# **Phenomenology of Fluid Turbulence**

*... and the early introduction of Gaussian Multiplicative Chaos*

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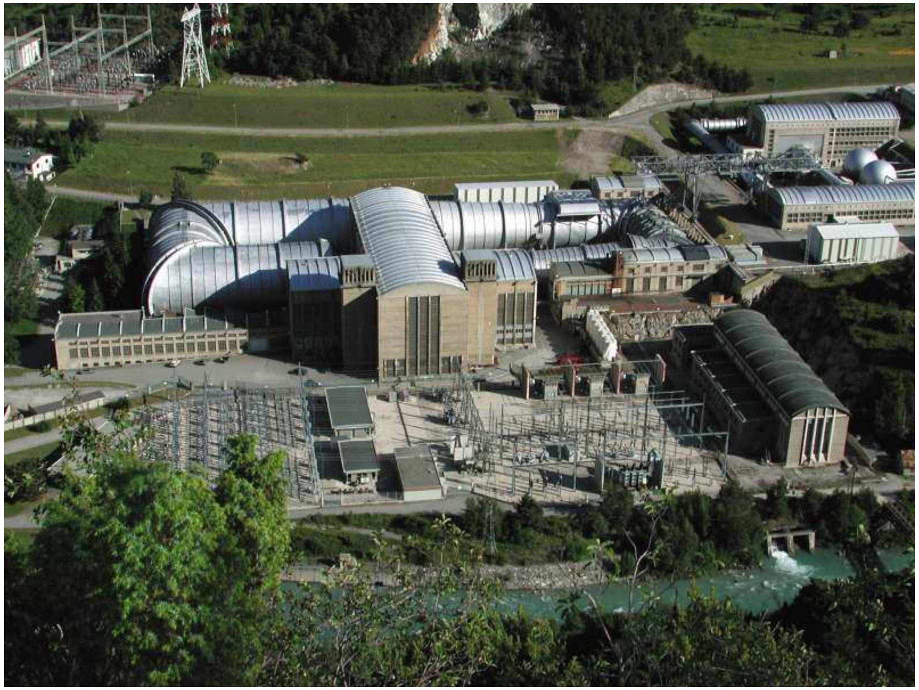
# *Plan of the Talk*

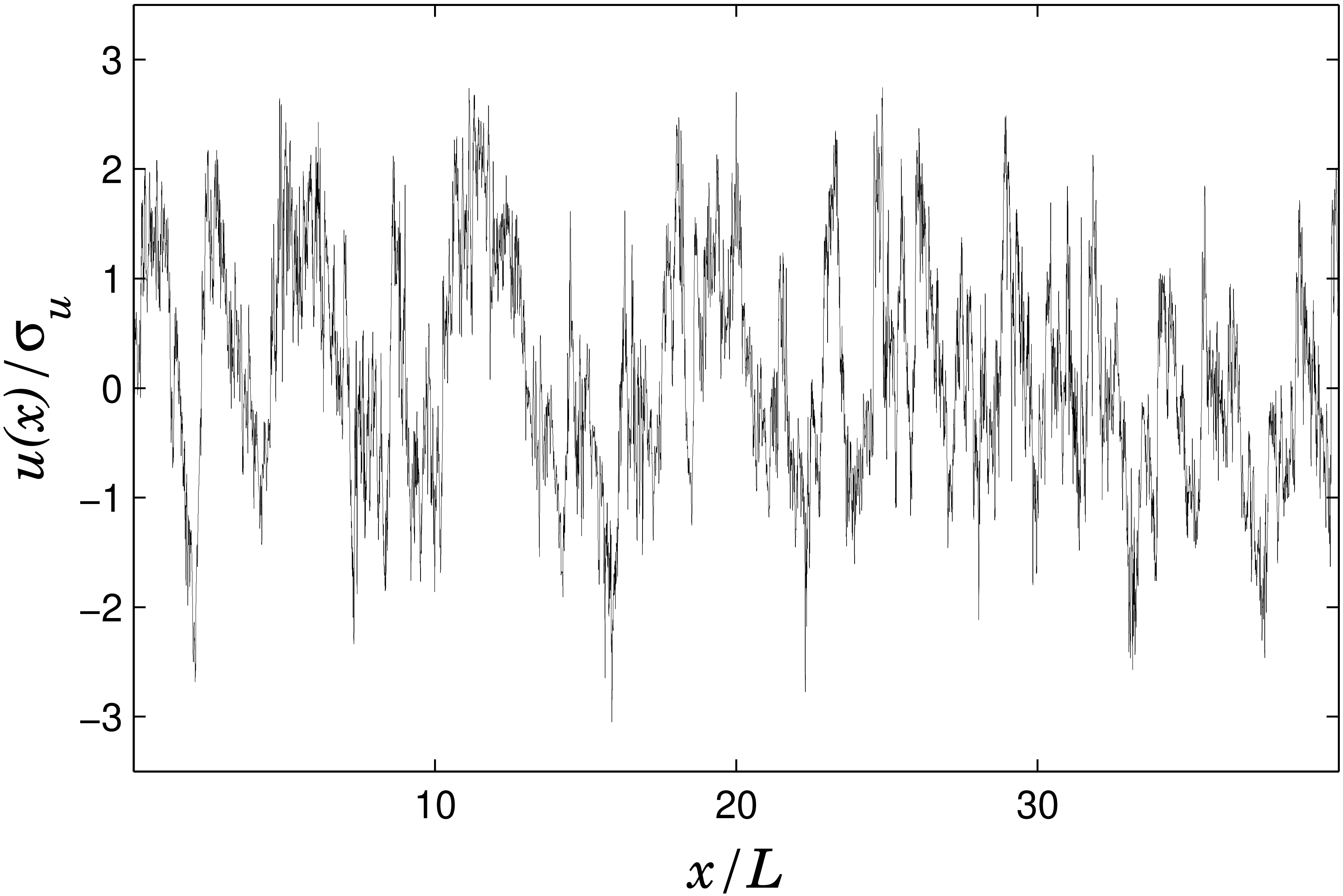
- Starting from a *Definition* (Statistical analysis of observations)
  - Some ingredients of the phenomenology of (mainly due to) Kolmogorov
  - First naive representation as Fractional Gaussian Fields
  - High order statistics (non Gaussianity/Intermittency/Multifractal)
- Early Introduction of GMC to model *intermittency* (i.e. multifractality)
- Modern formulation of things
- Building a realistic **stochastic** picture of three-dimensional fluid turbulence
  - which includes a matrix form of the GMC
- Defining these random fields as an invariant measure of some (simple) stochastic PDE

# *Wind tunnel at Modane*



See Gagne et al., Bourgoin et al.

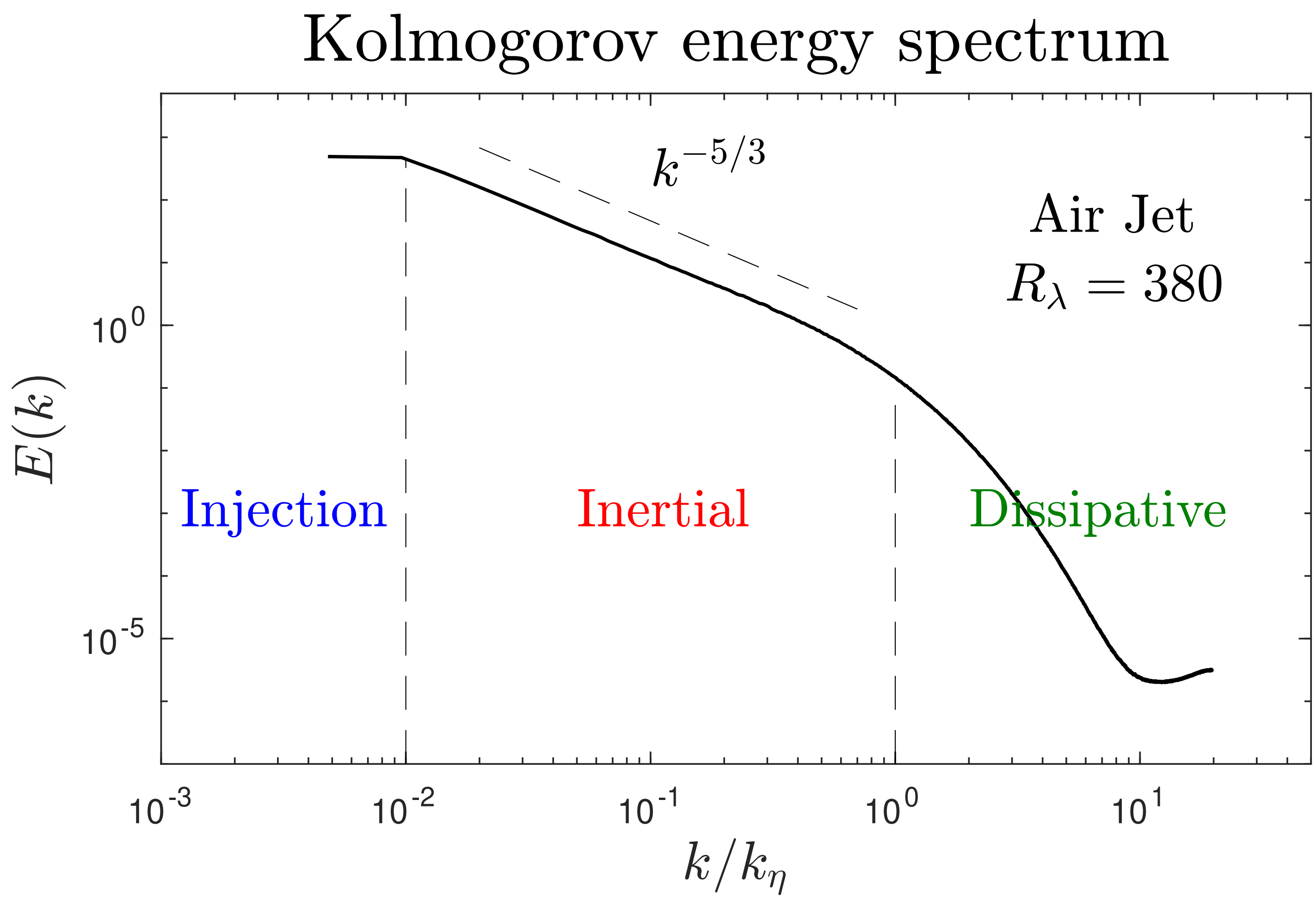
# *Wind tunnel at Modane*



# *Two-point statistical structure of turbulence*

Define the energy spectrum (Fourier transform of the correlation) as

$$E(k) = \int e^{-2i\pi k\ell} \langle u(x)u(x+\ell) \rangle \, d\ell$$

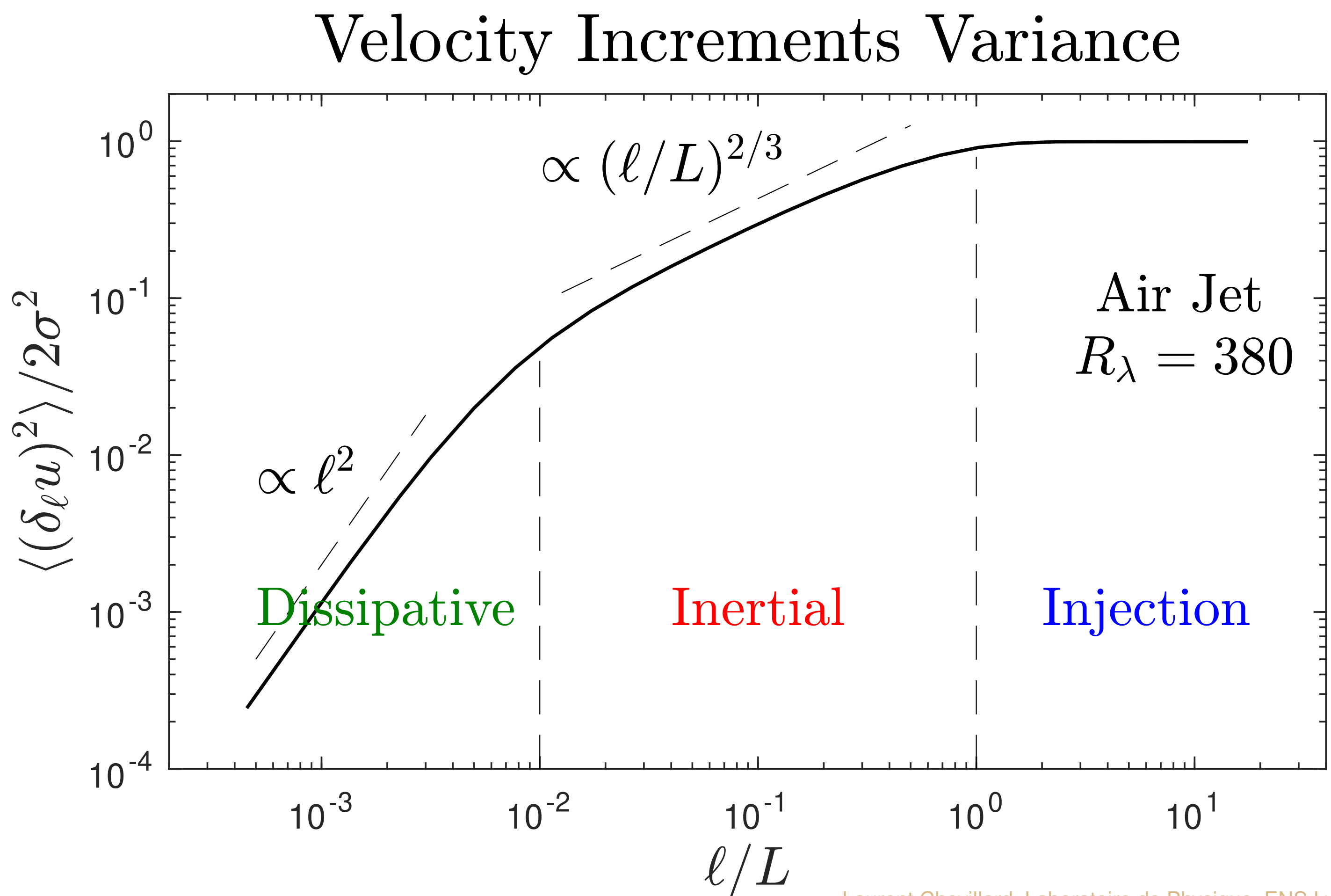


## *Two-point statistical structure of turbulence*

In an equivalent way, define the velocity increment as

$$\delta_\ell u(x) = u(x+\ell) - u(x),$$

and remark that $\langle(\delta_\ell u)^2\rangle = 2\sigma^2 - 2\langle u(x)u(x+\ell)\rangle$.

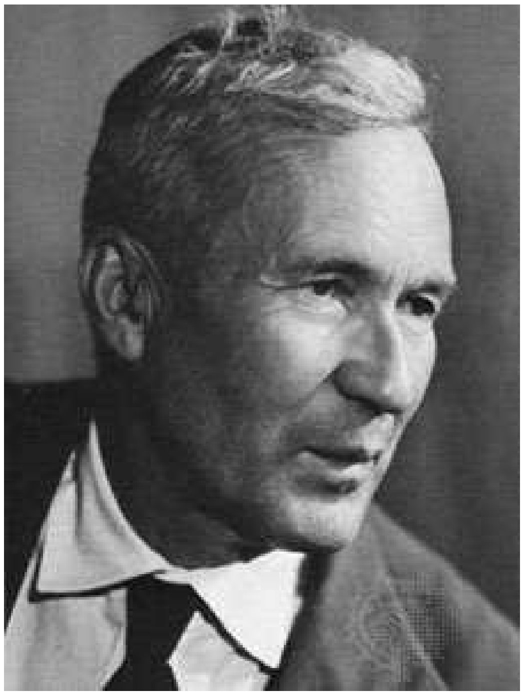


# *The Navier-Stokes equations*

In three-dimensional space, consider the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$, where $\boldsymbol{u} = (u_1, u_2, u_3)$, $\boldsymbol{x} \in \mathbb{R}^3$ and say $t > 0$. Given a (large-scale, divergence-free forcing) $\boldsymbol{f}$, it is solution of

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \Delta \boldsymbol{u} + \boldsymbol{f} \text{ and } \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0,$$

where $p$ is the pressure field, and $\nu$ the kinematic viscosity.



Kolmogorov 1903-1987

"I became interested in turbulent liquid and gas flows at the end of the thirties. From the very beginning it was clear that the theory of random functions of many variables (random fields), whose development only started at that time, must be the underlying mathematical technique. Moreover, I soon understood that there was little hope of developing a pure, closed theory, and because of the absence of such a theory the investigation must be based on hypotheses obtained by processing experimental data."

# Fractional Gaussian Fields

Can we give a probabilistic representation of these observed velocity fluctuations?

$$u_H(x) = \int_{y \in \mathbb{R}} P_H(x-y) W(dy)$$

- $W(dy)$ a (distributional) Gaussian white measure
- $P_H$ the fractional operator (Fourier multiplier) of Hurst $H \in ]0,1[$

$$P_H(x) = \int e^{2i\pi kx} |k|_{1/L}^{-H-1/2} dk.$$

It is a perfect representation of these former observations using the particular value $H = 1/3$:

- It is a zero-average and finite-variance **statistically** homogeneous field.
- Moreover,

$$E\left[(\delta_\ell u_H)^2\right] \underset{\ell \to 0^+}{\sim} c_2 \ell^{2H},$$

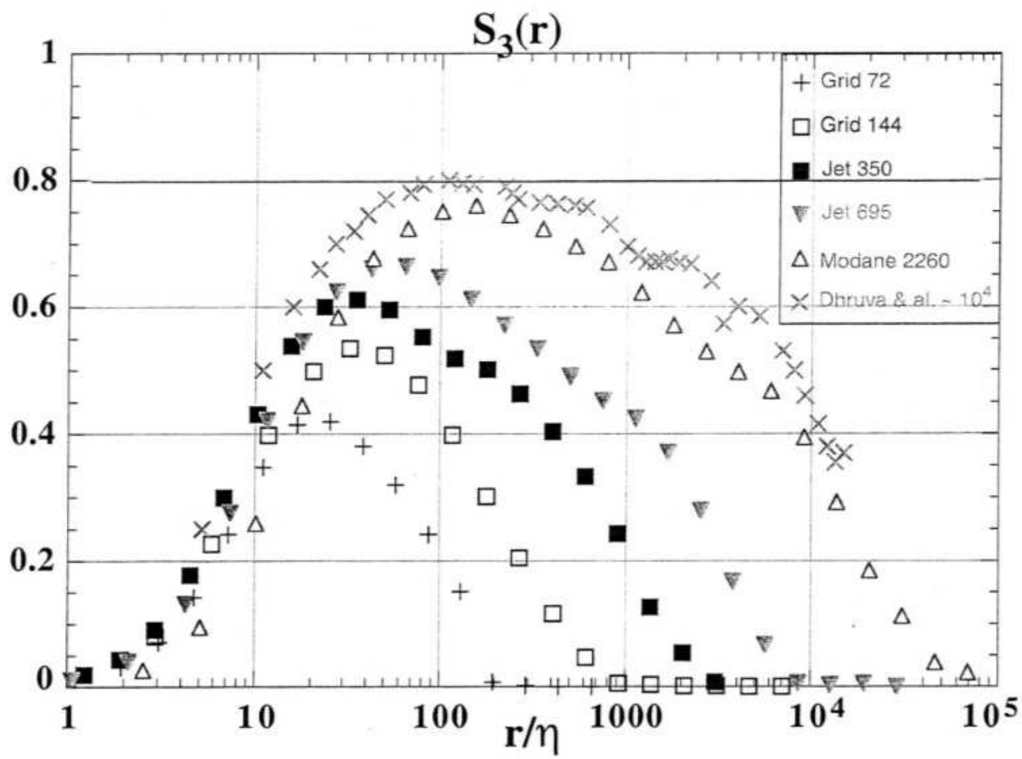
  with $c_2 > 0$, up to variance, independent of the cut-off at large scale $L$.
- easy generalization to a dissipative range (with proper phenomenology for $\varepsilon$) and to three-dimensional divergence-free fields.

# *Third-order statistics*

- Assuming former phenomenology, it can be shown from Navier-Stokes that

$$\lim_{\nu\to 0}\mathbb{E}\left[\left(\delta_\ell u(x)\cdot\frac{\ell}{|\ell|}\right)^3\right]\underset{|\ell|\to 0}{\sim}-\frac{4}{5}\varepsilon\ell.$$

where $\varepsilon=\lim_{\nu\to 0}\mathbb{E}\left[\nu|\nabla u|^2\right]$, which is finite and $>0$, is the mean energy dissipation (per unit of mass)

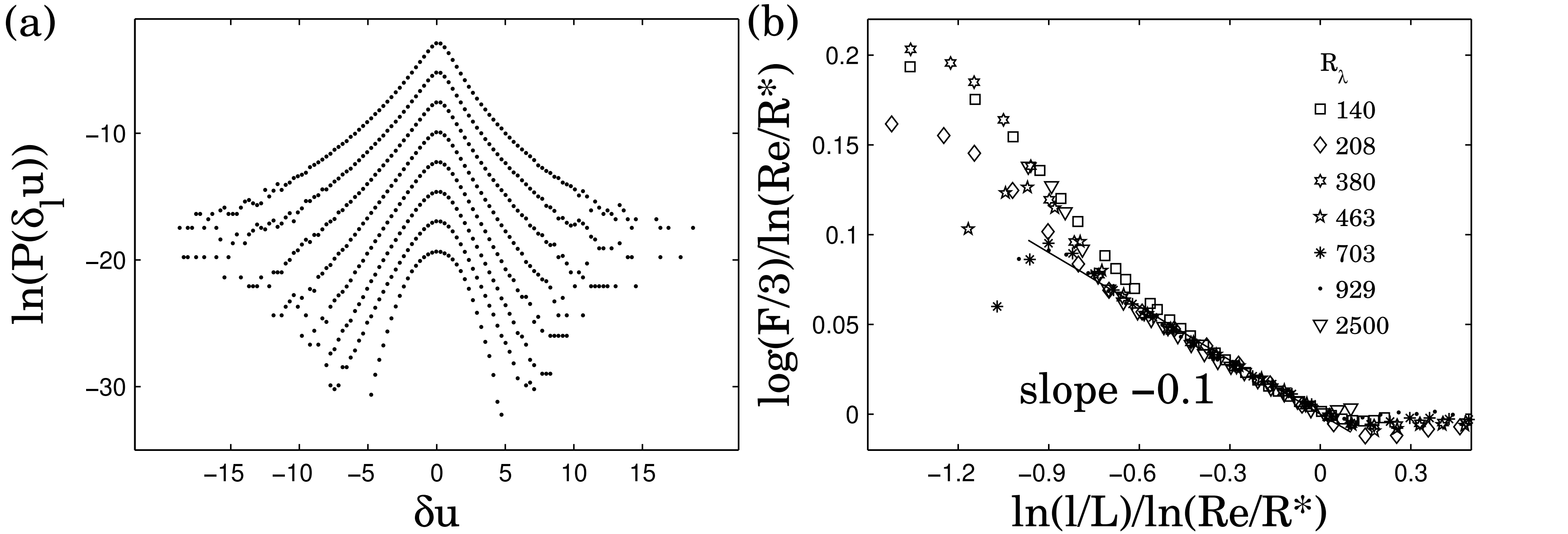



with $S_3$ the corresponding moment divided by $-\varepsilon\ell$, see Gagne et al. PoF (2004).

# *Intermittency in Eulerian fluctuations*

Eulerian longitudinal velocity increments: $\delta_\ell u(x) = u(x+\ell) - u(x)$

Flatness $F = \frac{\langle (\delta_\ell u)^4 \rangle}{\langle (\delta_\ell u)^2 \rangle^2}$



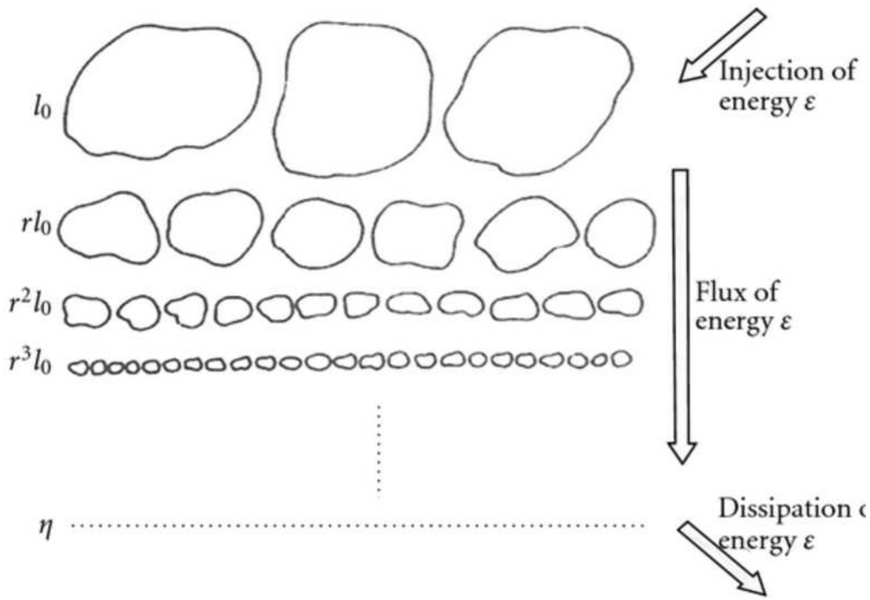

Actually, we observe that (Refined Similarity Hypothesis)

$$\langle (\delta_\ell u)^q \rangle \underset{\ell \to 0}{\propto} \langle \varepsilon_\ell^{q/3} \rangle \ell^{q/3} \underset{\ell \to 0}{\propto} \ell^{q/3 - \frac{\mu}{2}\frac{q}{3}\left(\frac{q}{3}-1\right)}$$ with $\mu \approx 0.22$

# Discrete Multiplicative Cascade Models

Consider the local dissipation field $\varepsilon(x,t) = \nu|\nabla u(x,t)|^2$
Then, following Yaglom (1962) and more recent modern formulations

(from Frisch (95))




$$\begin{cases} c_{0,0}=1, \\ c_{j,2k}=W^{(l)}_{j-1,k}c_{j-1,k}, \\ c_{j,2k+1}=W^{(r)}_{j-1,k}c_{j-1,k}, \end{cases}$$

(from Arneodo et al., JMP (98))

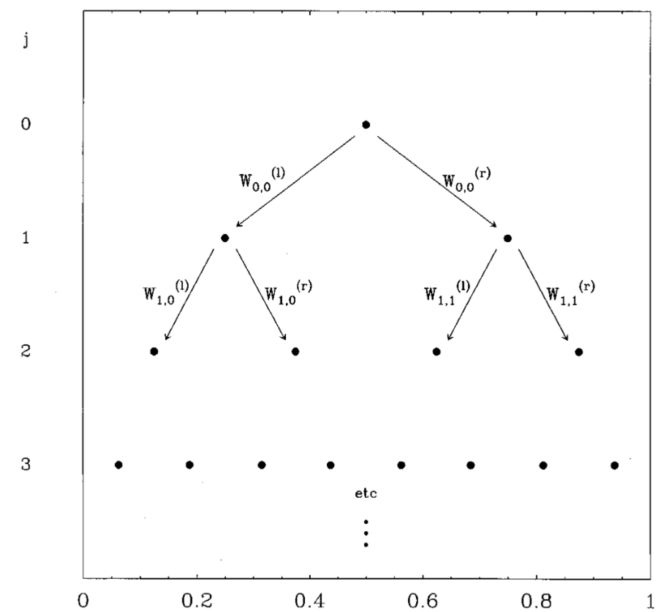
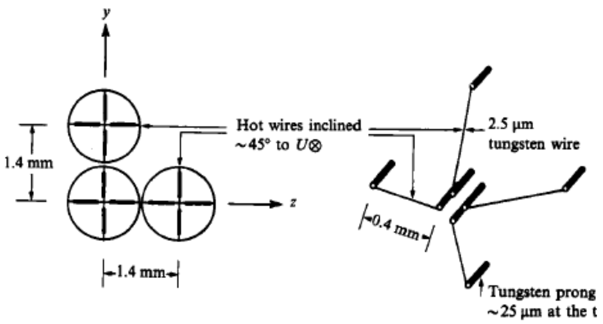



FIG. 1. Sketch of the construction rule of a $\mathscr{W}$-cascade. The wavelet coefficients $\{c_{j,k}\}_{j,k}$ lie on a dyadic grid. At each scale $a_j=2^{-j}$, the grid displays $2^j$ coefficients with abscissa $x_{j,k}=2^{-j}k$. The value of the wavelet coefficient $c_{j,2k}$ (resp. $c_{j,2k+1}$) is obtained from the value of the wavelet coefficient $c_{j-1,k}$ by multiplying it by $W^{(l)}_{j-1,k}$ (resp. $W^{(r)}_{j-1,k}$) as defined in Eq. (6).

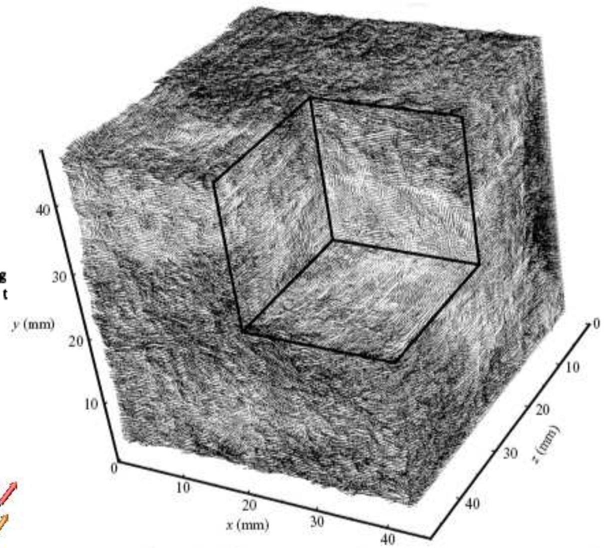
Then, $\langle \ln \varepsilon(x) \ln \varepsilon(x+\ell) \rangle \underset{\ell \to 0}{\propto} \ln \frac{1}{\ell}$

From there, Mandelbrot (72) introduced the limit lognormal model, which has been called GMC by Kahane (85).

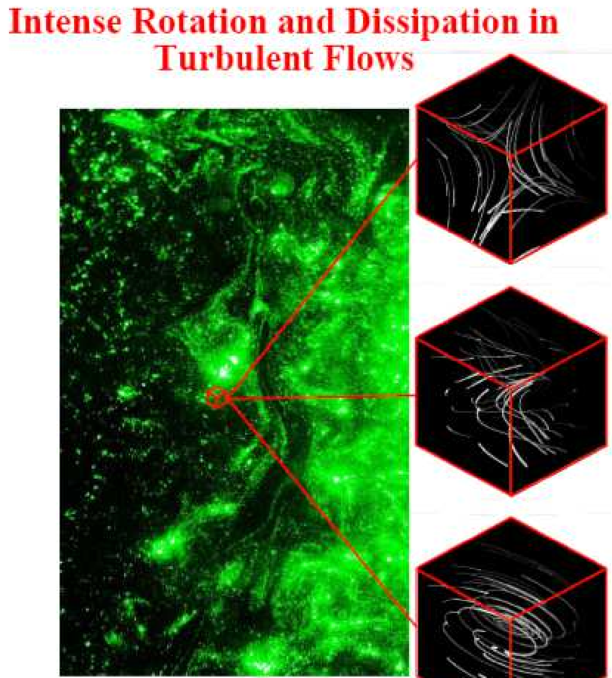
# *3D* *Fluid* ***Turbulence****:* *Full* *velocity gradients*



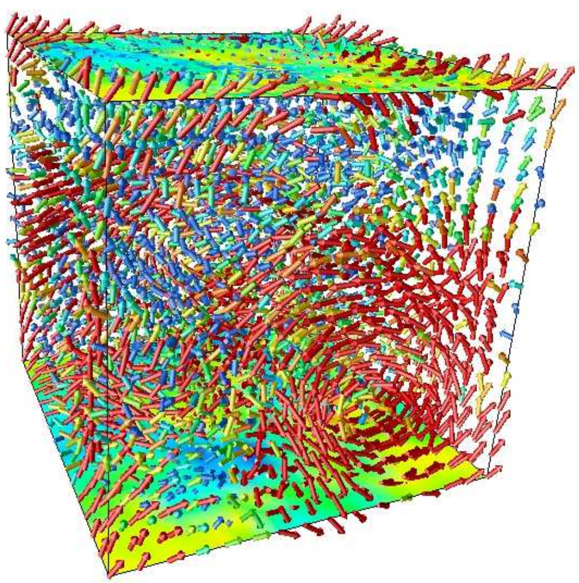

**Tsinober, Kit, Dracos J.F.M. 1992**




**Tao, Katz, Meneveau J.F.M. 2002**




**Zeff, et al., Nature 2003**



Direct Numerical Simulations
(picture by Toschi)

$$\mathbf{A} = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix}$$

See also Lüthi et al., Xu-Bodenshatz et al., etc.

# *Vorticity and Deformation matrix: Definitions*

Consider the (smooth) velocity field $\mathbf{v}(\mathbf{x}_0, t_0)$ at the point $(\mathbf{x}_0, t_0)$.

$$\mathbf{v}(\mathbf{x}_0 + \mathbf{h}, t_0) = \mathbf{v}(\mathbf{x}_0, t_0) + \underbrace{(\boldsymbol{\nabla}\mathbf{v})}_{\mathbf{A}}(\mathbf{x}_0, t_0)\mathbf{h} + \mathcal{O}(\mathbf{h}^2) \ , \forall \mathbf{h} \in \mathbb{R}^3$$

The 3×3 matrix $\boldsymbol{\nabla}\mathbf{v} = (\partial_j v_i)$ has a **symmetric** part $\mathcal{D}$ and an **antisymmetric** part $\boldsymbol{\Omega}$

$$\mathbf{A} = \boldsymbol{\nabla}\mathbf{v} = \mathcal{D} + \boldsymbol{\Omega}$$

- $\mathcal{D}$ is called the **deformation** or rate-of-**strain** matrix
  It has three (real) **eigenvalues** $\lambda_1 \geq \lambda_2 \geq \lambda_3$
  By the incompressibility condition tr($\mathbf{A}$) $=$ div($\mathbf{v}$) $= 0$

$$\rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$$

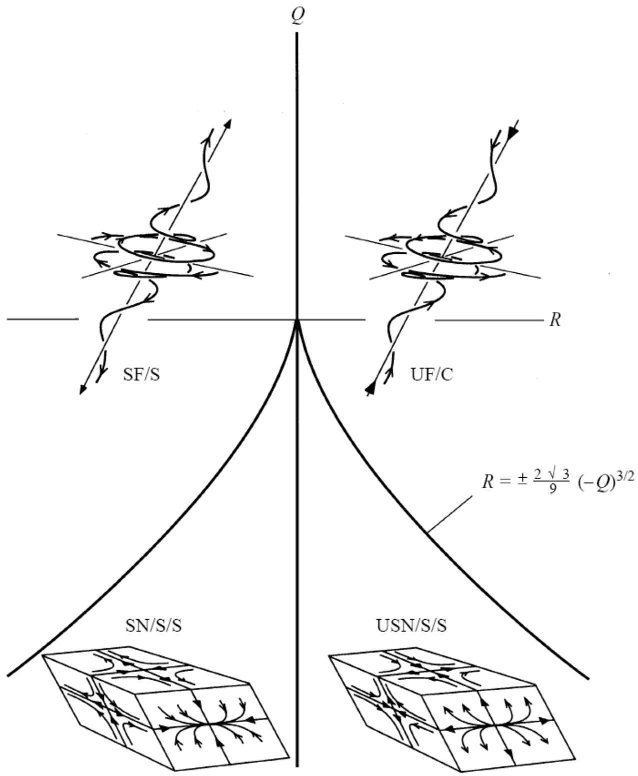
$$\rightarrow \lambda_1 \geq 0 \text{ and } \lambda_3 \leq 0$$

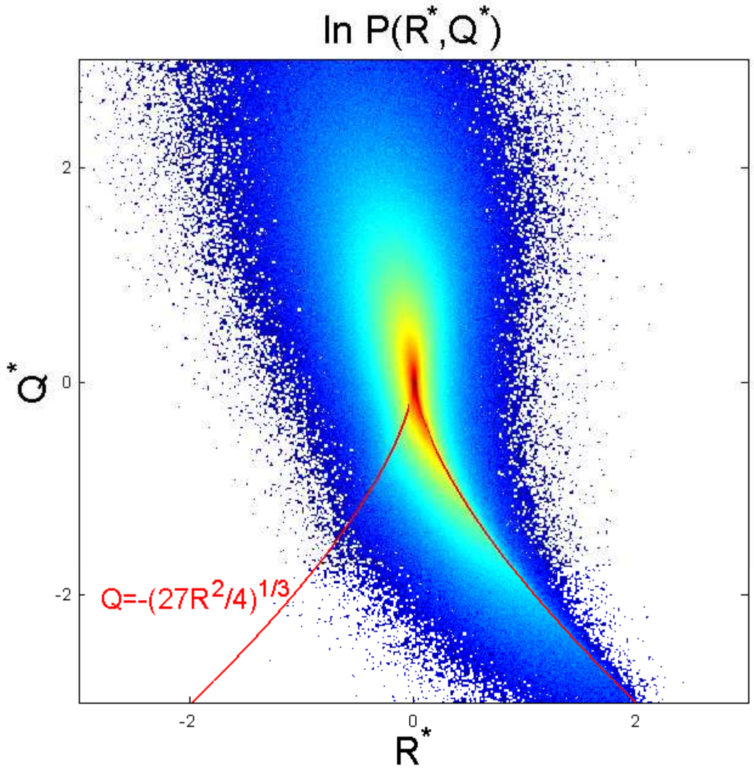
- $\boldsymbol{\Omega}$ is called the rate-of-**rotation** matrix
  The **vorticity** $\boldsymbol{\omega} =$ curl $\mathbf{v} = \boldsymbol{\nabla} \wedge \mathbf{v}$ satifies

$$\boldsymbol{\Omega}\mathbf{h} = \frac{1}{2}\boldsymbol{\omega} \wedge \mathbf{h} \ , \forall \mathbf{h} \in \mathbb{R}^3$$

# *The RQ plane - Local Topology*

See Chong, Perry and Cantwell (90) and Cantwell (93)

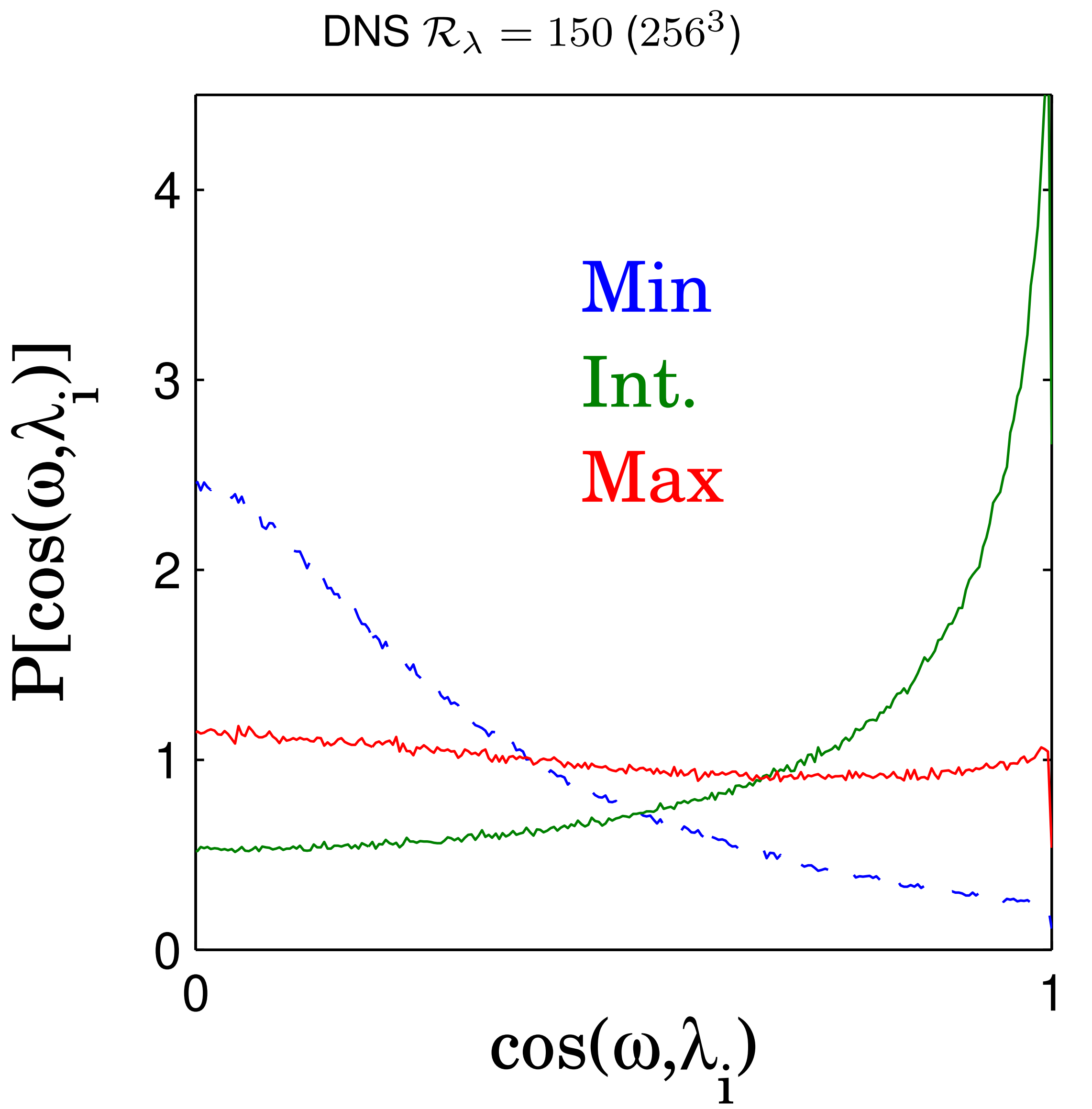



$\lambda_i = f(R,Q) \in \mathbb{C}$: Eigenvalues of **A**

- Second invariant: $Q = -\frac{1}{2}\text{Tr}(\mathbf{A}^2) = \frac{1}{4}\overbrace{|\omega|^2}^{\text{Enstrophy}} - \frac{1}{2}\overbrace{\text{Tr}(\mathbf{S}^2)}^{\text{Dissipation}}$
- Third invariant: $R = -\frac{1}{3}\text{Tr}(\mathbf{A}^3) = -\frac{1}{4}\underbrace{\omega_i S_{ij} \omega_j}_{\text{Enstrophy Production}} - \frac{1}{3}\underbrace{\text{Tr}(\mathbf{S}^3)}_{\text{Strain Skewness}}$

# *Vorticity Alignments*




**Preferential Alignment** with the intermediate eigendirection

(Ashurst et al. 87, Tsinober et al. 92)

# *The vorticity stretching mechanism*

From the Navier-Stokes equations for divergence-free vector fields

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}.\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + \nu \Delta \mathbf{v}$$

one gets,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\mathcal{S}}\boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega}$$

that relates the stretching of vorticity by the local Deformation matrix.

# *The vorticity stretching mechanism*

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one gets,

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\mathcal{S}}_{[\boldsymbol{\omega}]}\boldsymbol{\omega} + \nu \Delta \boldsymbol{\omega}$$

with,

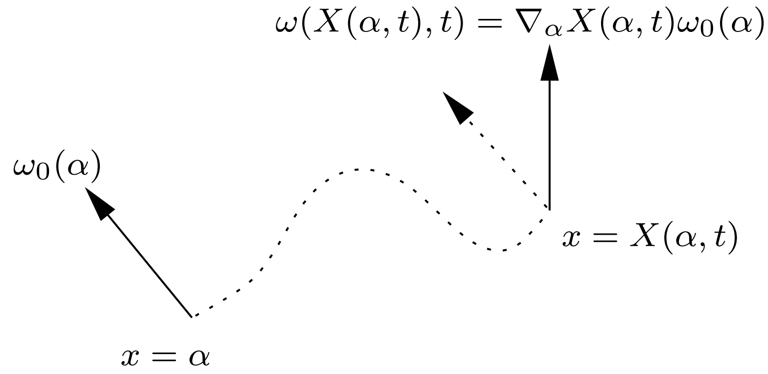
$$\boldsymbol{\mathcal{S}}_{[\boldsymbol{\omega}]}(\mathbf{x}) = \frac{3}{8\pi}\text{P.V.}\int \left[ \frac{(\mathbf{x}-\mathbf{y}) \otimes [(\mathbf{x}-\mathbf{y}) \wedge \boldsymbol{\omega}(\mathbf{y})]}{|\mathbf{x}-\mathbf{y}|^5} + \left( \quad \bullet \quad \right)^{\top} \right] d\mathbf{y}$$

See Constantin (1994) and Majda-Bertozzi (2002).

# *Euler = Advection and Stretching of Vorticity*

Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p$$




$$\mathbf{u}(\mathbf{x},t) = \underbrace{-\frac{1}{4\pi}\int \frac{\mathbf{x} - \overbrace{\mathbf{X}(\boldsymbol{\alpha},t)}^{\text{advection}}}{|\mathbf{x}-\mathbf{X}(\boldsymbol{\alpha},t)|^3}\wedge}_{\text{Biot-Savart}} \overbrace{\underbrace{\boldsymbol{\nabla}_{\boldsymbol{\alpha}}\mathbf{X}}_{\text{stretching}} \quad \boldsymbol{\omega}_0(\boldsymbol{\alpha})}^{\text{vorticity at } t} d\boldsymbol{\alpha}$$

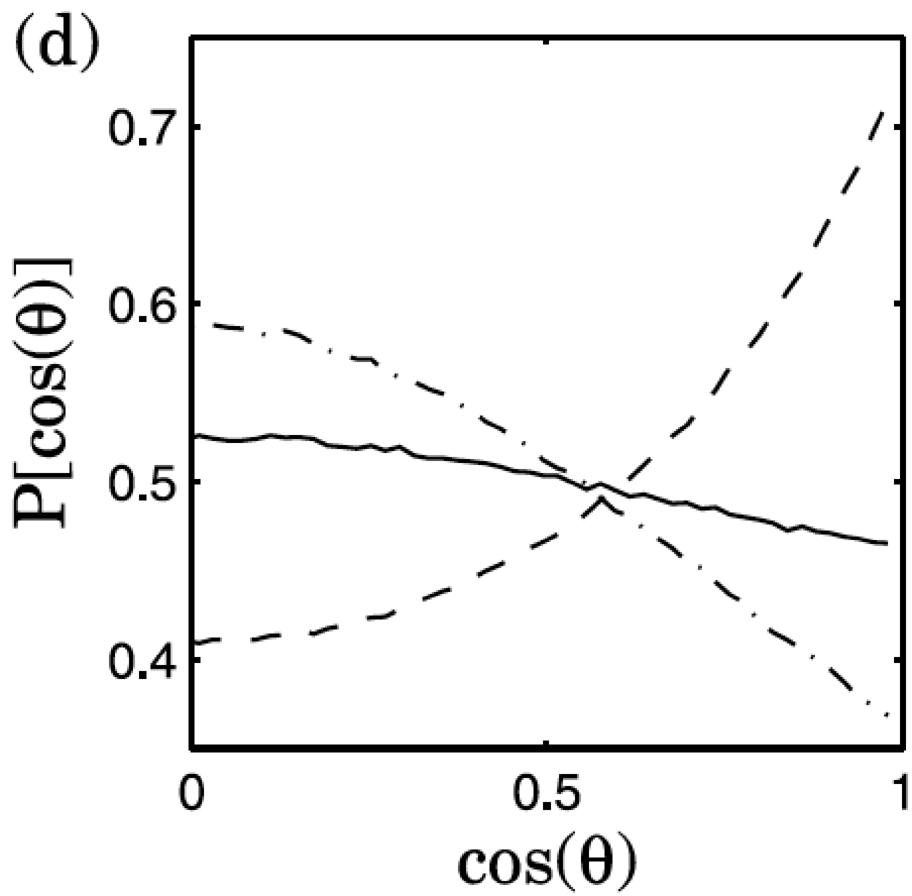
# *Recent Fluid Deformation Approximation*

During a small time scale $\tau(\mathbf{x}) = \frac{1}{\sqrt{\mathrm{tr}(\mathbf{S}_0^2)}}$

- Stretching by the local deformation: $\boldsymbol{\omega}(\tau) \approx e^{\tau \mathbf{S}_0} \boldsymbol{\omega}_0$
- Neglect of short-time Advection, i.e. $\mathbf{X}(\boldsymbol{\alpha}, t) \approx \boldsymbol{\alpha}$

$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\mathbf{x} - \boldsymbol{\alpha}}{|\mathbf{x} - \boldsymbol{\alpha}|^{\beta}} \wedge e^{\tau \mathbf{S}_0(\boldsymbol{\alpha})} d\mathbf{W}(\boldsymbol{\alpha})$$

- d$\mathbf{W}(\boldsymbol{\alpha})$: White noise
- $\beta = \frac{13}{6}$ to ensure Kolmogorov scalings




- Skewed longitudinal gradients (but not velocity increments!)
- Weakly non Gaussian (no intermittency)
- **Wrong** vorticity alignments (along $\mathbf{e}_1$)

with R. Robert and V. Vargas, EPJ (2010), also R. Pereira, C. Garban (2016)

# *Gaussian multiplicative chaos*

Mandelbrot (72), Kahane (85), Robert-Vargas(09).

$$\mathbf{x} \in \mathbb{R}^d \,,\, m_\epsilon(\mathbf{x},\ell) = \int_{B(\mathbf{x},\ell)} e^{X_\epsilon(\mathbf{y}) - \frac{1}{2}E(X_\epsilon^2)} d\mathbf{y}$$

- $X_\epsilon$ Gaussian scalar field $E(X_\epsilon) = 0$, $E(X_\epsilon(\mathbf{x})X_\epsilon(\mathbf{y})) \underset{\epsilon \ll |x-y| \ll L}{\sim} \lambda^2 \ln^+ \frac{L}{|x-y|}$
  - $\lambda$: intermittency coefficient
  - typically, $X_\epsilon(\mathbf{y}) = \lambda \int_{|\mathbf{y}-\mathbf{z}| \leq L} \frac{1}{|\mathbf{y}-\mathbf{z}|_\epsilon^{d/2}} dW(\mathbf{z})$
- $\lambda^2 < 2d$,

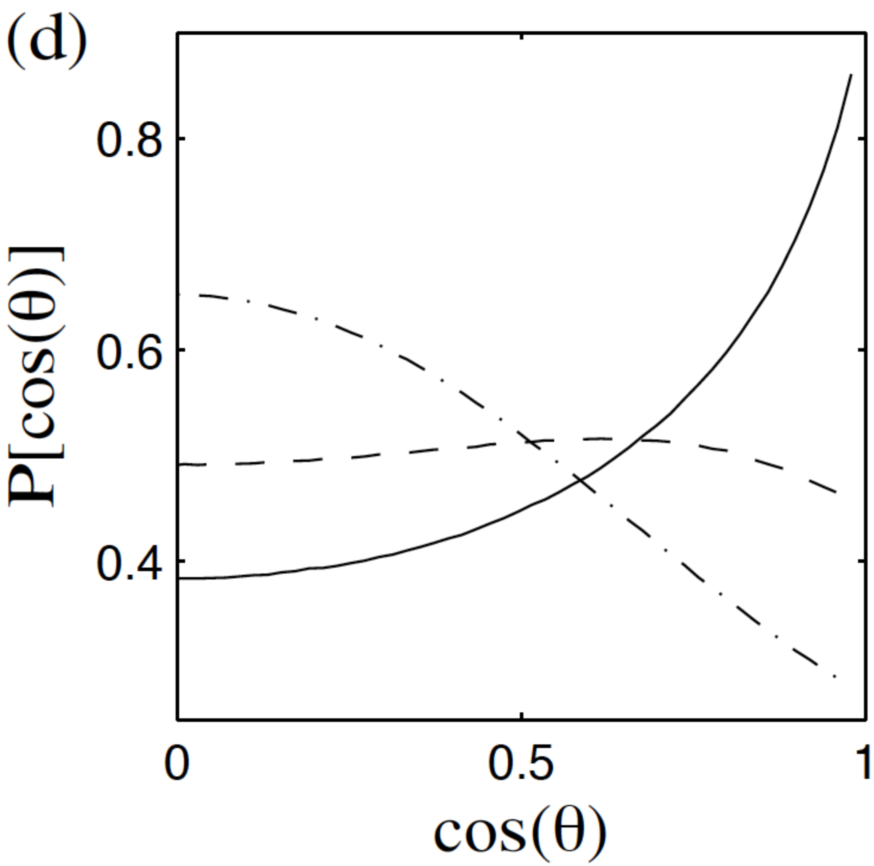
$$m(\mathbf{x},\ell) = \lim_{\epsilon \to 0} m_\epsilon(\mathbf{x},\ell) \text{ non trivial}$$

$$q < q^* \,, E(m^q) \underset{\ell \to 0}{\sim} C_q \left(\frac{\ell}{L}\right)^{\zeta_q} \text{ with } \zeta_q = (d + \frac{\lambda^2}{2})q - \lambda^2 \frac{q^2}{2}$$

# *A stochastic representation of the local structure of turbulence*

Multifractal (long-range correlated) “strain":

$$\widetilde{\mathbf{S}} = \sqrt{\frac{5}{4\pi}}\lambda \int \frac{(\mathbf{x}-\boldsymbol{\sigma}) \otimes [(\mathbf{x}-\boldsymbol{\sigma}) \wedge d\mathbf{W}(\boldsymbol{\sigma})] + [(\mathbf{x}-\boldsymbol{\sigma}) \wedge d\mathbf{W}(\boldsymbol{\sigma})] \otimes (\mathbf{x}-\boldsymbol{\sigma})}{|\mathbf{x}-\boldsymbol{\sigma}|^{7/2}}$$




$$\mathbf{u}(\mathbf{x}) = -\frac{1}{4\pi}\int \frac{\mathbf{x}-\boldsymbol{\alpha}}{|\mathbf{x}-\boldsymbol{\alpha}|^{\beta}} \wedge e^{\widetilde{\mathbf{S}}(\boldsymbol{\alpha})} d\mathbf{W}(\boldsymbol{\alpha})$$

- $\lambda^2 = 0.025$: intermittency parameter
  $\langle|\delta_\ell u|^q\rangle \sim \ell^{\zeta_q}$ with
  $\zeta_q = (\frac{1}{3} + \frac{3}{2}\lambda^2)q - \lambda^2 \frac{q^2}{2}$
- Karman-Howarth-Kolmogorov:
  $\langle(\delta_\ell u)^3\rangle \sim \ell$
- Correct vorticity alignments!!

$\rightarrow$ Realistic incompressible, intermittent (i.e. multifractal) and skewed (i.e. dissipative) turbulent velocity field.

# *Gaussian Multiplicative Matrix Chaos*

Consider

$$M^\epsilon(A) = \frac{1}{c_\epsilon}\int_A e^{X^\epsilon(x)}dx, \quad A \subset \mathbb{R}^d,$$

where

$$E[X^\epsilon_{i,i}(x)^2] = \gamma^2 \ln\frac{L}{\epsilon}, \quad E[X^\epsilon_{i,i}(x)X^\epsilon_{j,j}(x)] = -c\gamma^2\ln\frac{L}{\epsilon},\ i\neq j$$

and for $|y-x|>\epsilon$:

$$\begin{aligned} E[X^\epsilon_{i,i}(x)X^\epsilon_{i,i}(y)] =& K(x-y), \\ E[X^\epsilon_{i,i}(x)X^\epsilon_{j,j}(y)] =& -cK(x-y),\ i\neq j \end{aligned}$$

for some constant $L>0$ and some kernel of positive type $K(x)=\gamma^2\ln_+\frac{L}{|x|}$
Off diagonal terms independent of diagonal terms and mutually independent of variance $\bar{\sigma}^2_\epsilon = \frac{\sigma^2_\epsilon(1+c)}{2}$ and covariance for $|y-x|>\epsilon$:

$$E[X^\epsilon_{i,j}(x)X^\epsilon_{i,j}(y)] = \frac{1+c}{2}K(x-y),\ i<j.$$

# *Gaussian Multiplicative Matrix Chaos*

**theorem**: with Rhodes and Vargas (13)

Let $\gamma^2 < d$. Then there exists a random matrix $M$ such that for all $A \subset \mathbb{R}^d$:

$$E[\text{tr}(M^\epsilon(A) - M(A))^2] \underset{\epsilon \to 0}{\to} 0.$$

We also have the following asymptotic structure:

$$E[\text{tr}M(B(0,\ell))^2] \underset{\ell \to 0}{\sim} N^2 V_N \frac{\Gamma(N/2) e^{\gamma^2 \ln L}}{(1+c)^{(N-1)/2}\Gamma(1/2)} \frac{\ell^{2d-\gamma^2}}{(\gamma^2 \ln \frac{1}{\ell})^{(N-1)/2}} \tag{1}$$

with $V_N = \int_{|v|,|u| \leq 1} \frac{du dv}{|v-u|^{\gamma^2}}$. Furthermore, we get the following equivalent for $k \geq 2$:

$$\frac{\ln E[\text{tr}M(B(0,\ell))^k]}{\ln \ell} \underset{\ell \to 0}{\to} \zeta(k) \tag{2}$$

where $\zeta(k) = dk - \gamma^2 \frac{k(k-1)}{2}$.

# ***Dynamical Fractional Gaussian Fields (1)***

with G. Apolinário and J.-C. Mourrat (J. Stat. Phys. 2022)

Can we define these fractional fields as a statistically stationary solution of a PDE randomly stirred by a smooth forcing term? (thus, realizing a cascading of energy from a large scale towards smaller ones)

$\longrightarrow$ a proposition (i.e. a model): Give a meaning to a **transport** in **Fourier**

$$\partial_t \widehat{u}(t,k) + c\partial_k \widehat{u}(t,k) = \widehat{f}(t,k)$$

corresponding to $\partial_t u(t,x) = \mathcal{L}u(t,x) + f(t,x)$

with $\mathcal{L} = 2i\pi cx$

- $\mathcal{L}$ is an homogeneous operator of degree 0 that lies in the class pinpointed by Colin de Verdière and Saint-Raymond
- Take for instance $\mathbb{E}\left[f(t_1,x_1)f^*(t_2,x_2)\right] = \delta(t_1-t_2)\mathcal{C}_f(x_1-x_2)$ with $\mathcal{C}_f$ a smooth, even, bounded, rapidly decreasing function at large arguments.

# ***Dynamical Fractional Gaussian Fields (2)***

→Generation of a **white** noise

$$\partial_t \widehat{u}(t,k) + c\partial_k \widehat{u}(t,k) = \widehat{f}(t,k)$$

corresponding to $\partial_t u(t,x) = \mathcal{L}u(t,x) + f(t,x)$

with $\mathcal{L} = 2i\pi cx$

- The solution $u(t,x)$ is statistically homogeneous

$$\mathcal{C}_u(t,x_1,x_2) \equiv \mathbb{E}\left[u(t,x_1)u^*(t,x_2)\right] \equiv \mathcal{C}_u(t,x_1-x_2).$$

- Moreover, when tested against an <u>even</u> smooth function, we have:

$$\lim_{t\to\infty} \mathcal{C}_u(t,x) = \frac{\mathcal{C}_f(0)}{2|c|}\delta(x).$$

# ***Dynamical** Fractional Gaussian Fields (2bis)*

Consider the initial condition $u(0,x)=0$, take c>0, so the solution is

$$u(t,x)=\int_0^t e^{2i\pi cx(t-s)}f(s,x)ds,$$

such that

$$\mathcal{C}_u(t,x)=\mathcal{C}_f(x)\frac{e^{2i\pi cxt}-1}{2i\pi cx}.$$

Integration over a test function $g$ gives

$$\begin{aligned}\int g(x)\mathcal{C}_u(t,x)dx &= \int g(x)\mathcal{C}_f(x)\frac{e^{2i\pi cxt}-1}{2i\pi cx}dx \\ &= \frac{1}{c}\int \widehat{g\mathcal{C}_f}(k)\frac{\text{sign}(k)-\text{sign}(k-ct)}{2}dk \text{ by Parseval} \\ &= \frac{1}{c}\int_{k=0}^{ct}\widehat{g\mathcal{C}_f}(k)dk \underset{t\to\infty}{=} \frac{1}{c}\int_{k=0}^{\infty}\widehat{g\mathcal{C}_f}(k)dk.\end{aligned}$$

Taking $g$ and $\mathcal{C}_f$ real and even concludes the proof.

# ***Dynamical** Fractional Gaussian Fields (2bisbis)*

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$$u(t,x)=\int_0^t e^{2i\pi cx(t-s)}f(s,x)ds,$$

such that

$$\mathcal{C}_u(t,x)=\mathcal{C}_f(x)\frac{e^{2i\pi cxt}-1}{2i\pi cx}.$$

More general arguments for whatever test function $g$ developed by G. Beck, I. Gallagher and R. Grande give another (complex) bounded contribution, which is in particular $\neq 0$ when $g$ is odd, and is of the form

$$\lim_{t\to\infty}\mathcal{C}_u(t,x)=\frac{\mathcal{C}_f(0)}{2|c|}\delta(x)+i\text{ vp }\frac{\mathcal{C}_f(x)}{x}.$$

# ***Dynamical Fractional Gaussian Fields (3)***

→Generation of a **fractional** field

$$\partial_t u_H(t,x) = (P_H \mathcal{L} P_H^{-1}) u_H(t,x) + f(t,x)$$

with $\mathcal{L} = 2i\pi c x$ and $P_H(x) = \int e^{2i\pi k x} |k|_{1/L}^{-H-1/2} dk$.

- Again, the solution $u_H(t,x)$ is statistically homogeneous, but of finite variance in the limit $t \to \infty$. More generally, the spectral density is given by

$$\lim_{t\to\infty} \widehat{\mathcal{C}_{u_H}}(t,k) = \frac{1}{c} |k|_{1/L}^{-2H-1} \int_{-\infty}^{k} |s|_{1/L}^{2H+1} \widehat{\mathcal{C}_f}(s) ds.$$

- In particular, we have:

$$\lim_{t\to\infty} \mathbb{E}\left[|\delta_\ell u_H|^2\right] \underset{\ell \to 0^+}{\sim} c_H \ell^{2H}.$$

- and we can do funky simulations.

# *On the underlying cascading processes (2)*

The former budget exhibits *anomalous* (i.e. distributional) quantities. It is thus tempting to study the kinetic energy budget of a *coarse-grained* version of velocity, such as

$$u_\ell(x,t) = \int G_\ell(x-y)u(y,t)d^3y,$$

where $G_\ell$ is an appropriate mollifier of typical spatial extension $\ell$ (an approximation of the Dirac $\delta$ function at scale $\ell$).

This being said, the respective kinetic is determined by the **correlation** structure (once averaged) of velocity:

$$|u_\ell(x,t)|^2 = \iint G_\ell(x-y)G_\ell(x-z)\underbrace{u(y,t)\cdot u(z,t)}_{\text{two-points}}d^3yd^3z,$$

It is thus tempting to consider the budget of $u(x,t)\cdot u(x+\ell,t)$ instead of $|u(x,t)|^2$ for a given *scale* $\ell$.

# *On the underlying cascading processes (2)*

Start from the Navier-Stokes equations, and consider the following *velocity two-points* **budget**

$$\frac{1}{2}\frac{\partial u(x)\cdot u(x+\ell)}{\partial t}+\nabla_x\cdot\mathcal{J}=-\nu\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)_x\left(\frac{\partial u_i}{\partial x_j}\right)_{x+\ell}+\frac{1}{2}\left[u(x)\cdot f(x+\ell)+u(x+\ell)\cdot f(x)\right]$$

$$+\frac{1}{4}\nabla_\ell\cdot\left[\delta_\ell u(x)|\delta_\ell u(x)|^2\right]$$

where the current (vector) $\mathcal{J}(x,\ell)$ looks very much like $\mathcal{I}$ (ask me later), and where enters the velocity increment (vector)

$$\delta_\ell u(x)=u(x+\ell)-u(x).$$

# *On the underlying cascading processes (2)*

Start from the Navier-Stokes equations, and consider the following *velocity two-points* **budget**

$$\frac{1}{2}\frac{\partial u(x)\cdot u(x+\ell)}{\partial t}+\nabla_x\cdot\mathcal{J}=-\nu\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)_x\left(\frac{\partial u_i}{\partial x_j}\right)_{x+\ell}+\frac{1}{2}\left[u(x)\cdot f(x+\ell)+u(x+\ell)\cdot f(x)\right]$$

$$+\frac{1}{4}\nabla_\ell\cdot\left[\delta_\ell u(x)|\delta_\ell u(x)|^2\right]$$

- Similarly, for **statistically** homogeneity and stationary reasons, the left-hand side vanishes.
- **BUT**, the viscous contribution vanishes at a given (fixed) scale $|\ell|$ and $\nu\to 0$.
- For **statistically** homogeneity and stationary reasons, the contribution of the forcing term reduces to $\varepsilon>0$ in the double limit $\nu\to 0$ **AND** $\ell\to 0$.
- We are thus left with the following *energy transfer through scales*

$$\lim_{|\ell|\to 0}\lim_{\nu\to 0}\nabla_\ell\cdot\mathbb{E}\left[\delta_\ell u(x)|\delta_\ell u(x)|^2\right]=-4\varepsilon.$$

# A *Multifractal Ansatz*

Consider the random field $v_{H,\gamma}(t,x)$, defined as

$$v_{H,\gamma}(t,x) = \int P_H(x-y)e^{\gamma(\tilde{P}_0 u_0)(t,y)}u_0(t,y)dy,$$

where $u_0(t,x)$ goes towards complex white noise as $t \to \infty$.

- In the limit $t \to \infty$, this is multifractal process
- compared to the construction of a fractional gaussian field, notice the introduction of a random multiplier $e^{\gamma(\tilde{P}_0 u_0)(t,y)}$, called a complex multiplicative chaos, that makes $v_{H,\gamma}(t,x)$ non gaussian, and multifractal as $t \to \infty$.
- notice furthermore the intrinsic correlated structure of this field.
- in particular, this field is skewed as required by the phenomenology of turbulence

# *Induced nonlinear dynamics*

Start with

$$\partial_t u_0 = \mathcal{L} u_0,$$

so that

$$\begin{aligned}\partial_t \left(e^{\gamma \tilde{P}_0 u_0} u_0\right) &= e^{\gamma \tilde{P}_0 u_0}\partial_t u_0 + \gamma \tilde{P}_0 \partial_t u_0 \left(e^{\gamma \tilde{P}_0 u_0} u_0\right) \\ &= \mathcal{L}\left(e^{\gamma \tilde{P}_0 u_0} u_0\right) + \gamma \tilde{P}_0 \mathcal{L} u_0 \left(e^{\gamma \tilde{P}_0 u_0} u_0\right).\end{aligned}$$

From a formal point of view, note $\mathcal{W}$ a functional of some complex function $h : \mathbb{R} \to \mathbb{C}$, implicitly defined as $\mathcal{W}[h](x) e^{\gamma \tilde{P}_0 \mathcal{W}[h](x)} = h(x)$, if it exists, such that,

$$\begin{aligned}\partial_t v_{H,\gamma} \equiv P_H \partial_t \left(e^{\gamma \tilde{P}_0 u_0} u_0\right) &= P_H \mathcal{L} P_H^{-1} v_{H,\gamma} \\ &+ \gamma P_H \left[\left(\tilde{P}_0 \mathcal{L} \mathcal{W}\left[P_H^{-1} v_{H,\gamma}\right]\right)\left(P_H^{-1} v_{H,\gamma}\right)\right].\end{aligned}$$

Adopt a **closure** approach: $\mathcal{W}[h](x) \approx h(x)$. Doing so, we get a closed evolution that reads

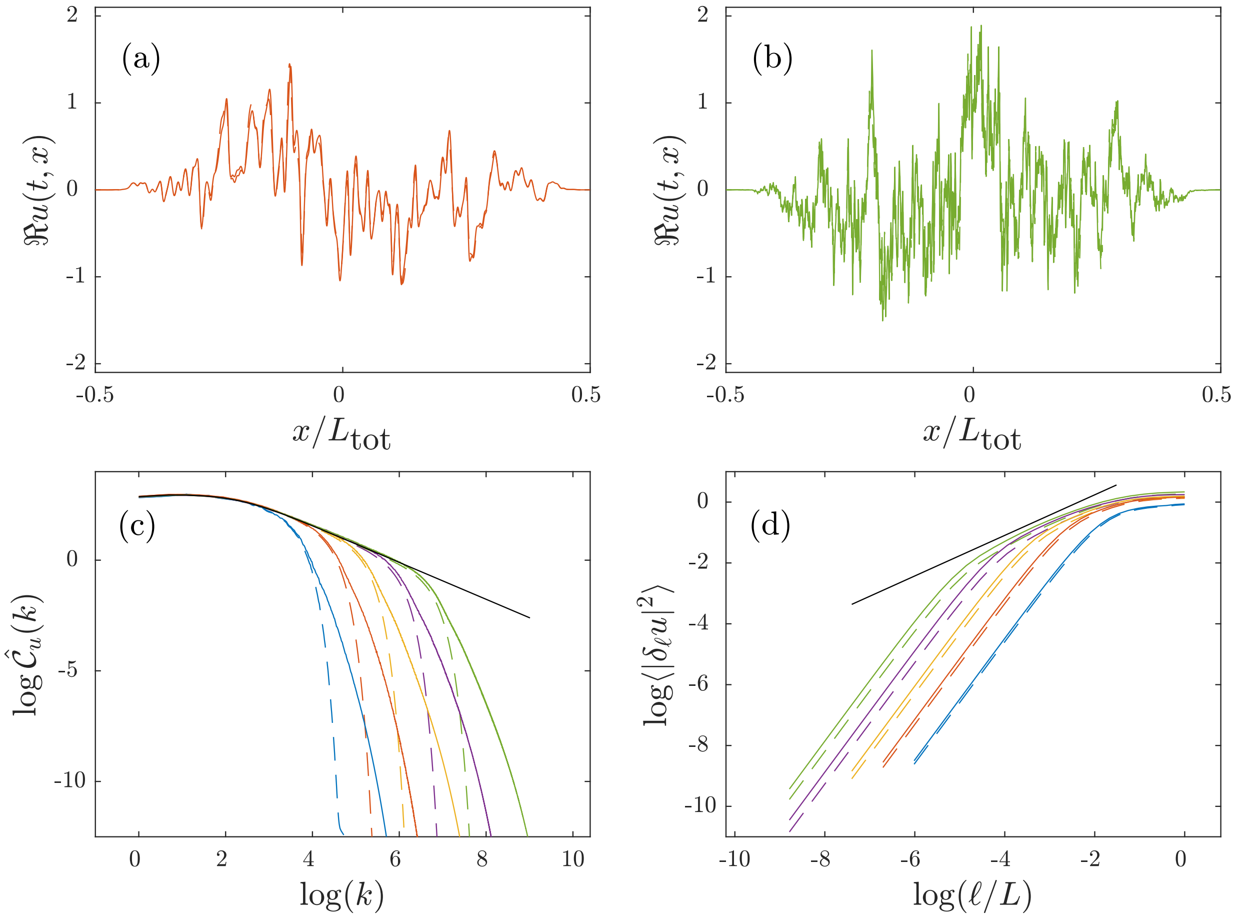
$$\partial_t v_{H,\gamma} \approx P_H \mathcal{L} P_H^{-1} v_{H,\gamma} + \gamma P_H \left[\left(\tilde{P}_0 \mathcal{L} P_H^{-1} v_{H,\gamma}\right)\left(P_H^{-1} v_{H,\gamma}\right)\right].$$

# *Simulations of the nonlinear dynamics*

Consider the numerical problem

$$du_{H,\gamma,\nu} = \left[ P_H \mathcal{L} P_H^{-1} u_{H,\gamma,\nu} + \gamma P_H \left[ \left( \tilde{P}_0 \mathcal{L} P_H^{-1} u_{H,\gamma,\nu} \right) \left( P_H^{-1} u_{H,\gamma,\nu} \right) \right] + \nu \partial_x^2 u_{H,\gamma,\nu} \right] \Delta t$$

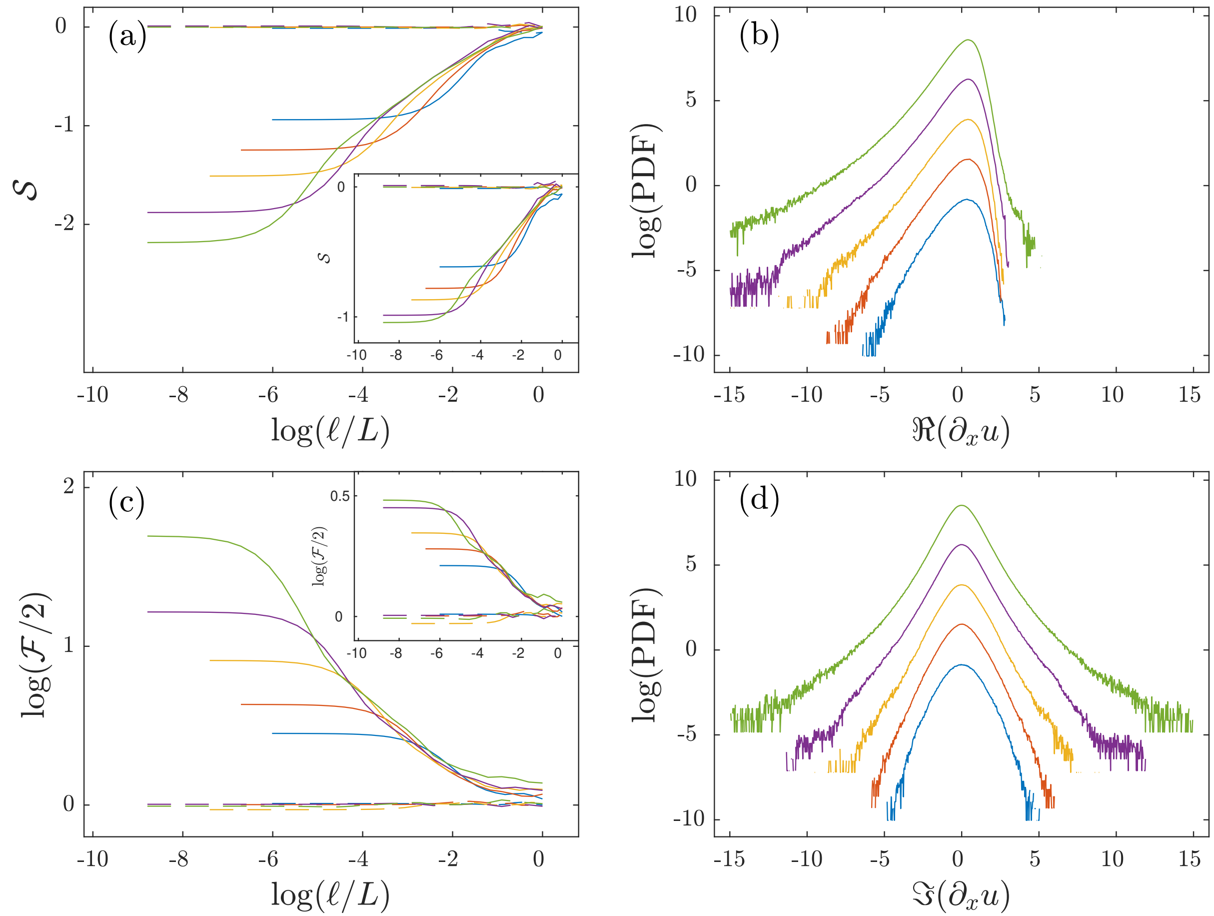
$$+ f_{\text{trunc}} \sqrt{\Delta t},$$



# *Simulations* of the *nonlinear* dynamics

Consider the numerical problem

$$du_{H,\gamma,\nu} = \left[P_H \mathcal{L} P_H^{-1} u_{H,\gamma,\nu} + \gamma P_H \left[\left(\tilde{P}_0 \mathcal{L} P_H^{-1} u_{H,\gamma,\nu}\right)\left(P_H^{-1} u_{H,\gamma,\nu}\right)\right] + \nu \partial_x^2 u_{H,\gamma,\nu}\right] \Delta t$$
$$+ f_{\text{trunc}} \sqrt{\Delta t},$$



# *On the underlying cascading processes (1)*

Start from the Navier-Stokes equations, and consider the following *kinetic energy* **budget**

$$\frac{1}{2}\frac{\partial|u|^2}{\partial t}+\nabla\cdot\mathcal{I}=-\nu\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2+u\cdot f,$$

where the current (vector) $\mathcal{I}$ is given by

$$\mathcal{I}=\frac{1}{2}|u|^2u+pu-\nu\nabla\left(\frac{1}{2}|u|^2\right).$$

# *On the underlying cascading processes (1)*

Start from the Navier-Stokes equations, and consider the following *kinetic energy* **budget**

$$\frac{1}{2}\frac{\partial |u|^2}{\partial t} + \nabla \cdot \mathcal{I} = -\nu \sum_{i,j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} \right)^2 + u \cdot f.$$

- As observed, the flow reaches a **statistically** stationary state of finite variance, and is **statistically** homogeneous

$$\mathbb{E}\left[ \frac{\partial |u|^2}{\partial t} \right] = 0 \text{ and } \mathbb{E}\left[ \nabla \cdot \mathcal{I} \right] = 0$$

- we are thus left with

$$\varepsilon = \mathbb{E}\left[ \nu \sum_{i,j=1}^{3} \left( \frac{\partial u_i}{\partial x_j} \right)^2 \right] = \mathbb{E}\left[ u \cdot f \right].$$

- Moreover (dissipative anomaly), it turns out that (i.e. as observed)

$$\lim_{\nu \to 0} \varepsilon > 0.$$