Imaginary chaos and Malliavin calculus

Janne Junnila

Probability and Conformal Field Theory
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Based on joint works with J. Aru, A. Jego, E. Saksman and C. Webb.

Introduction to imaginary chaos

Log-correlated Gaussian fields

- A Gaussian random field X is called **log-correlated** if its covariance (kernel) is of the form

$$C_X(x, y) = C(x, y) = \log |x - y|^{-1} + g(x, y)$$

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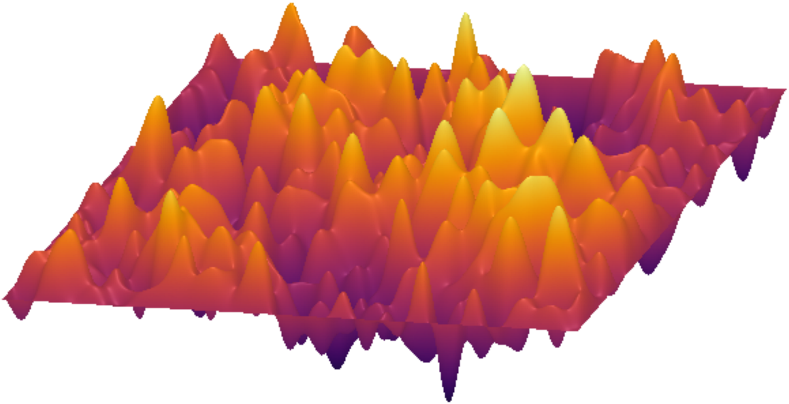
- **Note:** X is not pointwise defined since formally $\mathrm{Var}(X(z)) = \infty$. The random variables $\langle X, f\rangle = \int X(x) f(x)\,dx$ can however be defined for any test function f, and X can be given sense as a random Schwartz distribution.

2D Gaussian Free Field



An approximation of the zero-boundary GFF on the unit square.

(Complex) Gaussian multiplicative chaos

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- A distribution of the form $\mu^\gamma(x) \mathrel{=}: \exp(\gamma X(x)) := e^{\gamma X(x) - \frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}$ and $\gamma \in \mathbb{C}$ is a parameter.

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- As X is not a function, a rigorous definition requires a limiting procedure: $\mu^\gamma(x) \coloneqq \lim_{\varepsilon\to 0} \mu^\gamma_\varepsilon(x) \coloneqq \lim_{\varepsilon\to 0} e^{\gamma X_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(x)^2]}$, where X_ε is a smoothened version of X and the limit is taken in some suitable space of distributions on U (e.g. H^{-s} for large enough $s > 0$).

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- Appears e.g. in Liouville quantum gravity and as a limit of characteristic polynomials of random matrices.

Existence in L^2-phase

For a given test function f we may compute

$$\mathbb{E}[|\mu_\varepsilon(f)|^2] = \int \mathbb{E}[: e^{\gamma X_\varepsilon(x)} :: e^{\bar{\gamma} X_\varepsilon(y)} :]\, dx\, dy$$
$$\lesssim \int e^{|\gamma|^2 C(x,y)}\, dx\, dy \lesssim \int |x-y|^{-|\gamma|^2}\, dx\, dy.$$

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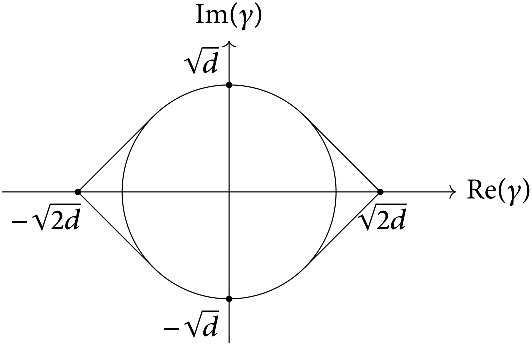
- This is bounded uniformly in ε if $|\gamma| < \sqrt{d}$.
- If $\mu_\varepsilon(f)$ is a martingale approximation we automatically obtain convergence in $L^2(\Omega)$.

The subcritical regime

With more careful analysis one can in fact show the existence of a non-trivial limit for μ_ε^γ when γ belongs to the interior of the eye-shaped region below. Moreover, the map $\gamma \mapsto \mu^\gamma$ is analytic. The disc corresponds to the L^2-phase.



The subcritical regime for γ in the complex plane.

Imaginary multiplicative chaos, $\gamma = i\beta$ $(|\beta| < \sqrt{d})$

- Heuristically in this case the modulus of $\mu(x)$ is an infinite normalising constant $e^{\frac{\beta^2}{2}\mathbb{E}[X(x)^2]}$ while the angle is given by the field $\beta X(x)$.

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- Compare this to the case when γ is real, where the angle is constant but the modulus is given by $e^{\gamma X(x)}$ times an infinitesimal constant $e^{-\frac{\gamma^2}{2}\mathbb{E}[X(x)^2]}$.

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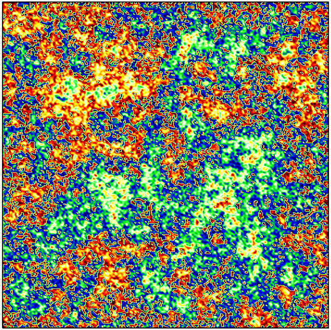

Simulation of imaginary chaos with X the GFF and $\beta = 1/\sqrt{2}$.

Analytic properties of imaginary chaos

- Tails: $\mathbb{P}[|\mu(f)| > \lambda]$ decays roughly like $\exp\left(-\lambda^{\frac{2d}{\beta^2}}\right)$.
  - Faster than Gaussian decay – in fact when $\beta = \sqrt{d}$, after a further renormalization the chaos becomes white noise.
- Regularity: $\mu \in H^s_{loc}$ if and only if $s < -\beta^2/2$.[1]
- μ is not a complex measure ($\|\mu\|_{TV} = \infty$ a.s.).
- The law of $\mu(f)$ has a smooth density w.r.t. the Lebesgue measure on $\mathbb{C}$.
  - The density is in fact everywhere positive (not yet published), which in particular implies $\mathbb{E}[|\mu(f)|^p] < \infty$ if and only if $p > -2$.
- Monofractality: A.s. for all $x \in U$ we have $\liminf_{r\to 0} \frac{\log|\mu(B(x,r))|}{\log r} = 2 - \beta^2/2$.[1]

[1] Case $s = -\beta^2/2$ for Sobolev regularity as well as the monofractality result are part of some unpublished work by Aru, Baverez, Jego and J.

Moments and Onsager's inequality

Many of the proofs rely on moment computations.

- Growth of moments: $\mathbb{E}[|\mu(f)|^{2N}] \leq C^N N^{\frac{\beta^2}{d}N}$
- Slow enough to determine the distribution of the random variable $\mu(f)$.

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A key tool in establishing sharp moment bounds is **Onsager's inequality**: For any $x_1, \ldots, x_{2N} \in U$ and $q_1, \ldots, q_{2N} \in \{\pm 1\}$ we have

$$-\sum_{1 \le j < k \le 2N} q_j q_k C(x_j, x_k) \le \frac{1}{2} \sum_{j=1}^{2N} \log \frac{1}{r_j} + CN$$

where $r_j = \frac{1}{2} \min_k |x_j - x_k|$ and $C \ge 0$ is a constant.

Moments and Onsager's inequality continued

Following [GP77], one can apply Onsager's inequality directly in the expansion

$$\mathbb{E}[|\mu(f)|^{2N}] = \int \prod_{j=1}^{N} f(x_j)\overline{f(x_{N+j})}e^{-\beta^2 \sum_{1\leq j<k\leq 2N} q_j q_k C(x_j,x_k)},$$

where $q_j = 1$ if $1 \leq j \leq N$ and $q_j = -1$ if $N+1 \leq j \leq 2N$ to get an upper bound of the form

$$\mathbb{E}[|\mu(f)|^{2N}] \lesssim C^N \sum_{\pi} \int \prod_{j=1}^{2N} |x_j - x_{\pi(j)}|^{-\beta^2/2}$$

where the sum runs over functions $\pi \colon \{1, \ldots, 2N\} \to \{1, \ldots, 2N\}$ designating a nearest neighbour $x_{\pi(j)}$ to every x_j.

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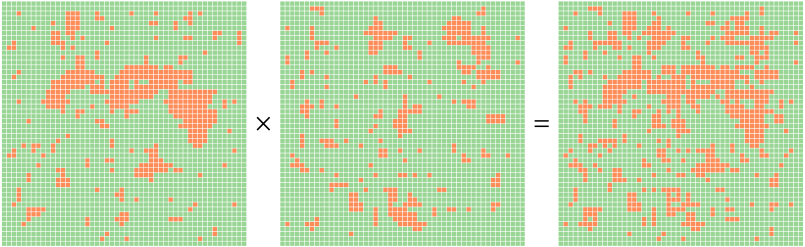
- Estimating the remaining integral can be done using a combinatorial argument.

Application: XOR-Ising model

- Two independent critical Ising models with spins (± 1) multiplied together.
- The spin field convergences to the real part of μ with $\beta = 1/\sqrt{2}$.
- Proof via method of moments: Correlation functions are known to converge pointwise [CHI15] to the right limit. To justify the convergence of the moments we prove a version of Onsager's inequality for the XOR-Ising model and use the dominated convergence theorem.



Interlude: Meaning of imaginary chaos?
Or is there any meaning?

Studying the field using multiplicative chaos?

- Real values of γ yield random measures that are supported on the so called γ-thick points of the field, i.e. those points for which $\lim_{\varepsilon \to 0} \frac{X_\varepsilon(x)}{\mathbb{E}[X_\varepsilon(x)^2]} = \gamma$.

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- Answer: Not yet completely understood, but see e.g. [SSV20] where imaginary chaos is used to study the Hausdorff dimension of two-valued sets of the GFF.
- A perhaps simpler fundamental question: Is it possible to recover the field X from μ? Or are we just studying some fancy noise?

Going from the chaos to the field

- When γ is real, one can recover the field X from μ as a simple limit [BSS14,Aru20]

$$X(x) = \lim_{\varepsilon \to 0} \gamma^{-1} (\log \mu(B(x, \varepsilon)) - \mathbb{E} \log \mu(B(x, \varepsilon))).$$

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- In the case g is differentiable a nice way of stating this tracking procedure is by saying that one can recover $\nabla g(x)$ from f via the explicit formula $\nabla g(x) = \frac{\nabla f(x)}{i\beta f(x)}$.

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- In reality a bit tricky because of renormalizations and infinities – dimension 2 is harder than larger dimensions and we do not know what happends when $d = 1$.

Existence of densities via Malliavin calculus

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Theorem

For any nonzero $f \in C_c^\infty(U)$ the random variable $\mu(f)$ has a density w.r.t. the Lebesgue measure on $\mathbb{C}$ and the density is a Schwartz function. Moreover, as $\beta \to \sqrt{d}$ the density tends to 0 pointwise and is uniformly bounded from above.

Application: Fyodorov–Bouchaud formula does not extend to imaginary chaos

Theorem (Remy, 2020)

Let X be the Gaussian field on S^1 with the covariance $\mathbb{E}[X(x)X(y)] = -\log|x-y|$ for $|x| = |y| = 1$. Let ν be the corresponding (real) GMC measure with parameter $\gamma \in (0, \sqrt{2})$. Then $\mathbb{E}[\nu(S^1)^p] = \frac{\Gamma(1-p\gamma^2/2)}{\Gamma(1-\gamma^2/2)^p}$.

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- Does this hold for $\gamma = i\beta$?
- Answer: Yes for $p \in \mathbb{N}$ but not in general.
- Idea: Let ρ_β be the p.d.f. of $\nu(S^1)$. Then $\mathbb{E}[\nu(S^1)^{-1}] \leq \int |x|^{-1} \rho_\beta(x)\, dx \to 0$ as $\beta \to 1$, but F–B does not tend to 0.

Preliminaries for proof: Isonormal Gaussian processes

Setup:

- Cameron–Martin space H: a (separable) Hilbert space.
- Isonormal Gaussian process $(\langle X, h\rangle_H)_{h\in H}$:
  $\mathbb{E}[\langle X, h\rangle_H \langle X, k\rangle_H] = \langle h, k\rangle_H$
- Formally $X = \sum_{k=1}^{\infty} X_k h_k$ where $(h_k)_{k=1}^{\infty}$ is an orthonormal basis and $(X_k)_{k=1}^{\infty}$ are i.i.d. standard Gaussians.

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In this talk: $H = H_0^1(U)$, X the GFF on U and $\sum_{k=1}^{\infty} X_k h_k$ converges in $H^{-\varepsilon}(U)$ for any $\varepsilon > 0$.

- $\langle f, g\rangle_H = \frac{1}{2\pi}\int_U \nabla f(x)\cdot\nabla g(x)\,dx = \int_U f(x)Lg(x)\,dx = \langle f, Lg\rangle_{L^2}$
- $L = -(2\pi)^{-1}\Delta$
- L^{-1} has the kernel $G(x, y) \sim \log\frac{1}{|x-y|}$.

Preliminaries for proof: Malliavin derivatives

- Suppose that $H = \mathbb{R}^n$ and that Y is an X-measurable random variable of the form $Y = F(X)$ with $F\colon \mathbb{R}^n \to \mathbb{R}$ differentiable.
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- In this case the *Malliavin derivative* of Y would simply be the $\mathbb{R}^n$-valued random vector $\nabla F(X)$.
- In general, if we have a random variable of the form $Y = F(\langle X, h_1 \rangle_H, \dots, \langle X, h_n \rangle_H)$ we define its Malliavin derivative as the H-valued random variable

$$DY = \sum_{k=1}^{n} \partial_k F(\langle X, h_1 \rangle_H, \dots, \langle X, h_n \rangle_H) h_k.$$

- For other suitable Y we may then define DY by taking limits of such *smooth random variables*.
- Completion under the natural norm $\|Y\|_{1,p}^p = \mathbb{E}[|Y|^p] + \mathbb{E}[\|DY\|^p]$ leads to the space $\mathbb{D}^{1,p}$.

Motivation: regularity of probability laws

Heuristic idea: $\|DY\|_H$ measures how easy it would have been to sample other values around Y.

Theorem

If $Y \in \mathbb{D}^{1,2}$ and $\|DY\|_H > 0$ a.s., then Y has a density w.r.t. the Lebesgue measure on $\mathbb{R}$.

Random vectors and smoothness of laws

To study the smoothness of laws we need higher order Malliavin derivatives which take values in $H^{\otimes k}$:

- For $Y = F(\langle X, h_1 \rangle_H, \dots, \langle X, h_n \rangle_H)$ we set

$$D^k Y = \sum_{\ell_1, \dots, \ell_k = 1}^{n} \partial_{\ell_1, \dots, \ell_k} F(\langle X, h_1 \rangle_H, \dots, \langle X, h_n \rangle_H) h_{\ell_1} \otimes \cdots \otimes h_{\ell_n}.$$

- Define $\mathbb{D}^{k,p}$ in a natural way and let $\mathbb{D}^{\infty} = \bigcap_{p>1} \bigcap_{k>1} \mathbb{D}^{k,p}$.

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Theorem

Let $(Y_1, \dots, Y_n)$ be a random vector in $\mathbb{R}^n$ such that $Y_k \in \mathbb{D}^\infty$ for all $k \in \{1, \dots, n\}$ and define the matrix $\gamma_Y = (\langle DY_i, DY_j\rangle_H)_{i,j=1}^n$. Suppose that $\mathbb{E}[(\det \gamma_Y)^{-p}] < \infty$ for all $p > 1$. Then $(Y_1, \dots, Y_n)$ has a density ρ w.r.t. the Lebesgue measure on $\mathbb{R}^n$ and ρ is a Schwartz function.

Complex valued random variables

All the definitions carry in a natural way to random variables taking values in $\mathbb{C}$.

- $DY = D\,\mathrm{Re}(Y) + iD\,\mathrm{Im}(Y)$

Also, if $Y = (\mathrm{Re}(Y), \mathrm{Im}(Y))$, we may write $\det \gamma_Y$ using complex notation as

$$\det \gamma_Y = \frac{1}{4}(\|DY\|_H^4 - |\langle DY, \overline{DY}\rangle_H|^2).$$

Malliavin derivative of $\mu(f)$

Note that if X would have pointwise values,

$$X(x) = \sum_{k=1}^{\infty} X_k h_k(x) = \sum_{k=1}^{\infty} \langle X, h_k \rangle_H h_k(x),$$

then

$$D(X(x))(y) = \sum_{k=1}^{\infty} h_k(x) h_k(y) = \mathbb{E}[X(x)X(y)] = G(x, y).$$

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Thus by chain rule the Malliavin derivative of

$$M = \mu(f) = \int :e^{i\beta X(x)}: f(x)\,dx$$

should be given by

$$DM(y) = i\beta \int :e^{i\beta X(x)}: f(x) G(x,y)\,dx = i\beta L^{-1}(f\mu)$$

and one can prove that this is indeed true and moreover $M \in \mathbb{D}^{\infty}$.

Malliavin determinant for $\mu(f)$

Writing $\mu(x) =: e^{i\beta X(x)}$: we have

$$
\begin{aligned}
\|DM\|_H^2 &= \beta^2 \langle L^{-1}(f\mu), L^{-1}(f\mu)\rangle_H = \beta^2 \langle L^{-1}(f\mu), f\mu\rangle_{L^2} \\
&= \beta^2 \int f(x)\overline{f(y)}\mu(x)\overline{\mu(y)}G(x,y)\,dx\,dy.
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Recall that we would like to show that

$$\det \gamma_M = \frac{1}{4}(\|DM\|_H^4 - |\langle DM, \overline{DM}\rangle_H|^2)$$

has negative moments of all orders. In this talk we will instead focus on the easier but morally equivalent problem of showing $\mathbb{E}[\|DM\|_H^{-p}] < \infty$ for all $p > 0$.

Projection bounds

Note that for any nonzero $h \in H$ we have

$$\|DM\|_H \geq \frac{|\langle DM, h \rangle_H|}{\|h\|_H} = \frac{\beta |\langle L^{-1}(f\mu), h \rangle_H|}{\|h\|_H} = \frac{\beta |\mu(f\overline{h})|}{\|h\|_H}.$$

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In this case

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Thus $\|DM\|_H \gtrsim \delta^{1-\frac{\beta^2}{2}+\varepsilon}$ with very high probability.

Projection bounds (continued)

More precisely, for a fixed small enough $\varepsilon > 0$ one can show that for all $\delta > 0$ small enough we have

$$\mathbb{P}[\|DM\|_H \geq \delta^{1-\frac{\beta^2}{2}+\varepsilon}] \geq 1 - e^{-c\delta^{-d}}$$

for some constants $c, d > 0$. This suffices to show that $\mathbb{E}[\|DM\|_H^{-p}] < \infty$ for all $p > 0$.

- One can notice here a nice general strategy: Showing that something is not too small with large probability by finding a sequence of lower bounds which concentrate better and better, the point being that one can again work with positive moments instead of negative ones to show the concentration.

Some remarks

- This choice of h for the projection is not good enough to show that the density tends to 0 as $\beta \to \sqrt{d}$. Instead we use something like

$$h(x) = e^{i\beta X_\delta(x) - \frac{\beta^2}{2}\mathbb{E}[X_\delta(x)^2]} \int f(y) : e^{i\beta \hat{X}_\delta(x)} : \mathbb{E}[\hat{X}_\delta(x)\hat{X}_\delta(y)]\, dy$$

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  (with some additional technical tricks).
- There is also a direct projection bound for $\det \gamma_M$.
- Getting pointwise bounds on the density requires bounding $\mathbb{E}[|\delta(A)|^p]$, where δ is the so-called *divergence operator* and

$$A = \frac{\|DM\|_H^2 DM - \langle DM, D\overline{M}\rangle_H D\overline{M}}{\|DM\|_H^4 - |\langle DM, D\overline{M}\rangle_H|^2}.$$

- To this end we show that

$$\delta(A) \lesssim \frac{\|DM\|_H^2(|\delta(DM)| + \|D^2 M\|_{H\otimes H})}{\|DM\|_H^4 - |\langle DM, D\overline{M}\rangle_H|^2}.$$

General log-correlated fields and decompositions

The methods can in fact be generalized to a large class of log-correlated Gaussian fields.

- Some of the nicest log-correlated fields/approximations to work with are so-called star-scale invariant fields:
  - Nice scaling properties.
  - Nice independence structure both in space and in the level of approximation.
- We prove a general decomposition theorem which lets one express **any non-degenerate** log-correlated Gaussian field (satisfying some mild regularity conditions) as the sum of an almost star-scale invariant field and an **independent** regular field.

Thanks!

Papers

- J. Aru, A. Jego and J. Junnila: Density of imaginary multiplicative chaos via Malliavin calculus, Probability Theory and Related Fields, 2022.
- J. Aru and J. Junnila: Reconstructing the base field from imaginary multiplicative chaos, Bulletin of the London Mathematical Society, 2021.
- J. Junnila, E. Saksman and C. Webb: Imaginary multiplicative chaos: Moments, regularity and connections to the Ising model, Annals of Applied Probability, 2020.
- J. Junnila, E. Saksman and C. Webb: Decompositions of log-correlated fields with applications, Annals of Applied Probability, 2019.