Multiplicative chaos of the Brownian loop soup

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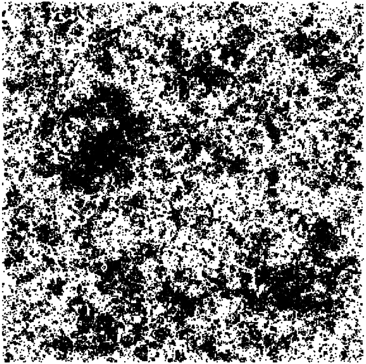
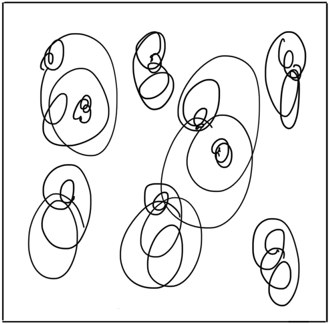
Based on joint work with E. Aïdékon, N. Berestycki and T. Lupu

Brownian loop soup

Countable infinite collection of Brownian-like loops in a domain $D \subset \mathbb{R}^2$





Introduced by Lawler and Werner

Related to many random conformal invariant objects

Definition

Poisson point process $\mathcal{L}_D^\theta \sim \mathsf{PPP}(\theta\mu_D^{\text{loop}})$

- $\theta > 0$: intensity parameter
- μ_D^{loop}: loop measure

$$\mu_D^{\mathrm{loop}}(d\wp) = \int_D \int_0^{+\infty} \mathbb{P}_{D,t}^{z,z}(d\wp) p_D(t,z,z) \frac{\mathrm{d}t}{t} \mathrm{d}z.$$

Brownian bridge   heat kernel

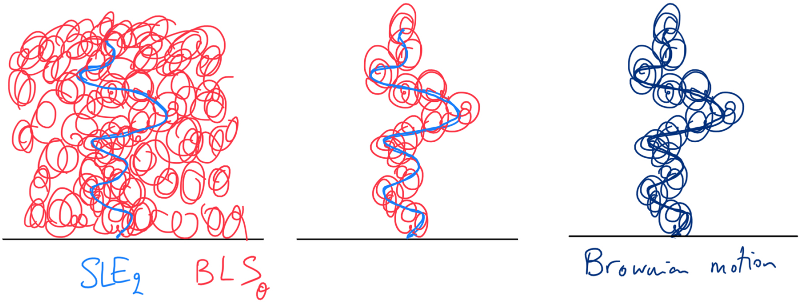
Fact

- *Infinitely many small loops*
- *Conformally invariant*
- *Restriction property*

Loop-erased random walk:

(erase chronologically each loop)

$$\begin{array}{rcl} \text{random walk} & = & \text{loop erasure} + \text{loops} \\ & \downarrow & \\ \text{Brownian motion} & = & \text{SLE}_2 + \text{loops} \\ & & \updownarrow \\ & & \text{Brownian loop soup } (\theta = 1) \end{array}$$



Lawler–Schramm–Werner

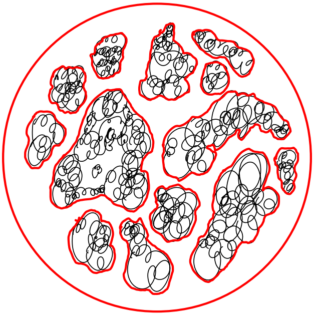
Clusters

Cluster = chain of intersecting loops

Phase transition: (Sheffield–Werner)

• $\theta > 1/2$: one big cluster

• $\theta \leq 1/2$: infinitely many clusters
Outer boundaries of outermost clusters
$= \text{CLE}_\kappa$ where $\kappa = \kappa(\theta) \in (8/3, 4]$
Conformal Loop Ensemble



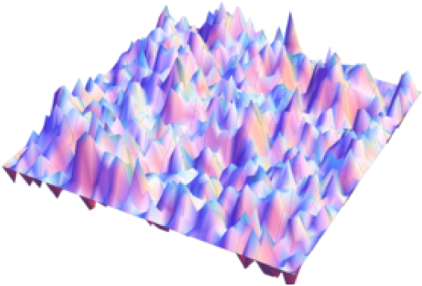
Le Jan's isomorphism

When $\theta = 1/2$,

" the occupation field of the Brownian loop soup $\overset{(d)}{=} \frac{1}{2}$(Gaussian free field)^2 "

The Gaussian free field

The unique random generalised function satisfying some domain Markov property (+ moment condition)
(Berestycki–Powell–Ray, Aru–Powell)



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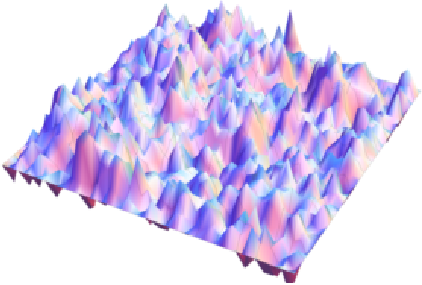

Universal random height function: Scaling limit of

- Height function in dimers
- Ginzburg-Landau $\nabla\phi$ interface
- Characteristic polynomial of large random matrices

The Gaussian free field

$h =$ GFF

Formally, $h = (h_x)_{x \in D} \sim \mathcal{N}(0, G_D)$ where

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)\mathrm{d}t = \text{ Green function}$$

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In 2D, $G_D(x,y) \sim -\log|x-y|$ as $x - y \to 0$
$\rightsquigarrow \mathrm{var}(h_x) = +\infty$
Rigorously, random generalised function

$$\mathrm{var}((h,f)) = \int_{D \times D} f(x) G_D(x,y) f(y) \, \mathrm{d}x\mathrm{d}y$$

Exponential of the GFF

h : GFF
$\gamma \in (-2, 2)$

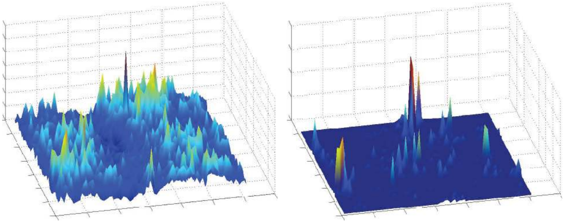
Informally: random measure

$$\mu_\gamma = e^{\gamma h(x)} \mathrm{d}x$$

Instance of **Gaussian multiplicative chaos** measure

Rigorously,

$$\mu_\gamma = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(x)} \mathrm{d}x$$



(a) $\gamma = 0.2$ (b) $\gamma = 1$

Simulation by Rhodes–Vargas

Thick points

$$\mu_\gamma = e^{\gamma h} \mathrm{d}x$$

x fixed deterministic point $\rightsquigarrow h_\varepsilon(x) = O(\sqrt{|\log \varepsilon|})$

x is μ^γ − typical point $\rightsquigarrow \lim_{\varepsilon \to 0} \frac{h_\varepsilon(x)}{|\log \varepsilon|} = \gamma$

In fact,

Theorem (Discrete: Biskup–Louidor)

$\mathcal{T}_\varepsilon(\gamma) := \{x \in D : h_\varepsilon(x) \geq \gamma |\log \varepsilon|\}$ *(γ-thick points)*

$$\sqrt{|\log \varepsilon|} \varepsilon^{-\gamma^2/2} \mathbf{1}_{\{x \in \mathcal{T}_\varepsilon(\gamma)\}} \mathrm{d}x \xrightarrow[\varepsilon \to 0]{} \mu_\gamma$$

"μ_γ = uniform measure on γ-thick points"

This talk

Loop soup: $\mathcal{L}_D^\theta \sim \mathrm{PPP}(\theta \mu_D^{\mathrm{loop}})$
GFF: h

Couple $(\mathcal{L}_D^\theta, h)$ such that $(L_x)_{x \in D} = \frac{1}{2} h^2$ $\theta = 1/2$

Sample $z \sim e^{\gamma h}$.

Questions:

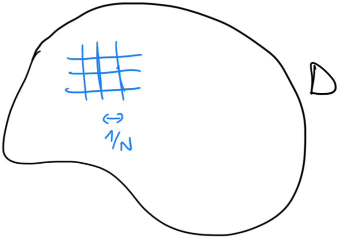
- What does the loop soup look like near z?
- How does the loop soup create a thick local time?
  - $\hookrightarrow$ A few very thick loops?
  - $\hookrightarrow$ Many loops w/ typical local time?
- What about $\theta \neq 1/2$? What is the associated chaos?

Multiplicative chaos construction

$D_N = \frac{1}{N}\mathbb{Z}^2 \cap D$ discrete approximation of D

$\mathcal{L}^\theta_{D_N}$ random walk loop soup

$$\ell_x = \text{local time at } x = \sum_{\wp \in \mathcal{L}^\theta_{D_N}} \int_0^{\tau_\wp} \mathbf{1}_{\{\wp_t = x\}} \mathrm{d}t$$



typical point: $\mathbb{E}[\ell_x] \sim \frac{\theta}{2\pi} \log N$

Multiplicative chaos construction

$\theta > 0$ intensity

typical point: $\mathbb{E}[\ell_x] \sim \frac{\theta}{2\pi} \log N$ $a = \frac{\gamma^2}{2}$ thickness parameter

a-thick points: $\mathcal{T}_N(a) := \left\{ x \in D_N : \ell_x \geq \frac{1}{2\pi} a (\log N)^2 \right\}$

Uniform measure on $\mathcal{T}_N(a)$: $\mathcal{M}_a^N := \frac{(\log N)^{1-\theta}}{N^{2-a}} \sum_{x \in \mathcal{T}_N(a)} \delta_x$

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Theorem (E. Aïdékon, N. Berestycki, A. J., T. Lupu 21)

$(\mathcal{M}_a^N, \mathcal{L}_{D_N}^\theta) \to (\mathcal{M}_a, \mathcal{L}_D^\theta)$ *as* $N \to \infty$.

$\rightsquigarrow \mathcal{M}_a =$ multiplicative chaos associated to $\mathcal{L}_D^\theta$.

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- $\theta = 1/2$: discrete GFF, Biskup–Louidor
- $\theta \to 0$: random walk thick points, Jego
- related result: random walk close to cover time, Abe, Biskup, Lee

Multiplicative chaos and loop soup

$\theta > 0$ intensity
$a = \frac{\gamma^2}{2}$ thickness

- $\theta = 1/2 \rightsquigarrow \mathcal{M}_a \overset{(d)}{=} e^{\gamma \mathsf{GFF}} + e^{-\gamma \mathsf{GFF}}$
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Sample $z \sim \mathcal{M}_a(\cdot)/\mathcal{M}_a(D)$: typical a-thick point.
How many loops go through z? What are the thicknesses associated to each individual loop?

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1) Infinitely many loops.
2) Denote $a_1, a_2, \dots$ thicknesses of loops going through z.
$\{a_1, a_2, \dots\} \sim$ *Poisson-Dirichlet(θ) on the interval* $[0, a]$.

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$\{a_1, a_2, \dots\} \sim$ Poisson-Dirichlet(θ) on the interval $[0, a]$.
3) Cond. on $\{a_1, a_2, \dots\}$, the loops that go through z are indep. and distributed like the concatenation of the excursions in PPP($a_i \mu_D^z$).

Construction from the continuum

Brownian multiplicative chaos: multiplicative chaos associated to **finitely** many trajectories

$$e^{\gamma\sqrt{2L_x}}\mathrm{d}x$$

Bass–Burdzy–Koshnevisan 94
Aïdékon–Hu–Shi 20
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- Kill each loop independently of each other at rate $K > 0$

$\mathbb{P}(\wp \text{ killed}) = 1 - e^{-KT(\wp)}$

- $\mathcal{M}_a^K :=$ multiplicative chaos associated to **killed** loops

Theorem (E.A., N.B., A.J., T.L. 21)

$$(\log K)^{-\theta}\mathcal{M}_a^K \xrightarrow[K\to\infty]{\mathbb{P}} \mathcal{M}_a$$

Exact solvability

$\mathcal{M}_a^K$ = multiplicative chaos associated to K-killed loops (cutoff)
Define $C_K(z) = \int_0^\infty (1 - e^{-Kt}) p_D(t, z, z)$

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Proposition (E.A., N.B., A.J., T.L.)

$$\mathbb{E}[\mathcal{M}_a^K(\mathrm{d}z)] = \frac{1}{a}\mathsf{F}(C_K(z)a)\mathrm{CR}(z, D)^a \mathrm{d}z$$

where

$$\mathsf{F}(u) = \theta \int_0^u e^{-t} {}_1F_1(\theta, 1, t)\mathrm{d}t$$

with ${}_1F_1$ *= Kummer's confluent hypergeometric function.*

Exact solvability

Proof in two steps:

- Step 1 (could treat any cutoff)

$$\mathbb{E}[\mathcal{M}_a^K(\mathrm{d}z)] = \frac{1}{a}\hat{\mathsf{F}}(C_K(z)a)\mathrm{CR}(z,D)^a \mathrm{d}z$$

where

$$\hat{\mathsf{F}}(u) := \sum_{n\geq 1} \frac{\theta^n}{n!} \int_{E(1,n)} \mathrm{d}\mathbf{a} \prod_{i=1}^n \frac{1-e^{-ua_i}}{a_i}$$

and

$$E(1,n) = \{\mathbf{a} = (a_1,\dots,a_n) \in (0,1)^n : a_1 + \cdots + a_n = 1\}$$

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- Step 2 (**very much cutoff-sensitive!**): $\hat{\mathsf{F}}$ is "nice"

$$(1-\theta)\hat{\mathsf{F}}'(u) + \hat{\mathsf{F}}''(u) + u(\hat{\mathsf{F}}''(u) + \hat{\mathsf{F}}'''(u)) = 0$$

A martingale

Proposition (E.A., N.B., A.J., T.L. 21)

$$\frac{1}{a^{1-\theta}} \mathrm{CR}(z,D)^a e^{-aC_K(z)} dz + \int_0^a \frac{\mathrm{d}\alpha}{(a-\alpha)^{1-\theta}} \mathrm{CR}(z,D)^{a-\alpha} e^{-(a-\alpha)C_K(z)} \mathcal{M}_\alpha^K(dz)$$

is a measure-valued martingale (as a function of K)

A martingale: heuristics when $\theta = 1/2$

$\mathcal{L}_K^\theta = \{K\text{-killed loops}\}$

$$L_x(\mathcal{L}^\theta) \overset{\text{(d)}}{=} \frac{1}{2} h^2 \qquad \text{and} \qquad L_x(\mathcal{L}^\theta \setminus \mathcal{L}_K^\theta) \overset{\text{(d)}}{=} \frac{1}{2} {h_K}^2$$

GFF massive GFF

(massive Green function associated to $-\Delta + K$)

$$L_x(\mathcal{L}_K^\theta) \overset{\perp\!\!\!\perp}{+} \frac{1}{2} h_K^2 = \frac{1}{2} h^2$$

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$$f(0) e^{\gamma |h_K|} + \int_0^a \mathrm{d}\alpha \; f(\alpha) e^{\sqrt{2(a-\alpha)} |h_K|} \mathcal{M}_\alpha^K = e^{\gamma |h|}$$

Multiplicative chaos and Wick powers

Define $: e^{\gamma h(x)} \mathrm{d}x : \quad = \quad \lim_{\varepsilon \to 0} e^{\gamma h_\varepsilon(x)} / \mathbb{E}\left[e^{\gamma h_\varepsilon(x)} \right] \mathrm{d}x$

Theorem

When $\gamma \in [0, \sqrt{2})$, $: e^{\gamma h(x)} \mathrm{d}x := \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} : h(x)^k : \mathrm{d}x$

$: h(x)^k : =$ k-th Wick power of h

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Scaling limit of $\qquad h_N =$ discrete GFF

$$: h_N(x)^k : \quad = \quad G_N(x,x)^{k/2} H_k \left(\frac{h_N(x)}{\sqrt{G_N(x,x)}} \right)$$

discrete Green function Hermite polynomial

$(H_k)_{k \geq 0}$ orthogonal w.r.t. $e^{-t^2/2} \mathrm{d}t$

Wick powers of local time (Le Jan)

ℓ_x local time of discrete loop soup

$$:\ell_x^k: \quad = \quad G_N(x,x)^k L_k^{(\theta-1)}\left(\frac{\ell_x}{G_N(x,x)}\right)$$

$(L_k^{(\theta-1)})_{k\geq 0}$ generalised Laguerre polynomials, orthogonal for Gamma(θ) distribution.

Non degenerate scaling limit: $:L_x^k:$

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Theorem (Le Jan)

When $\theta = 1/2$, $(: L_x^k :)_{k \geq 0} \overset{\mathrm{(d)}}{=} (2^{-k} : h_x^{2k} :)_{k \geq 0}$

Expansion of the multiplicative chaos of the loop soup

Renormalise $\mathcal{M}_\gamma$ so that $\mathbb{E}[\mathcal{M}_\gamma(\mathrm{d}x)] = 2\mathrm{d}x$.
When $\theta = 1/2$, $\gamma \in (0, \sqrt{2})$,

$$\mathcal{M}_\gamma(\mathrm{d}x) = 2 \sum_{k=0}^{\infty} \frac{2^k \gamma^{2k}}{(2k)!} (2\pi)^k : L_x^k : \mathrm{d}x$$

Theorem (J. – Lupu – Qian 22+)

For all $\theta > 0$, $\gamma \in (0, \sqrt{2})$,

$$\mathcal{M}_\gamma(\mathrm{d}x) = 2 \sum_{k=0}^{\infty} \frac{\gamma^{2k}}{2^k} \frac{\Gamma(\theta)}{k!\Gamma(\theta + k)} (2\pi)^k : L_x^k : \mathrm{d}x$$

Key identity

Hermite polynomials:

$$\sum_{n\geq 0} \frac{\gamma^n t^{n/2}}{n!} H_n(u/\sqrt{t}) = e^{\gamma u - \gamma^2 t/2}, \qquad t, u \in \mathbb{R}.$$

Laguerre polynomials: (J. – Lupu – Qian)

$$\sum_{n\geq 0} \left(\frac{\gamma^2 t}{2}\right)^n \frac{1}{n!\Gamma(\theta+n)} L_n^{(\theta-1)}\left(\frac{u^2}{2t}\right) = e^{-\gamma^2 t/2} \left(\frac{\gamma^2 ut}{4}\right)^{\frac{1-\theta}{2}} I_{\theta-1}(\gamma u)$$

$I_{\theta-1}$ = modified Bessel function
$I_{\theta-1}(\gamma u) \sim \frac{1}{\sqrt{2\pi\gamma u}} e^{\gamma u}$ as $u \to \infty$.

Thank you for your attention!