Stress-Energy Tensor in Liouville CFT

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Probability and CFT
Agay
2022

Outline

1. SE-tensor in classical Liouville CFT
   - Uniformisation of Riemann surfaces
   - Accessory parameters
   - Toy computation for the 2nd part
2. SE-tensor in probabilistic Liouville CFT by varying the background metric
   - Conformal Ward identities on the sphere (joint work with A. Kupiainen)
   - Higher genus surfaces

Conformal Symmetry

- Fix a compact orientable Riemannian surface (Σ, g) and consider an action functional

$$\varphi \mapsto S(\varphi, g),$$

satisfying the diffeomorphism covariance property

$$S(\psi^* \varphi, \psi^* g) = S(\varphi, g), \quad \psi \in \mathrm{Diff}(\Sigma).$$

and Weyl invariance

$$S(\varphi, e^{\omega} g) = S(\varphi, g), \quad \omega \in C^{\infty}(\Sigma, \mathbb{R}).$$

- These properties imply that for $\psi : (\Sigma, g) \to (\Sigma, g)$ conformal

$$S(\psi^* \varphi, g) = S(\varphi, g).$$

- The group of conformal maps $\psi : \Sigma \to \Sigma$ is finite-dimensional (possibly 0). Weyl invariance gives a more general definition of conformal symmetry.

Holomorphic Stress-Energy Tensor

- We define the **Stress-Energy Tensor** T by

$$T_{\alpha\beta}(z) := -4\pi \frac{\delta S(\varphi_g, g)}{\delta g^{\alpha\beta}(z)},$$

where φ_g is the minimiser of the action. I.e. if $g_\varepsilon = g + \varepsilon f$ for f smooth symmetric 2-tensor, then T is the distribution

$$\int_\Sigma \langle f, T\rangle_g dv_g := -4\pi \partial_\varepsilon|_0 S(\varphi_g, g_\varepsilon).$$

- Weyl Invariance $\implies \mathrm{Tr}_g(T) = 0$
- Diffeomorphism covariance $\implies \mathrm{Div}_g(T) = 0.$
- In 2D these two properties imply that, in conformal coordinates,

$$T_{z\bar{z}} = T_{\bar{z}z} = 0, \quad \partial_{\bar{z}} T_{zz} = 0, \quad \partial_z T_{\bar{z}\bar{z}} = 0.$$

Classical Liouville Field Theory

- Liouville Action on a Riemannian surface (Σ, g)

$$S_L(\varphi, g, \mathbf{x}) = \int_\Sigma \left(\tfrac{1}{2}|\nabla^g \varphi|_g^2 + R_g \varphi + 2e^\varphi\right) dv_g - 4\pi \sum_{j=1}^n \alpha_j \varphi(x_j)\,.$$

  - R_g the scalar curvature
  - $\alpha_j \in (-\infty, 1]$, $\sum_{j=1}^n \alpha_j > 2(1 - \text{genus})$.
  - $\mathbf{x} = (x_1, \dots, x_n) \in \Sigma^n$, $x_i \neq x_j$ for $i \neq j$.
- Euler–Lagrange equation $\frac{\delta}{\delta\varphi} S_L = 0$

$$\Delta_g \varphi = R_g + 2e^\varphi - 4\pi \sum_{j=1}^n \alpha_j \delta_{g,x_j}\,. \tag{1}$$

- If $\varphi_{g,\mathbf{x}}$ solves (1), then $R_{e^{\varphi_{g,\mathbf{x}}}g}(z) = -2$ for $z \in \Sigma \setminus \{x_1, \dots x_n\}$.
- $e^{\varphi_{g,\mathbf{x}}} g$ describes a hyperbolic surface $\Sigma_{\alpha,\mathbf{x}}$ with
  - conical singularity at x_j of angle $2\pi(1 - \alpha_j)$ when $\alpha_j < 1$
  - cusp (puncture) at x_j when $\alpha_j = 1$

Minimiser

- Liouville Action on a Riemannian surface (Σ, g)

$$S_L(\varphi, g, \mathbf{x}) = \int_\Sigma \left(\tfrac{1}{2}|\nabla^g \varphi|_g^2 + R_g \varphi + 2e^\varphi\right) dv_g - 4\pi \sum_{j=1}^n \alpha_j \varphi(x_j)\,, \qquad (2)$$

$$\Delta_g \varphi = R_g + 2e^\varphi - 4\pi \sum_{j=1}^n \alpha_j \delta_{g,x_j}\,.$$

- The source term $-\alpha_j \delta_{g,x_j}$ forces the solution $\varphi_{g,\mathbf{x}}$ to have the asymptotic behaviour (for $\alpha_j < 1$)

$$\varphi_{g,\mathbf{x}}(z) = -2\alpha_j \log d_g(z, x_j) + O(1)\,, \quad z \to x_j\,. \qquad (3)$$

- The function $\varphi_{g,\mathbf{x}}$ is the minimiser of $\varphi \mapsto S_L(\varphi, g, \mathbf{x})$ over functions φ that are smooth on $\Sigma \setminus \{x_1, \ldots x_n\}$ and satisfy (3), which means that (2) has to be regularised (substract infinity).

Classical symmetries

- **Diffeomorphism Covariance**: for $\psi \in \mathrm{Diff}(\Sigma)$

$$S_L(\psi^* \varphi, \psi^* g, \psi^{-1}(\mathbf{x})) = S_L(\varphi, g, \mathbf{x})$$

- **Weyl Anomaly**: for $\omega \in C^\infty(\Sigma, \mathbb{R})$

$$S_L(\varphi, e^\omega g, \mathbf{x}) = S_L(\varphi + \omega, g, \mathbf{x}) - S_L^0(\omega, g) - \sum_{j=1}^{n} \Delta_{\alpha_j} \omega(x_j),$$

  where $S_L^0(\varphi, g) = \int (\frac{1}{2} |\nabla^g \omega|_g^2 + R_g \omega) dv_g$ and $\Delta_{\alpha_j} = \alpha_j (1 - \frac{\alpha_j}{2})$.

- The shift $\varphi \to \varphi + \omega$ is natural because $e^\varphi (e^\omega g) = e^{\varphi + \omega} g$.

SE-tensor in classical LCFT

- Recall

$$T_{zz} = -4\pi \frac{\delta S_L(\varphi_{g,\mathbf{x}}, g, \mathbf{x})}{\delta g^{zz}}$$

- Varying the Dirichlet energy and the curvature w.r.t. g one gets

$$T_{zz} = \nabla_z^2 \varphi_{g,\mathbf{x}} - \tfrac{1}{2}(\nabla_z \varphi_{g,\mathbf{x}})^2 .$$

- The Liouville equation $\Delta_g \varphi_{g,\mathbf{x}} = R_g + 2e^{\varphi_{g,\mathbf{x}}}$ and the behaviour $\varphi_{g,\mathbf{x}}(z) \overset{z \to x_j}{\simeq} -2\alpha_j \log|z - x_j|$ imply

$$T_{zz}(z) = \sum_{j=1}^{n} \left(\frac{\Delta_{\alpha_j}}{(z-x_j)^2} + \frac{c_j}{z - x_j} \right) + \text{smooth}$$

where $\Delta_{\alpha_j} = \alpha_j(1 - \frac{\alpha_j}{2})$ and $c_j \in \mathbb{C}$.

Accessory parameters

- Set $\Sigma = \mathbb{S}^2$, $g = |dz|^2$, $\varphi_{\mathbf{x}}(z) \sim -4 \ln |z|$ when $|z| \to \infty$. Then

$$T_{zz}(z) = \sum_{j=1}^{n} \left(\frac{\Delta_{\alpha_j}}{(z - x_j)^2} + \frac{c_j}{z - x_j} \right).$$

- Residues c_j are the **accessory parameters**. Geometric objects that are related to uniformisation and Weil–Petersson metric ($\alpha_j = 1$).
- Polyakov conjectured in the 80's that

$$c_j = -\partial_{x_j} S_L(\varphi_{\mathbf{x}}, |dz|^2, \mathbf{x}) \tag{4}$$

- Polyakov's argument was to take a semi-classical limit of the first Conformal Ward identity of quantised Liouville.
  - Proven rigorously using probabilistic LCFT by Lacoin–Rhodes–Vargas, 2019.
- Takhtajan–Zograf proved (4) in a classical setting (1988, 2002).
  - Based on a relation between the $\varphi_{\mathbf{x}}$ and the uniformising map of $\Sigma_{\alpha,\mathbf{x}}$.

Accessory parameters

- Uniformisation theorem: there is a biholomorphism $\Sigma_{\alpha,\mathbf{x}} \to \mathbb{D}/\Gamma$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a subgroup (Fuchsian when $\alpha_j = 1 - \frac{1}{n}$, $n \in \mathbb{N} \cup \{\infty\}$).
- The covering map $J : \mathbb{D} \to \Sigma$ defines a hyperbolic metric $J_* g_{\mathbb{D}}$ on $\Sigma_{\alpha,\mathbf{x}}$.
- How are $e^{\varphi_{\mathbf{x}}}|dz|^2$ and $J_* g_{\mathbb{D}}$ related? A computation shows that $e^{\varphi_{\mathbf{x}}}|dz|^2 = J_* g_{\mathbb{D}}$ if

$$\mathscr{S}(J^{-1}) = \partial_z^2 \varphi_{\mathbf{x}} - \tfrac{1}{2}(\partial_z \varphi_{\mathbf{x}})^2 = T_{zz}\,,$$

  where $\mathscr{S}$ is the Schwarzian derivative $\mathscr{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$.
- Using this, Takhtajan–Zograf compute, in the conical case, that

$$\partial_{x_j} \varphi_{\mathbf{x}}(z) = -\frac{\alpha_j}{z - z_j} + \frac{c_j}{\alpha_j} + O((z - z_j)^{1-2\alpha_j}) \quad z \to x_j\,.$$

- Using this, they eventually get $c_j = -\partial_{x_j} S_L(\varphi_{\mathbf{x}}, |dz|^2, \mathbf{x})$.

- We want to compute T_{zz} by using the symmetries
- **Diffeomorphism Covariance**: for $\psi \in \mathrm{Diff}(\Sigma)$

$$S_L(\psi^* \varphi, \psi^* g, \psi^{-1}(\mathbf{x})) = S_L(\varphi, g, \mathbf{x})$$

- **Weyl Anomaly**: for $\omega \in C^\infty(\Sigma, \mathbb{R})$

$$S_L(\varphi, e^\omega g, \mathbf{x}) = S_L(\varphi + \omega, g, \mathbf{x}) - A(\omega, g) - \sum_{j=1}^{n} \Delta_{\alpha_j} \omega(x_j),$$

- We want to do the following:
  1. Perturb the metric: $g_\varepsilon = g + \varepsilon f$
  2. Apply the symmetries to $S_L(\varphi_{g,\mathbf{x}}, g_\varepsilon, \mathbf{x})$
  3. Get a formula for T_{zz} by differentiation
- This leads to the "classical Conformal Ward identity", and especially the formula for the accessory parameters.

Moduli Space

- Every smooth metric g on Σ has a decomposition

$$g = e^{\omega} \psi^* \hat{g}(\tau),$$

where $\omega \in C^\infty(\Sigma, \mathbb{R})$, $\psi \in \mathrm{Diff}(\Sigma)$ and $\tau \in \mathrm{Mod}(\Sigma)$.

- $\{\hat{g}(\tau)\}_{\tau \in \mathrm{Mod}(\Sigma)}$ is a fixed family of constant curvature metrics.
- $\mathrm{Mod}(\Sigma)$ is the space of conformal (or complex) structures

$$\dim_{\mathbb{R}} \mathrm{Mod}(\Sigma) = \begin{cases} 0, & \text{genus} = 0, \\ 2, & \text{genus} = 1, \\ 6\,\text{genus} - 6, & \text{genus} \geq 2. \end{cases}$$

- If $\Sigma = \mathbb{S}^2$, then $g = e^{\omega} \psi^* \hat{g}$ for a single fixed metric $\hat{g}$.

Sphere: trivial moduli space

- Let $\Sigma = \mathbb{S}^2$ and $g_\varepsilon^{zz} = g^{zz} + \varepsilon f^{zz}$ for some smooth f^{zz} with support away from $\{x_1, \dots, x_n\}$.
- Rewrite this as $g_\varepsilon = e^{\omega_\varepsilon} \psi_\varepsilon^* g$, where ψ_ε solves the Beltrami equation

$$\partial_{\bar z} \psi_\varepsilon = \mu_\varepsilon \partial_z \psi_\varepsilon \,,$$

  where $\mu_\varepsilon = -\frac{\varepsilon}{2\pi} f^{zz} g_{z\bar z} + O(\varepsilon^2)$, $\psi_\varepsilon(z) = z + O(\varepsilon)$.
- Classical theory implies that

$$\psi_\varepsilon(z) = z + \mathscr{C} \sum_{k=0}^{\infty} (\mu_\varepsilon \partial_z \mathscr{C})^k \mu_\varepsilon(z) \,,$$

  where $\mathscr{C}$ is the Cauchy transform $(\mathscr{C} h)(w) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{h(z)}{w-z} dv_g(z)$.
- It follows that $(\varepsilon, z) \mapsto \psi_\varepsilon(z)$ is C^∞ and

$$\dot\psi(w) := \partial_\varepsilon|_0 \psi_\varepsilon(w) = -\tfrac{1}{4} \mathscr{C}(f^{zz})(w) \,,$$
$$\dot\omega(w) := \partial_\varepsilon|_0 \omega_\varepsilon(w) = -\partial_w \dot\psi(w) - \partial_w \sigma(w) \dot\psi(w) \,, \quad (g = e^{\sigma} |dz|^2) \,.$$

- For $g_\varepsilon = e^{\omega_\varepsilon}\psi_\varepsilon^* g$ the symmetries lead to

$$S_L(\varphi_{g,\mathbf{x}}, g_\varepsilon, \mathbf{x}) = S_L(\varphi_{g,\mathbf{x}} \circ \psi_\varepsilon + \omega_\varepsilon \circ \psi_\varepsilon, g, \psi_\varepsilon^{-1}(\mathbf{x})) \\ - S_L^0(\omega_\varepsilon \circ \psi_\varepsilon, g) - \sum_{j=1}^n \Delta_{\alpha_j}\omega_\varepsilon(x_j)\,.$$

- Now we can compute everything. The relevant terms are:
  1. $\partial_\varepsilon|_0 S_L(\varphi_{g,\mathbf{x}} \circ \psi_\varepsilon + \omega_\varepsilon \circ \psi_\varepsilon, g, \psi_\varepsilon^{-1}(\mathbf{x}))$,
  2. $\partial_\varepsilon|_0 \sum_{j=1}^n \Delta_{\alpha_j}\omega_\varepsilon(x_j)$.
- To compute the first term, we first write ($\mathbf{x} \mapsto \varphi_{g,\mathbf{x}}$ is smooth)

$$\varphi_{g,\mathbf{x}} \circ \psi_\varepsilon + \omega_\varepsilon \circ \psi_\varepsilon = \varphi_{g,\psi_\varepsilon^{-1}(\mathbf{x})} + O(\varepsilon)$$

Then we use the fact that $\varphi_{g,\psi_\varepsilon^{-1}}$ is the minimiser to get

$$\partial_\varepsilon|_0 S_L\big(\varphi_{g,\psi_\varepsilon^{-1}(\mathbf{x})} + O(\varepsilon), g, \psi_\varepsilon^{-1}(\mathbf{x})\big) = \partial_\varepsilon|_0 S_L(\varphi_{g,\psi_\varepsilon^{-1}(\mathbf{x})}, g, \psi_\varepsilon^{-1}(\mathbf{x})) \\ = -\sum_{j=1}^n \dot{\psi}(x_j)\partial_{x_j} S_L(\varphi_{g,\mathbf{x}}, g, \mathbf{x})\,.$$

- Recall also that $\dot{\omega}(x_j) = \partial_{x_j}\dot{\psi}(x_j) + \partial_{x_j}\sigma(x_j)\dot{\psi}(x_j)$.

- We get

$$4\pi\partial_\varepsilon|_0 S_L(\varphi_{g,\mathbf{x}}, g_\varepsilon, \mathbf{x}) = \int_{\mathbb{C}} f^{zz}(z) \sum_{j=1}^n \frac{\partial_{x_j} S_L(\varphi_{g,\mathbf{x}}, g, \mathbf{x})}{x_j - z} dv_g(z)$$
$$+ \int_{\mathbb{C}} f^{zz}(z) \sum_{j=1}^n \Big(\frac{\Delta_{\alpha_j}}{(z-x_j)^2} + \frac{\Delta_{\alpha_j}\partial_{x_j}\sigma(x_j)}{x - z_j} \Big) dv_g(z)$$
$$- \partial_\varepsilon|_0 S_L^0(\omega_\varepsilon, g).$$

- Left-hand side is equal to $\int f^{zz} T_{zz} dv_g(z)$ by definition so we get

$$T_{zz}(z) = \sum_{j=1}^n \Big(\frac{\Delta_{\alpha_j}}{(z-x_j)^2} + \frac{\Delta_{\alpha_j}\partial_{x_j}\sigma(x_j) - \partial_{x_j} S_L(\varphi_{g,\mathbf{x}}, g, \mathbf{x})}{z - x_j} \Big) + \dots$$

- In the $g = |dz|^2$ case (no background metric) we recover the formula $c_j = -\partial_{x_j} S_L(\varphi_{g,\mathbf{x}}, g, \mathbf{x})$.

Quantized Liouville

$S_L(\varphi, g) = \frac{1}{4\pi}\int_\Sigma \left(|\nabla^g \varphi|_g^2 + QR_g\varphi + 4\pi\mu e^{\gamma\varphi}\right) dv_g,$
$\gamma \in (0,2),\ Q = \frac{2}{\gamma} + \frac{\gamma}{2},\ \mu > 0.$

- Path integral: (Euclidean) Liouville QFT

$$\langle F \rangle_g := \int F(\varphi) e^{-S_L(\varphi, g)} D\varphi$$

- Probabilistic definition of the path integral

$$\langle F \rangle_g := \int_{\mathbb{R}} \mathbb{E}[F(c + X_g) e^{-\frac{Q}{4\pi}\int (c+X_g) R_g dv_g - \mu e^{\gamma c}\int e^{\gamma X_g} dv_g}] dc,$$

  where $\mathbb{E}$ is expectation w.r.t. the zero-mean GFF X_g and $e^{\gamma X_g} dv_g$ is the GMC measure of X_g.

- Primary Fields $V_\alpha(x) = e^{\alpha(c + X(x))}$.

Symmetries of the Path Integral

- Diffeomorphism covariance $\langle \prod_{i=1}^{N} V_{\alpha_i}(x_i) \rangle_{\psi^* g} = \langle \prod_{i=1}^{N} V_{\alpha_i}(\psi(x_i)) \rangle_g$.
- Weyl Anomaly

$$\langle \prod_{i=1}^{N} V_{\alpha_i}(x_i) \rangle_{e^{\omega} g} = e^{c_L S_L^0(\omega, g) - \sum_{i=1}^{N} \Delta_{\alpha_i} \omega(x_i)} \langle \prod_{i=1}^{N} V_{\alpha_i}(x_i) \rangle_g \, .$$

- **Central Charge** $c_L = 1 + 6Q^2$.
- **Conformal Dimension** $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.
- Correlation functions of the SE-tensor

$$\langle T_{\alpha\beta}(z) F \rangle_g := 4\pi \frac{\delta}{\delta g^{\alpha\beta}(z)} \langle F \rangle_g \, .$$

- I.e. for $g_\varepsilon^{\alpha\beta} = g^{\alpha\beta} + \varepsilon f^{\alpha\beta}$

$$\sum_{\alpha,\beta} \int f^{\alpha\beta} \langle T_{\alpha\beta} F \rangle_g dv_g(z) := 4\pi \partial_\varepsilon |_0 \langle F \rangle_{g_\varepsilon} \, .$$

Sphere: Outline

1. Perturb the metric $g_\varepsilon = g + \sum_k \varepsilon_k f_k = e^{\omega_\varepsilon} \psi_\varepsilon^* g$
2. ψ_ε solves a Beltrami equation, so it has a series expansion in terms of iterated integral transforms
3. Use this series expansion to show that
   1. $(f_1, \dots, f_n) \mapsto \prod_k \partial_{\varepsilon_k}|_0 \langle \prod_i V_{\alpha_i}(x_i) \rangle_{g_\varepsilon}$ defines a distribution
   2. When supports of f_k's are disjoint, we show that this distribution is given in terms of a point-wise defined function, which we denote by $\prod_k \frac{\delta}{\delta g^{z_k z_k}} \langle \prod_i V_{\alpha_i}(x_i) \rangle_g$
4. Show that $\prod_k \frac{\delta}{\delta g^{z_k z_k}} \langle \prod_i V_{\alpha_i}(x_i) \rangle_g$ has a Weyl anomaly and diffeomorphism covariance relations
5. Derive the Conformal Ward identities by varying the metric and using these relations

Sphere: First Ward identity

- First Conformal Ward identity: $g_\varepsilon^{zz} = g^{zz} + \varepsilon f^{zz}$ and $g_\varepsilon = e^{\omega_\varepsilon}\psi_\varepsilon^* g$

$$\partial_\varepsilon|_0 \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{g_\varepsilon} = \partial_\varepsilon|_0 \Big(e^{c_L S_L^0(\omega_\varepsilon, \psi_\varepsilon^* g) - \sum_{i=1}^N \Delta_{\alpha_i}\omega_\varepsilon(x_i)} \langle \prod_{i=1}^N V_{\alpha_i}(\psi_\varepsilon(x_i))\rangle_g \Big)$$

$$= \Big(-\sum_{i=1}^N \Delta_{\alpha_i}\dot{\omega}_\varepsilon(x_i) + \sum_{i=1}^N \dot{\psi}(x_j)\partial_{x_j} \Big) \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g$$

$$+ \partial_\varepsilon|_0 c_L S_L^0(\omega_\varepsilon, \psi_\varepsilon^* g) \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g$$

$$= \int f^{zz}(z) \Big(\sum_{j=1}^N \big(\tfrac{\Delta_{\alpha_j}}{(z-x_j)^2} + \tfrac{\Delta_{\alpha_j}\partial_{x_j}\sigma(x_j) + \partial_{x_j}}{z-x_j} \Big) \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g dv_g(z) + \dots$$

Theorem (J.O. 2019)

$(x_1, \dots, x_N) \mapsto \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g$ *is* C^∞ *when* $x_i \neq x_j$ *for* $i \neq j$.

Sphere: Higher Ward identities

- For higher Ward identities (more T-insertions) one takes $g_\varepsilon = g + \sum_k \varepsilon_k f_k$ with f_k having disjoint supports.
- After showing that $\prod_k \frac{\delta}{\delta g^{z_k z_k}} \langle \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g$ is defined point-wise, we derive the Weyl anomaly

$$\langle T_{zz}(z) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_{e^\omega g}$$

$$= e^{c_L S_L^0(\omega, g) - \sum_i \Delta_{\alpha_i} \omega(x_i)} \langle \big(T_{zz}(z) + 4\pi c_L \frac{\delta}{\delta g^{zz}(z)} S_L^0(\omega, g) \big) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g \,.$$

and diffeomorphism covariance

$$\langle T_{zz}(z) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_{\psi^* g} = \langle (\psi^* T)_{zz}(z) \prod_{i=1}^m V_{\alpha_i}(\psi_\varepsilon(x_i)) \rangle_g \,,$$

where $(\psi^* T)_{\mu\nu} = \sum_{\alpha,\beta} (D\psi^T)_{\mu\alpha} (T_{\alpha\beta} \circ \psi)(D\psi)_{\beta\nu}$.
- With these relations, deriving the higher conformal Ward identities works analogously to the case of the first Ward identity.

Result

Theorem (A. Kupiainen, J.O. 2020)

For $\Sigma = \mathbb{S}^2$, the correlation functions $\langle \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g$ are smooth with respect to g. The functional derivatives with respect to g are smooth functions for non-coinciding z_j, x_i, and they satisfy the Conformal Ward Identities

$$
\begin{aligned}
\langle \prod_{j=1}^n T_{zz}(z_j) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g &= \sum_{k=1}^n \frac{c_L/2}{(z_1-z_k)^4} \langle \prod_{j\neq 1,k}^n T_{zz}(z_j) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g \\
&+ \sum_{k=2}^n \left(\frac{2}{(z_1-z_k)^2} + \frac{\partial_{z_k}}{z_1-z_k} \right) \langle \prod_{j=2}^n T_{zz}(z_j) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g \\
&+ \sum_{k=1}^m \left(\frac{\Delta_{\alpha_k}}{(z_1-x_k)^2} + \frac{\partial_{z_k} + \Delta_{\alpha_k} \partial_{z_k} \sigma(z_k)}{z_1-x_k} \right) \langle \prod_{j=2}^n T_{zz}(z_j) \prod_{i=1}^m V_{\alpha_i}(x_i) \rangle_g
\end{aligned}
$$

Higher genus

- In general $g_\varepsilon = g + \varepsilon f$ takes the form $g_\varepsilon = e^{\omega_\varepsilon}\psi_\varepsilon^* \hat{g}(\tau_\varepsilon)$ for some moduli parameter $\tau_\varepsilon \in \mathrm{Mod}(\Sigma)$.
- After Weyl anomaly and diffeo covariance we still have

$$\partial_\varepsilon|_0 \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{\hat{g}(\tau_\varepsilon)}$$

- No symmetries to use, have to compute explicitly by varying the underlying Gaussian measure

$$\partial_\varepsilon|_0 \mathbb{E}_{g_\varepsilon}[F(X)] = \frac{1}{2}\int_\Sigma \dot{G}_g(x,y)\mathbb{E}_g[\frac{\delta}{\delta X(x)}\frac{\delta}{\delta X(y)}F(X)]dv_g(x)dv_g(y),$$

where $\dot{G}_g(x,y) = \partial_\varepsilon|_0 \mathbb{E}_{g_\varepsilon}[X(x)X(y)]$.
- Leads to complicated expressions, which can be shown to be well-defined using properties of Liouville correlation functions.

Higher genus: First Ward identity

- We have $g^{zz}_\varepsilon = g^{zz} + \varepsilon f^{zz}$, $g_\varepsilon = e^{\omega_\varepsilon}\psi_\varepsilon^* \hat{g}(\tau_\varepsilon)$ and want to compute

$$\partial_\varepsilon \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{g_\varepsilon} = \partial_\varepsilon \left(e^{c_L S_L^0(\omega_\varepsilon, \psi_\varepsilon^* \hat{g}(\tau_\varepsilon) - \sum_{i=1}^N \Delta_{\alpha_i}\omega_\varepsilon(x_i)} \langle \prod_{i=1}^N V_{\alpha_i}(\psi_\varepsilon(x_i))\rangle_{\hat{g}(\tau_\varepsilon)} \right)$$

- Again $\partial_{\bar{z}}\dot{\psi} = -\frac{1}{2\pi} f^{zz} g_{z\bar{z}}$, but we don't have a Cauchy transform.
- Instead, it holds that

$$\dot{\psi} = -(P_g)^{-1} f$$

where P_g maps vector fields to symmetric traceless 2-tensors given by

$$P_g u = 2S(\nabla^g u^\flat) - \mathrm{Tr}_g(S\nabla^g u^\flat) g = 2L_u g - \mathrm{Tr}_g(L_u g) g\,,$$

where S denotes symmetrisation and L denotes the Lie derivative.
- $(P_g)^{-1}$ sends the perturbation f to the vector field u for which $\psi_\varepsilon(z) = z + \varepsilon u(z) + O(\varepsilon^2)$ (it kills the part of f that deforms the complex structure). Appears (in some form) in Eguchi–Ooguri, 1987[1].

[1] "Standard differential operator taking a vector into traceless symmetric tensor"

Higher genus: Higher Ward identities

- For higher Ward identities we set
$$g_\varepsilon = g + \sum_{k=1}^{n} \varepsilon_k f_k \,,$$
with f_k having disjoint supports.
- To show that $\prod_k \partial_{\varepsilon_k}|_0 \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{g_\varepsilon}$ is well-defined need to consider e.g.
$$\partial_{\varepsilon_1}\partial_{\varepsilon_2}\langle \prod_{i=1}^N V_{\alpha_i}(\psi_{\varepsilon_1}(x_i))\rangle_{\hat{g}(\tau_{\varepsilon_2})} = \sum_j \big(\partial_{\varepsilon_1}\psi_{\varepsilon_1}(x_j)\big)\,\partial_{x_j}\big(\partial_{\varepsilon_2}\langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{\hat{g}(\tau_{\varepsilon_2})}\big)$$
- I.e. need smoothness of $\mathbf{x} \mapsto \partial_\varepsilon|_0 \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_{\hat{g}(\tau_\varepsilon)}$. Can be proven with the method used for smoothness of $\mathbf{x} \mapsto \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g$ (technical).
- Decompose $f_k = f_k^{\psi,\omega} + f_k^\tau$ (no more disjoint supports!)
- Showing that $\prod_k \frac{\delta}{\delta g^{z_k z_k}} \langle \prod_{i=1}^N V_{\alpha_i}(x_i)\rangle_g$ is defined point-wise is more technical than on the sphere.

Result

Theorem (J.O. 2021)

The correlation functions are smooth w.r.t. g, and the functional derivatives are smooth functions for non-coinciding z_j, x_i, and they satisfy the Conformal Ward Identities (in terms of the integral kernel $\mathscr{G}$ of P_g^{-1})

$$\langle\prod_{j=1}^{n} T_{z_j z_j}(z_j)\prod_{i=1}^{m} V_{\alpha_i}(x_i)\rangle_g = \tfrac{c_L}{12}\sum_{k=1}^{n}\nabla^3_{z_k}\mathscr{G}^{z_k}_{z_1 z_1}(z_k,z_1)\langle\prod_{j\neq 1,i}^{n} T_{z_j z_j}(z_j)\prod_{i=1}^{m} V_{\alpha_i}(x_i)\rangle_g$$

$$+\sum_{k=2}^{n}\left(2\nabla_{z_k}\mathscr{G}^{z_k}_{z_1 z_1}(z_k,z_1)+\mathscr{G}^{z_k}_{z_1 z_1}(z_k,z_1)\nabla_{z_k}\right)\langle\prod_{j=2}^{n} T_{z_j z_j}(z_j)\prod_{i=1}^{m} V_{\alpha_i}(x_i)\rangle_g$$

$$+\sum_{k=1}^{m}\left(\Delta_{\alpha_k}\nabla_{x_k}\mathscr{G}^{x_k}_{z_1 z_1}(x_k,z_1)+\mathscr{G}^{x_k}_{z_1 z_1}(x_k,z_1)\nabla_{x_k}\right)\langle\prod_{j=2}^{n} T_{z_j z_j}(z_j)\prod_{i=1}^{m} V_{\alpha_i}(x_i)\rangle_g$$

$$+\langle T_m(z_1)\prod_{k=2}^{n} T_{z_k z_k}(z_k)\prod_{i=1}^{m} V_{\alpha_i}(x_i)\rangle_g\,, \qquad \mathscr{G}^{w}_{zz}(w,z)=\frac{1}{z-w}+\text{smooth}$$

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Thank you for your attention!