Geometric Applications of the Conformal Bootstrap

Dalimil Mazáč, IAS Princeton

Probability and CFT 2022, Agay

Plan

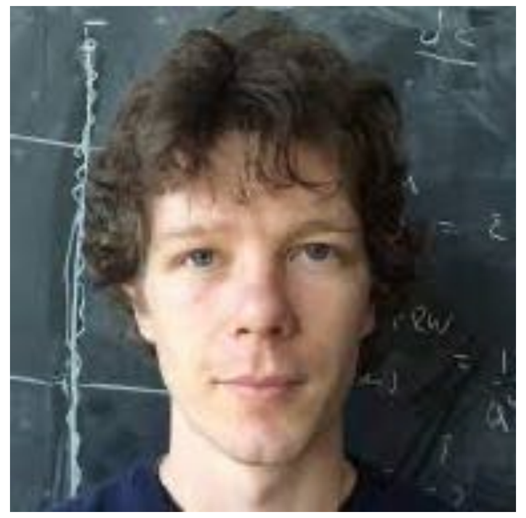
1. Spectra of hyperbolic manifolds and the conformal bootstrap.

2. The conformal bootstrap and the sphere packing problem.

1. Conformal Bootstrap and Hyperbolic Geometry

Based on **arXiv:2111.12716** with **Petr Kravchuk** and **Sridip Pal**.

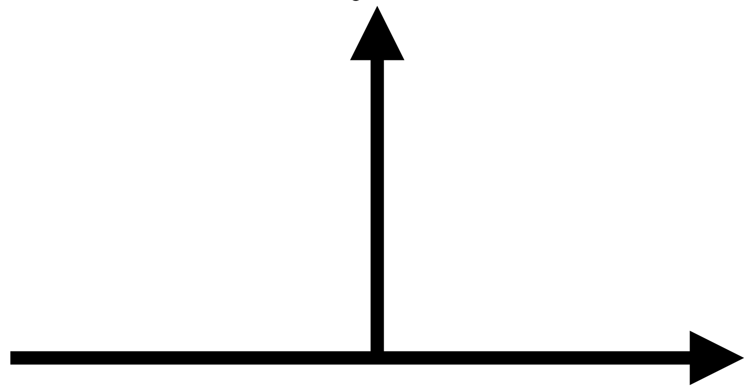
Similar results appeared in **arXiv:2111.13215** by **James Bonifacio**.



I will also mention some ongoing work with all of the above.

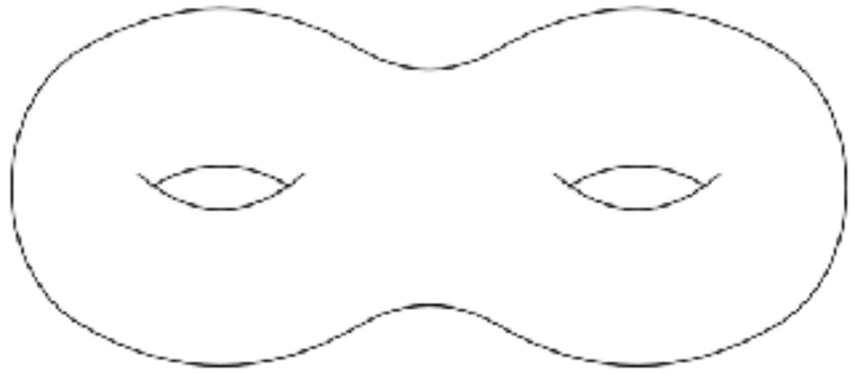
2D Hyperbolic Orbifolds

1. Upper half-plane with the hyperbolic metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$



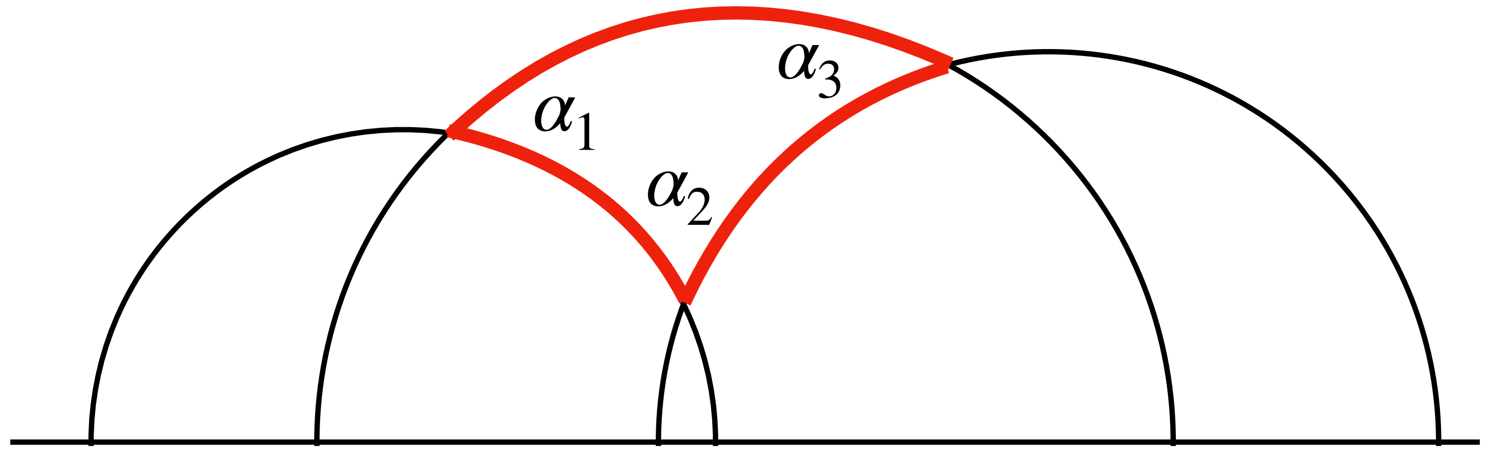
   - $G = \mathrm{PSL}_2(\mathbb{R})$ acts on $z = x + iy \in \mathbb{H}^2$ $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \ z \mapsto \dfrac{az + b}{cz + d}$

2. Γ = discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ $\Leftrightarrow$ $\Gamma \backslash \mathbb{H}^2$ = a hyperbolic orbifold.



   - Will assume $\Gamma \backslash \mathbb{H}^2$ has finite volume.
   - Γ only has hyperbolic elements $\Leftrightarrow$ $\Gamma \backslash \mathbb{H}^2$ is a compact surface.
   - Γ only has hyperbolic and elliptic elements $\Leftrightarrow$ $\Gamma \backslash \mathbb{H}^2$ is a compact orbifold.

Example 1: Hyperbolic Triangle Groups

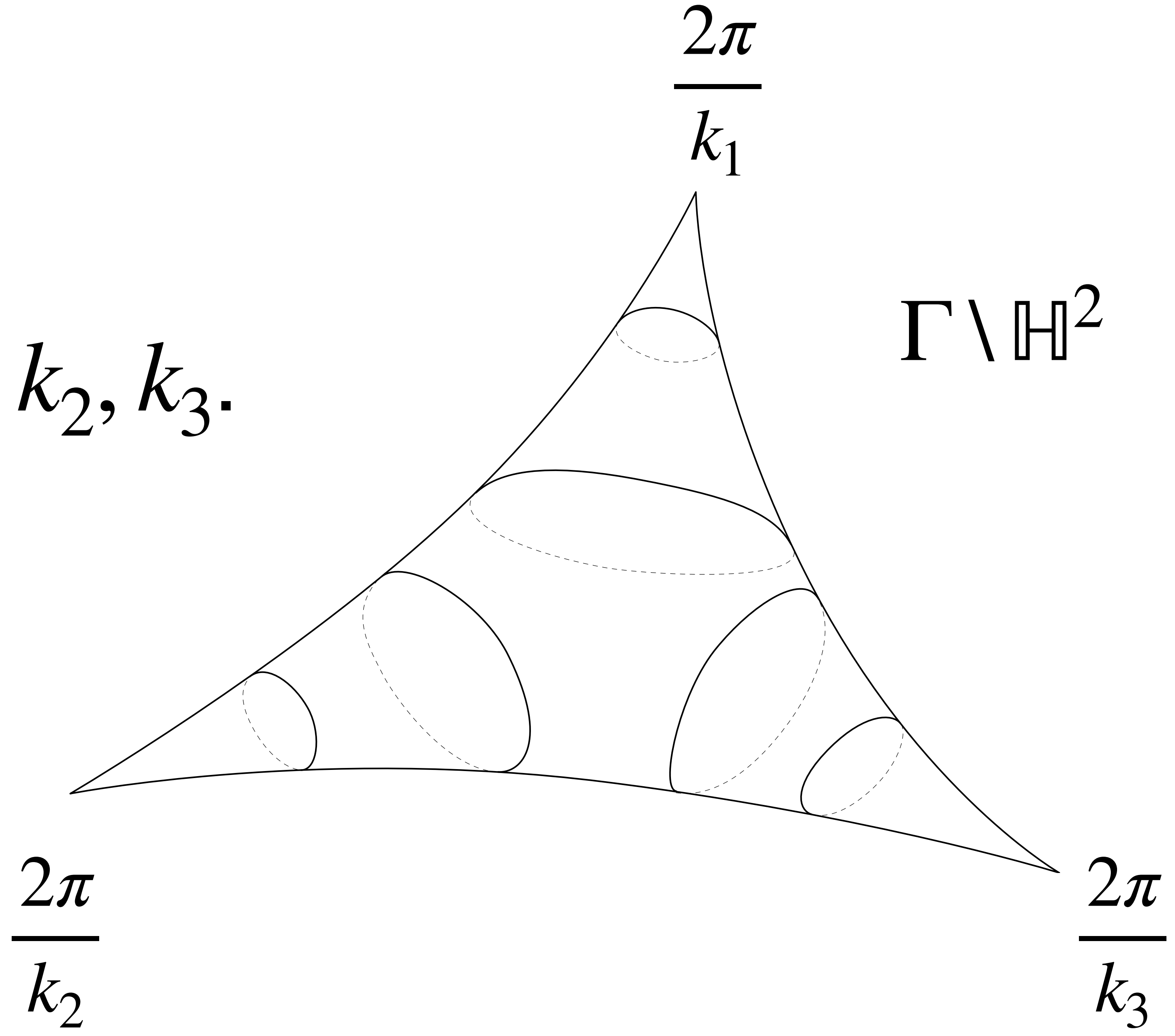


$$\alpha_i = \frac{\pi}{k_i} \qquad k_i \in \mathbb{N}_{\geq 2} \qquad \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} < 1$$

$$\Updownarrow$$

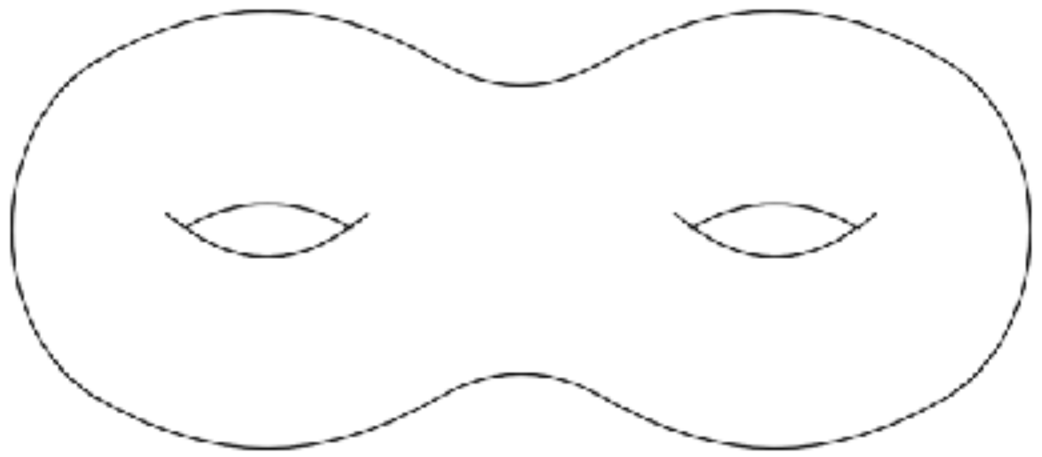
area > 0

- Γ generated by rotations around vertices by angles $\dfrac{2\pi}{k_i}$.
- A fundamental domain of Γ consists of two adjacent triangles.
- $\Gamma \backslash \mathbb{H}^2$ is an orbifold of genus 0 with 3 orbifold points of orders k_1, k_2, k_3.
- Orbifold of minimal area: $[k_1, k_2, k_3] = [2,3,7]$.



Example 2: The Bolza Surface

A hyperbolic surface without orbifold points must have genus ≥ 2.

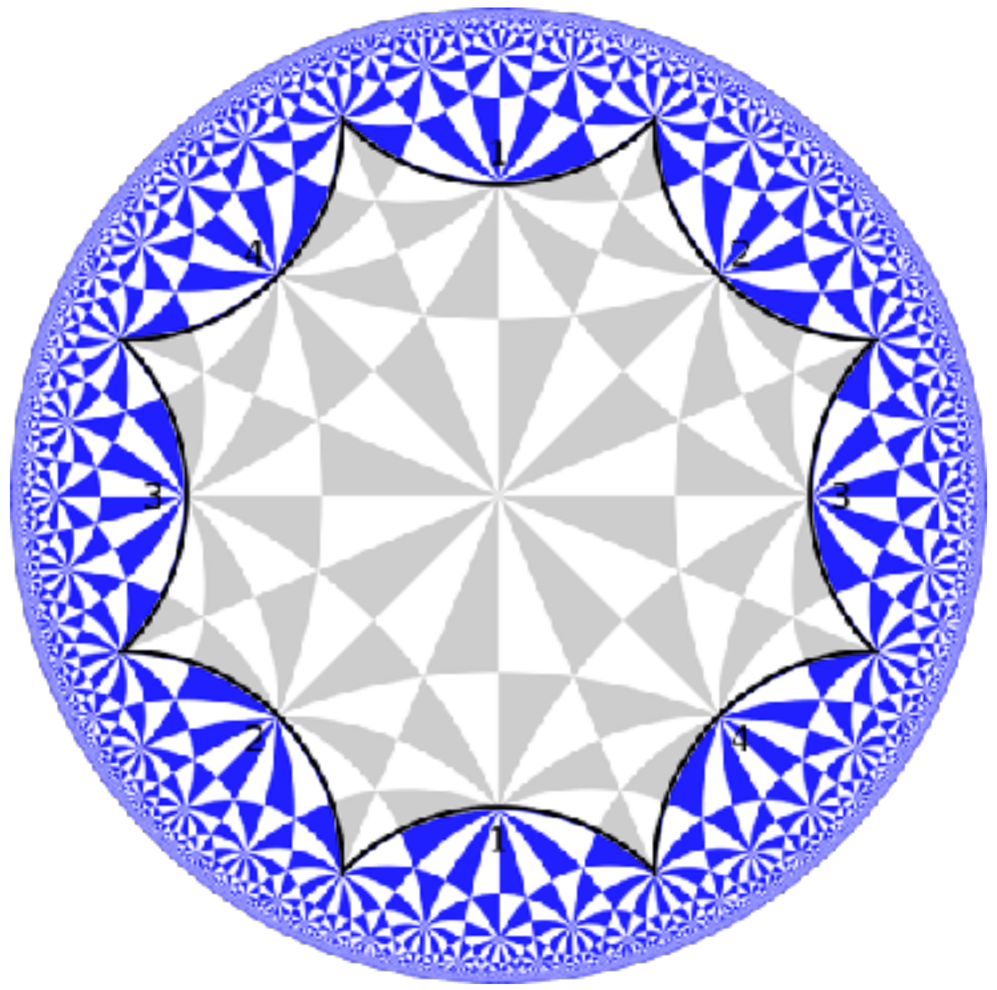


- Genus $= 2$: six-dimensional moduli space.

Bolza surface: the genus-two surface with the largest group of isometries.

- Iso(Bolza) = $\mathrm{GL}_2(\mathbb{F}_3)$, a group of order 48.

- Bolza = $\Gamma \backslash \mathbb{H}^2$, where Γ is a normal subgroup of index 48 of the [2,3,8] triangle group.



General Orbifolds

Topological type of $\Gamma \backslash \mathbb{H}^2$: $[g; k_1, \ldots, k_r]$ $\Leftrightarrow$ isomorphism type of Γ

genus   orders of orbifold points

Laplacian Spectrum of $\Gamma \backslash \mathbb{H}^2$

The Laplacian on $\mathbb{H}^2$: $\nabla^2 = y^2(\partial_x^2 + \partial_y^2)$

$$-\nabla^2 \varphi(x, y) = \lambda \, \varphi(x, y)$$

$\varphi(x, y)$: a smooth real function on $\mathbb{H}^2$ satisfying $\varphi(\gamma \cdot (x, y)) = \varphi(x, y)$ for all $\gamma \in \Gamma$.

Spectrum: $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$

- no closed expression for λ_i in general
- a useful model for studying classical and quantum chaos

Today: New upper bounds on λ_1.

Main results

[Kravchuk, DM, Pal ’21] [Bonifacio ’21]

Theorem:

1. Every hyperbolic orbifold satisfies: $\lambda_1 \leq 44.8883537$.

   $[2,3,7]$ triangle orbifold: $\lambda_1 \approx 44.88835$

2. Every hyperbolic orbifold of genus two satisfies: $\lambda_1 \leq 3.8388977$.

   Bolza surface: $\lambda_1 \approx 3.838887258$

   previous bound: $\lambda_1 \leq 4$ [Yang, Yau ’80] [Soufi, Ilias ’83]

3. Every hyperbolic orbifold of genus three satisfies: $\lambda_1 \leq 2.6784824$.

   Klein quartic: $\lambda_1 \approx 2.6779$

   previous bound: $\lambda_1 \leq 2(4 - \sqrt{7}) \approx 2.7085$ [Ros ’20]

Spectrum of the Spectrum

Conjecture (Selberg 1965): If Γ is a congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$, then $\lambda_1 = 1/4$.

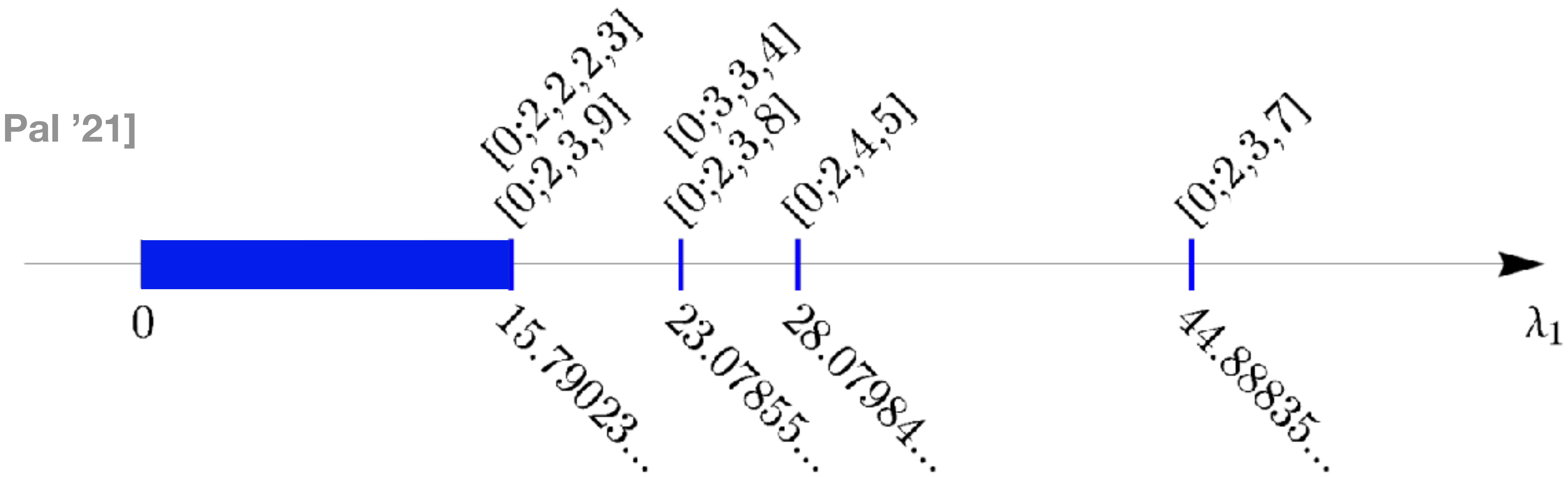
$$\Leftrightarrow$$

If X ranges over **congruence** orbifolds, the image of the map $X \mapsto \lambda_1(X)$ is the set $\{1/4\}$.

Question: What is the image of the map $X \mapsto \lambda_1(X)$ when X ranges over **all** orbifolds?

Answer:

[Kravchuk, DM, Pal '21]



The Method

1. The Hilbert space and local operators

2. Operator product expansion

3. Associativity

4. Bounds from linear programming

Previous Work

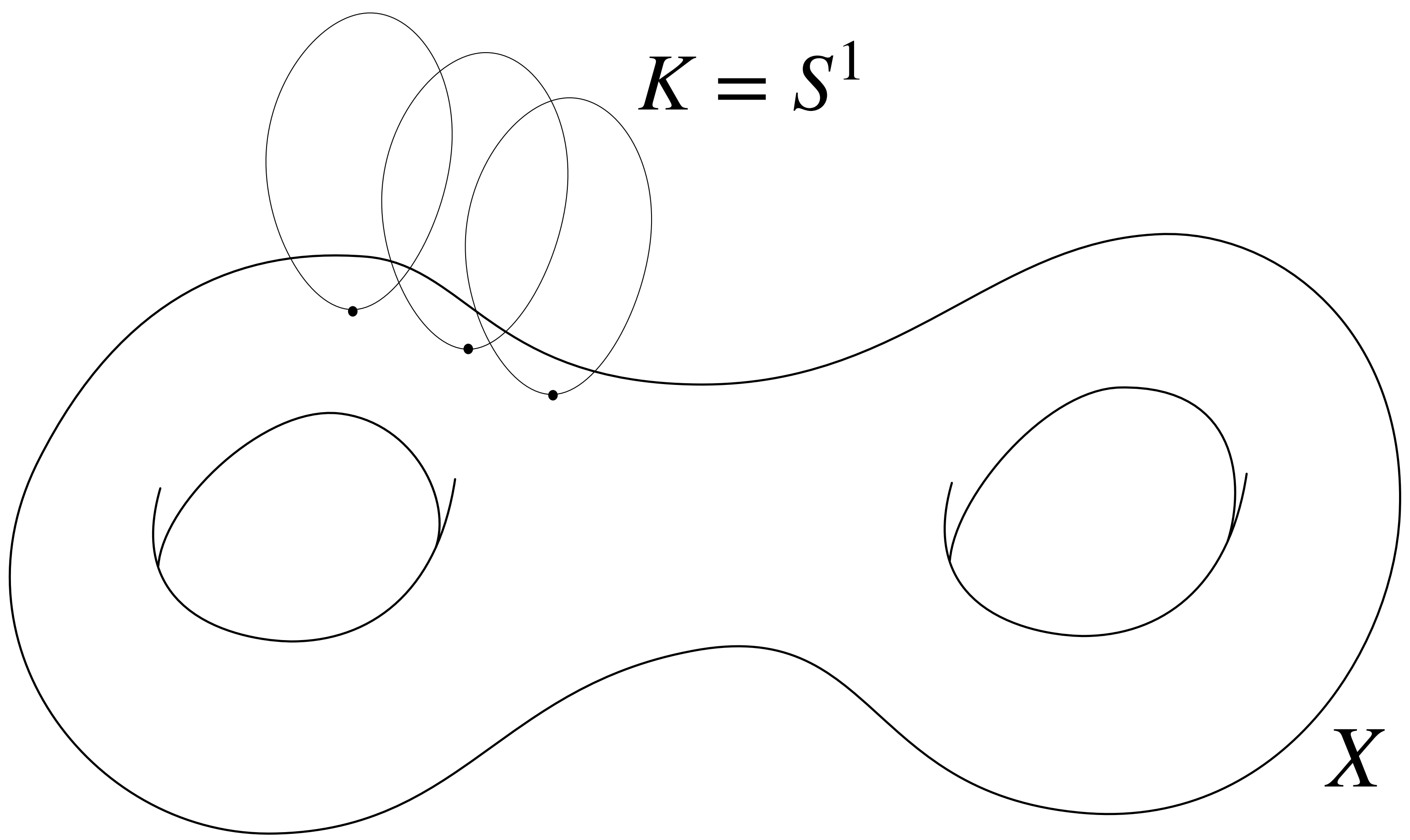
Bonifacio+Hinterbichler (2020): Einstein manifolds $R_{ab} = \frac{R}{d} g_{ab}$

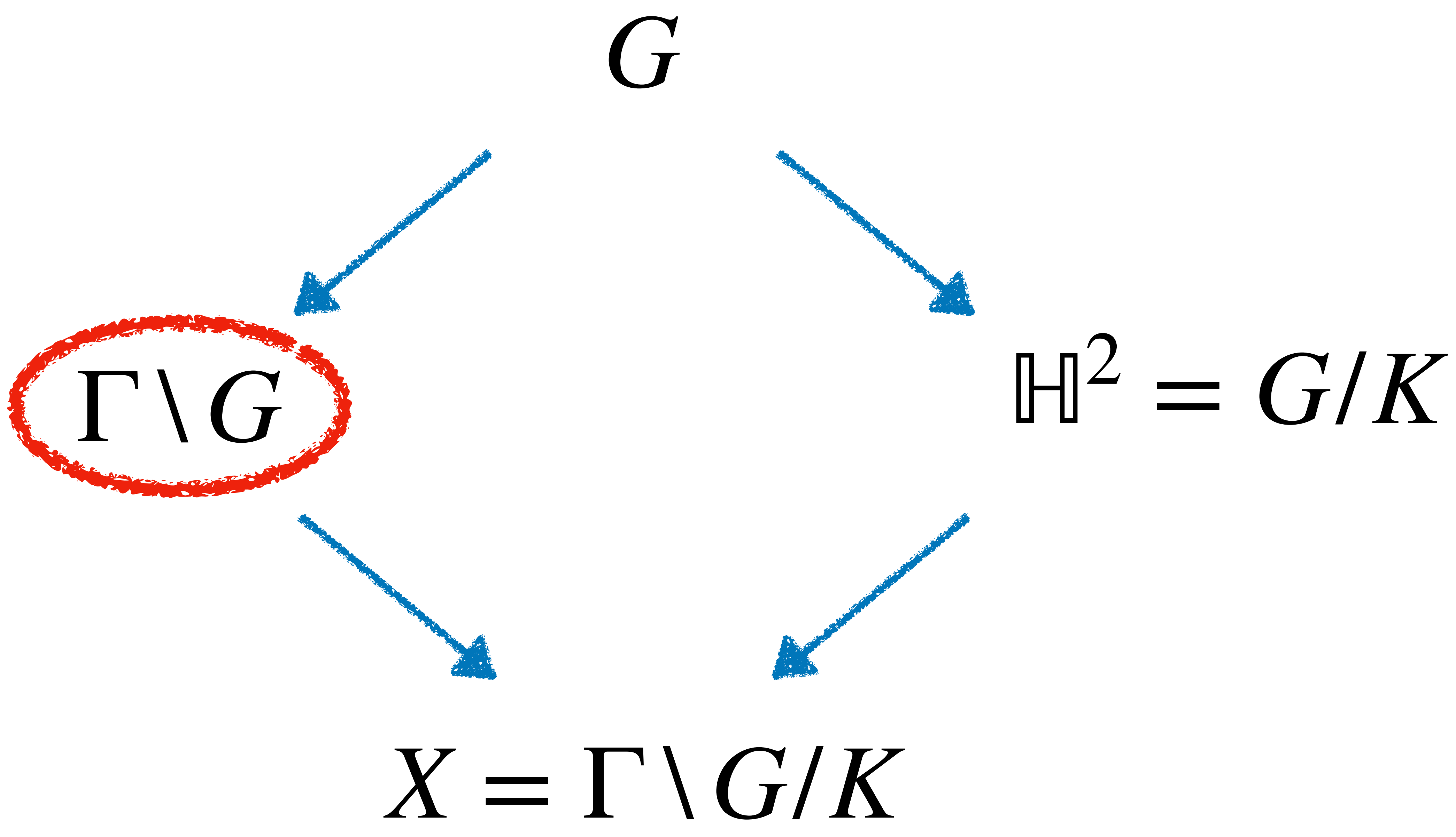
Bonifacio (2021): Hyperbolic manifolds $R_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}$

Kravchuk, DM, Pal (2021): Pointed out the role played by $\mathrm{SO}(1,d)$ in the case of hyperbolic manifolds, and systematized the ideas using its representation theory.

The Coset Space

- $G = \mathrm{PSL}_2(\mathbb{R})$
- $K = \mathrm{PSO}_2(\mathbb{R})$, maximal compact subgroup of G
- $\Gamma =$ discrete co-compact subgroup of G

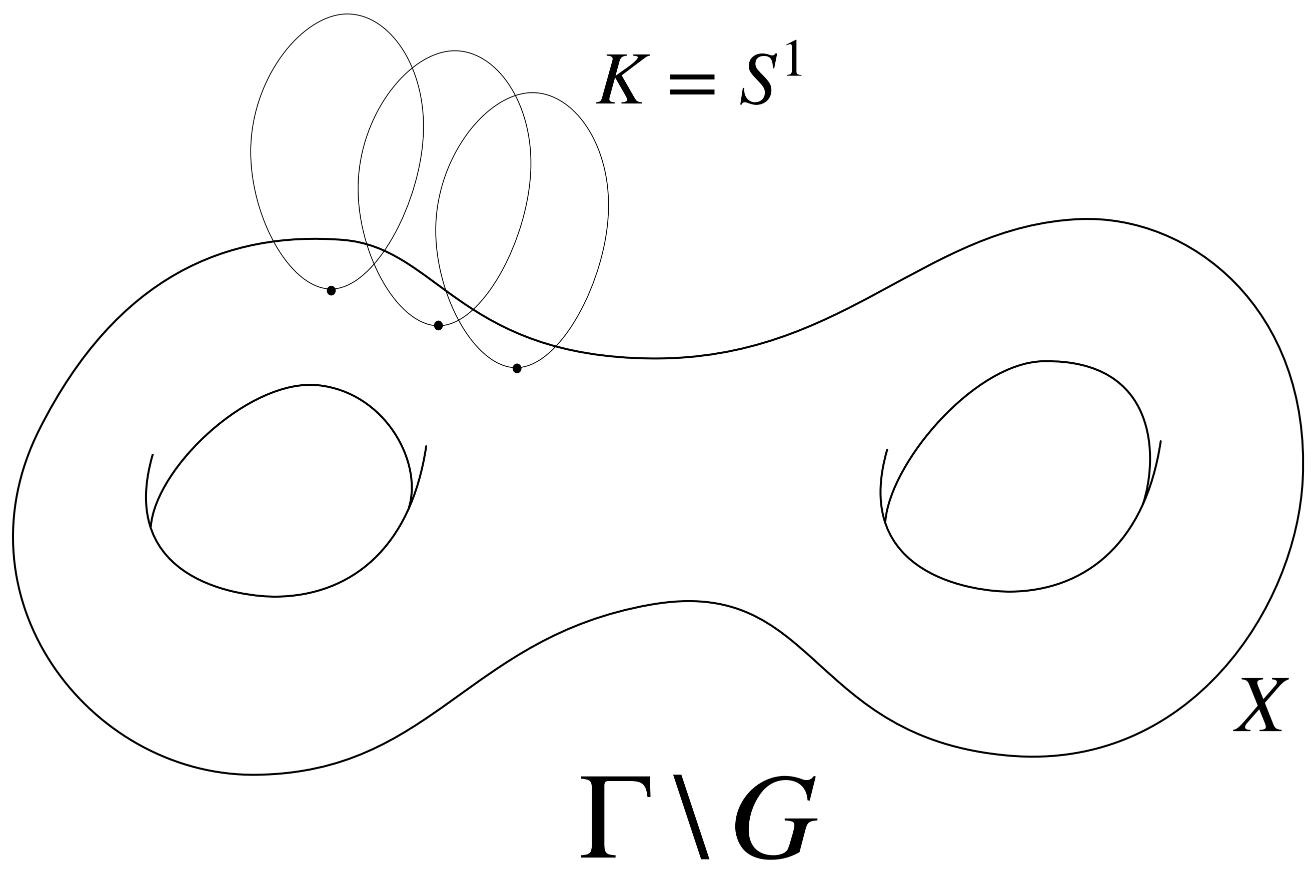




The Hilbert Space: $L^2(\Gamma \backslash G)$

Consider the space $L^2(\Gamma \backslash G)$

- a representation of G: $F(g) \overset{\tilde{g}}{\mapsto} F(g\tilde{g})$
- unitary, with inner product: $\|F(g)\|^2 = \int_{\Gamma \backslash G} dg \, |F(g)|^2$



Decomposition under K: $L^2(\Gamma \backslash G) = \bigoplus_{n \in \mathbb{Z}} V_n$

- $V_0 = L^2(X)$
- $V_n = L^2(n\text{-forms})$: $h(x, y) \, dz^n$ such that $\forall \gamma \in \Gamma$: $h(z) = (cz + d)^{-2n} h\left(\frac{az + b}{cz + d}\right)$
- Generators of G act as follows: $L_0|_{V_n} = n \, \mathrm{id}$, $L_{\pm 1} : V_n \to V_{n \mp 1}$

The Spectral Decomposition

Decompose $L^2(\Gamma\backslash G)$ into irreducible representations of $G = \mathrm{PSL}_2(\mathbb{R})$:

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

1. Trivial representation $\mathbb{C}$: constant functions.

2. Principal and complementary series P_λ: Laplace eigenfunction with eigenvalue λ.
   - principal series: $\lambda \in [1/4,\infty)$, complementary series: $\lambda \in (0,1/4)$.
   - Casimir$\big|_{V_0}$ = Laplacian $\Rightarrow v \in P_\lambda \cap V_0$ is a Laplace eigenfunction of eigenvalue λ.

3. Holomorphic discrete series D_n: holomorphic modular forms of weight $2n \in 2\mathbb{N}_{>0}$.
   - $L_1 = \bar{\partial}$, $L_1\big|_{D_n \cap V_n} = 0 \Rightarrow v \in D_n \cap V_n$ is a holomorphic modular form of weight $2n$.
   - Antiholomorphic discrete series $\overline{D}_n$: complex conjugates of modular forms.

Terminology: The Laplace eigenfunctions and holomorphic modular forms are examples of *automorphic forms.*

$$L^2(\Gamma\backslash G) = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} P_{\lambda_i} \oplus \bigoplus_{j=1}^{\infty} (D_{n_j} \oplus \overline{D}_{n_j})$$

Question: What are the constraints on the set of representations on the RHS?

Ingredients:

1. Riemann-Roch theorem: The topology of Γ determines the spectrum of holomorphic forms = discrete series. Namely, for $[g; k_1, \ldots, k_r]$, we have

$$\text{multiplicity}(D_n) = (2n-1)(g-1) + \sum_{i=1}^{r} \left\lfloor n\frac{k_i - 1}{k_i} \right\rfloor + \delta_{n,1}$$

$\Rightarrow$ Can focus on specific topology by making simple assumptions about the spectrum of D_n.

2. Consider the pointwise product $C^\infty(\Gamma\backslash G) \times C^\infty(\Gamma\backslash G) \to C^\infty(\Gamma\backslash G)$

$$(F_1(g), F_2(g)) \mapsto F_1(g)F_2(g)$$

Associativity and G-invariance $\Rightarrow$ bounds on the Laplacian spectrum.

Local Operators

Definition (local operator):

Let $F(g) \in L^2(\Gamma \backslash G)$ be a holomorphic modular form of weight $2n$. Define

$$\mathcal{O}(w) = e^{wL_{-1}} \cdot F(g) = F(g) + wL_{-1} \cdot F(g) + \frac{w^2}{2} L_{-1}^2 \cdot F(g) + \dots$$

Properties:

- $\mathcal{O}(w) \in L^2(\Gamma \backslash G) \cap D_n$ for $|w| < 1$.
- As w ranges over the unit disk, $\mathcal{O}(w)$ generates $L^2(\Gamma \backslash G) \cap D_n$.
- $\mathcal{O}(w)$ transforms like a conformal primary operator of scaling dimension n.

$$L_m \cdot \mathcal{O}(w) = [w^{m+1}\partial_z + (m+1)nw^m]\mathcal{O}(w)$$

Similarly, define the conjugate operator $\overline{\mathcal{O}}(w) = w^{-2n} e^{-L_1/w} \cdot \overline{F(g)}$.

- $\overline{\mathcal{O}}(w) \in L^2(\Gamma \backslash G) \cap \overline{D}_n$ for $|w| > 1$.

Correlation Functions

Definition (correlation function):

Given $F_1, \ldots, F_N \in C^\infty(\Gamma \backslash G)$, their correlation function is given by

$$\langle F_1 \ldots F_N \rangle = \frac{1}{\mathrm{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} d\mu \, F_1(g) \ldots F_2(g)$$

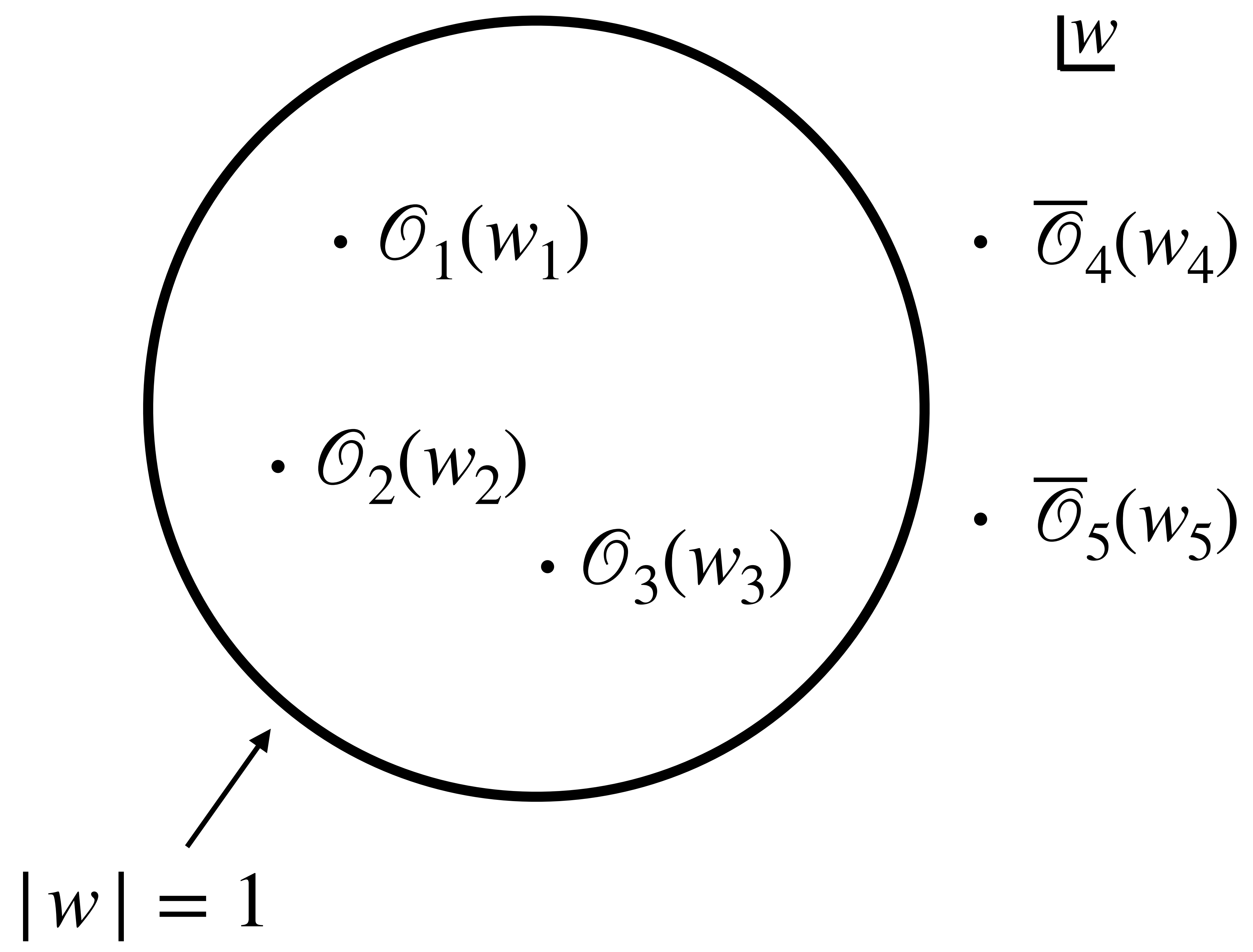
Since μ is G-invariant, so are the correlation functions.

Properties:

- one-point functions: $\langle 1 \rangle = 1, \langle \mathcal{O}_i(w) \rangle = \langle \overline{\mathcal{O}}_i(w) \rangle = 0$
- two-point functions: $\langle \mathcal{O}_i(w_1) \overline{\mathcal{O}}_j(w_2) \rangle = \dfrac{\delta_{ij}}{(w_1 - w_2)^{2n}}$

Each hyperbolic orbifold defines a large class of observables:

$$\langle \mathcal{O}_1(w_1)\dots\mathcal{O}_N(w_N)\overline{\mathcal{O}}_{N+1}(w_{N+1})\dots\overline{\mathcal{O}}_{N+M}(w_{N+M})\rangle$$



The Operator Product Expansion

Express products $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2)$, $\mathcal{O}(w_1)\mathcal{O}(w_2)$ using the spectral decomposition of $L^2(\Gamma \backslash G)$.

- $\mathcal{O}(w_1)\overline{\mathcal{O}}(w_2) = \dfrac{1}{(w_1 - w_2)^{2n}} + \sum_i f_i K_i(w_1, w_2)$, where $K_i(w_1, w_2) \in P_{\lambda_i}$.
- $\mathcal{O}(w_1)\mathcal{O}(w_2) = \sum_j \tilde{f}_j \widetilde{K}_j(w_1, w_2)$, where $\widetilde{K}_j(w_1, w_2) \in D_{n_j}$.

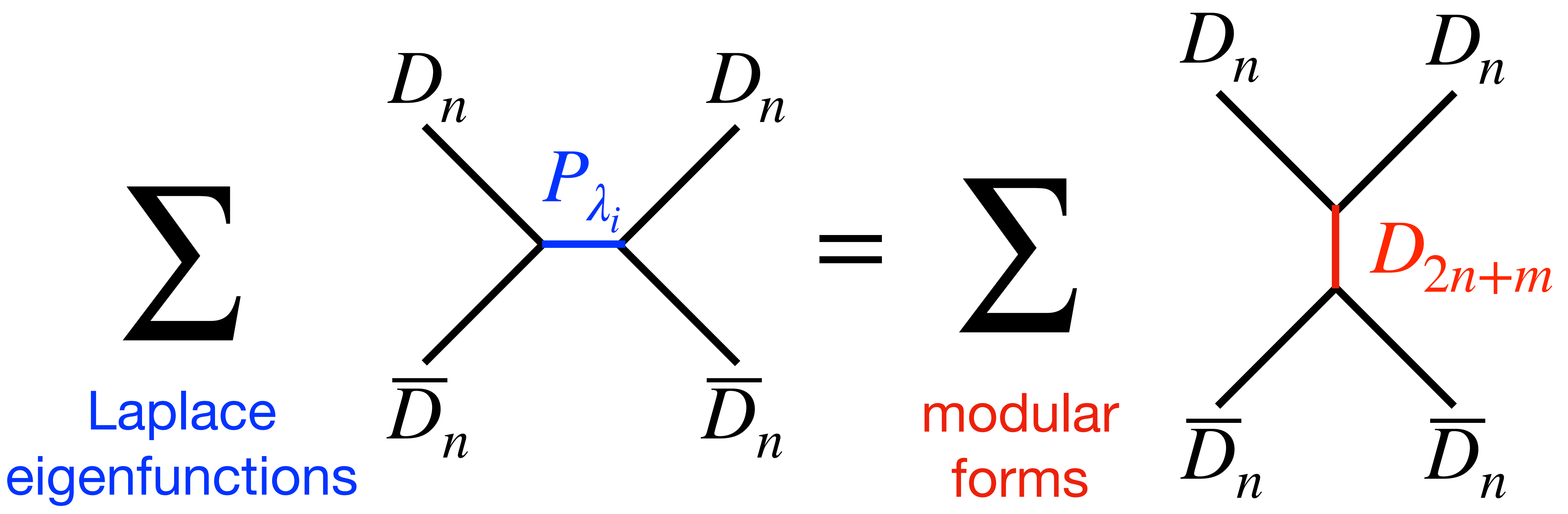
Crucial fact: $K_i(w_1, w_2)$ and $\widetilde{K}_j(w_1, w_2)$ are universal = fixed by G-invariance.

- The space of G-invariant maps $D_n \times \overline{D}_n \to P_\lambda$ and $D_n \times D_n \to D_m$ is one-dimensional.
- $f_i \sim \langle h\,\overline{h}\,\varphi_i \rangle, \tilde{f}_j \sim \langle h\,h\,\overline{h}_j \rangle$, integrals of triple products of automorphic forms.

Imposing Associativity

Suppose $L^2(\Gamma\backslash G)$ contains D_n and let $\mathcal{O}_n(w)$ be the corresponding local operator.

$$\langle \mathcal{O}_n(w_1)\mathcal{O}_n(w_2)\overline{\mathcal{O}}_n(w_3)\overline{\mathcal{O}}_n(w_4)\rangle$$



$$(1-\chi)^{-2n}\sum_i |f_i|^2 k_{\lambda_i}(\chi) = \chi^{-2n}\sum_{\substack{m\geq 0\\ m \text{ even}}} |\tilde{f}_m|^2 \tilde{k}_{2n+m}(\chi)$$

$$k_{s(1-s)}(\chi) = {}_2F_1(s, 1-s; 1; \tfrac{\chi}{\chi-1}) \qquad \tilde{k}_m(\chi) = \chi^m {}_2F_1(m, m; 2m; \chi)$$

conformal blocks

$\Rightarrow$ Get an infinite number of spectral identities by expanding around $\chi = 0$.

Spectral Bounds from Linear Programming

Spectral identities: $\sum_i |f_i|^2 P_{n,m}(\lambda_i) = |\tilde{f}_m|^2$ for all even $m \geq 0$, $\sum_i |f_i|^2 P_{n,m}(\lambda_i) = 0$ for all odd $m > 0$

Proposition: Fix $M \in \mathbb{N}$ and suppose $Q(\lambda) = \sum_{m=0}^{M} x_m P_{n,m}(\lambda)$ with $x_m \in \mathbb{R}$, such that

1. $x_m \leq 0$ for all even m
2. $Q(0) = 1$
3. $Q(\lambda) \geq 0$ for all $\lambda \geq \lambda_*$.

Then there is an upper bound on the Laplace spectral gap $\lambda_1 < \lambda_*$ for every hyperbolic orbifold with a holomorphic form of weight $2n$.

Proof: Consider $\sum_i |f_i|^2 Q(\lambda_i)$, exchange order of summations and use the spectral identities. □

Strategy: Minimize λ_* by optimizing over x_m satisfying 1.-3. Increase M to improve the bound.
We used the semidefinite programming solver SDPB. [Simmons-Duffin ’15] [Simmons-Duffin, Landry ’19]

Results

Let $2n_1(\Gamma)$ be the minimal weight of a modular form for Γ.

Fact: We have $n_1(\Gamma) \in \{1, 2, 3, 4, 6\}$ for every hyperbolic orbifold.

n_1	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	[1; 2] at the $\mathbb{Z}_6$-symmetric point
2	15.79144	15.79023	[0; 2,2,2,3] at the $\mathbb{Z}_3$-symmetric point
3	23.07917	23.07855	[0; 3,3,4]
4	30.35432	28.07984	[0; 2,4,5]
6	44.8883537	44.88835	[0; 2,3,7]

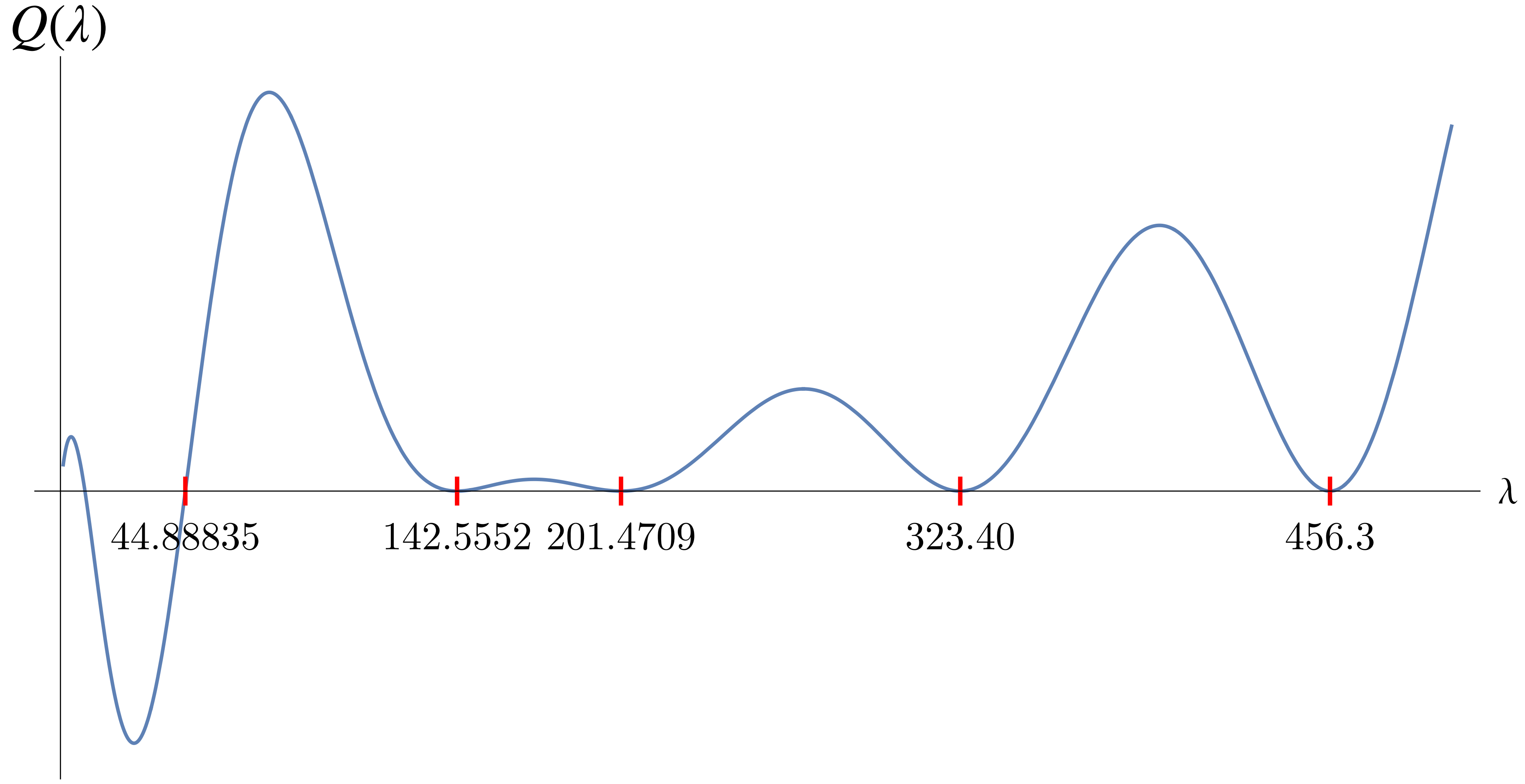
Corrolary: Every hyperbolic orbifold satisfies: $\lambda_1 \leq 44.8883537$.

Sharp Bounds

Question: Is the linear-programming upper bound on λ_1 sharp for $M \to \infty$?

If yes, the linear program must reconstruct the full Laplace spectrum of the $[0; 2,3,7]$ orbifold!

- $Q(\lambda_i) = 0$ for all $\lambda_i \in$ spectrum.
- Output of the linear program for $M = 41$ →
- Zeros agree with the $[0; 2,3,7]$ spectrum!
- Proof would amount to a construction of $Q(\lambda)$ for $M = \infty$.



This is precisely what happens for the Cohn-Elkies bound on sphere packing in $d = 8, 24$.

- Viazovska (2016): Construction of optimal $Q(\lambda)$ for sphere packing.
- DM (2016), DM+Paulos (2018): Construction of optimal $Q(\lambda)$ for the gap problem in 1D CFTs.
- Hartman+DM+Rastelli (2019): Precise mapping between Viazovska (2016) and DM (2016).

Challenge: Construct the optimal $Q(\lambda)$ for the Laplacian spectral gap problem.

Bounds at Fixed Genus

Bounds on λ_1 of genus-g orbifolds: Use g linearly independent holomorphic 1-forms.

Associativity implemented by the system of coupled equations:

$$\langle \mathcal{O}_i(w_1)\mathcal{O}_j(w_2)\overline{\mathcal{O}}_k(w_3)\overline{\mathcal{O}}_l(w_4)\rangle \qquad n_i = n_j = n_k = n_l = 1 \qquad i,j,k,l = 1,\ldots,g$$

This is a matrix generalization of the original linear program ⇒ need semidefinite programming.

genus	our bound on λ_1	largest known λ_1	orbifold
1	8.47032	8.46776	$[1;2]$ at the $\mathbb{Z}_6$-symmetric point
2	3.83890	3.83889	Bolza surface
3	2.67849	2.67793	Klein quartic

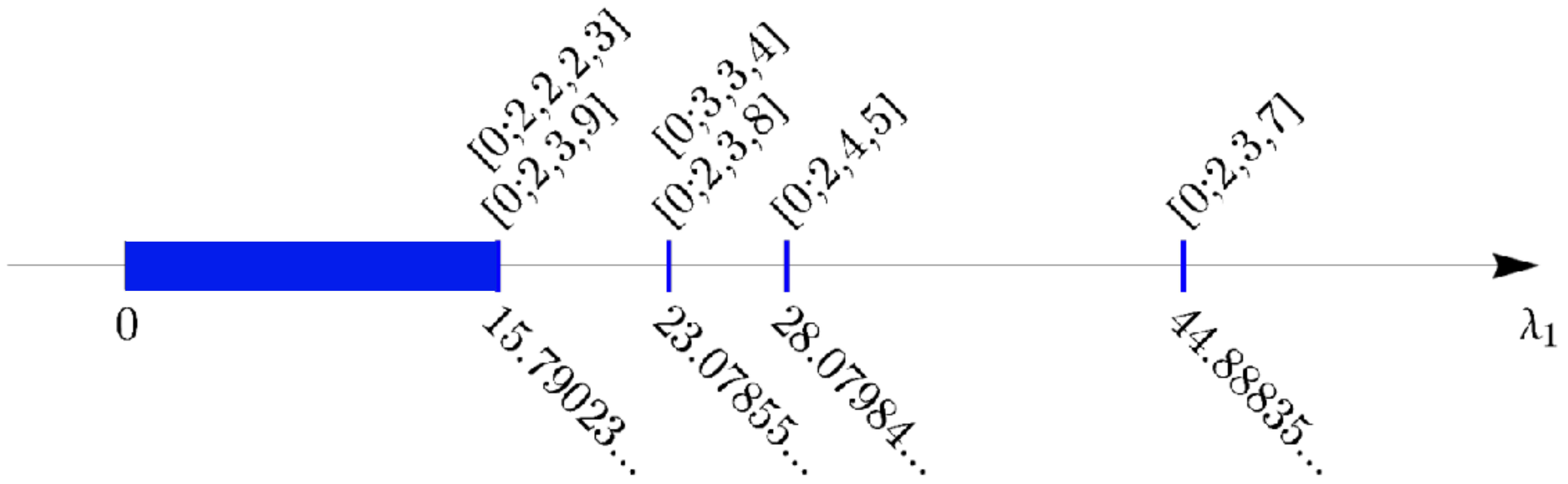
Values of λ_1 Attained by All Orbifolds

Idea: Topological type is uniquely identified by the spectrum of weights of modular forms.
Only finitely many weigths are needed to identify each topological type.

Study associativity for **two** holomorphic forms of minimal weight $2 \leq 2n_1 < 2n_2$

$$\langle \mathcal{O}_{n_1}(w_1)\mathcal{O}_{n_2}(w_2)\overline{\mathcal{O}}_{n_1}(w_3)\overline{\mathcal{O}}_{n_2}(w_4)\rangle$$

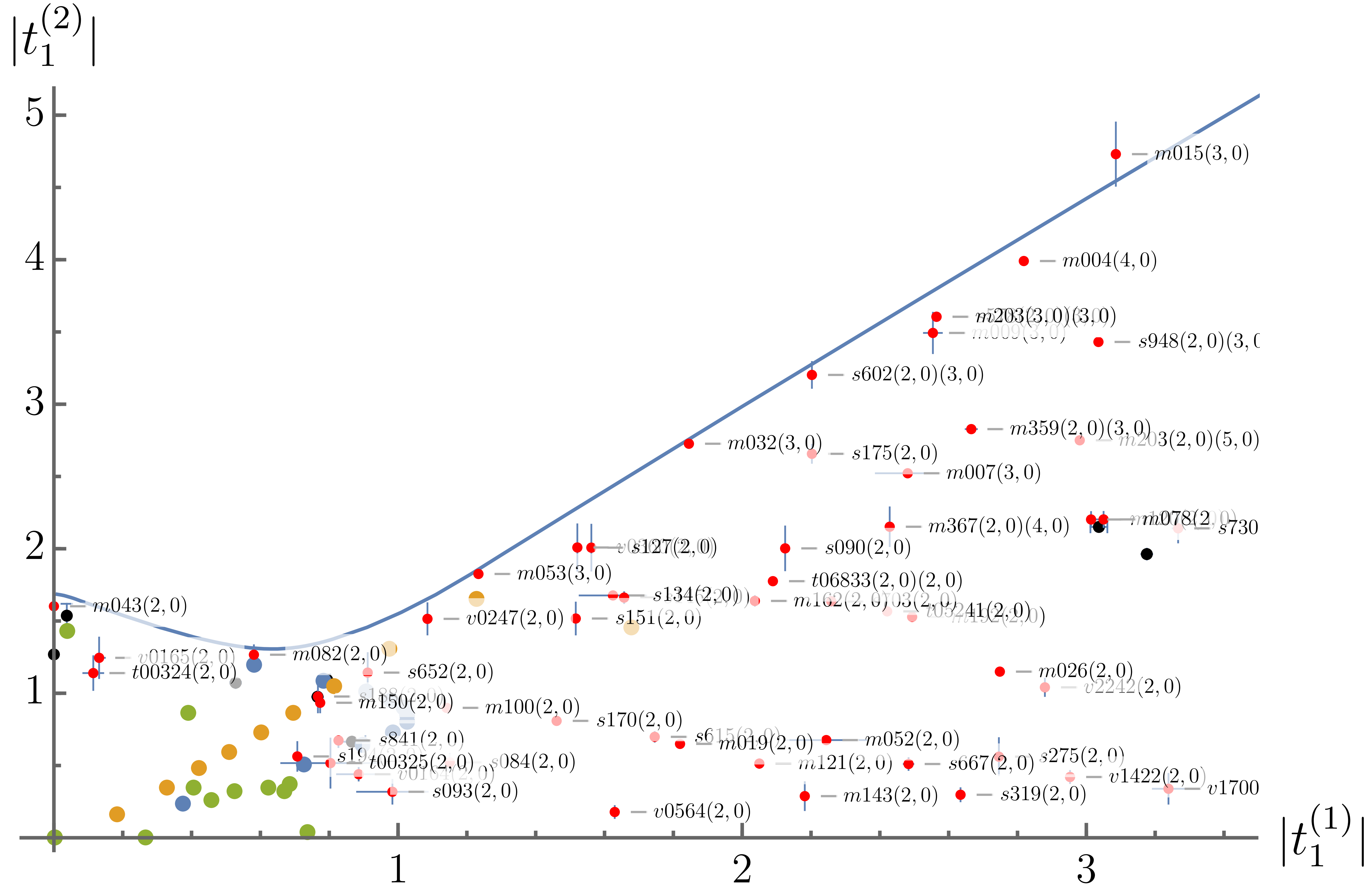
theorem: If X ranges over all orbifolds, $\lambda_1(X)$ takes the following values:



Example: $n_1 = 6, n_2 = 8 \Rightarrow \lambda_1 \leq 23.0997$ unless the orbifold is $[0; 2,3,7]$ or $n_1 \leq 4$.

Hyperbolic Three-Manifolds

work in progress with J. Bonifacio, P. Kravchuk and S. Pal



$|t_1^{(J)}|^2 + 1$ = the lowest Laplace eigenvalue on symmetric tensors of rank J.

Summary

- There is a close analogy between conformal field theories and hyperbolic manifolds.
- This leads to an infinite set of identities satisfied by the Laplacian spectra of hyp. manifolds.
- Linear/semidefinite programming turns the identites into bounds on the spectral gap λ_1.
- The bounds on λ_1 for 2D hyperbolic orbifolds are often nearly sharp.
- They allow us to (more or less) identify the set of λ_1 realized by all 2D hyperbolic orbifolds.

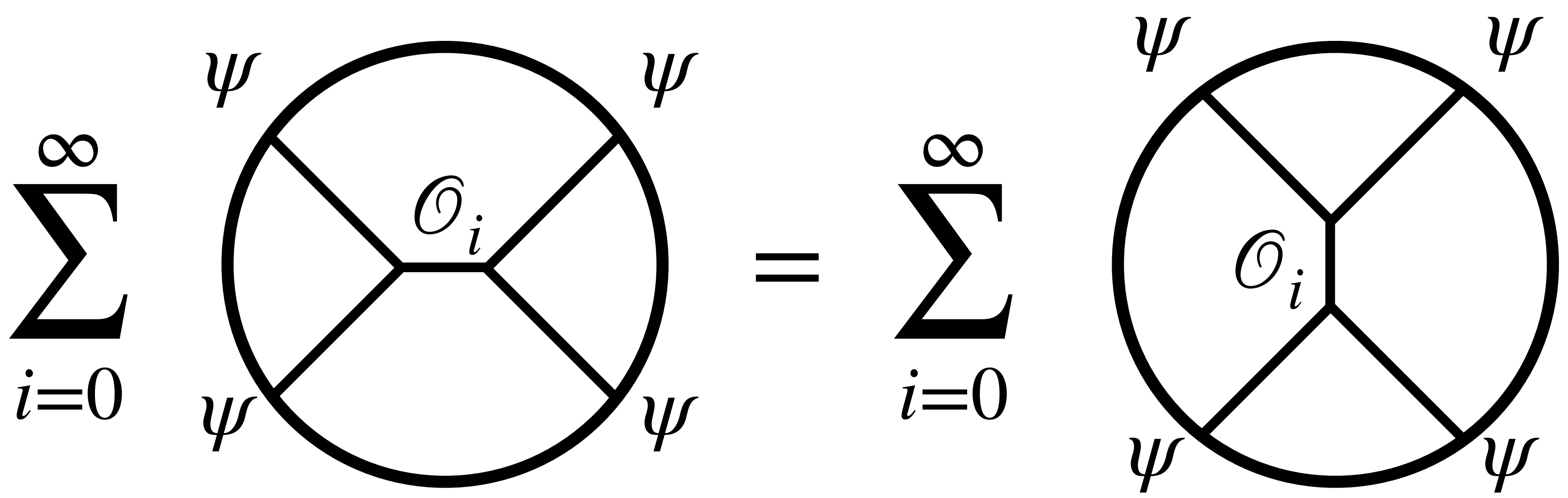
2. Conformal Bootstrap and Sphere Packing

Overview

- Problems arising in the conformal bootstrap naturally take the form of infinite-dimensional linear programs (Rattazzi+Rychkov+Tonni+Vichi 2008).
- This type of problem is hard to solve exactly in general, but examples of exact solutions have appeared in the conformal bootstrap literature (DM 2016).
- A closely related type of an infinite-dimensional linear program has been used to prove upper bounds on sphere-packing density (Cohn+Elkies 2001).
- In this context, Viazovska (2016) found an exact solution of the problem, leading to the solution of the sphere-packing problem in dimensions 8 and 24.
- Viazovska's solution can be exactly mapped to the exact solution found in the conformal bootstrap (Hartman+DM+Rastelli 2019).

Four-Point Bootstrap

- Consider the four-point correlation function in 1D: $\langle \psi(x_1)\psi(x_2)\psi(x_3)\psi(x_4)\rangle$.
- OPE: $\psi \times \psi = \sum_{i=0}^{\infty} f_i \mathcal{O}_i$, where $f_i \in \mathbb{R}$ and $\mathcal{O}_0 = 1$.
- Spectrum: $0 = \Delta_0 < \Delta_1 \leq \Delta_2 \leq \dots$



$$\sum_{i=0}^{\infty} |f_i|^2 G_{\Delta_i}^{\Delta_\psi}(z) = \sum_{i=0}^{\infty} |f_i|^2 G_{\Delta_i}^{\Delta_\psi}(1-z) \qquad (\star)$$

- Conformal blocks for the $sl_2(\mathbb{R})$ algebra: $G_{\Delta}^{\Delta_\psi}(z) = z^{\Delta - 2\Delta_\psi} {}_2F_1(\Delta, \Delta; 2\Delta; z)$.

Question: What is the maximal possible Δ_1 compatible with (★), for a given Δ_ψ?

Bounds from Functionals

[Rattazzi, Rychkov, Tonni, Vichi '08]

$$\sum_{i=0}^{\infty} |f_i|^2 \underbrace{[G_{\Delta_i}^{\Delta_\psi}(z) - G_{\Delta_i}^{\Delta_\psi}(1-z)]}_{F_{\Delta_i}^{\Delta_\psi}(z)} = 0$$

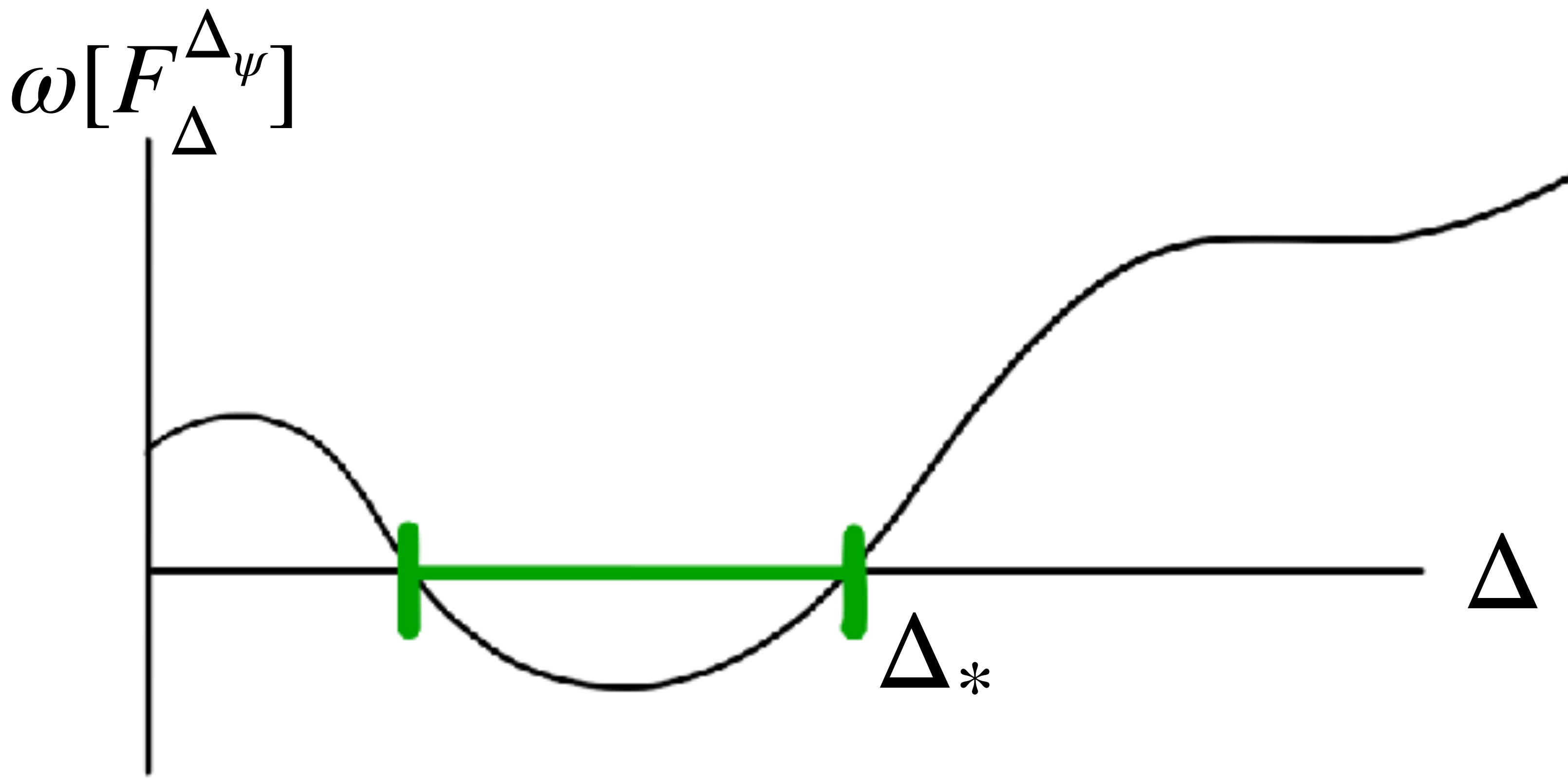
- Apply linear functionals $\omega : F(z) \to \mathbb{R}$ to rule out possible spectra.

  If $\exists \omega$ such that $\omega[F_{\Delta_i}^{\Delta_\psi}] > 0$ for all Δ_i in a putative theory, then the theory is ruled out.

Example: To get an upper bound on Δ_1, suppose

1. $\omega[F_0^{\Delta_\psi}] = 1$
2. $\omega[F_\Delta^{\Delta_\psi}] \geq 0$ for all $\Delta \geq \Delta_*$

then $\Delta_1 \leq \Delta_*$ in all consistent theories.



Analytic Functionals

[DM '16], [DM, Paulos '18]

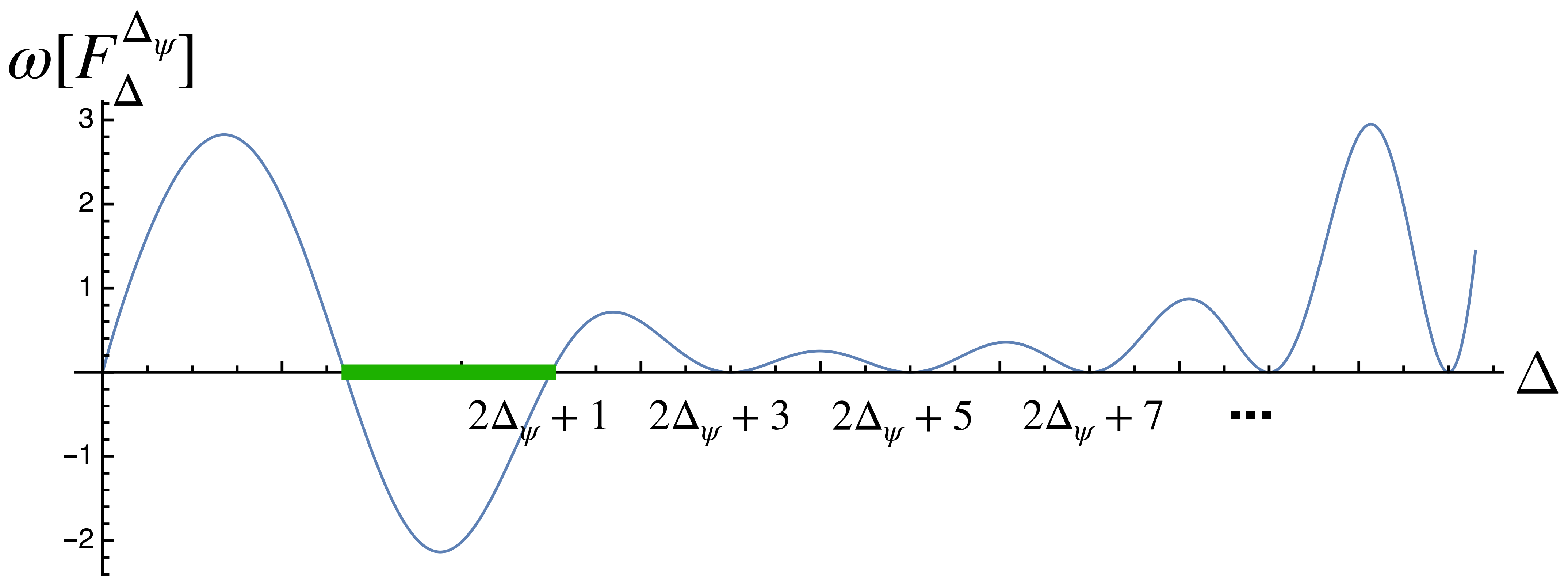
Question: What is the maximal possible Δ_1 for a given Δ_ψ?

Theorem: The gap-maximizing solution is the fermionic mean field theory, with $\Delta_1 = 2\Delta_\psi + 1$.

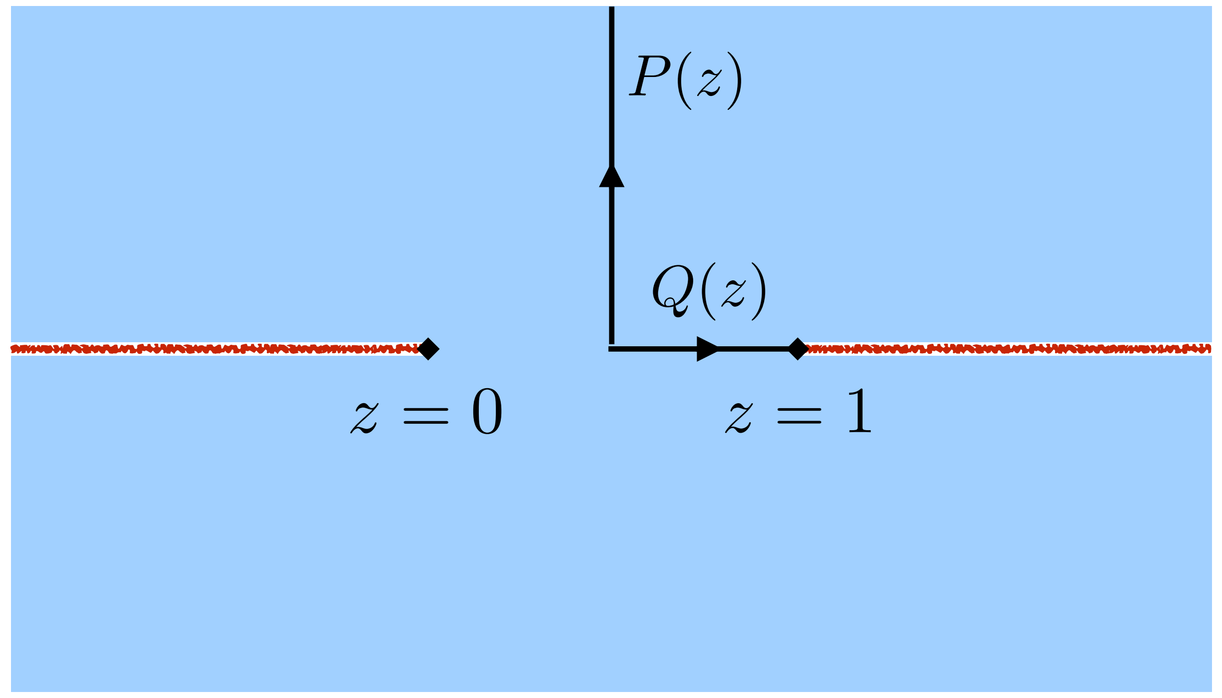
$$\psi \times \psi = 1 + \psi\partial\psi + \psi\partial^3\psi + \ldots$$

Spectrum: $\Delta = 0, 2\Delta_\psi + 1, 2\Delta_\psi + 3\,, \ldots$

Proof: Construct a linear functional ω with double zeros on the extremal spectrum.



$$\omega[F_\Delta^{\Delta_\psi}] = \sin^2\left[\frac{\pi}{2}(\Delta - 2\Delta_\psi - 1)\right]\int_0^1 dz\, Q(z)\, G_\Delta^{\Delta_\psi}(z)$$



$P(z), Q(z)$ are subject to a system of functional equations which admits a unique solution.

Functionals and the 2D Modular Bootstrap

Observable: Torus partition function of a 2D CFT: $Z(\beta) = \sum_i e^{-\beta(\Delta_i - c/12)}$.

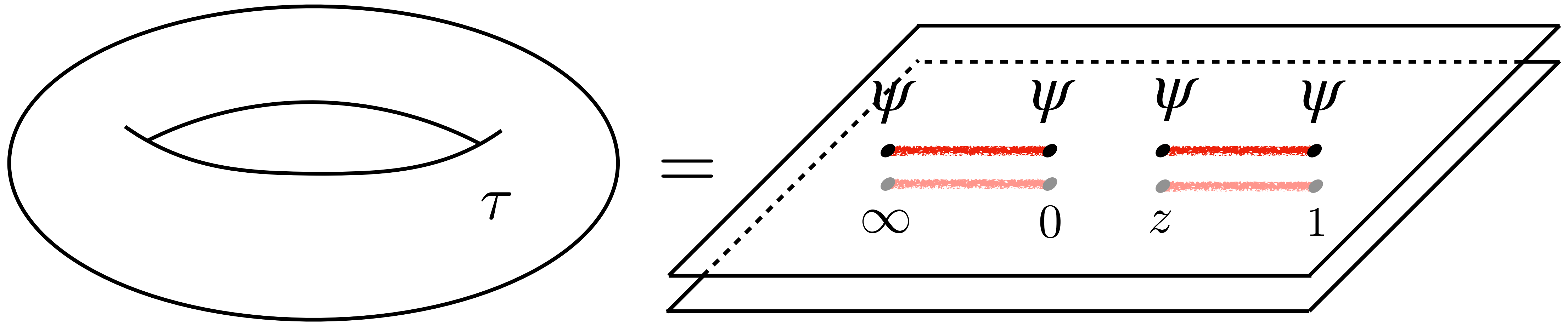
Constraint: Modular invariance $Z(\beta) = Z(4\pi^2/\beta)$.

Question: What is the maximal Δ_1 (first Virasoro primary) subject to this constraint?

Equivalent to a four-point correlator bootstrap

$$Z(\beta) \sim \langle 0 | \psi(0)\psi(z)\psi(1)\psi(\infty) | 0 \rangle_{(T\times T)/\mathbb{Z}_2}$$

$$\Delta_\psi = \frac{c}{8} \qquad z = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}$$



$\Rightarrow$ Can uplift the analytic functionals from the four-point function to the modular bootstrap!

$\Rightarrow$ **Theorem:** Every compact unitary 2D CFT with $c \notin (4,12)$ contains a Virasoro primary with

$$\Delta \leq \frac{c}{8} + \frac{1}{2}.$$

[Hartman, DM, Rastelli '19]

Sphere Packing Problem

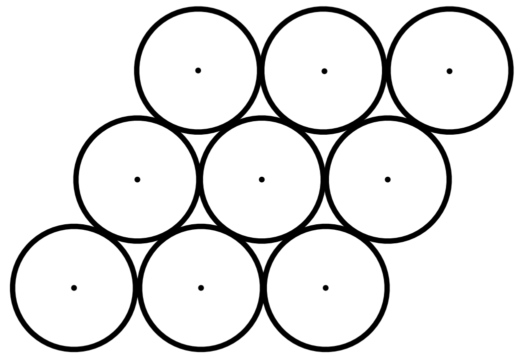
Task : Find the densest arrangement of identical, non-overlapping spheres in $\mathbb{R}^d$.

Applications: error-correcting codes, stacking of oranges

Known solutions:

$d = 2$:

[Toth '40]



$d = 3$:

[Hales '98]

$d = 8$: E_8 lattice

[Viazovska '16]

$d = 24$: Leech lattice

[Cohn, Kumar, Miller, Radchenko, Viazovska '16]

Sphere Packing Bounds

[Cohn, Elkies '01]
[Hartman, DM, Rastelli '19]

Idea: Use bootstrap-like constraints to prove an upper bound on the sphere-packing density.

- For a periodic packing with sphere centers at $x_i \in \mathbb{R}^d$, define the partition function

$$Z(\tau) = \sum_{(ij)} \frac{e^{i\pi|x_i - x_j|^2\tau}}{\eta(\tau)^d} = \sum_{(ij)} \chi_{\Delta_{ij}}(\tau)$$

- Here $\chi_\Delta(\tau) = \dfrac{e^{2\pi i\Delta\tau}}{\eta(\tau)^d}$ is a character of the $U(1)^d$ chiral algebra, and $\Delta_{ij} = |x_i - x_j|^2/2$.

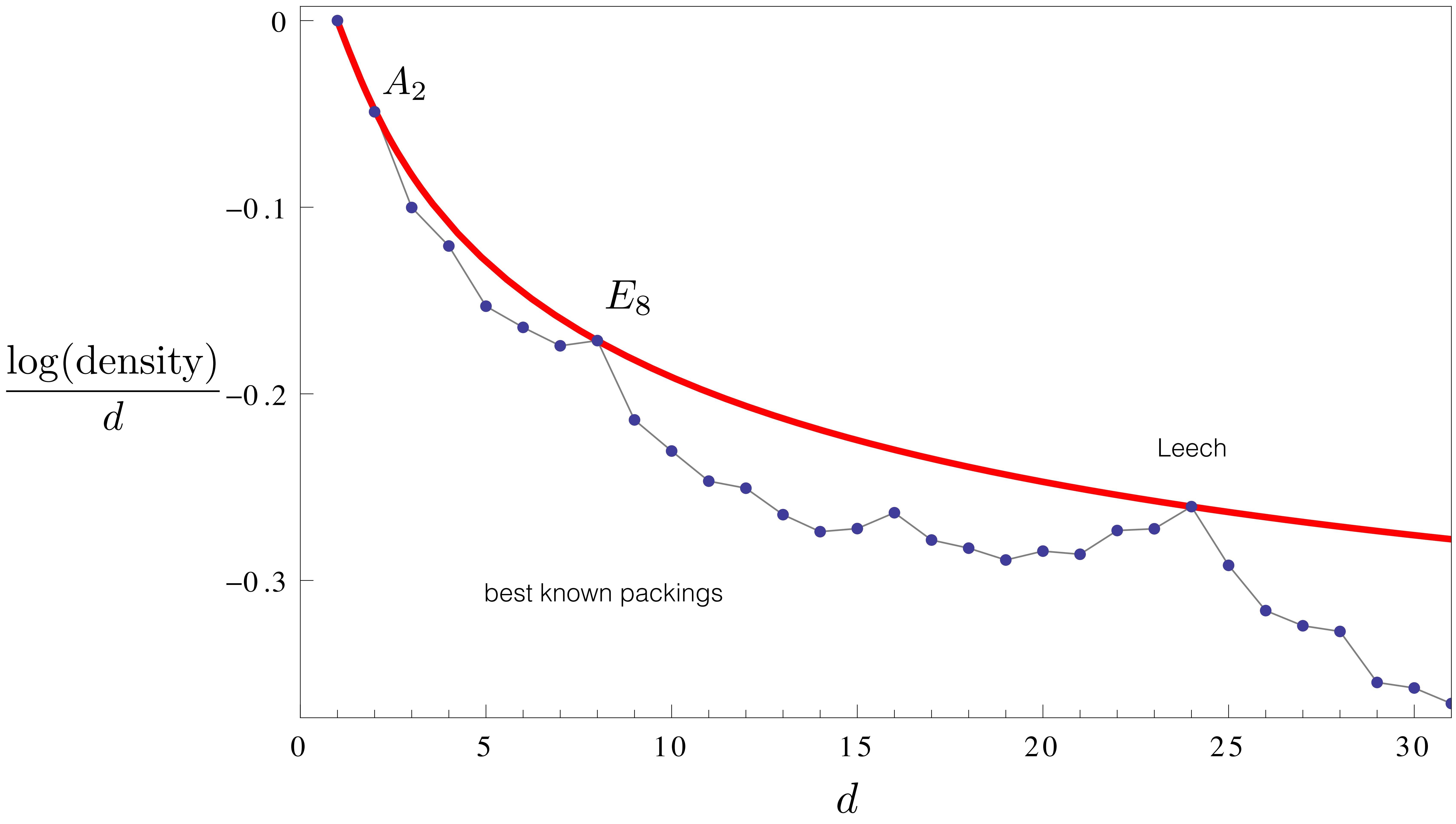
- Upper bound on the smallest Δ_{ij} ⇔ upper bound on the density of all sphere packings in $\mathbb{R}^d$.

- Poisson summation implies an alternative expression $Z(\tau) = \sum_i p_i\, \chi_{\Delta_i}(-1/\tau)$.

⇒ Can use standard bootstrap techniques to prove upper bounds on sphere packing density.

Resulting Bound

[Cohn, Elkies '01]



Sharp Bounds for d=8 and d=24

- Viazovska (2016): the Cohn-Elkies upper bound is sharp in $d = 8$, saturated by the E_8 lattice.
- Similarly, the Cohn-Elkies upper bound is sharp in $d = 24$, saturated by the Leech lattice.

[Cohn, Kumar, Miller, Radchenko, Viazovska '16]

Method of proof: (translated to CFT language)

- Construct the analytic functional for the bootstrap problem with $U(1)^d$ characters.
- Can recover Viazovska's magic function from the analytic functional for 1D conformal bootstrap.

[Viazovska '16]

[Cohn, Kumar, Miller, Radchenko, Viazovska '16]

$d = 8, 24$ $\Leftrightarrow$ $\Delta_\psi = \frac{1}{2}, \frac{3}{2}$

[DM '16]

Summary

General problem: $0 = \Delta_0 \leq \Delta_1 \leq \Delta_2 \leq \ldots$, maximize Δ_1, subject to:

1. **Four-point bootstrap:** $\sum_{i=0}^{\infty} |f_i|^2 \, G^{\Delta_\psi}_{\Delta_i}(z) = \sum_{i=0}^{\infty} |f_i|^2 \, G^{\Delta_\psi}_{\Delta_i}(1-z)$ $\qquad G^{\Delta_\psi}_{\Delta}(z) = z^{\Delta - 2\Delta_\psi} {}_2F_1(\Delta, \Delta; 2\Delta; z)$

   Optimal solution: $\Delta_1 = 2\Delta_\psi + 1$ for all $\Delta_\psi > 0$. **[DM '16], [DM, Paulos '18]**

2. **Virasoro modular bootstrap:** $\sum_{i=0}^{\infty} |f_i|^2 \, \chi^c_{\Delta_i}(\tau) = \sum_{i=0}^{\infty} |f_i|^2 \, \chi^c_{\Delta_i}(-1/\tau)$ $\qquad \chi^c_\Delta(\tau) = \dfrac{e^{2\pi i \tau (\Delta - \frac{c-1}{12})}}{\eta(\tau)^2}$

   Optimal solution: $\Delta_1 = 1$ for $c = 4$ and $\Delta_1 = 2$ for $c = 12$. **[Hartman, DM, Rastelli '19]**

3. **Sphere packing bootstrap:** $\sum_{i=0}^{\infty} |f_i|^2 \, \chi^d_{\Delta_i}(\tau) = \sum_{i=0}^{\infty} |f_i|^2 \, \chi^d_{\Delta_i}(-1/\tau)$ $\qquad \chi^d_\Delta(\tau) = \dfrac{e^{2\pi i \tau \Delta}}{\eta(\tau)^d}$

   Optimal solution: $\Delta_1 = 1$ for $d = 8$ and $\Delta_1 = 2$ for $d = 24$.

   [Viazovska '16] **[Cohn, Kumar, Miller, Radchenko, Viazovska '16]**

- Solution of 2. and 3. ⇔ solution of 1. for $\Delta_\psi = 1/2$ and $\Delta_\psi = 3/2$. **[Hartman, DM, Rastelli '19]**

Thank you!

References

First part:

- P. Kravchuk, DM, S. Pal: Automorphic Spectra and the Conformal Bootstrap, [arXiv:2111.12716]

Second part:

- DM: Analytic Bounds and Emergence of AdS_2 Physics from the Conformal Bootstrap, [arXiv:1611.10060]
- DM, M. Paulos: The Analytic Functional Bootstrap I: 1D CFTs and 2D S-matrices, [arXiv:1803.10233]
- T. Hartman, DM, L. Rastelli: Sphere Packing and Quantum Gravity, [arXiv:1905.01319]