is our spin chain representation of this loop model faithful ?

Yes ! $\mathrm{B}_{\delta,y,N}$ has exactly $N+1$ irreps $\mathcal{W}_j$, $-N/2 \leq j \leq N/2$ and we can show complete Schur-Weyl duality :

Theorem (Ch.-Gainutdinov-Saleur '22)

i) $U_{\mathfrak{q}}\mathfrak{sl}_2$ and $\mathrm{B}_{\delta,y,N}$ are mutual centralisers on $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N}$.

ii) The Hilbert space decomposes as a $(U_{\mathfrak{q}}\mathfrak{sl}_2, \mathrm{B}_{\delta,y,N})$-bimodule

$$\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} = \bigoplus_{j=-N/2}^{N/2} \mathcal{V}_{\alpha+2j} \otimes \mathcal{W}_j \, .$$

iii) The partition function decomposes as

$$Z_{N\times M}(\delta, y, \mu) = \mathrm{qtr}\ T_b^M = \sum_{j=-N/2}^{N/2} \frac{[\alpha+2j]_{\mathfrak{q}}}{[\alpha]_{\mathfrak{q}}} \mathrm{tr}_{\mathcal{W}_j}\, T_b^M \, .$$

Idea of the proof

- Use the fusion rule $\mathcal{V}_\alpha \otimes \mathbb{C}^2 = \mathcal{V}_{\alpha+1} \oplus \mathcal{V}_{\alpha-1}$ iteratively to obtain the $U_\mathfrak{q}\mathfrak{sl}_2$-decomposition

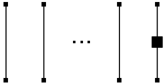
$$\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} = \bigoplus_{j=-N/2}^{N/2} \binom{N}{j+N/2} \mathcal{V}_{\alpha+2j} \, .$$

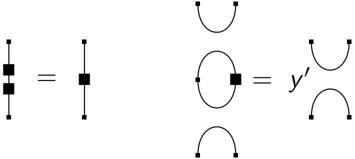
- Identify the multiplicity spaces with $\mathrm{B}_{\delta,y,N}$ irreps by induction on N.
- Use abstract (categorical) interpretation of the quantum trace and the loop model to compute the partition function.

What about the two-boundary case ?

We have a new generator b', which, unsurprisingly,

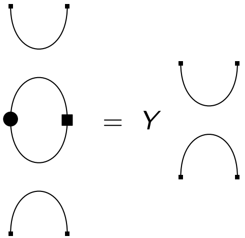
- Commutes with the $U_{\mathfrak{q}}\mathfrak{sl}_2$ action,
- Commutes with the e_i for all $1 \leq i \leq N-2$,
- Is a projector so $b'^2 = b'$
- Satisfies $e_{N-1} b' e_{N-1} = y' e_1$ with $y' := \dfrac{[\beta+1]_{\mathfrak{q}}}{[\beta]_{\mathfrak{q}}}$.

Graphically, we set $b' =$  and so



This is not enough however, because we also need to assign a weight to a loop carrying both b and b'.

In the simplest case $N = 2$, we need to find a Y such that



It turns out Y is not a number but a central element of $U_{\mathfrak{q}}\mathfrak{sl}_2$!

$U_\mathfrak{q}\mathfrak{sl}_2$ admits a central Casimir element

$$\mathsf{C} := (\mathfrak{q} - \mathfrak{q}^{-1})^2 \mathsf{FE} + \mathfrak{q}\mathsf{K} + \mathfrak{q}^{-1}\mathsf{K}^{-1}\,.$$

Evaluated on our spin chain $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta$ it commutes with the $U_\mathfrak{q}\mathfrak{sl}_2$ action and also the e_i, b and b'.

With

$$Y = \frac{\mathfrak{q}^{\alpha+\beta+1} + \mathfrak{q}^{-\alpha-\beta-1} - \mathsf{C}}{(\mathfrak{q}^{\alpha} - \mathfrak{q}^{-\alpha})(\mathfrak{q}^{\beta} - \mathfrak{q}^{-\beta})}$$

e_i, b and b' define a representation of the **universal two-boundary Temperley-Lieb algebra** $2\mathsf{B}^{\mathrm{uni}}_{\delta,y,y',N}$ with loop weights δ, y and y'.

What values can Y take ?

The Casimir C is constant on any irrep of $U_{\mathfrak{q}}\mathfrak{sl}_2$. We just need to compute the decomposition of $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta$ into $U_{\mathfrak{q}}\mathfrak{sl}_2$-irreps !

Using the fusion rules

$$\mathcal{V}_\alpha \otimes \mathcal{V}_\beta = \bigoplus_{n\geq 0} \mathcal{V}_{\alpha+\beta-1-2n} \qquad \text{and} \qquad \mathcal{V}_\alpha \otimes \mathbb{C}^2 = \mathcal{V}_{\alpha+1} \oplus \mathcal{V}_{\alpha-1}$$

we obtain

$$\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta = \bigoplus_{-N/2\leq m} \mathcal{V}_{\alpha+\beta-1-2m} \otimes \mathcal{Z}_m$$

where the $\mathcal{Z}_m$ are some multiplicity spaces of dimension

$$\dim \mathcal{Z}_m = d_m := \sum_{k=0}^{m+N/2} \binom{N}{k}.$$

For $m \geq N/2$ $d_m = 2^N$.

Now since $C_{\mathcal{V}_\alpha} = \mathfrak{q}^\alpha + \mathfrak{q}^{-\alpha}$,

$$Y_{\mathcal{V}_{\alpha+\beta-1-2m}\otimes\mathcal{Z}_m} = \frac{[m+1]_\mathfrak{q}[\alpha+\beta-m]_\mathfrak{q}}{[\alpha]_\mathfrak{q}[\beta]_\mathfrak{q}} := Y_m \,.$$

Therefore :

Theorem (Ch.-Gainutdinov-Saleur '22)

i) $U_\mathfrak{q}\mathfrak{sl}_2$ *and* $\mathrm{B}_{\delta,y,N}$ *commute on* $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta$.

ii) *The Hilbert space decomposes as a* $(U_\mathfrak{q}\mathfrak{sl}_2, 2\mathrm{B}^{\mathrm{uni}}_{\delta,y,y',N})$*-bimodule*

$$\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta = \bigoplus_{-N/2\leq m} \mathcal{V}_{\alpha+\beta-1-2m} \otimes \mathcal{Z}_m$$

iii) Y *acts as a scalar* Y_m *on the* $2\mathrm{B}^{\mathrm{uni}}_{\delta,y,y',N}$*-module* $\mathcal{Z}_m$*. In other words,* $\mathcal{Z}_m$ *is a representation of the usual two-boundary TL algebra* $2\mathrm{B}_{\delta,y,y',Y_m,N}$.

Do the $\mathcal{Z}_m$ faithfully represent the two-boundary loop model ?

No :

- Y can only take a discrete set of values
- Even at fixed $Y = Y_m$, $\mathcal{Z}_m$ cannot possibly contain all the $2\mathrm{B}_{\delta,y,y',Y_m,N}$ irreps (its dimension is too small).

So what are the representations $\mathcal{Z}_m$?

Conjecture

i) For $m \geq N/2$, $\mathcal{Z}_m$ is the irreducible 2^N-dimensional vacuum module of $2\mathrm{B}_{\delta,y,y',Y_m,N}$.

ii) For $-N/2 \leq m < N/2$, $\mathcal{Z}_m$ is an irreducible d_m-dimensional subfactor of the reducible 2^N-dimensional vacuum module of $2\mathrm{B}_{\delta,y,y',Y_m,N}$

iii) $U_{\mathfrak{q}}\mathfrak{sl}_2$ and $2\mathrm{B}^{\mathrm{uni}}_{\delta,y,y',N}$ are mutual centralizers on $\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} \otimes \mathcal{V}_\beta$

Hard because no induction argument available and the representation theory of $2\mathrm{B}_{\delta,y,y',Y_m,N}$ is non-generic.

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4. Summary and Outlook

Can we compute the large N limit of these models ?

- For $|\mathfrak{q}| = 1$ these loop models are known to be critical and we can use CFT techniques.
- Their conformal partition functions have already been computed using the Coulomb gas approach (Dubail, Jacobsen, Saleur, 2008).

We can now rigorously derive some of these results by working directly on the spin chain and computing the scaling limits of the ground state and the low-lying excitations.

To obtain the spectrum :

- Free fermion mapping at $\mathfrak{q} = i$,
- Bethe ansatz for all the other values of $\mathfrak{q}$.

1. XXZ spin chains
   - The open $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant XXZ spin chain
   - New $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant boundary conditions
2. Loop Models and their lattice algebras
3. Conformal scaling limit
   - Free fermions at $\mathfrak{q} = i$
   - General case for $|\mathfrak{q}| = 1$
4. Summary and Outlook

- If $\mathfrak{q}$ is $2p$-th root of unity, we can (and must) truncate the infinite-dimensional Verma modules $\mathcal{V}_\alpha$ to irreducible p-dimensional representations.
- In the case $\mathfrak{q} = i$, these are of dimension $p = 2$ so our new one-boundary Hilbert space is just $(\mathbb{C}^2)^{\otimes(N+1)}$.
- $\delta := \mathfrak{q} + \mathfrak{q}^{-1} = 0$ so H_{XXZ} is just the Hamiltonian of the XY model, which can be mapped to free fermions.

The Jordan-Wigner transform gives

$$e_i = -c_j c_{j+1}^\dagger - c_{j+1} c_j^\dagger - i(c_j^\dagger c_j - c_{j+1}^\dagger c_{j+1})$$

for $1 \leq j \leq N-1$, and

$$b = c_0^\dagger c_0 + \cot\frac{\pi\alpha}{2}\left(e^{-\frac{i\pi\alpha}{2}} c_1 c_0^\dagger + \frac{1}{\cos\frac{\pi\alpha}{2}} c_0 c_1^\dagger + i(c_0^\dagger c_0 - c_1^\dagger c_1)\right) .$$

with fermionic anti-commutation relations

$$\{c_j^\dagger, c_{j'}^\dagger\} = 0\,, \qquad \{c_j, c_{j'}\} = 0\,, \qquad \{c_j^\dagger, c_{j'}\} = \delta_{j,j'}\,.$$

H_b is quadratic in c's and so let us introduce plane waves of the form

$$\theta^\dagger = \sum_{j=0}^{N} (a_+ x^j + a_- x^{-j}) c_j^\dagger$$

for some x and $a_\pm$ such that $[H_b, \theta^\dagger] = \lambda\theta^\dagger$ with $\lambda := x + x^{-1}$.

The boundary conditions then impose a quantization condition on λ

$$U_N(\lambda/2) + \mu U_{N-1}(\lambda/2) + (1 - \mu y) U_{N-2}(\lambda/2) = 0$$

where U_n is the n-th Chebyshev polynomial of degree n.

- This polynomial equation have exactly N solutions $\lambda_1, \ldots, \lambda_N$ from which we construct N fermionic modes $\theta_1^\dagger, \ldots, \theta_N^\dagger$.
- There is a zero-mode $\theta_0 :=$ E with coming from the $U_\mathfrak{q}\mathfrak{sl}_2$-symmetry.

The spectrum of H_b is given by

$$|S\rangle := \prod_{k\in S} \theta_k^\dagger |{\downarrow} \dots {\downarrow}\rangle \ , \qquad E_S = \sum_{k\in S} \lambda_k$$

for all $S \subseteq \{0, \dots, N\}$.

Recall the $(U_\mathfrak{q}\mathfrak{sl}_2, \mathrm{B}_{\delta,y,N})$-bimodule decomposition

$$\mathcal{V}_\alpha \otimes (\mathbb{C}^2)^{\otimes N} = \bigoplus_{j=-N/2}^{N/2} \mathcal{V}_{\alpha+2j} \otimes \mathcal{W}_j \qquad \text{with} \qquad \dim \mathcal{W}_j = \binom{N}{j+N/2} .$$

The spectrum of H_b in the representation $\mathcal{W}_j$ is given by all E_S such that $|S| = j + N/2$ and $0 \notin S$.

This solves the spectral problem for H_b.

To obtain the conformal spectrum in the continuum, one has to compute the $1/N$ correction to the ground state and the first excited states.

Denote E_j the ground state in the $\mathcal{W}_j$ representation of $\mathrm{B}_{\delta,y,N}$.

For all $j \in \mathbb{Z}$

$$E_j = Ne_{\mathrm{b}} + E_{\mathrm{s}} + \frac{\pi v_{\mathrm{F}}}{N}\left(-\frac{c}{24} + h_{\alpha,\alpha+2j}\right) + o(1/N^2)\,,$$

where

- $e_{\mathrm{b}} = -\frac{2}{\pi}$ is the bulk energy per site,
- E_{s} is the surface energy,
- $v_{\mathrm{F}} = 2$ is the Fermi velocity,
- $c = -2$ is the central charge,
- $h_{r,s} = \frac{(2r-s)^2-1}{8}$ are conformal weights.

Taking into account the excitations above E_j in each $\mathcal{W}_j$ sector

$$\lim_{N\to\infty} \mathrm{tr}_{\mathcal{W}_j}\, q^{\frac{N}{\pi v_F}(H_b - Ne_b - E_s)} = \frac{q^{-\frac{c}{24}+h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)}.$$

This is exactly the Virasoro character for a representation of conformal weight $h_{\alpha,\alpha+2j}$.

The partition function of our boundary loop model on a cylinder of parameter $\tau = M/N$

$$Z_\tau(\delta=0,y) = \lim_{N\to\infty} \mathrm{qtr}\, T_b^M = \sum_{j\in\mathbb{Z}} (-1)^j \frac{q^{-\frac{c}{24}+h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)}$$

where $q = e^{-\tau}$, $y = \cot\frac{\pi\alpha}{2}$.

- It does not explicitly depend on the boundary coupling μ.
- It describes spanning forests rooted at one of the boundaries of the cylinder.

- The result for general $\mathfrak{q} = e^{\frac{i\pi}{p}}$, $p \in]1, +\infty[$, is very similar but is much harder to derive.
- We use Bethe ansatz and a distribution-based method developed by Granet, Jacobsen and Saleur to compute the $1/N$ corrections.

$$E_j = N e_{\mathrm{b}} + E_{\mathrm{s}} + \frac{\pi v_{\mathrm{F}}}{N} \left(-\frac{c}{24} + h_{\alpha,\alpha+2j} \right) + o(1/N^2)\,,$$

where

- e_{b} is the bulk energy per site,
- E_{s} is the surface energy,
- $v_{\mathrm{F}} = p \sin \frac{\pi}{p}$ is the Fermi velocity,
- $c = 1 - \frac{6}{p(p-1)}$ is the central charge,
- $h_{r,s} = \frac{(pr-(p-1)s)^2-1}{4p(p-1)}$ are conformal weights.

We recover the previous result for $p = 2$.

Taking into account "descendants" we have again

$$\lim_{N\to\infty} \mathrm{tr}_{\mathcal{W}_j}\, q^{\frac{N}{\pi v_F}(H_b - Ne_b - E_s)} = \frac{q^{-\frac{c}{24} + h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)} .$$

The partition function on a cylinder of parameter $\tau = M/N$

$$Z_\tau(\delta, y) = \lim_{N\to\infty} \mathrm{qtr}\, T_b^M = \sum_{j\in\mathbb{Z}} \frac{\sin\frac{\pi(\alpha+1)}{p}}{\sin\frac{\pi\alpha}{p}} \frac{q^{-\frac{c}{24} + h_{\alpha,\alpha+2j}}}{\prod_{n=1}^{+\infty}(1-q^n)}$$

where $q = e^{-\tau}$, $\delta = 2\cos\frac{\pi}{p}$ and $y = \dfrac{\sin\frac{\pi(\alpha+1)}{p}}{\sin\frac{\pi\alpha}{p}}$.

Again, it does not explicitly depend on the coupling constant μ.

Summary

- Starting from the $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant open XXZ spin chain we have constructed new $U_{\mathfrak{q}}\mathfrak{sl}_2$-invariant boundary conditions using infinite-dimensional Verma modules $\mathcal{V}_\alpha$.
- We have used the new boundary couplings b and b' to construct representations of the blob and two-boundary Temperley-Lieb algebras, extending the known Temperley-Lieb case.
- We showed that the blob algebra representation is faithful and computed its Schur-Weyl decomposition with respect to $U_{\mathfrak{q}}\mathfrak{sl}_2$.
- We computed the conformal scaling limit of the corresponding boundary loop model in the critical regime $|\mathfrak{q}| = 1$ using free fermions at $\mathfrak{q} = i$ and Bethe ansatz for general $\mathfrak{q}$.

Outlook

- What about the spectrum and scaling limit of the two-boundary case ?
- Can we build a faithful spin chain for the two-boundary loop model ?
- The $\mathfrak{q} = i$ case can be identified in the continuum with some symplectic fermion QFT with special boundary conditions. Can we find a similar QFT formulation for any $|\mathfrak{q}| = 1$?
- What about the periodic case ?
- Can we learn something about fusion of generic Virasoro representations ?
- Can we learn something about integrable non-compact spin chains ?