Exact probabilities for some topological events for metric graph GFF

Titus Lupu
CNRS/Sorbonne Université, LPSM, Paris

Probability and CFT, September 22nd, 2022

Discrete GFF on an electrical network

$\mathcal{G} = (V, E)$ finite undirected graph. Conductances $C(x, y) = C(y, x)$ for $\{x, y\} \in E$.

V divided into 2 parts: interior vertices V_{int} and boundary V_{∂}.

Boundary condition $u : V_{\partial} \to \mathbb{R}$.

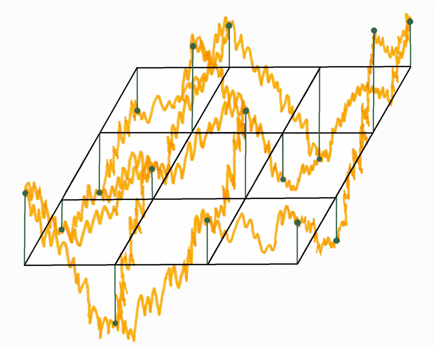
Discrete GFF with boundary condition u on V_{∂}:

$$\frac{1}{Z^u} \exp\Big(-\frac{1}{2} \sum_{\{x,y\}\in E} C(x, y)(\varphi(y) - \varphi(x))^2 \Big) \prod_{z\in V_{\text{int}}} d\varphi(z).$$

GFF on metric graph

Metric graph $\widetilde{\mathcal{G}}$: replace each edge $\{x, y\} \in E$ by a continuous line of length $C(x, y)^{-1}$ (length = resistance).

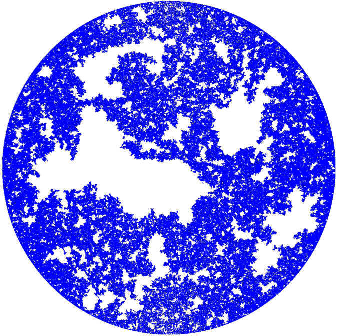
$(\phi(x))_{x \in V}$ discrete GFF.
We extrapolate ϕ to a continuous Gaussian field $(\tilde{\phi}(x))_{x \in \widetilde{\mathcal{G}}}$ with conditionally independent Brownian bridges inside the edge-lines.
$\tilde{\phi}$ is the GFF on the metric graph. Satisfies the Markov property.

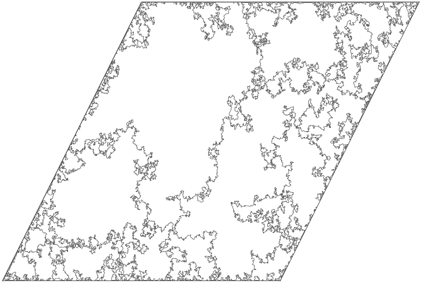


Why metric graph GFF?

- The metric graph GFF $\tilde{\phi}$, just as a discrete GFF ϕ, approximates the continuum GFF in the scaling limit.
- $\tilde{\phi}$ satisfies some exact identities that ϕ does not.
- $\tilde{\phi}$ has more interfaces than ϕ. For instance in 2D the outer boundaries of sign clusters of $\tilde{\phi}$ converge to CLE_4 (L. 2015, relation to Miller-Sheffield coupling) while the outer boundaries of sign clusters of ϕ converge to something else (towards ALE).
- The relation between ϕ and $\tilde{\phi}$ is very analogous to the relation between the spin Ising field and the FK-Ising random cluster model.
- The random walk representations of the GFF have stronger versions for $\tilde{\phi}$.
- The connected components of the level sets $\{\tilde{\phi} \geq a\}$ are easier to study than those for $\{\phi \geq a\}$, cf works of L., Ding-Wirth and Drewitz-Prévost-Rodriguez.

CLE_4 and ALE





Left: Computer simulation of the CLE_4 by David B. Wilson.
Right: Computer simulation of ALE by Brent Werness.

Some exact identities for the metric graph GFF (1)

$\tilde{\phi}$ with 0 boundary conditions. $x, y \in \widetilde{\mathcal{G}}$.

$$\mathbb{P}(x, y \text{ in the same connected component of } \{\tilde{\phi} \neq 0\}) =$$

$$\mathbb{E}[\text{sign}(\tilde{\phi}(x))\,\text{sign}(\tilde{\phi}(y))] = \frac{2}{\pi} \arcsin\left(\frac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\right).$$

Some exact identities for the metric graph GFF (2)

Boundary V_∂ divided into 3 parts $V_{\partial,1}$, $V_{\partial,2}$ and $V_{\partial,0}$, with $V_{\partial,0}$ possibly empty.
u boundary condition, $u > 0$ on $V_{\partial,1} \cup V_{\partial,2}$, $u = 0$ on $V_{\partial,0}$.
u^* boundary condition, $u^* = u$ on $V_{\partial,1}$, $u^* = -u$ on $V_{\partial,2}$, $u^* = 0$ on $V_{\partial,0}$.
$\tilde{\phi}_u$ with b.c. u. $\tilde{\phi}_{u^*}$ with b.c. u^*.

$$\mathbb{P}(V_{\partial,1} \overset{\tilde{\phi}_u > 0}{\leftrightarrow} V_{\partial,2}) = \frac{Z^{u*}}{Z^u} = \exp\Big(- 2 \sum_{x \in V_{\partial,1}} \sum_{y \in V_{\partial,2}} u(x) H(x,y) u(y) \Big),$$

$H(x,y)$ boundary Poisson kernel.

Conditionally on $V_{\partial,1} \overset{\tilde{\phi}_u > 0}{\leftrightarrow} V_{\partial,2}$, $|\tilde{\phi}_u|$ is distributed as $|\tilde{\phi}_{u^*}|$.

No similar formula if the boundary condition u mixes both positive and negative values.

Some exact identities for the metric graph GFF (3)

$\tilde{\rho}_u$, resp. $\tilde{\rho}_{u^*}$ interacting fields on $\widetilde{\mathcal{G}}$ with interaction $\exp\Big(-\int_{\widetilde{\mathcal{G}}}\mathcal{V}(|\varphi|)dx\Big)$ and b.c. u, resp. u^*.

$$\mathbb{P}(V_{\partial,1} \overset{\tilde{\rho}_u>0}{\leftrightarrow} V_{\partial,2}) = \frac{Z^{u*}\mathbb{E}\Big[\exp\Big(-\int_{\widetilde{\mathcal{G}}}\mathcal{V}(|\tilde{\phi}_{u^*}|)dx\Big)\Big]}{Z^{u}\mathbb{E}\Big[\exp\Big(-\int_{\widetilde{\mathcal{G}}}\mathcal{V}(|\tilde{\phi}_{u}|)dx\Big)\Big]}.$$

Conditionally on $V_{\partial,1} \overset{\tilde{\rho}_u>0}{\leftrightarrow} V_{\partial,2}$, $|\tilde{\rho}_u|$ is distributed as $|\tilde{\rho}_{u^*}|$.

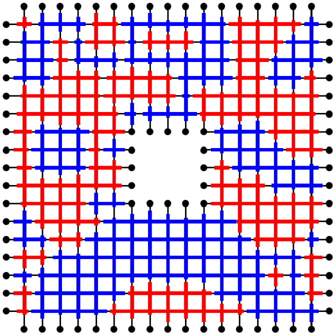
Examples: $\mathcal{V}(|\varphi|)=\varphi^4$, $\mathcal{V}(|\varphi|)=\cosh(\gamma\varphi)$.
Not true for $e^{\gamma\varphi}$ because then the interaction depends on the sign of the field, not just the absolute value.

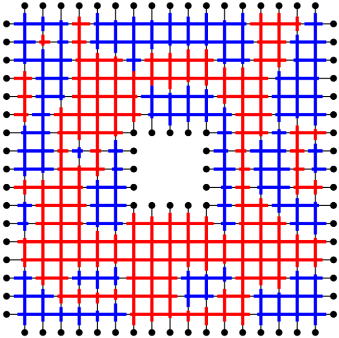
Some exact identities for the metric graph GFF (4)

Drewitz, Prévost, Rodriguez, 2021:
Exact law of the effective conductance between the boundary V_∂ and the connected component of x_0 of the level set $\{\tilde{\phi} \geq a\}$, with $x_0 \in \widetilde{\mathcal{G}}$ and $a \geq 0$ fixed.

Exact probabilities for some topological events

L. 2022: There are some exact probabilities for some topological events for $\tilde{\phi}$, related to $\{-1,1\}$-valued gauge fields.
For instance, the probability that a connected component of $\{\tilde{\phi} \neq 0\}$ surrounds the inner hole of a planar two-connected domain.





Gauge field, gauge equivalence and holonomy

Gauge group $\{-1, 1\}$.

Gauge field $\sigma \in \{-1, 1\}^E$.

Gauge transformation: $\sigma \in \{-1, 1\}^E$ and $\hat{\sigma} \in \{-1, 1\}^V$.

$$(\hat{\sigma} \cdot \sigma)(x, y) = \hat{\sigma}(x)\sigma(x, y)\hat{\sigma}(y).$$

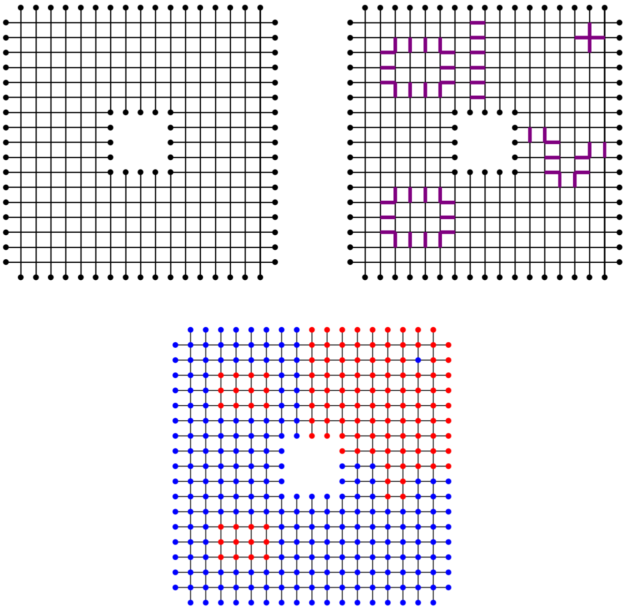
$\hat{\sigma} \cdot \sigma$ and σ are gauge equivalent.

Nearest-neighbor path in $\mathcal{G}$, $\wp = (x_1, x_2, \dots, x_n)$.

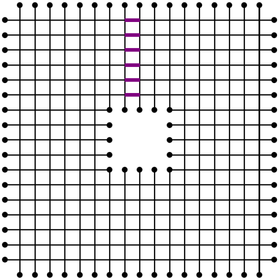
$$\mathrm{hol}^\sigma(\wp) = \sigma(x_1, x_2)\sigma(x_2, x_3)\dots\sigma(x_{n-1}, x_n).$$

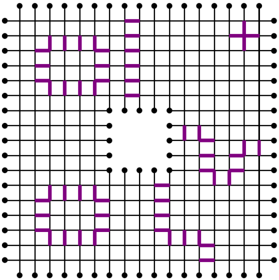
The gauge equivalence classes are characterized by the holonomies along closed loops.

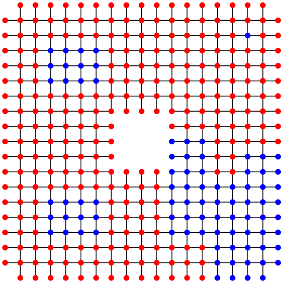
Example of gauge transformation (1)



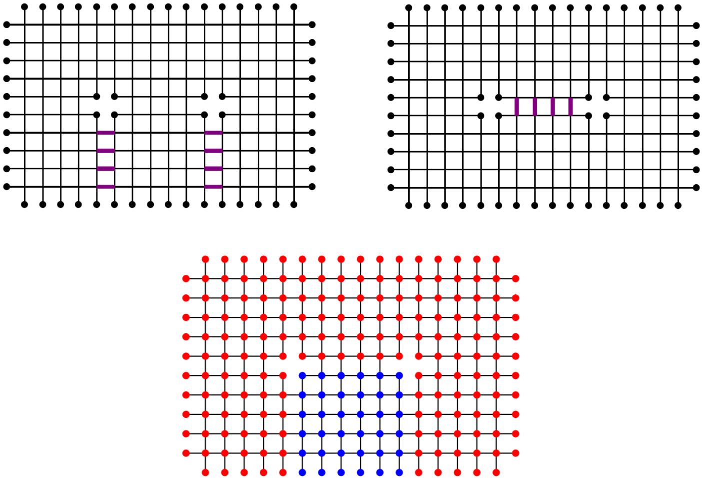
Example of gauge transformation (2)







Example of gauge transformation (3)



Gauge-twisted discrete GFF

ϕ discrete GFF on $\mathcal{G}$ with 0 boundary conditions:

$$\frac{1}{Z^0} \exp\Big(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\varphi(y) - \varphi(x))^2 \Big) \prod_{z \in V_{\text{int}}} d\varphi(z).$$

$\sigma \in \{-1,1\}^E$. ϕ_σ discrete σ-twisted GFF on $\mathcal{G}$ with 0 boundary conditions:

$$\frac{1}{Z^0_\sigma} \exp\Big(-\frac{1}{2} \sum_{\{x,y\} \in E} C(x,y)(\sigma(x,y)\varphi(y) - \varphi(x))^2 \Big) \prod_{z \in V_{\text{int}}} d\varphi(z).$$

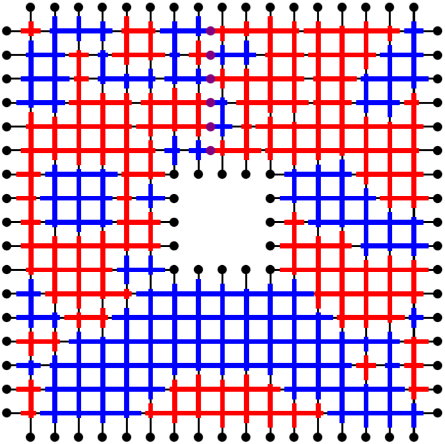
If $\sigma' = \hat{\sigma} \cdot \sigma$ in the same gauge equivalence class, $\phi_{\sigma'} \overset{(d)}{=} \hat{\sigma} \phi_\sigma$.

ϕ has a natural extension $\tilde{\phi}$ to the metric graph $\widetilde{\mathcal{G}}$.
ϕ_σ also has a natural extension $\tilde{\phi}_\sigma$ to $\widetilde{\mathcal{G}}$. Unlike $\tilde{\phi}$, $\tilde{\phi}_\sigma$ has discontinuities: one discontinuity per edge $e \in \{e \in E | \sigma(e) = -1\}$, placed in the middle of the edge x_e^{m}.

$$\lim_{x \to x^{\mathrm{m}}_{e,-}} \tilde{\phi}_\sigma(x) = - \lim_{x \to x^{\mathrm{m}}_{e,+}} \tilde{\phi}_\sigma(x).$$

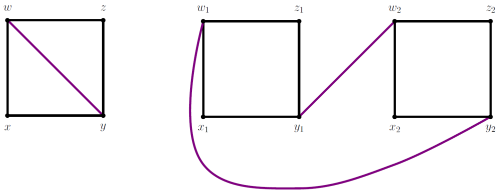
The absolute value $|\tilde{\phi}_\sigma|$ is continuous on $\widetilde{\mathcal{G}}$.

Conceptual picture for $\tilde{\phi}_\sigma$



Double cover of $\widetilde{\mathcal{G}}$ induced by sigma

V_1 and V_2 two copies of the set of vertices V.
$\widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma$ double cover of $\widetilde{\mathcal{G}}$ induced by σ. $\pi_\sigma : \widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma \to \widetilde{\mathcal{G}}$ cover map.



$\tilde{\phi}^{\mathrm{db}}_\sigma$ GFF on $\widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma$ with 0 boundary conditions.
$\psi_\sigma : \widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma \to \widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma$ automorphism of the covering map π_σ (interchanges the two sheets).
$\mathtt{s} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}^{\mathrm{db}}_\sigma$ section of π_σ ($\pi_\sigma \circ \mathtt{s} = Id$). $\mathtt{s}$ has discontinuities inside the edges $e \in E$ with $\sigma(e) = -1$.

$$\tilde{\phi}_\sigma \overset{(d)}{=} \frac{1}{\sqrt{2}}(\tilde{\phi}^{\mathrm{db}}_\sigma - \tilde{\phi}^{\mathrm{db}}_\sigma \circ \psi_\sigma) \circ \mathtt{s}.$$

The topological event

$$\mathcal{T}_\sigma = \{f \in \mathcal{C}(\widetilde{\mathcal{G}}) | \forall U \text{ connected component of } \{f \neq 0\}, \pi_\sigma^{-1}(U) \text{ not connected}\}$$

$\mathbb{P}(|\tilde{\phi}_\sigma| \in \mathcal{T}_\sigma) = 1.$
If U connected component of $\{|\tilde{\phi}_\sigma| \neq 0\}$, and $x \in U$, then $\mathtt{s}(x)$ and $\psi_\sigma(\mathtt{s}(x))$ cannot be connected inside $\pi_\sigma^{-1}(U)$ because of the change of sign.

Main result

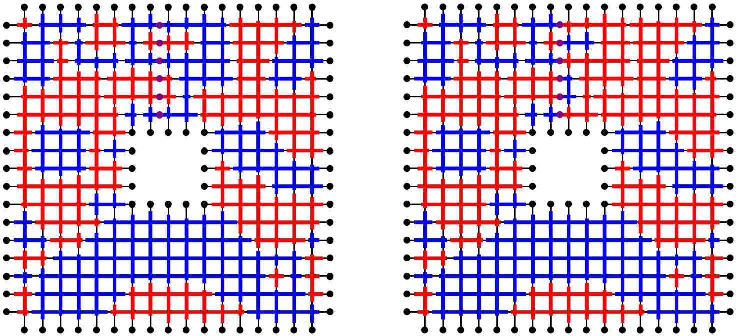
$$\mathbb{P}(\tilde{\phi} \in \mathcal{T}_\sigma) = \frac{Z_\sigma^0}{Z^0} = \frac{\det(-\Delta^{\mathcal{G}})^{1/2}}{\det(-\Delta_\sigma^{\mathcal{G}})^{1/2}}.$$

Conditionally on the event $\{\tilde{\phi} \in \mathcal{T}_\sigma\}$, the field $|\tilde{\phi}|$ is distributed as $|\tilde{\phi}_\sigma|$.

$\tilde{\rho}$ interacting field on $\widetilde{\mathcal{G}}$ with interaction $\exp\Big(- \int_{\widetilde{\mathcal{G}}} \mathcal{V}(|\varphi|)dx \Big)$ and 0 boundary condition.

$$\mathbb{P}(\tilde{\rho} \in \mathcal{T}_\sigma) = \frac{Z_\sigma^0 \mathbb{E}\Big[\exp\Big(- \int_{\widetilde{\mathcal{G}}} \mathcal{V}(|\tilde{\phi}_\sigma|)dx \Big) \Big]}{Z^0 \mathbb{E}\Big[\exp\Big(- \int_{\widetilde{\mathcal{G}}} \mathcal{V}(|\tilde{\phi}|)dx \Big) \Big]}.$$

Illustration



Left: $\tilde{\phi}$ conditioned on $\tilde{\phi} \in \mathcal{T}_\sigma$.
Right: $\tilde{\phi}_\sigma$.

Thank you for your attention!