conformal bootstrap
4-point connectivities in 2d percolation

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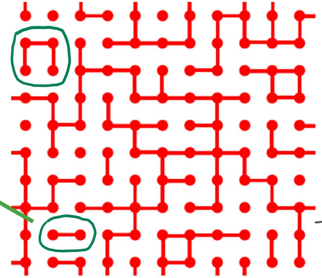
Based on:

- e-Print: 2005.07258, *JHEP 12 (2020) 019*, w/ Jacobsen, Saleur
- e-Print: 2002.09071, *JHEP 05 (2020) 156*, w/ Grans-Samuelsson, Jacobsen, Saleur

Agay les roches rouges, septembre 2022

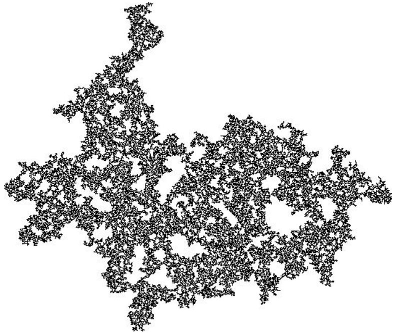
Critical percolation and cluster connectivities

clusters



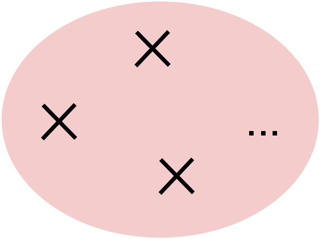
bond
- *close:* p
- *open:* $1-p$

p_c



cluster connectivities

geometrical phase transition

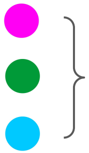


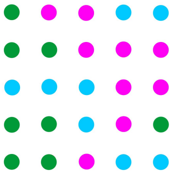
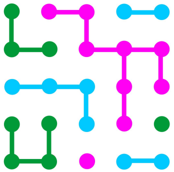
CFT description?

non-unitary conformal field theory

non-locality

Q-state Potts model

 Q

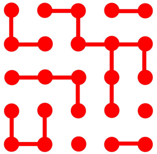
 $Z_{\text{spin}} \longrightarrow Z_{\text{cluster}}$ 

cluster weight $Q \in \mathbb{R}$

[Fortuin, Kasteleyn, 1972]

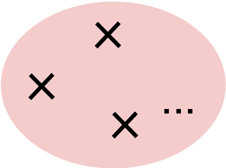
critical cluster model $0 < Q < 4$ → Potts CFT $-2 < c < 1$

$Q \to 1 \; (c \to 0)$ percolation



Potts spin: order parameter

cluster connectivities



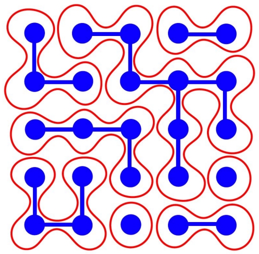
↔

CFT correlator of spin operator $\Phi_{1/2,0}$

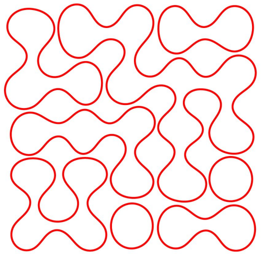
$$\langle \Phi_{1/2,0} \Phi_{1/2,0} \cdots \Phi_{1/2,0} \rangle$$

[Delfino, Viti, 2011]

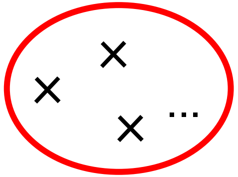
Loop representation



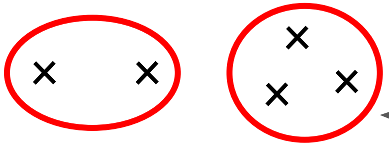
$$Z_{\text{cluster}} \to Z_{\text{loop}}$$



loop weight $\sqrt{Q}$



$$\langle \Phi_{1/2,0} \Phi_{1/2,0} \dots \Phi_{1/2,0} \rangle$$



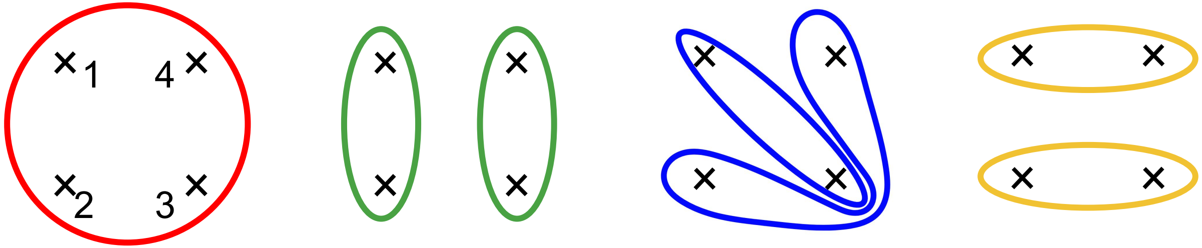
connectivities are understood

[Delfino, Viti, 2010]
[Picco, Santachiara, Viti, Delfino, 2013]
[Ikhlef, Jacobsen, Saleur, 2015]

DOZZ formula

Four-point connectivities

non-trivial, probe the spectrum of the CFT



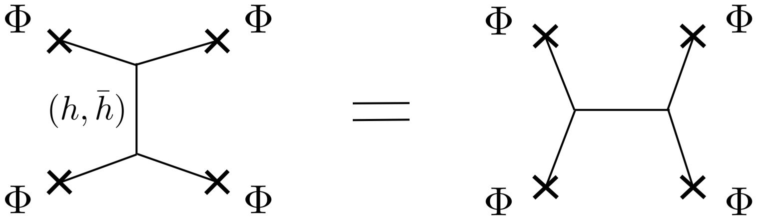
P_{aaaa} P_{aabb} P_{abab} P_{abba}

Potts CFT: $\langle \Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\rangle$ — **compute using CFT**

fractional Kac indices

cannot use BPZ

Conformal bootstrap approach



★ amplitudes

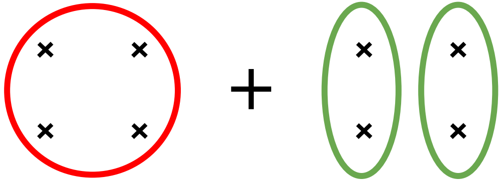
$$\langle \Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\Phi_{1/2,0}\rangle = \sum_{(h,\bar{h})\in\text{s-channel}} A(h,\bar{h})\mathcal{F}^{(s)}_{h,\bar{h}} = \sum_{(h,\bar{h})\in\text{t-channel}} A(h,\bar{h})\mathcal{F}^{(t)}_{h,\bar{h}}$$

★ spectrum

conformal block ✓

$$\left(C^{(h,\bar{h})}_{\Delta_{1/2,0}\Delta_{1/2,0}}\right)^2$$

first attempt to bootstrap connectivity:

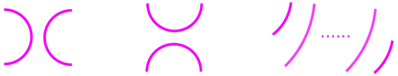


[Picco, Ribault, Santachiara, 2016]

Spectrum of connectivities

[Jacobsen, Saleur, 2018]

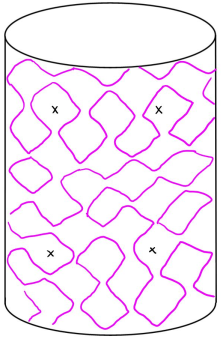
eigenvalues λ_i of transfer matrix → irreducible modules $\mathcal{W}$ of affine Temperley-Lieb algebra

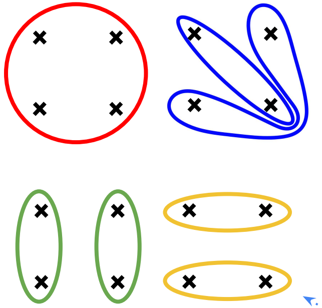
generators: 

↓ *continuum limit*

scaling dimensions $(h, \bar{h})$ in the spectrum

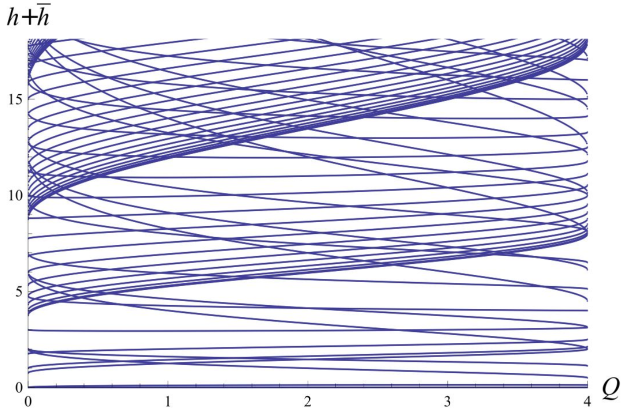
each $\mathcal{W}$ *: infinite tower of Virasoro conformal families*



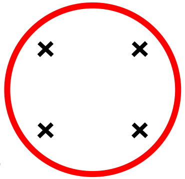
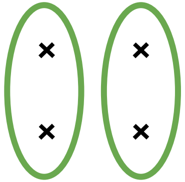


e.g. $\mathcal{W}_{j, e^{2i\pi p/M}}$

$(r, s): \quad (p/M + \mathbb{Z}, j)^N$



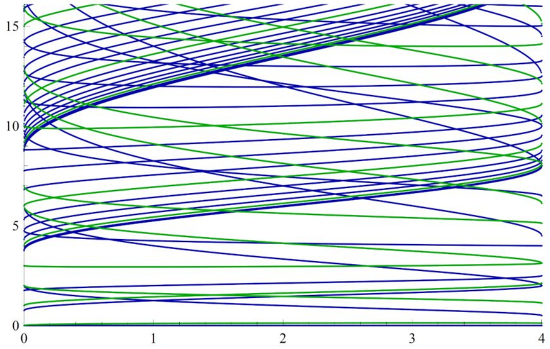
Non-diagonal Liouville theory

$$\left\langle \Phi_\Delta \Phi_\Delta \Phi_\Delta \Phi_\Delta \right\rangle \underset{\text{Monte-Carlo}}{\propto}$$  $$+ \frac{2}{Q-2}$$  [Picco, Ribault, Santachiara, 2016]

$$\Delta = \Delta_{\frac{1}{2},0}$$

bootstrap: simple spectrum (subset of Potts)

$$(r,s): \quad (1/2+\mathbb{Z}, 2\mathbb{Z})^N$$



non-diagonal Liouville

[Migliaccio, Ribault, 2017][Ribault, 2018]
[Picco, Ribault, Santachiara, 2019][Ribault, 2019]

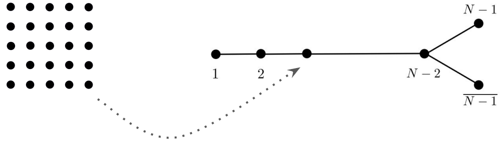
- analytic continuation of D series of MM to irrational c
- using degeneracy of $\Phi_{1,2}, \Phi_{2,1}$, analytical solution A^L

  [Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2018]

- poles in Q $\quad Q = 4\cos^2\left(\frac{\pi}{4}\right),\ 4\cos^2\left(\frac{\pi}{8}\right),\ 4\cos^2\left(\frac{3\pi}{8}\right),\ \ldots$

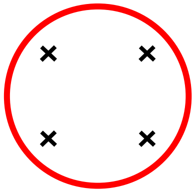
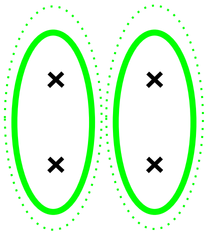
Loop interpretation [YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]

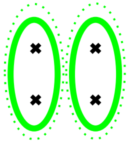
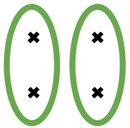
MM on the lattice: ADE RSOS model [Pasquier, 1987] [Kostov, 1989]



partition function, correlation functions have natural loop expansion

loop weights depending on topology
- *contractible:* $\sqrt{Q}$
- *non-contractible:* complicated, $\neq \sqrt{Q}$

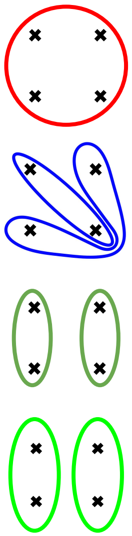
$\langle \Phi_\Delta \Phi_\Delta \Phi_\Delta \Phi_\Delta \rangle \propto$  $+$  **pseudo-probabilities** $\tilde{P}$

 vs $\frac{2}{Q-2}$  *difference involves unlikely configurations*

Universal amplitude ratios on the lattice

[YH, Grans-Samuelsson, Jacobsen, Saleur, 2020]

measure the spectrum of pseudo-probabilities $\tilde{P}$ as in [Jacobsen, Saleur, 2018] ⟶ $\lambda_i \in \mathcal{W}$



$A_{aaaa}(\lambda_i)$

$A_{abab}(\lambda_i)$

$A_{aabb}(\lambda_i)$

$\tilde{A}_{aabb}(\lambda_i)$

⋮

$$\frac{A_{abab}}{A_{aaaa}}(\lambda_i), \quad \frac{A_{aabb}}{A_{aaaa}}(\lambda_i), \quad \frac{\tilde{A}_{aabb}}{A_{aabb}}(\lambda_i), \quad \ldots$$

depend only on Q and $\lambda_i \in \mathcal{W}$

do not depend on lattice size

ATL modules

$$\frac{\tilde{A}_{abab}}{A_{abab}}(\mathcal{W}_{2,-1}) = \frac{2}{Q-2} \qquad \frac{\tilde{A}_{abab}}{A_{abab}}(\mathcal{W}_{4,-1}) = -\frac{4}{(Q-1)(Q-2)(Q^2-4Q+2)}$$

$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{1}{1-Q} \qquad \frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,-1}) = \frac{(Q-1)(Q-4)}{4}$$

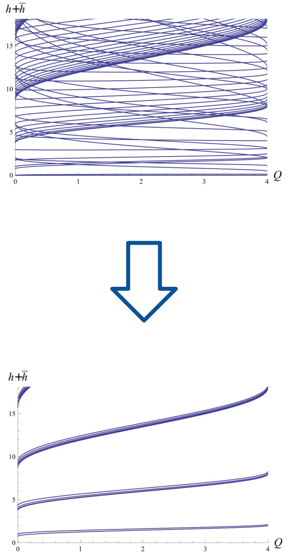
$$\frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{2-Q}{2} \qquad \frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{4,1}) = -\frac{Q^5-7Q^4+15Q^3-10Q^2+4Q-2}{2(Q^2-3Q+1)}$$

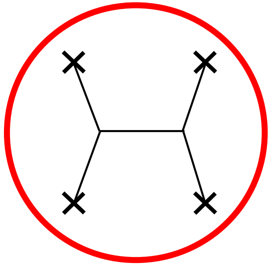
$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{4,-1}) = \frac{2-Q}{2} \qquad \frac{A_{abab}}{A_{aaaa}}(\mathcal{W}_{4,1}) = -\frac{(Q^2-4Q+2)(Q^2-3Q-2)}{4}$$

Interchiral conformal blocks [YH, Jacobsen, Saleur, 2020]

$$\lambda_i \xrightarrow{\text{continuum}} (h, \bar{h})$$

organize the CFT states according to $\mathcal{W}$





P_{aaaa}

$$= \sum_{(h,\bar{h})\in s-\text{channel}} A_{aaaa}(h,\bar{h}) \mathcal{F}_{h,\bar{h}}$$

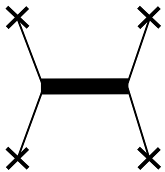
$$= \sum_{\mathcal{W}\in s-\text{channel}} A_{aaaa}(\mathcal{W}) \sum_{(h,\bar{h})\in\mathcal{W}} \frac{A_{\cancel{aaaa}}(h,\bar{h})}{A_{\cancel{aaaa}}(\mathcal{W})} \mathcal{F}_{h,\bar{h}}$$

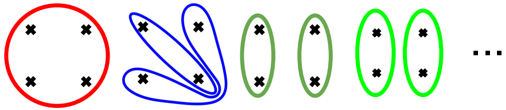
$$= \sum_{\mathcal{W}\in s-\text{channel}} A_{aaaa}(\mathcal{W})\, \mathbb{F}_{\mathcal{W}}$$

interchiral conformal block

interchiral algebra

[Gainutdinov, Read, Saleur, 2012]





From non-diag Liouville to Potts amplitudes [YH, Jacobsen, Saleur, 2020]

$$\sum_{\mathcal{W}} \left[A_{aaaa}(\mathcal{W}) + \tilde{A}_{abab}(\mathcal{W}) \right] \mathbb{F}_{\mathcal{W}} = P_{aaaa} + \tilde{P}_{abab} = \sum_{\mathcal{W}} A^L(\mathcal{W}) \mathbb{F}_{\mathcal{W}}$$

non-diagonal Liouville amplitude

$$A^L = A_{aaaa} + \tilde{A}_{abab}$$

$$= A_{aaaa} \left(1 + \frac{A_{abab}}{A_{aaaa}} \frac{\tilde{A}_{abab}}{A_{abab}} \right)$$

✓ ✓

$\mathcal{W} \in \mathcal{S}^L, \mathcal{S}_{aaaa}$ analytic expression

$$A_{aaaa}(\mathcal{W}_{0,-1}) = A^L(\mathcal{W}_{0,-1})$$

$$A_{abab}(\mathcal{W}_{2,-1}) = \frac{Q-2}{2} A^L(\mathcal{W}_{2,-1})$$

$$A_{aaaa}(\mathcal{W}_{4,-1}) = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2} A^L(\mathcal{W}_{4,-1})$$

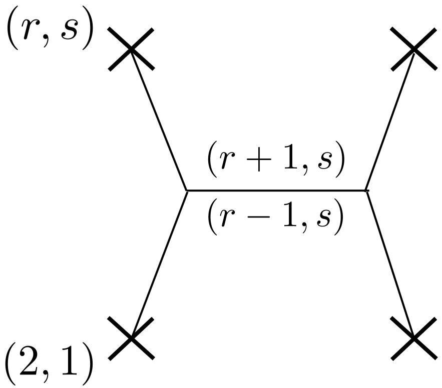
$\mathcal{W} \notin \mathcal{S}^L$ $\mathcal{W} \in \mathcal{S}_{aaaa}$

$A^L = 0$ bootstrap A_{aaaa}

Degeneracy ⟶ recursion

e.g. $\Phi_{21} : \ (h_{2,1}, h_{2,1})$ degenerate ✓

$$\Phi_{21} \times \phi_{r,s} \to \phi_{r+1,s} + \phi_{r-1,s}$$



$$\frac{A_{r+1,s}}{A_{r-1,s}}$$

technique in Liouville bootstrap

[Zamolodchikov^2, 1995] [Teschner, 1995] [Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2017]

Potts

~~in Liouville: $\Phi_{12} : \ (h_{1,2}, h_{1,2})$ degenerate~~

~~$$\frac{A_{r,s+1}}{A_{r,s-1}}$$~~

DOZZ, A^L

Constructing interchiral blocks

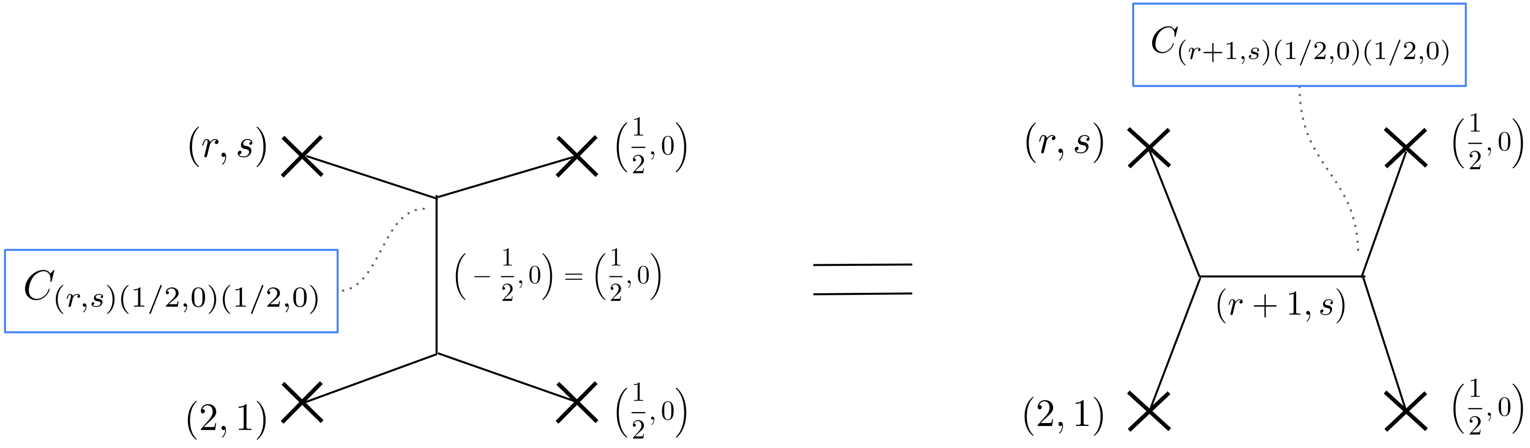
[YH, Jacobsen, Saleur, 2020]

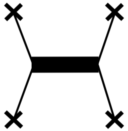
degenerate $\Phi_{21} \longrightarrow \mathbb{F}_{\mathcal{W}}$

$$\frac{A_{r+1,s}}{A_{r,s}} = \frac{C^2_{(r+1,s)(1/2,0)(1/2,0)}}{C^2_{(r,s)(1/2,0)(1/2,0)}}$$

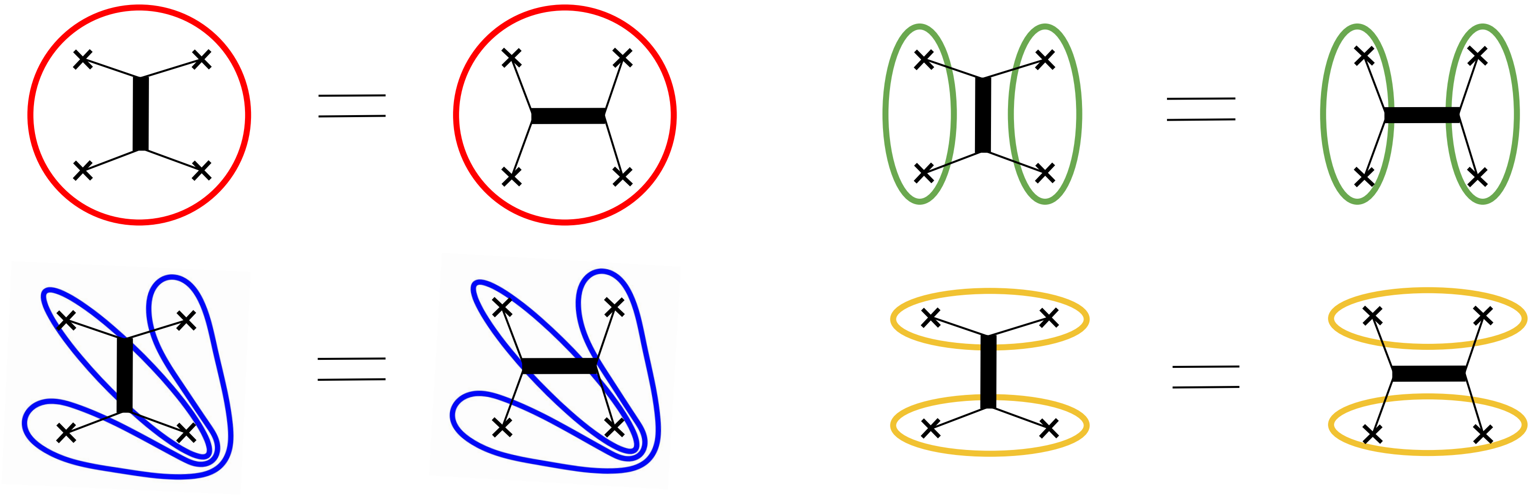
$\mathcal{W}_{j,e^{2i\pi p/M}}$

$(r,s): \quad (p/M+\mathbb{Z}, j)$

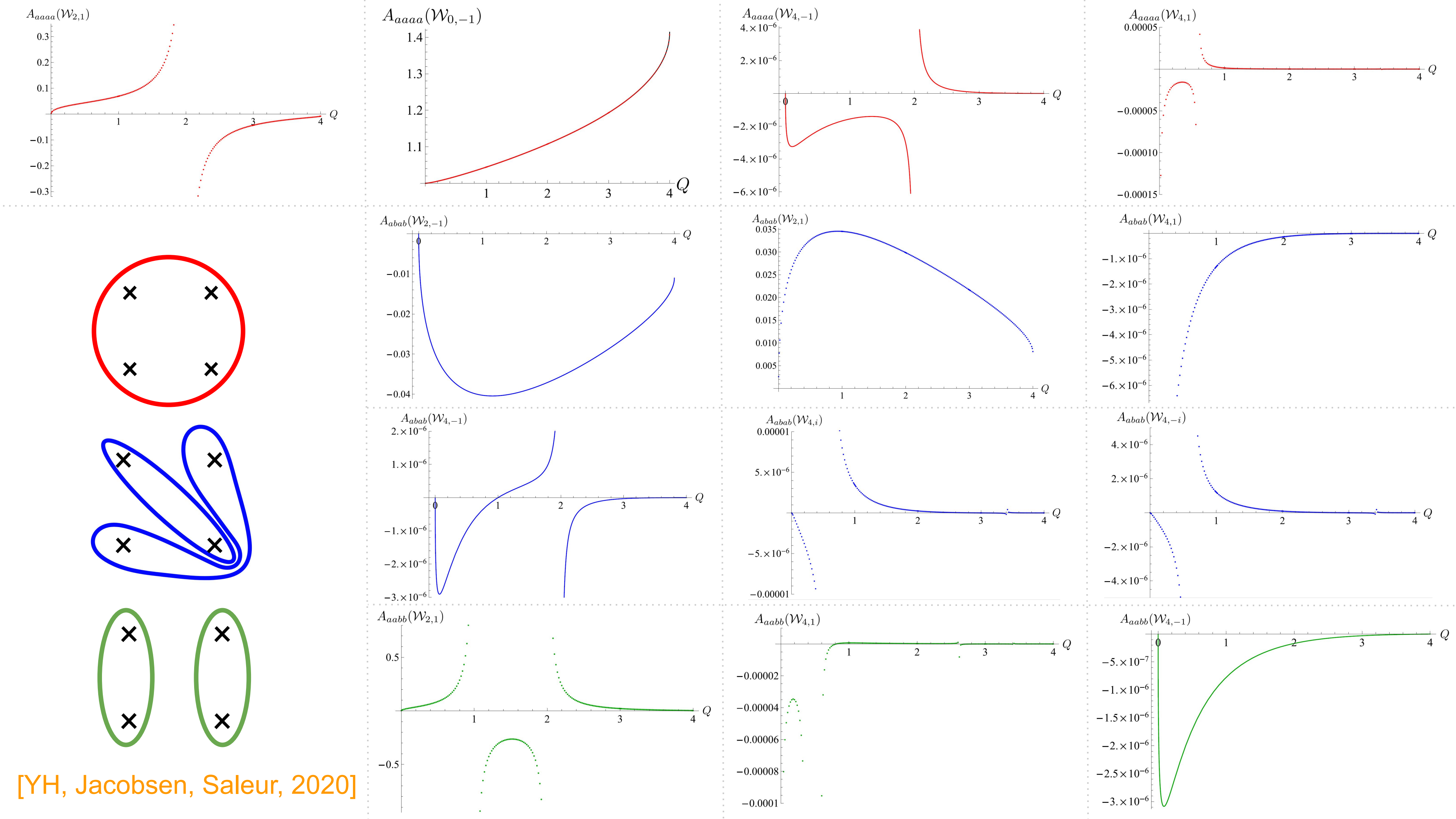


recursion & Virasoro block → $\mathbb{F}_{\mathcal{W}}$ 

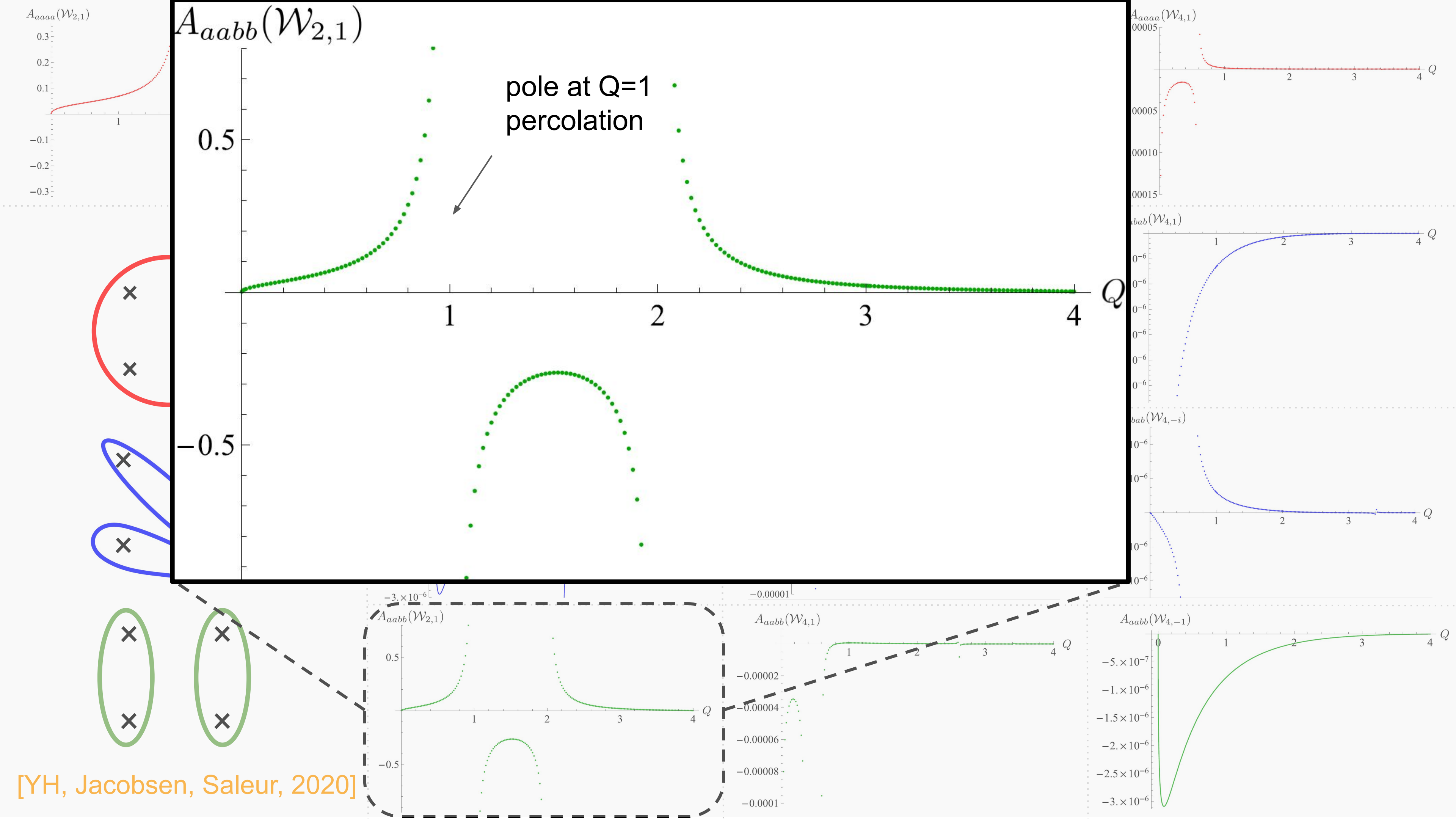
Interchiral conformal bootstrap [YH, Jacobsen, Saleur, 2020]



solve $A_{aaaa}(\mathcal{W}),\quad A_{abab}(\mathcal{W}),\quad A_{aabb}(\mathcal{W}),\quad A_{abba}(\mathcal{W})$



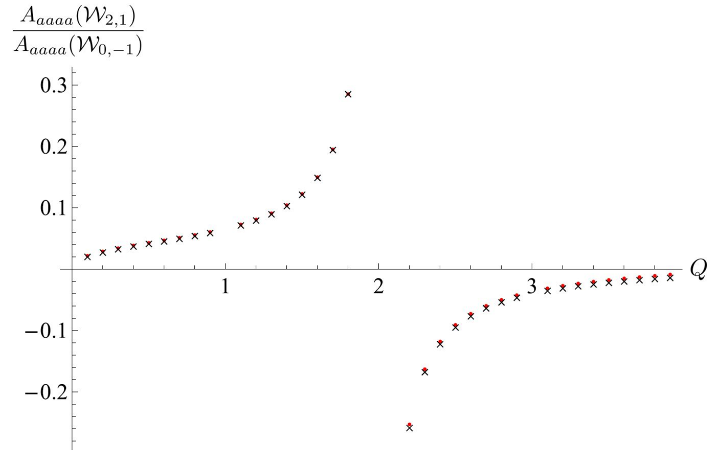
[YH, Jacobsen, Saleur, 2020]

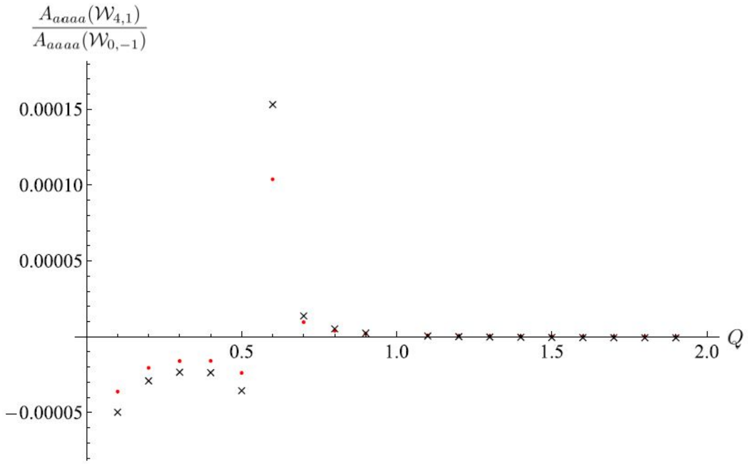

$A_{aabb}(\mathcal{W}_{2,1})$
pole at Q=1
percolation
Q
$A_{aaaa}(\mathcal{W}_{2,1})$
$A_{aaaa}(\mathcal{W}_{4,1})$
$A_{aabb}(\mathcal{W}_{2,1})$
$A_{aabb}(\mathcal{W}_{4,1})$
$A_{aabb}(\mathcal{W}_{4,-1})$
[YH, Jacobsen, Saleur, 2020]

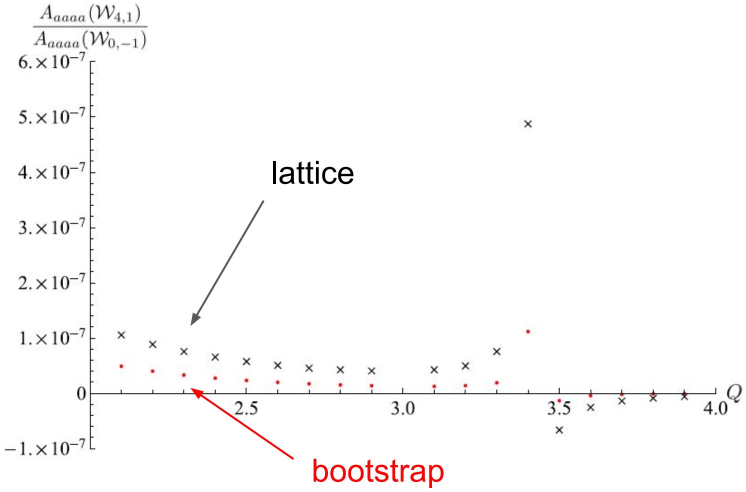
Comparison with lattice

[YH, Jacobsen, Saleur, 2020]

- *order of magnitude*
- *behavior as a function of Q*
- *analytic structure*





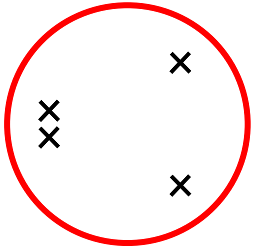


Four- & three-point connectivities

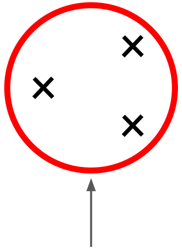
[YH, Jacobsen, Saleur, 2020]

$$\mathcal{W}_{0,-1}:\ (1/2,0),\dots$$

$$\Phi_{1/2,0}\times\Phi_{1/2,0}\sim C^{(1/2,0)}_{(1/2,0)(1/2,0)}\,\Phi_{1/2,0}$$



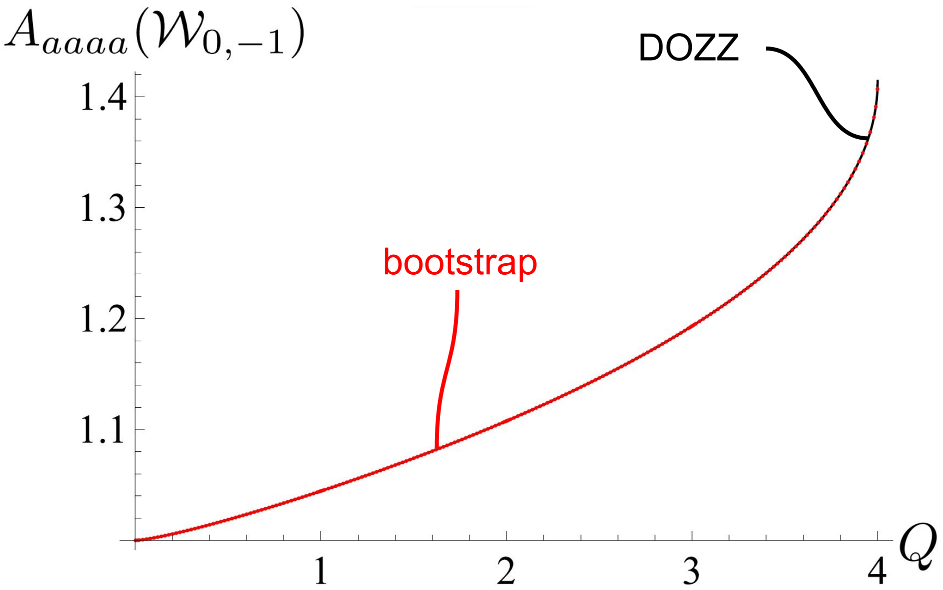
$$\sim C^{(1/2,0)}_{(1/2,0)(1/2,0)}$$



$$C^{(1/2,0)}_{(1/2,0)(1/2,0)}$$

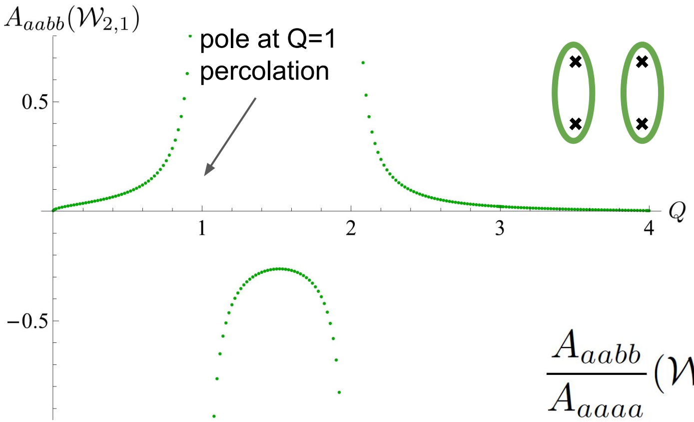
$$A_{aaaa}(\mathcal{W}_{0,-1})\sim\left(C^{(1/2,0)}_{(1/2,0)(1/2,0)}\right)^2$$

DOZZ

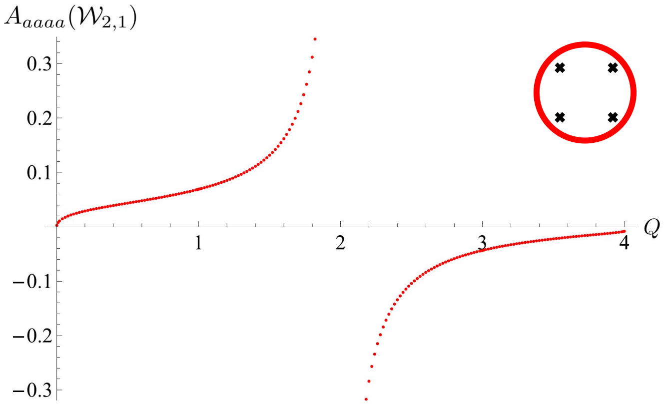


Singularities in the amplitudes

[YH, Jacobsen, Saleur, 2020]



$$\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1}) = \frac{1}{1-Q}$$



P_{aabb} spectrum: $\mathcal{W}_{2,1}$ $\overline{\mathcal{W}}_{0,\mathfrak{q}^2}$ …

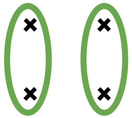
P_{aaaa} spectrum: $\mathcal{W}_{2,1}$ …

$(r,s) : (1,1),\ (2,1),\ (3,1), \ldots$

analytic structure in Q ⟷ **difference in spectrum**

Singularities cancellation & exact amplitudes [YH, Jacobsen, Saleur, 2020]

$Q=1 \qquad h_{1,1}=\bar{h}_{1,1}=h_{1,2} \qquad (1,1)\in\overline{\mathcal{W}}_{0,\mathfrak{q}^2}$

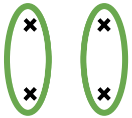


$$\mathcal{F}_{h_{1,1}}(z)=\ldots+\frac{R_{11}}{h_{1,1}-h_{1,2}}\mathcal{F}_{h_{1,-2}}(z)$$

similarly for $\bar{\mathcal{F}}_{h_{1,1}}(\bar{z})$

$\mathcal{F}^{\text{reg}}_{h_{1,2}}(z) \qquad \frac{\#}{Q-1}$

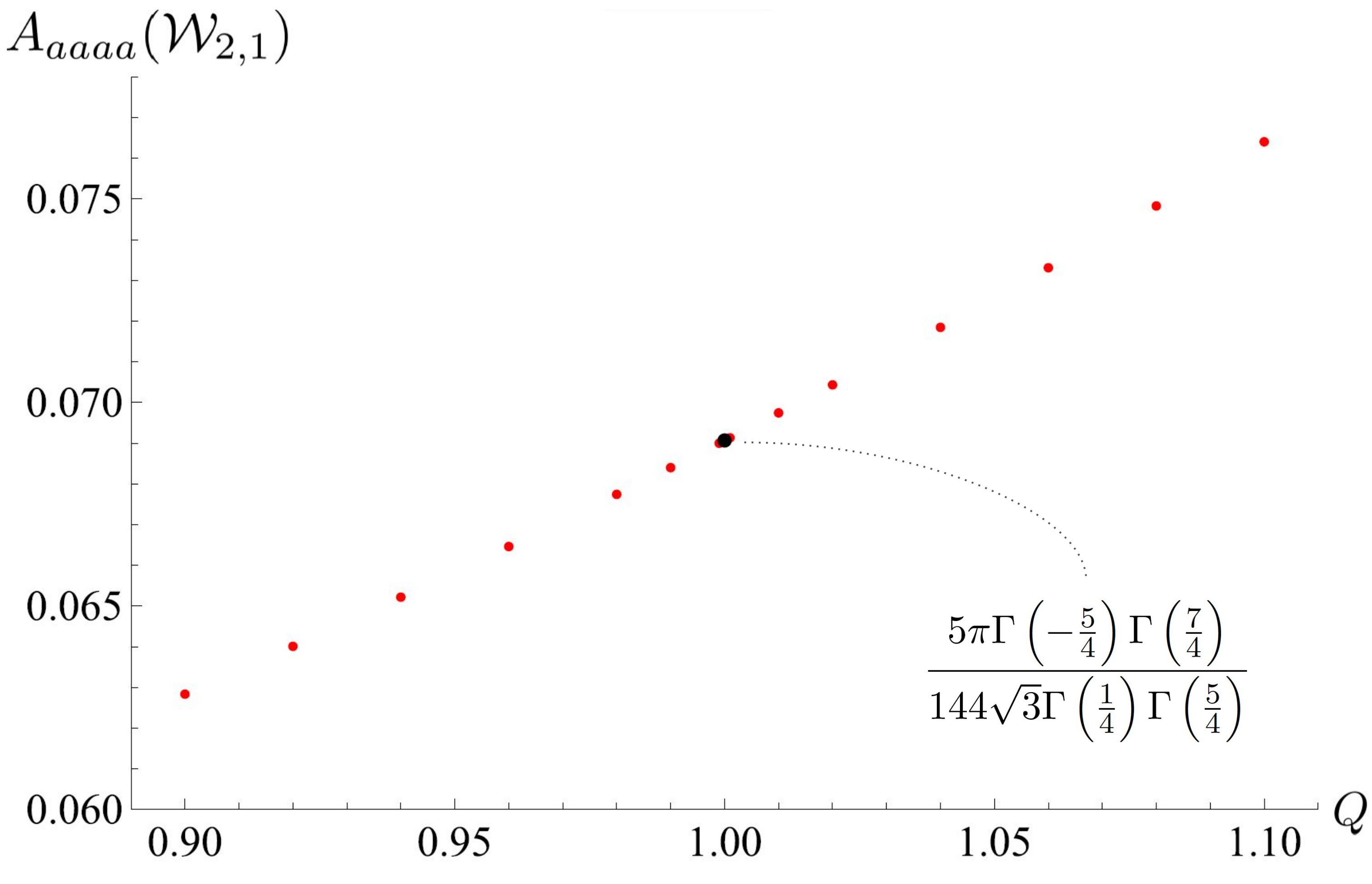
$$\mathcal{F}_{h_{1,1}}(z)\bar{\mathcal{F}}_{h_{1,1}}(\bar{z})=\ldots+\frac{\#}{Q-1}\left(\mathcal{F}^{\text{reg}}_{h_{1,2}}(z)\mathcal{F}_{h_{1,-2}}(\bar{z})+c.c.\right)$$

P_{aabb}  should be smooth in Q

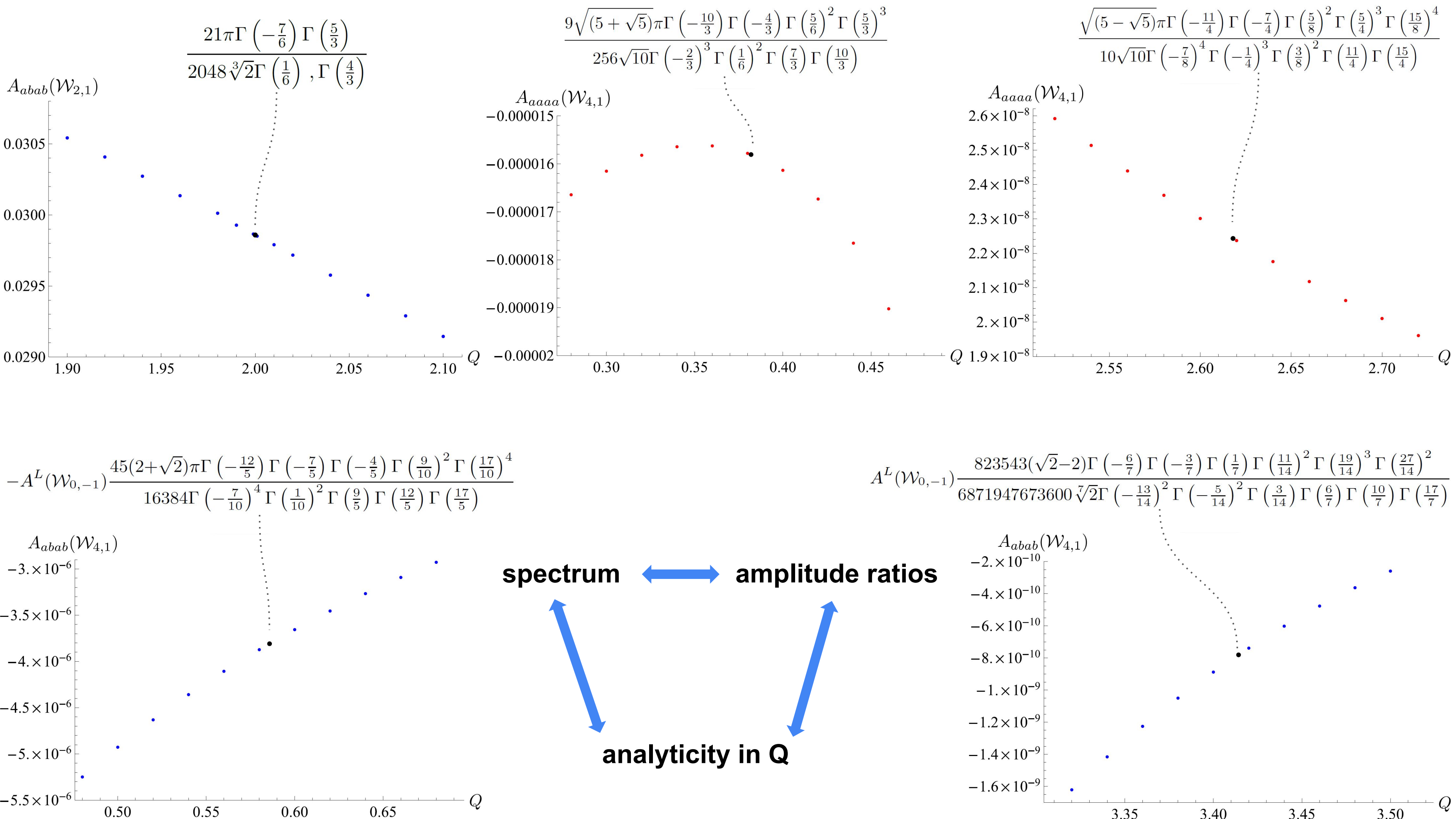
$$A_{aabb}(\mathcal{W}_{2,1})=\frac{\#}{Q-1} \qquad (h_{1,2},h_{1,-2})\in\mathcal{W}_{2,1}$$

in contrast, $A_{aaaa}(\mathcal{W}_{2,1})$ *has no pole at Q=1*

explains the amplitude ratios $\frac{A_{aabb}}{A_{aaaa}}(\mathcal{W}_{2,1})=\frac{1}{1-Q}$ → extract $A_{aaaa}(\mathcal{W}_{2,1})|_{Q=1}$



$A_{aaaa}(\mathcal{W}_{2,1})$
0.075
0.070
0.065
0.060
0.90
0.95
1.00
1.05
1.10
Q
$\frac{5\pi\Gamma\left(-\frac{5}{4}\right)\Gamma\left(\frac{7}{4}\right)}{144\sqrt{3}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{4}\right)}$



$\frac{21\pi\Gamma\left(-\frac{7}{6}\right)\Gamma\left(\frac{5}{3}\right)}{2048\sqrt[3]{2}\Gamma\left(\frac{1}{6}\right),\Gamma\left(\frac{4}{3}\right)}$
$A_{abab}(\mathcal{W}_{2,1})$
Q
$\frac{9\sqrt{(5+\sqrt{5})}\pi\Gamma\left(-\frac{10}{3}\right)\Gamma\left(-\frac{4}{3}\right)\Gamma\left(\frac{5}{6}\right)^2\Gamma\left(\frac{5}{3}\right)^3}{256\sqrt{10}\Gamma\left(-\frac{2}{3}\right)^3\Gamma\left(\frac{1}{6}\right)^2\Gamma\left(\frac{7}{3}\right)\Gamma\left(\frac{10}{3}\right)}$
$A_{aaaa}(\mathcal{W}_{4,1})$
Q
$\frac{\sqrt{(5-\sqrt{5})}\pi\Gamma\left(-\frac{11}{4}\right)\Gamma\left(-\frac{7}{4}\right)\Gamma\left(\frac{5}{8}\right)^2\Gamma\left(\frac{5}{4}\right)^3\Gamma\left(\frac{15}{8}\right)^4}{10\sqrt{10}\Gamma\left(-\frac{7}{8}\right)^4\Gamma\left(-\frac{1}{4}\right)^3\Gamma\left(\frac{3}{8}\right)^2\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{15}{4}\right)}$
$A_{aaaa}(\mathcal{W}_{4,1})$
Q
$-A^L(\mathcal{W}_{0,-1})\frac{45(2+\sqrt{2})\pi\Gamma\left(-\frac{12}{5}\right)\Gamma\left(-\frac{7}{5}\right)\Gamma\left(-\frac{4}{5}\right)\Gamma\left(\frac{9}{10}\right)^2\Gamma\left(\frac{17}{10}\right)^4}{16384\Gamma\left(-\frac{7}{10}\right)^4\Gamma\left(\frac{1}{10}\right)^2\Gamma\left(\frac{9}{5}\right)\Gamma\left(\frac{12}{5}\right)\Gamma\left(\frac{17}{5}\right)}$
$A_{abab}(\mathcal{W}_{4,1})$
Q
$A^L(\mathcal{W}_{0,-1})\frac{823543(\sqrt{2}-2)\Gamma\left(-\frac{6}{7}\right)\Gamma\left(-\frac{3}{7}\right)\Gamma\left(\frac{1}{7}\right)\Gamma\left(\frac{11}{14}\right)^2\Gamma\left(\frac{19}{14}\right)^3\Gamma\left(\frac{27}{14}\right)^2}{6871947673600\sqrt[7]{2}\Gamma\left(-\frac{13}{14}\right)^2\Gamma\left(-\frac{5}{14}\right)^2\Gamma\left(\frac{3}{14}\right)\Gamma\left(\frac{6}{7}\right)\Gamma\left(\frac{10}{7}\right)\Gamma\left(\frac{17}{7}\right)}$
$A_{abab}(\mathcal{W}_{4,1})$
Q
spectrum
amplitude ratios
analyticity in Q

“Renormalized” Liouville recursion

[YH, Jacobsen, Saleur, 2020]

in Liouville degenerate $\Phi_{1,2}, \Phi_{2,1}$ → analytic bootstrap solution DOZZ, A^L

[Zamolodchikov^2, 1995] [Teschner, 1995]
[Estienne, Ikhlef, 2015] [Migliaccio, Ribault, 2017]

Potts: only $\Phi_{2,1}$ degenerate

$$\frac{A_{aaaa}(\mathcal{W}_{4,-1})}{A_{aaaa}(\mathcal{W}_{0,-1})} = \frac{(Q-2)(Q^2-4Q+2)}{Q(Q-3)^2}\frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{0,-1})}$$

$$\frac{A_{abab}(\mathcal{W}_{4,-1})}{A_{abab}(\mathcal{W}_{2,-1})} = \frac{(Q-1)(Q-4)(Q^2-4Q+2)}{2Q(Q-3)^2}\frac{A^L(\mathcal{W}_{4,-1})}{A^L(\mathcal{W}_{2,-1})}$$

$$\frac{A_{aaaa}(\mathcal{W}_{4,1})}{A_{aaaa}(\mathcal{W}_{2,1})} = \frac{(Q-2)^2}{(Q-1)^2(Q^2-4Q+2)}\frac{A^L(\mathcal{W}_{4,1})}{A^L(\mathcal{W}_{2,1})}$$

dressed by factors -- rational functions of Q

analytic bootstrap solution?

Liouville recursion
IF $\Phi_{1,2}$ is degenerate

what structure replaces the $\Phi_{1,2}$ degeneracy?

Thank you!