Conformal Bootstrap

(or at least the thing that I know by this name)

based on work with D. Mazáč and S. Pal

[arXiv: 2111.12716]

and also J. Bonifacio

[WIP]

+ partially on work with J. Qiao & S. Rychkov

[arXiv: 2104.02090]

Petr Kravchuk, King's College London

Probability and Conformal Field Theory

Agay les Roches Rouges September 20, 2022

Outline

1. Bootstrap point of view on CFTs
2. Numerical Bootstrap
3. New Crossing Equations
4. Solutions from CFTs and Hyperbolic Manifolds

…to be continued in the talk by Dalimil Mazáč

Conformal Bootstrap

Part 1

Conformal Field Theory

↓

A solution to a rich set of self-consistency conditions

↓ **Numerics**, exact functionals, etc…

Useful results

Conformal Bootstrap

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Part 2

Hyperbolic Manifolds
Conformal Field Theory

↓

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$\approx$

Part 2

Hyperbolic Manifolds Conformal Field Theory

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A solution to a rich set of self-consistency conditions

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Useful results

Bootstrap from Axiomatics

Setup: correlation functions of local operators on $\mathbb{R}^d, S^d$ or $\mathbb{R}^{d-1,1}, \mathbb{R} \times S^{d-1}$, all related by analytic continuation

Less established: boundaries, defects, other manifolds…

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- Wightman/OS Axioms
- Conformal Symmetry
- Asymptotic OPE

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \underset{x\to 0}{\sim} \sum_i c_i(x)\mathcal{O}_i(0)$$

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- Asymptotic OPE

[Mack, Luscher, …] → Convergent OPE in the vacuum state

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \underset{x\to 0}{\sim} \sum_i c_i(x)\mathcal{O}_i(0)$$

$$\mathcal{O}_1(x)\mathcal{O}_2(0)\,|\,0\rangle = \sum_i c_i(x)\mathcal{O}_i(0)\,|\,0\rangle$$

Bootstrap Axiomatics [Qiao, Rychkov, PK'21]

- A set of primary local operators $\{\mathcal{O}_i(x)\}$ with $\Delta_i \in \mathbb{R}$ and $\rho_i \in \widehat{\mathrm{SO}(d)}$

$$(g \cdot \mathcal{O}_i)(x) = \Omega_g(x)^\Delta \rho(R_g(x))\mathcal{O}_i(g^{-1}x) \qquad g \in \mathrm{SO}(1,d+1)$$

$$\mathcal{O}_i(x),\ \partial_\mu \mathcal{O}_i(x),\ \partial_\mu \partial_\nu \mathcal{O}_i(x), \cdots$$

- A collection of conformally-invariant correlation (Schwinger) functions

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle, \qquad x_i \in \mathbb{R}^d,\ x_i \neq x_j$$

$$\langle \mathcal{O}_{i_1}(x_1) \cdots \mathcal{O}_{i_n}(x_n) \rangle = \langle (g \cdot \mathcal{O}_{i_1})(x_1) \cdots (g \cdot \mathcal{O}_{i_n})(x_n) \rangle$$

$$g \in \mathrm{SO}(1,d+1)$$

Bootstrap Axiomatics

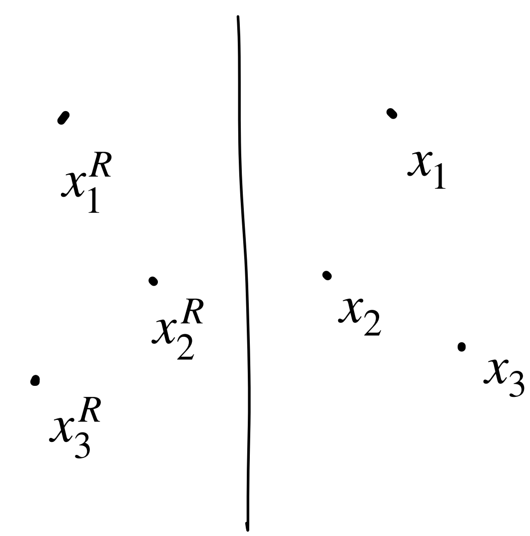
- Convergent OPE

$$\langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle = \sum_k f^k_{i_1 i_2} c_{i_1,i_2,k}(x_1, x_2, \partial_{x_2})\langle \mathcal{O}_k(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle$$

- Reflection positivity

$$\langle \mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\mathcal{O}_{i_1}(x_1^R)\cdots\mathcal{O}_{i_n}(x_n^R)\rangle \geq 0$$

$$\int d^{n\times d}x\, d^{n\times d}y\, f(x) f^*(y^R)\langle \mathcal{O}_{i_1}(x_1)\cdots\mathcal{O}_{i_n}(x_n)\mathcal{O}_{i_1}(y_1)\cdots\mathcal{O}_{i_n}(y_n)\rangle \geq 0$$



is often imposed only on 2-point functions

$$\int d^d x\, d^d y\, f(x) f^*(y^R)\langle \mathcal{O}_i(x)\mathcal{O}_i(y^R)\rangle \geq 0$$

Bootstrap Axiomatics

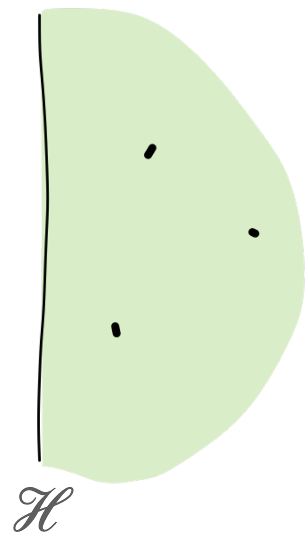
General idea: if "CFT data" $\{(\Delta_k, \rho_k), f_{ij}^k\}$ leads to associative OPE then it should define a QFT (i.e. O-S axioms should follow etc…)

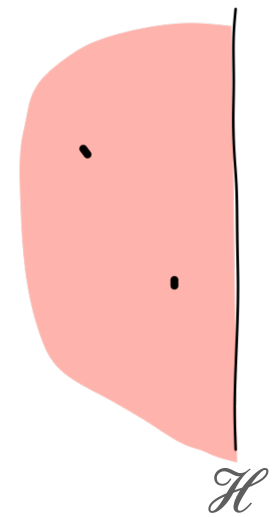
Theorem: if "CFT data" $\{(\Delta_k, \rho_k), f_{ij}^k\}$ leads to associative OPE then O-S and Wightman axioms are satisfied by n-point functions with $n \leq 4$. [Qiao, Rychkov, PK'21]

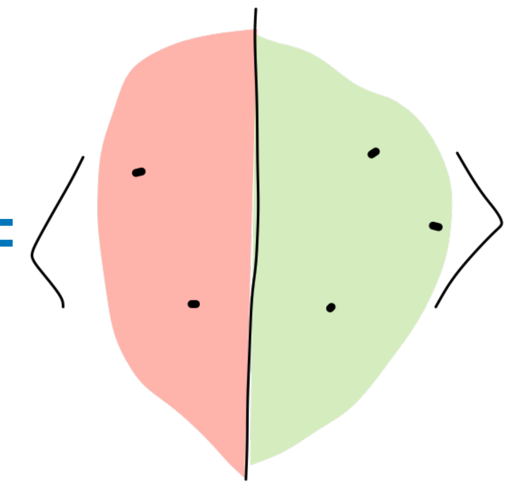
Higher-point functions are more subtle (e.g. precise form of the OPE convergence statement).

Bootstrap Axiomatics

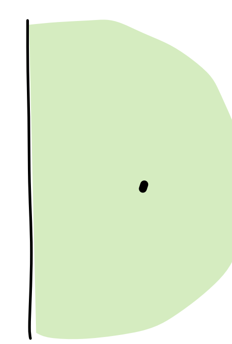
- Reflection positivity can be used to construct the Hilbert space $\mathscr{H}$

$|\Psi\rangle =$ 

$\langle\Phi| =$ 

$\langle\Phi|\Psi\rangle =$ 

- The conformal group in $\mathbb{R}^d$ is $\mathrm{SO}(d+1,1)$
- Due to x^R in the positivity condition, $\widetilde{\mathrm{SO}}(d,2)$ is represented unitarily

$$\mathscr{H} = \bigoplus_i R[\mathcal{O}_i] = \mathrm{Span}\{$$  $$\}$$

- The OPE is the partial wave expansion for $\widetilde{\mathrm{SO}}(d,2)$

Crossing equations

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$$

Crossing equations

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$\widetilde{\mathrm{SO}}(d,2)$ symmetry

$$z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$$

$$(1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

$$\sum_k f_k^2 G_{\Delta_k,\rho_k}(z,\bar{z}) = \sum_k f_k^2 G_{\Delta_k,\rho_k}(1-z,1-\bar{z})$$

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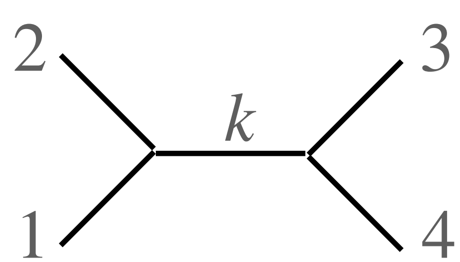
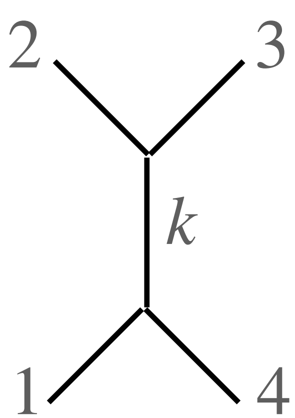
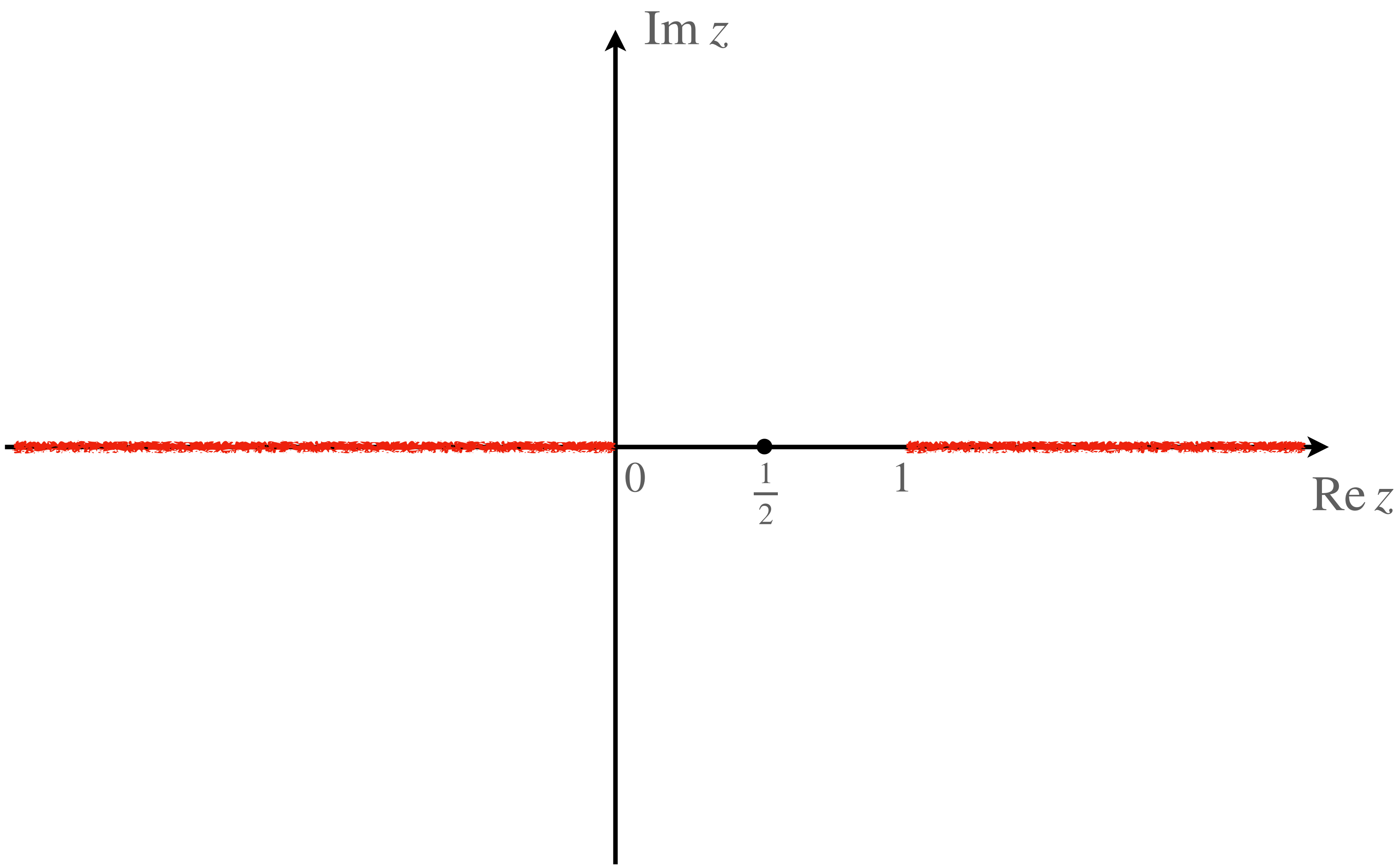




Crossing equations

Example: $d = 1$ $\qquad G_\Delta(z) = z^{\Delta - 2\Delta_\phi} {}_2F_1(\Delta, \Delta, 2\Delta, z)$

$$\sum_k f_k^2 G_{\Delta_k}(z) = \sum_k f_k^2 G_{\Delta_k}(1 - z)$$



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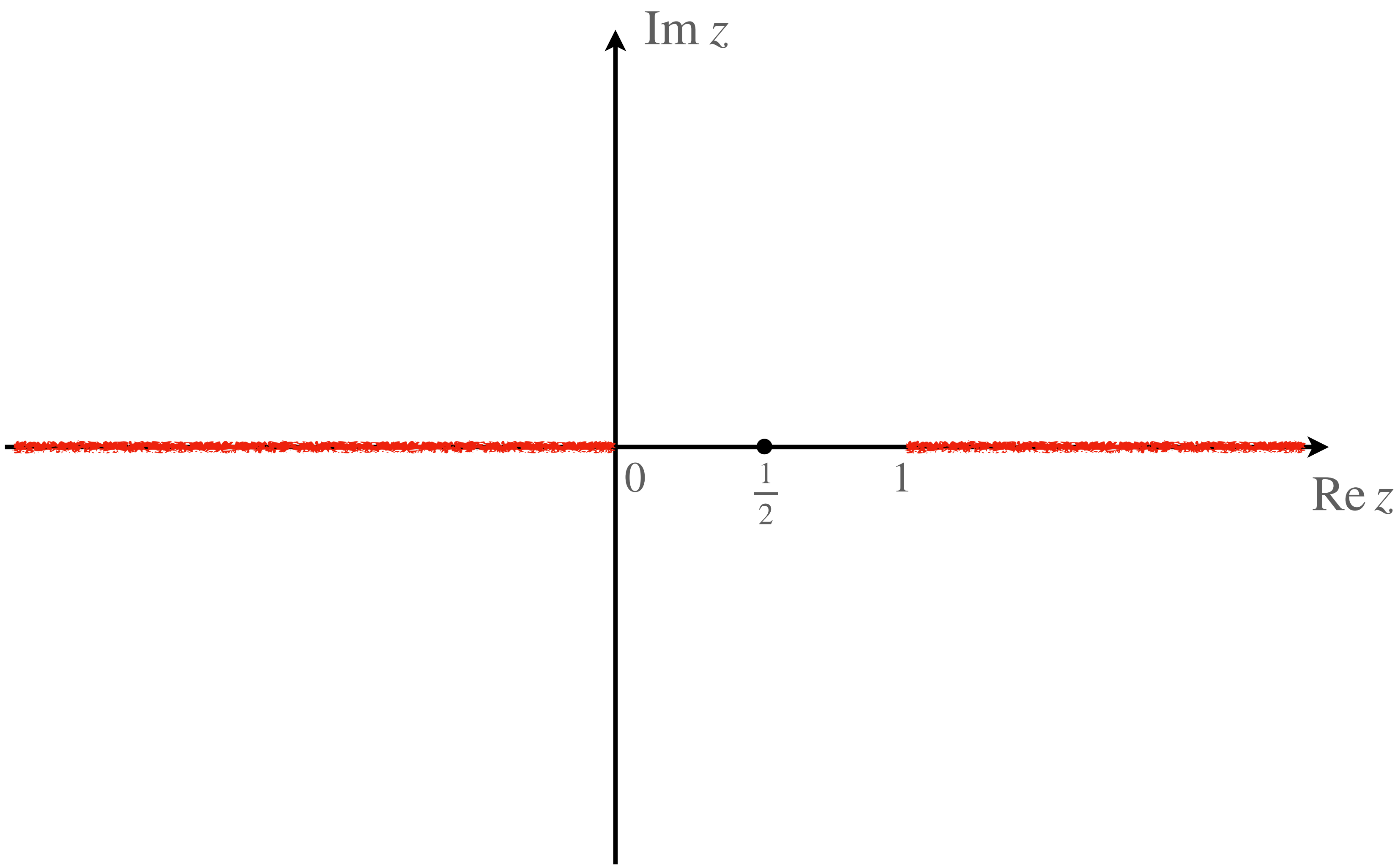
Taylor expansion at $z = \frac{1}{2}$

$$\sum_k f_k^2 \vec{F}_{\Delta_k} = 0$$



Numerical Bootstrap

[Rattazzi,Rychkov,Tonni,Vichi'08]

$$\sum_k f_k^2 \overrightarrow{F}_{\Delta_k} = 0 \qquad f_k^2 \geq 0$$

(Reflection positivity)

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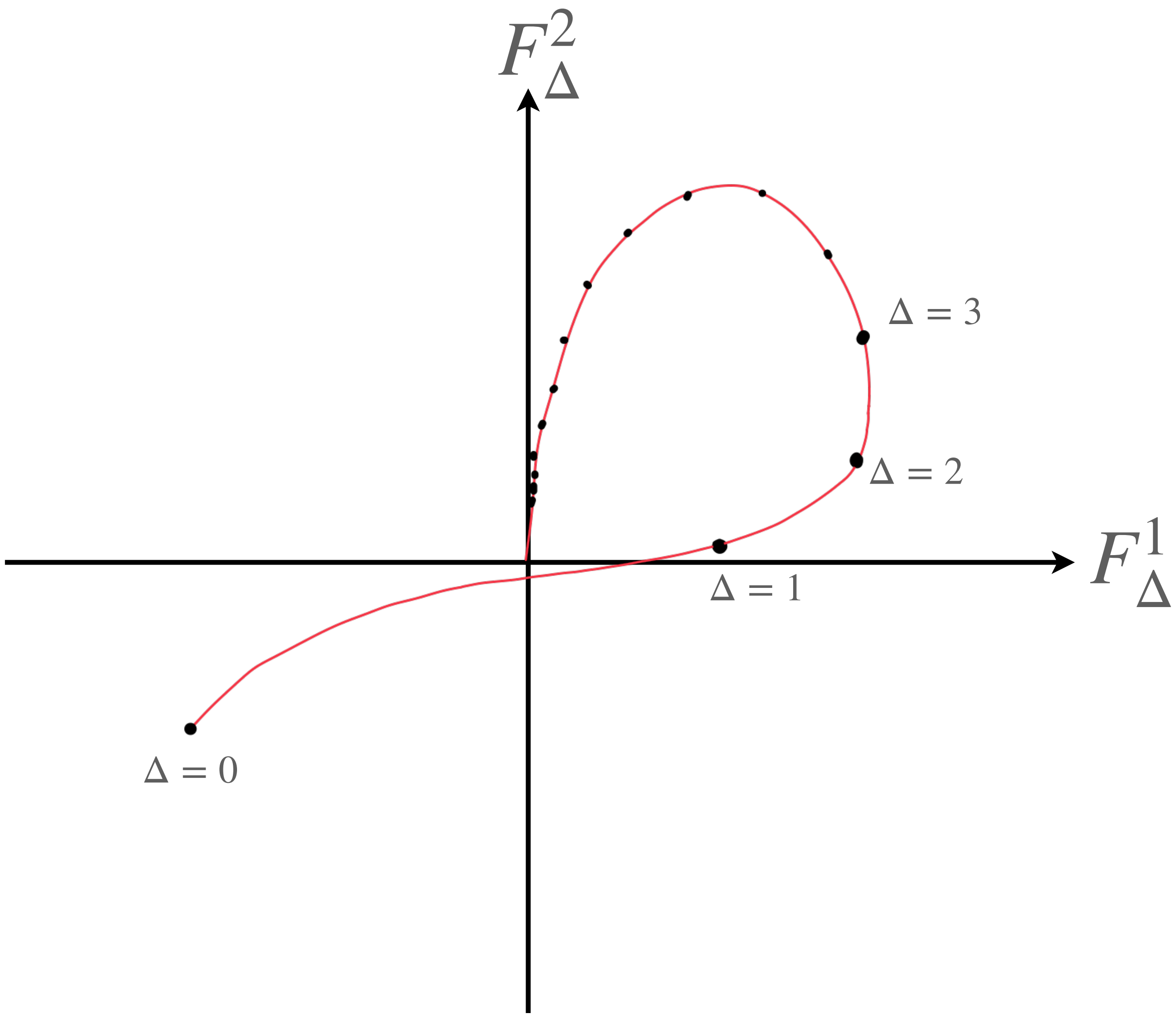

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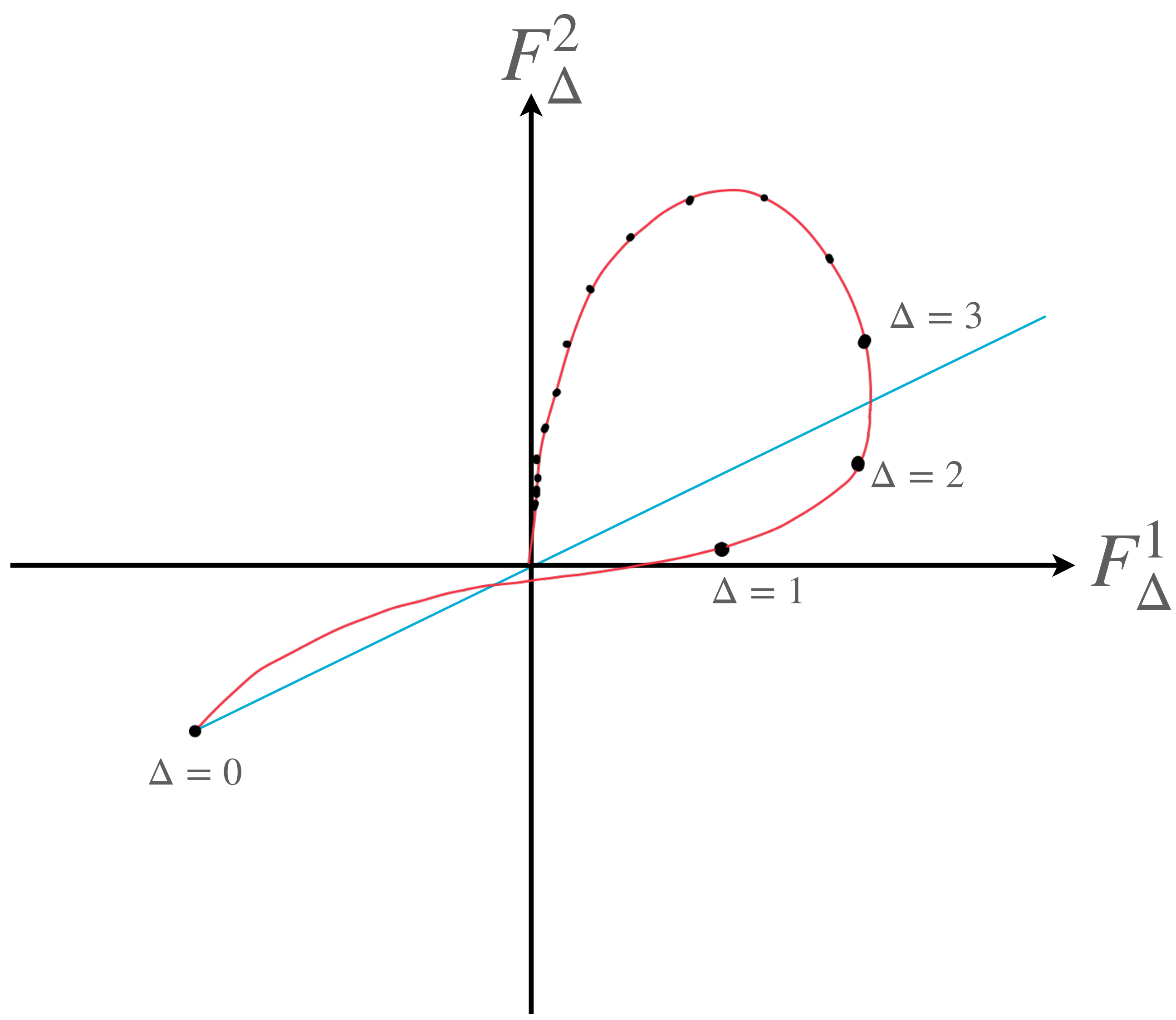

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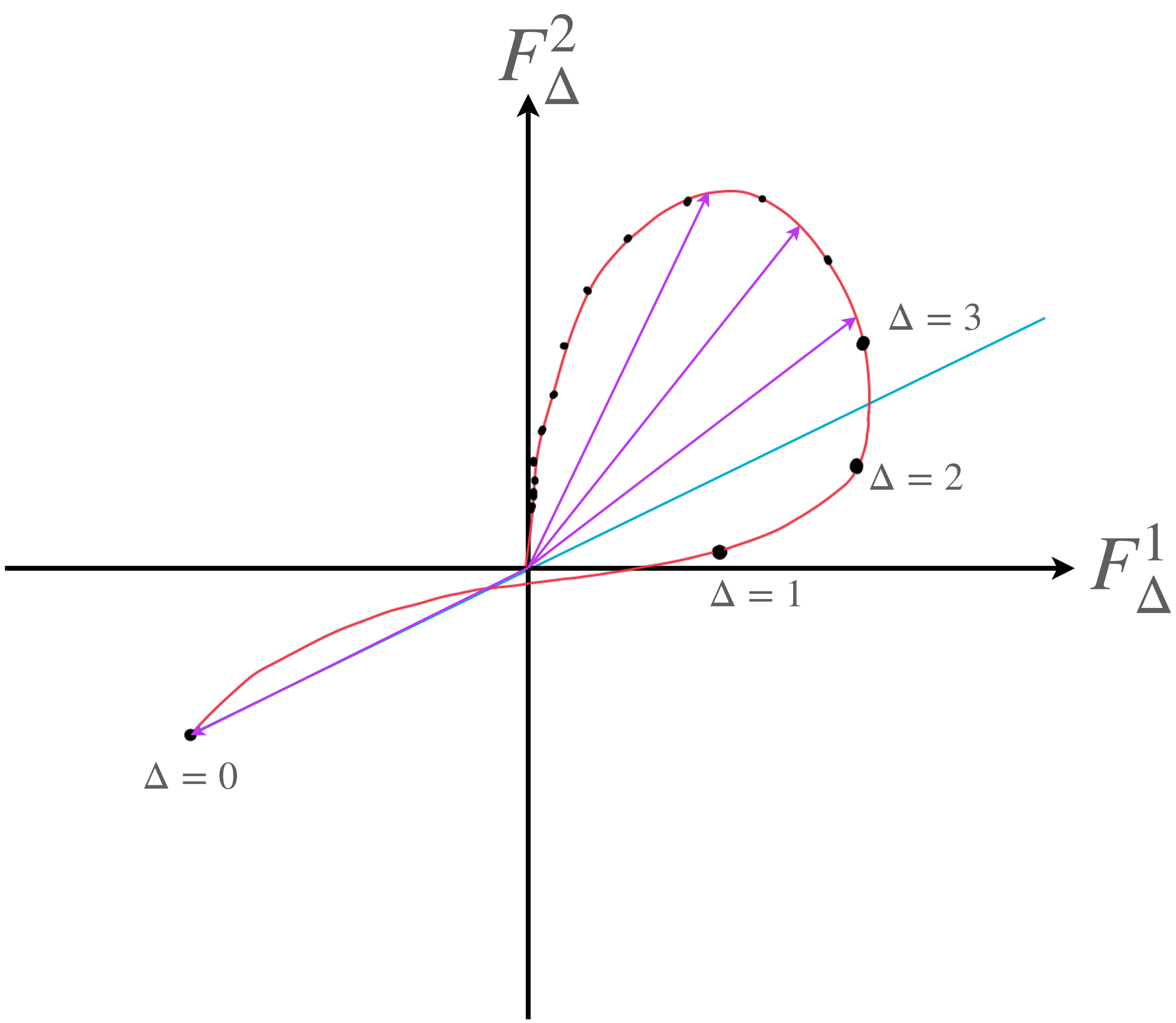

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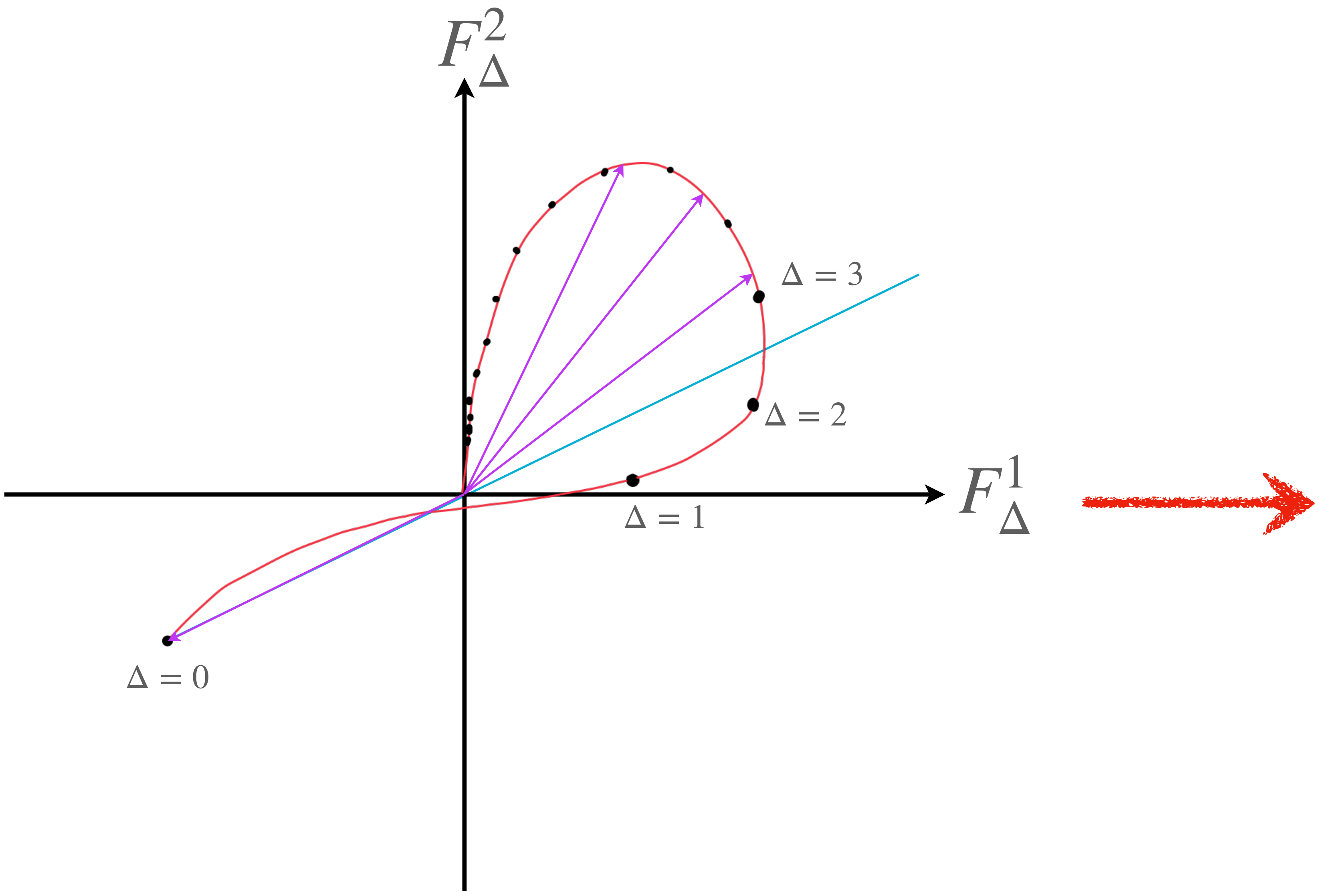

$$\exists k : \Delta_k \in [0.5, 2.5]$$

Numerical Bootstrap

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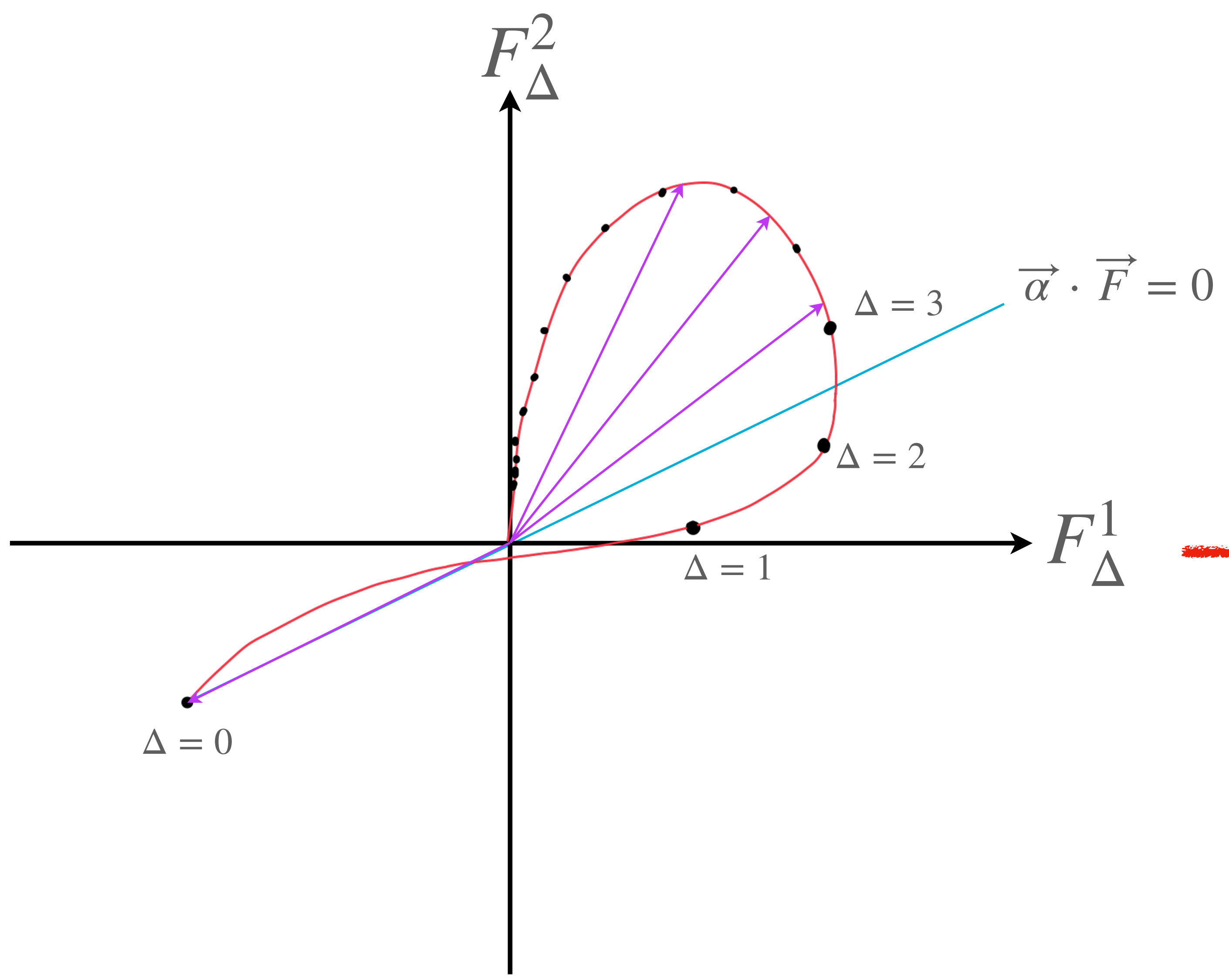

$$\longrightarrow \quad \exists k : \Delta_k \in [0.5, 2.5]$$

Numerical bootstrap

- In general try to find $\vec{\alpha}$ such that $\vec{\alpha} \cdot \vec{F}_\Delta \geq 0$ for all Δ in a trial spectrum — a linear program.

- Several correlation functions can be studied at the same time using semidefinite programming ($\vec{\alpha} \cdot \vec{F}_\Delta \succeq 0$)

$$\langle \phi\phi\phi\phi \rangle \longrightarrow \langle \phi\phi\phi\phi \rangle \quad \langle \phi\phi\epsilon\epsilon \rangle \quad \langle \epsilon\epsilon\epsilon\epsilon \rangle$$

$$f_k^2 = f_{\phi\phi k}^2 \longrightarrow f_{\phi\phi k} f_{\epsilon\epsilon k}, f_{\phi\phi k}^2, f_{\epsilon\epsilon k}^2, f_{\phi\epsilon k}^2$$

$$f_{\phi\phi k}^2 \geq 0 \longrightarrow \begin{cases} \begin{pmatrix} f_{\phi\phi k}^2 & f_{\phi\phi k} f_{\epsilon\epsilon k} \\ f_{\phi\phi k} f_{\epsilon\epsilon k} & f_{\epsilon\epsilon k}^2 \end{pmatrix} \succeq 0 \\ f_{\phi\epsilon k}^2 \geq 0 \end{cases}$$

- We now have efficient and general algorithms for semidefinite programming (SDPB) and conformal blocks (blocks_3d, …)

- Mostly numerical, but some exact $\vec{\alpha}$ are known

Numerical bootstrap

3d Ising

$\langle \sigma\sigma\sigma\sigma \rangle$

$\langle \sigma\sigma\epsilon\epsilon \rangle$

$\langle \epsilon\epsilon\epsilon\epsilon \rangle$

Numerical bootstrap

3d Ising

$\langle \sigma\sigma\sigma\sigma \rangle$

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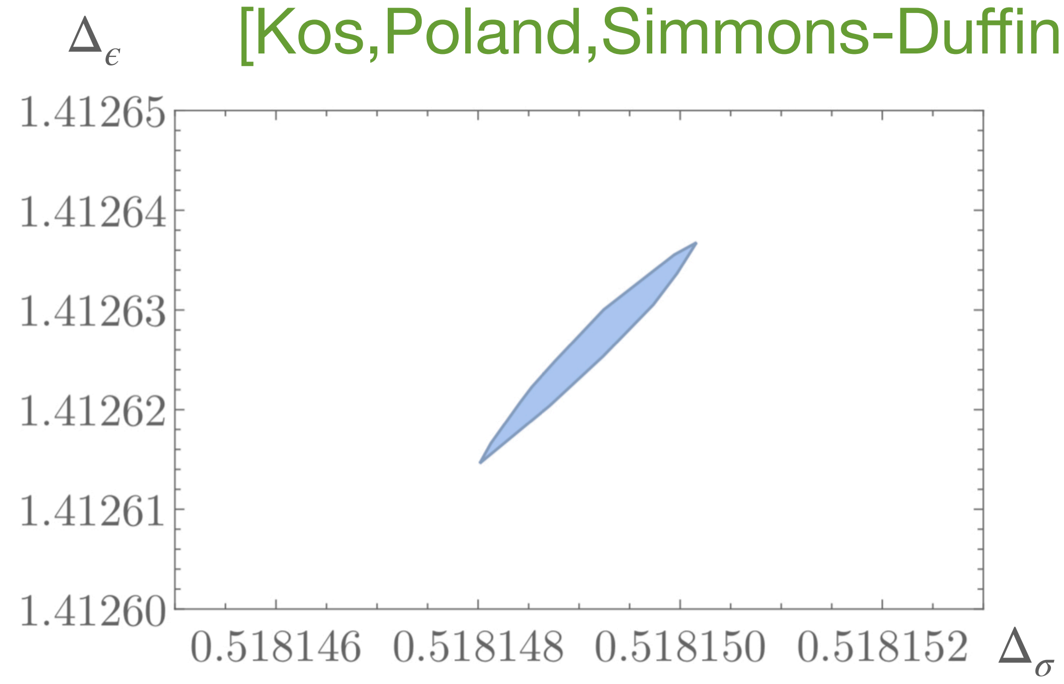
$\langle \epsilon\epsilon\epsilon\epsilon \rangle$

[Kos,Poland,Simmons-Duffin,Vichi'16]



Numerical bootstrap

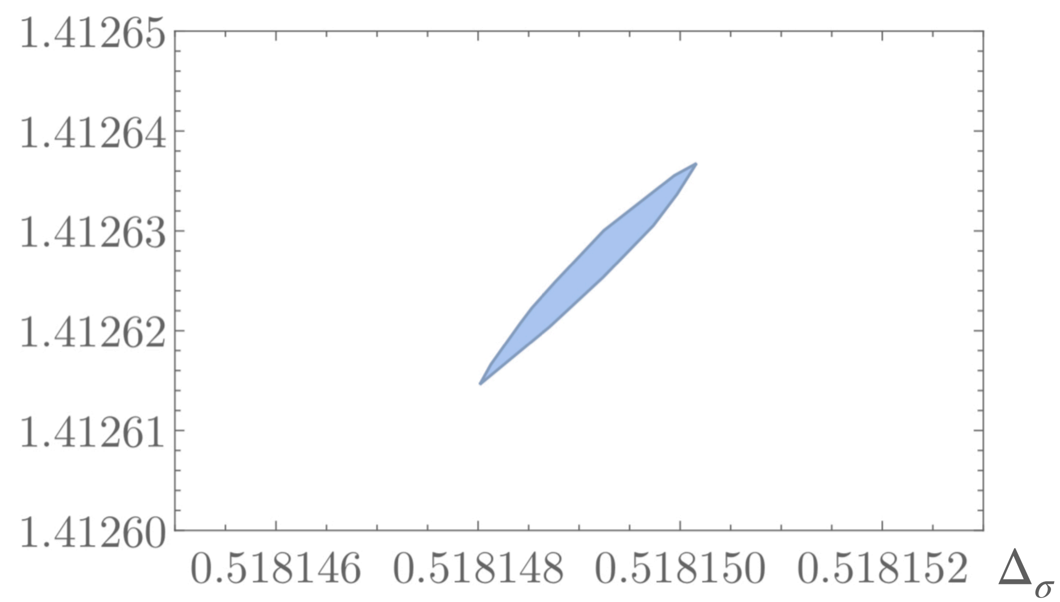
3d Ising

$\langle \sigma\sigma\sigma\sigma \rangle$

$\langle \sigma\sigma\epsilon\epsilon \rangle$

$\langle \epsilon\epsilon\epsilon\epsilon \rangle$

[Kos,Poland,Simmons-Duffin,Vichi'16]



3d Gross-Neveu-Yukawa

$$(\partial\sigma)^2 + \sigma^4 + \psi_a \partial \psi_a + \sigma \psi_a \psi_a$$

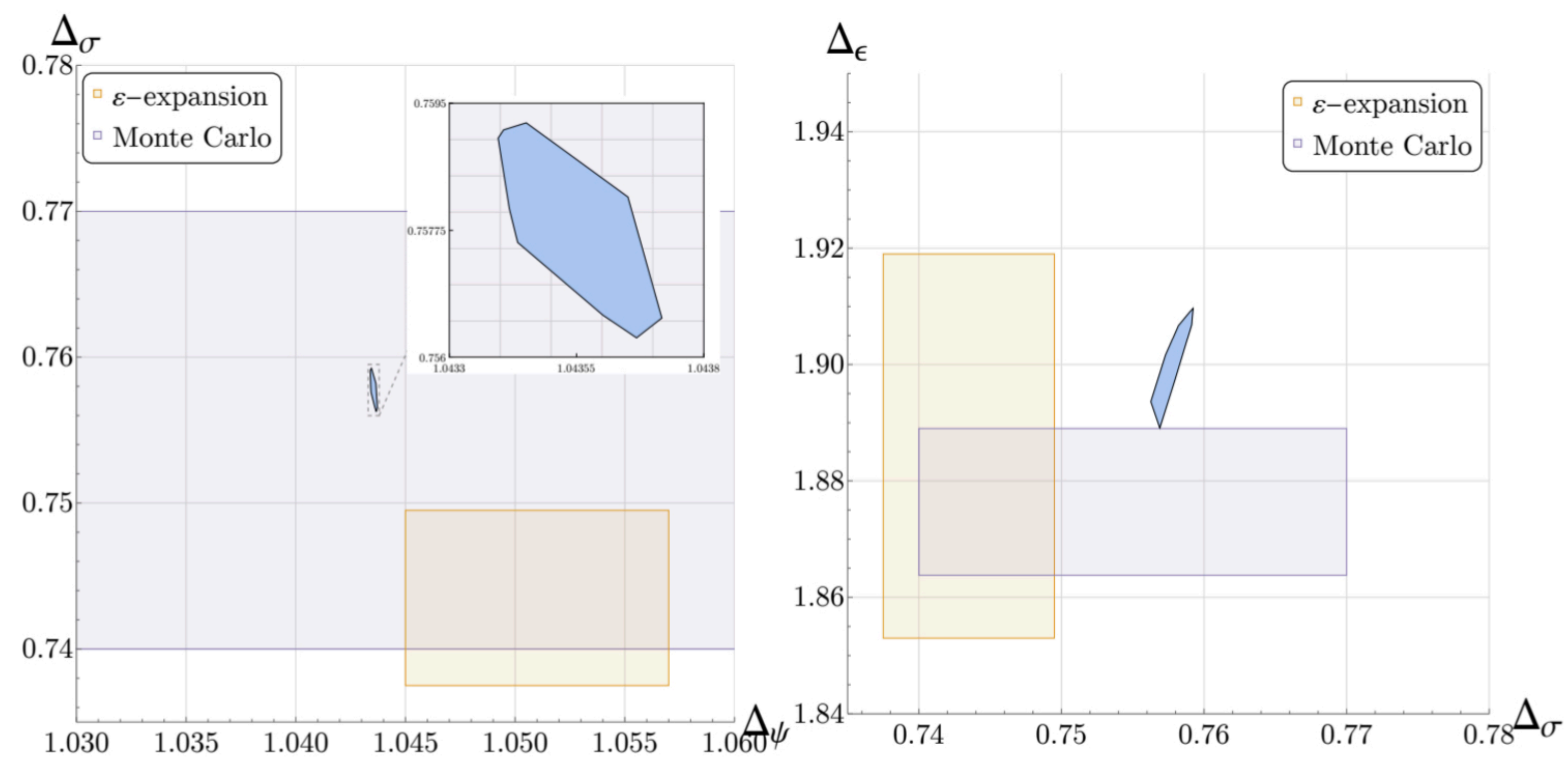
$\langle \sigma\sigma\sigma\sigma \rangle$ $\langle \psi\psi\psi\psi \rangle$

$\langle \sigma\sigma\epsilon\epsilon \rangle$ $\langle \psi\psi\sigma\sigma \rangle$

$\langle \epsilon\epsilon\epsilon\epsilon \rangle$ $\langle \psi\psi\epsilon\epsilon \rangle$

$\langle \psi\psi\sigma\epsilon \rangle$

[Erramilli, Iliesiu, Liu, Poland, Simmons-Duffin, PK]



Part 2: Euclidean positivity

Reflection positivity

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$\widetilde{\mathrm{SO}}(2,d)$ unitary action on $\mathscr{H}$

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OPE from irreps

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OPE from irreps

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Useful results

Part 2: Euclidean positivity

Reflection positivity

↓

$\widetilde{\mathrm{SO}}(2,d)$ unitary action on $\mathscr{H}$

↓

OPE from irreps

↓

Useful results

Euclidean positivity
(probability ≥ 0)

↓

$\mathrm{SO}(1,d+1)$ unitary action on $\mathscr{H}_E$

↓

OPE from irreps

↓

Useful results?

Euclidean positivity

$$\langle \mathcal{O} \rangle = \int_{\Phi} d\mu(\phi) \mathcal{O}(\phi) \qquad\qquad d\mu(\phi) \approx D\phi e^{-S[\phi]} \geq 0$$

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If $\langle \, | \, \mathcal{O} \, |^2 \, \rangle$ makes sense then

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Direct analogue of O-S reconstruction:

$$\mathcal{H}_E = \{\mathcal{O} \,|\, \langle\, |\, \mathcal{O}\, |^2 \rangle \text{ makes sense}\} = L^2(\Phi, d\mu)$$

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Direct analogue of O-S reconstruction:

$$\mathscr{H}_E = \{\mathcal{O}\, |\, \langle\, |\, \mathcal{O}\, |^2 \rangle \text{ makes sense}\} = L^2(\Phi, d\mu)$$

$\mathrm{SO}(1,\mathrm{d}+1)$ invariance of $d\mu(\phi)$ => unitary action on $L^2(\Phi, d\mu)$

Examples

Conformal Field Theories: $\Phi = \{\text{distributions}\}$

- GFF
- 2d Ising [Camia,Garban,Newman'12]
- 3d Ising?
- Statistical models without RP?

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Hyperbolic $(d+1)$-manifolds: $\Phi = \Gamma\backslash G$

- $G = \mathrm{SO}(1,\mathrm{d}+1)$
- $\Gamma \simeq \pi_1(M)$
- $M = \Gamma\backslash G/\mathrm{SO}(d+1) = \Gamma\backslash\mathbb{H}^{d+1}$

Hyperbolic manifolds

Consider $d+1=2$ and compact M

$G = \mathrm{SL}(2,\mathbb{R})$

$$L^2(\Gamma\backslash G) \simeq \mathbb{C} \oplus \sum_n (D_n \oplus \bar{D}_n) \oplus \bigoplus_i P_{\lambda_i}$$

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Principal series representations $P_\lambda,\ \lambda \geq \frac{1}{4}$

$$P_\lambda = L^2(S^1)$$

$$(gf)(\theta) = \Omega_g(\theta)^{1-\Delta} f(g^{-1}\theta)$$

$\lambda = \Delta(1 - \Delta)$

Hyperbolic manifolds

Consider $d+1=2$ and compact M

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Principal series representations P_λ, $\lambda \geq \frac{1}{4}$

$$P_\lambda = L^2(S^1)$$

$$(gf)(\theta) = \Omega_g(\theta)^{1-\Delta} f(g^{-1}\theta)$$

$\lambda = \Delta(1-\Delta)$

Define $\mathbb{O}_i : C^\infty(S^1) \to (L^2(\Gamma\backslash G))^\infty$ by

$$\mathbb{O}_i(f) = f \in P^\infty_{\lambda_i}$$

Correlation functions

In fact,

$$\mathbb{O}_i(f) \in (L^2(\Gamma\backslash G))^\infty = C^\infty(\Gamma\backslash G)$$

[Borel, Wallach]

Therefore the correlators are well-defined as distributions

$$\langle \mathbb{O}_{i_1}(f_1)\mathbb{O}_{i_2}(f_2)\mathbb{O}_{i_3}(f_3)\mathbb{O}_{i_4}(f_4)\rangle \equiv \int_{\Gamma\backslash G} d\mu(g)\mathbb{O}_{i_1}(f_1)\mathbb{O}_{i_2}(f_2)\mathbb{O}_{i_3}(f_3)\mathbb{O}_{i_4}(f_4)$$

Decomposing into irreps gives the OPE

$$\mathbb{O}_{i_1}(f_1)\mathbb{O}_{i_2}(f_2) = \sum_k \mathbb{O}_k(\tau_{i_1,i_2;k}(f_1, f_2)) + \cdots$$

$$\tau_{i,j;k} : P^\infty_{\lambda_i} \times P^\infty_{\lambda_j} \to P^\infty_{\lambda_k}$$

Discrete series

Principal series story generalizes to $d+1 \geq 2$ and to complementary series

In even $d+1$ need to consider discrete series. For $d+1=2$

$$\mathcal{O}_n(z) \in D_n^\infty \qquad \bar{\mathcal{O}}_n(w) \in \bar{D}_n^\infty$$

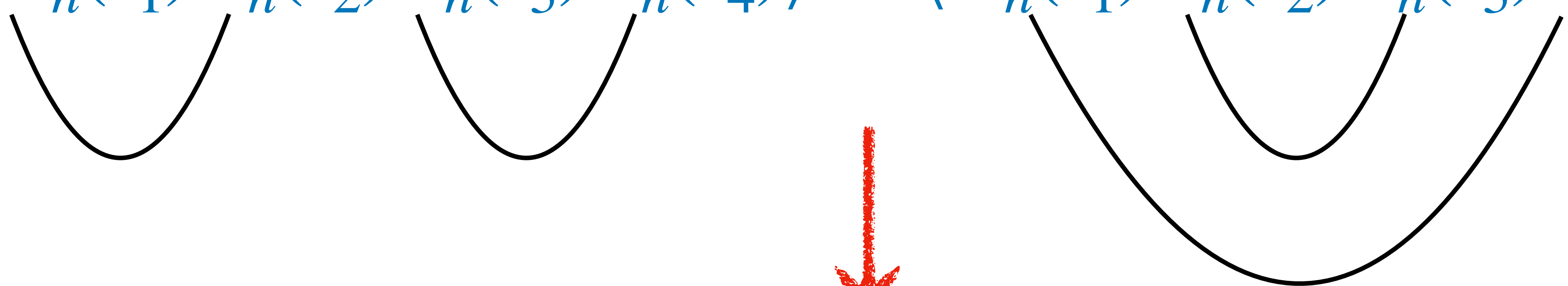
$$|z| < 1 \qquad |w| > 1$$

The correlators are holomorphic functions (boundary values are distributions)

$$\langle \mathcal{O}_{n_1}(z_1)\mathcal{O}_{n_2}(z_2)\bar{\mathcal{O}}_{n_3}(z_3)\bar{\mathcal{O}}_{n_4}(z_4)\rangle$$

Crossing equation

$$\langle \mathcal{O}_n(z_1)\mathcal{O}_n(z_2)\bar{\mathcal{O}}_n(z_3)\bar{\mathcal{O}}_n(z_4)\rangle = \langle \mathcal{O}_n(z_1)\mathcal{O}_n(z_2)\bar{\mathcal{O}}_n(z_3)\bar{\mathcal{O}}_n(z_4)\rangle$$



SO(1,2) symmetry

$$\sum_p |f_p|^2 G_p(z) = \sum_i c_i^2 H_{\Delta_i}(z) \qquad z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

Crossing equation

$$\langle \mathbb{O}_k(h_{m_1})\mathbb{O}_k(h_{m_2})\mathbb{O}_k(h_{m_3})\mathbb{O}_k(h_{m_4})\rangle = \langle \mathbb{O}_k(h_{m_1})\mathbb{O}_k(h_{m_2})\mathbb{O}_k(h_{m_3})\mathbb{O}_k(h_{m_4})\rangle$$

$$h_m(\theta) = e^{im\theta}$$

$SO(1,2)$ symmetry

$$\sum_k p_k^2 G_{m_1 m_2 m_3 m_4}(\Delta_k) = 0$$

Numerical bootstrap

$$\sum_p |f_p|^2 G_p(z) = \sum_i c_i^2 H_{\Delta_i}(z)$$

$$\sum_k p_k^2 G_{m_1 m_2 m_3 m_4}(\Delta_k) = 0$$

$$p_k^2, c_k^2, |f_k|^2 \geq 0$$

+ extensions to mixed correlators

Same numerics*

→ **Dalimil's talk**

*in fact can be truncated to exact polynomials in Δ => fully rigorous numerics

Application to CFTs?

- We checked the sum rules for the fundamental fields in GFF and 2d Ising
- Reflection-positive CFTs necessarily have continuous spectrum in $L^2(\Phi, \mu)$.
- It seems reasonable to conjecture that sum rules also apply to 3d Ising and other CFTs
- Are there CFTs which are not RP for which these sum rules hold?
- Are there CFTs with discrete spectra in $L^2(\Phi, \mu)$?

Application to CFTs?

- We checked the sum rules for the fundamental fields in GFF and 2d Ising
- Reflection-positive CFTs necessarily have continuous spectrum in $L^2(\Phi, \mu)$.
- It seems reasonable to conjecture that sum rules also apply to 3d Ising and other CFTs
- Are there CFTs which are not RP for which these sum rules hold?
- Are there CFTs with discrete spectra in $L^2(\Phi, \mu)$?

Fin

Merci!