The sheaf of (irregular) (co)vacua in the WZNW model. (G. Rembado)

Main message | There are purely algebro-geometric constructions of conformal blocks (CBs) in the WZNW model, using:

- Rep. theory of affine Lie algebras ($\hat{\mathfrak{g}}$).

($\Rightarrow$ related to VOA's).

[illegible]

• Geometric deformation quantization of moduli spaces of meromorphic connections ($\mathcal{M}_{dR}$). (*)

And these can be related.

(*) The merom. connections can have simple poles (= tame/regular singularities) or high-order poles (= wild/irregular singularities).

$\Longrightarrow$ Irregular CBs come from 'quantum' moduli spaces of irregular connections on surfaces.

[Viewpoint taken in work w/ G. Felder: in genus zero get an 'irregular' Knizhnik–Zamolodchikov connection (KZ) (cf. Reshetikhin, '92).]

Let us mainly review the old story (for rational CFTs).

Choose a Riemann surface Σ, let

$$\underline{a} = (a_1 - a_m) \subseteq \Sigma$$

be marked points. There is a set of labels Λ (think: states of some particles). It's finite, and it carries an involution

$\lambda \mapsto \lambda^* : \Lambda \longrightarrow \Lambda$ (think:

opposite state). There is an element $\lambda_0 \in \Lambda$ fixed by it (think: vacuum).

Finally choose loc. coordinates

$$\underline{z} = (z_1 - z_m) \text{ on } \Sigma, \text{ around}$$

$\underline{a} \subseteq \Sigma$, s.t. $z_i(a_i) = 0$.

We want a space

$$V = V_\Sigma(\underline{a}, \underline{\lambda}, \underline{z}), \text{ where}$$

$\underline{\lambda} = (\lambda_1 - \lambda_m) \in \Lambda^m$ are labels @ $\underline{a} \subseteq \Sigma$.

Should satisfy axioms/conditions

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- Conformal invariance.

An infinitesimal conf. transformation (in dim. 2) is given by (local) holo. vector fields on Σ:

$$X = f(z)\, d/dz, \quad f \text{ holomorphic.}$$

In particular $L_n := z^{n+1}\, d/dz$, $n \geq -1$, act w/ commutation rule

$$[L_m, L_n] = (m-n) L_{m+n}.$$

Rk| $\bigoplus_{i \geq -1} \mathbb{C} L_i \supseteq sl_2(\mathbb{C})$, via

the inclusion of (L_{-1}, L_0, L_1).

It's the Lie algebra of (global) conf. transf. of $\mathbb{CP}^1$.

$\Rightarrow$ V is indep. of $\underline{z}$, i.e.

$$V = V_{\Sigma}(\underline{a}, \underline{\lambda}).$$

But <u>more</u> is true: impose invariance under <u>meromorphic</u> vector fields.

$\Rightarrow$ Action of

$$\mathrm{Witt} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} L_i \; ; \text{ and}$$

actually of Virasoro:

I.e. $Vir = Witt \oplus \mathbb{C}\cdot c$ as vector space, w/ Lie bracket:

$$\begin{cases} [L_m, L_n] = (m-n)L_{m+n} \\ \qquad\qquad + \frac{c}{12}(m^3-m)\cdot\delta_{m+n,0}; \\ [Witt, c] = (0). \end{cases}$$

$\boxed{c = \text{'central' charge.}}$

This corresponds to infinitesimal deformations of $(\Sigma, \underline{a})$. (!)

$\Rightarrow$ The spaces $V_\Sigma(\underline{a}, \underline{\lambda})$ should

be locally constant as $(\Sigma, \underline{q})$
is deformed (w.r.t $\underline{1}$ fixed).

This means the following ...
Let $\mathbb{B}$ be a space parameterising a family of n-pointed, genus-g Riemann surfaces. I.e. :

1) we have a holo. projection
$$p : \tilde{\Sigma} \longrightarrow \mathbb{B}, \quad \text{s.t.}$$
$$\Sigma_b := p^{-1}(b) \subseteq \tilde{\Sigma} \quad \text{is a}$$
R. surface of genus g, $b \in \mathbb{B}$;

2) we have n nonintersecting holo. sections

$$\tilde{a}_1 - \tilde{a}_m : \mathbb{B} \longrightarrow \tilde{\Sigma}, \text{ so}$$

$$\underline{a}(b) := (\tilde{a}_1(b) - \tilde{a}_m(b)) \subseteq \Sigma_b$$

are always distinct.

Then we can assemble the spaces $V_b := V_{\Sigma_b}(\underline{a}(b), \underline{\lambda})$ into a family:

$$\pi : V_{\underline{\lambda}} \longrightarrow \mathbb{B}, \quad \pi^{-1}(b) = V_b.$$

The fact these are loc. constant means the following:

1) all points $b_0 \in \mathbb{B}$ have a

neighbourhood $U_0 \subseteq \mathbb{B}$ s.t.

$$\pi^{-1}(U_0) \xrightarrow[\varphi_{U_0}]{\simeq} U_0 \times V_0, \quad V_0 := V_{b_0};$$

(I.e. $V_b \simeq V_0$ smoothly for $b \in U_0$)

2) if $U_0' \subseteq \mathbb{B}$ is another nbhd. of b_0, then the composition

$$(U_0 \cap U_0') \times V_0 \xrightarrow[\varphi_{U_0}^{-1}]{\simeq} \pi^{-1}(U_0 \cap U_0') \xrightarrow[\varphi_{U_0'}]{\simeq} (U_0 \cap U_0') \times V_0$$

reads $(b, v) \mapsto (b, \phi(v))$, for $b \in U_0 \cap U_0'$, $v \in V_0$ and

$\phi_{U_0 \cap U_0'} \in GL(V_0)$ constant.

How to think this globally?

Suppose $s: \mathbb{R} \to V_1$ is a

section, i.e. a collection of maps $s_{U_0} : U_0 \longrightarrow V_0$ for a choice of points $b_0 \in \mathbb{B}$ s.t.

$$\bigcup_{b_0} U_{b_0} = \mathbb{B} \text{ (w./ compatibility)}.$$

If X is a v. field on $\mathbb{B}$ define:

$$(\nabla_X s)_{U_0} := \langle ds_{U_0}, X|_{U_0} \rangle : U_0 \longrightarrow V_0 .$$

Ah: these glue to a global section

$$\nabla_X s : \mathbb{B} \longrightarrow V_\lambda .$$

Hence locally trivial covariant derivatives yield a global (nontrivial) covariant derivative

$\Rightarrow \nabla$ is a flat connection.

Rk| Conversely the trivialisations are constructed from local horizontal frames of ∇, i.e. locally solving $\nabla s = 0$.

Hence we also want to have such flat v. bundles in CFT.

Rk | If genus ≥ 2 then only get a projective connection.

⌈ Ie. replace V w/

I.e. replace V_b w/

$$\mathbb{P}(V_b) := V_b \setminus \{0\} / \mathbb{C}^\times$$

throughout.

$\dashv$

Constructions of such bundles:

- KZ ('84) $(g=0)$.
- Tsuchiya–Kanie ('87) $(g=0)$.
- Bernard ('88) $(g \geq 1)$.
- Tsuchiya–Ueno–Yamada ('89) ($g \geq 0$ + singularities)

Important point: correlation functions should be (local) horizontal sections.

i.e. again solutions of

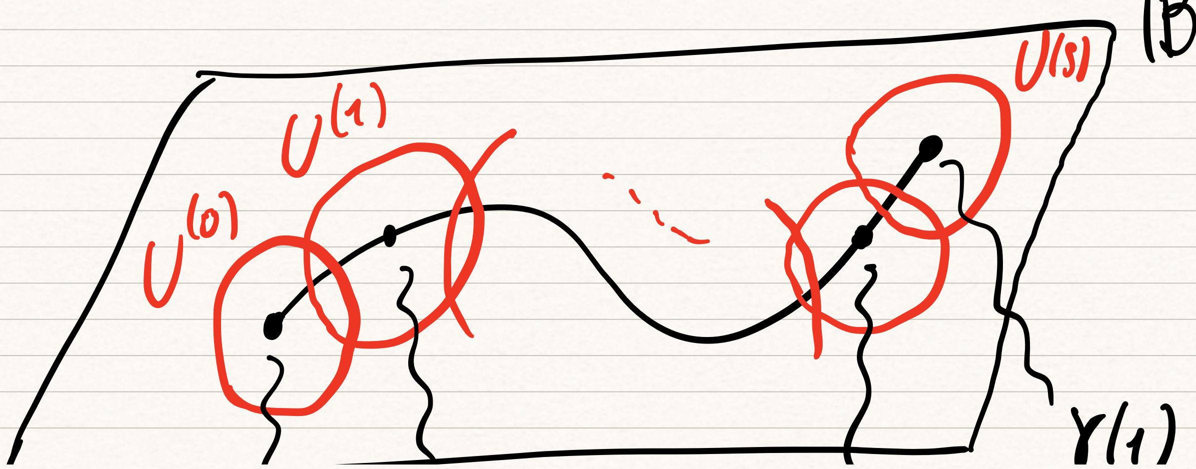
$$\nabla s = 0 .$$

Application : <u>monodromy</u>.

Choose a path $\gamma : [0,1] \to \mathbb{B}$.

Cover $\gamma(I) \subseteq \mathbb{B}$ by trivialising open sets $U^{(0)} \dots U^{(s)} \subseteq \mathbb{B}$, around points

$$\underset{\gamma(0)}{\underset{\shortparallel}{\gamma(t_0)}}, \gamma(t_1) \dots, \gamma(t_{s-1}), \underset{\gamma(1)}{\underset{\shortparallel}{\gamma(t_s)}} \in \mathbb{B} :$$



$\gamma(0) \quad \gamma(t_1) \quad \quad \gamma(t_{j-1})$

Now use the trivialisations

$$\pi^{-1}(U^{(i)}) \xrightarrow{\cong} U^{(i)} \times V_{\gamma(t_i)}$$

to map any $v \in V_{\gamma(0)}$ to $V_{\gamma(1)}$.

<u>Fact</u> Only depends on homotopy <u>class</u> of γ, and is compatible w.r.t composition/concatenation.

In particular taking loops @ $b_0 \in B$:

$$\rho_\nabla : \pi_1(B, b_0) \to GL(V_{b_0}),$$

a group morphism (i.e. a

representation of $\pi_1(B)$)

representation of $\pi_1(B)$.

Back to CFT: when

$$\mathbb{B} = \mathcal{M}_{g,n} \quad (\text{carrying}$$

the universal family) get a repr. of braid/mapping class groups.

What about other axioms?

- (Duality)

$$V_\Sigma(\underline{a}, \underline{\lambda}) \simeq V_\Sigma(\underline{a}, \underline{\lambda}^\vee), \quad \text{for}$$

$$\underline{\lambda}^\vee := (\lambda_1^\vee - \lambda_n^\vee).$$

• (Vacua propagation)

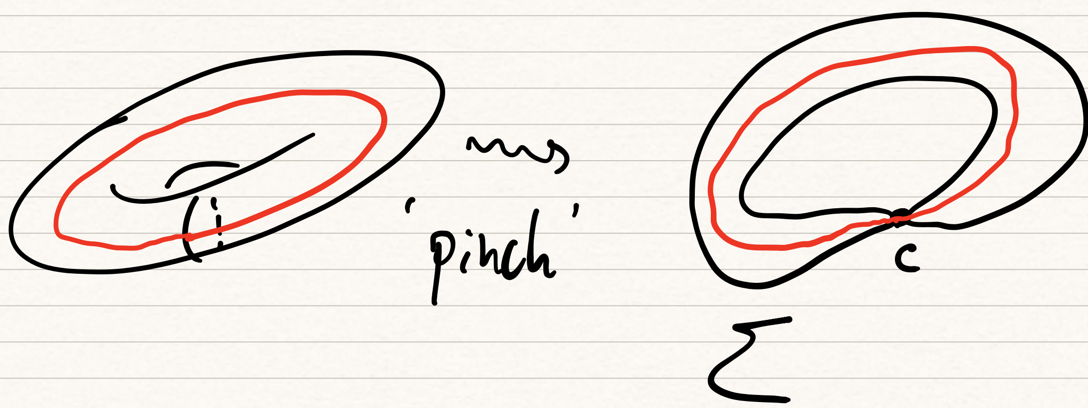
Let a_{m+1} be a new marked point. Then

$$V_\Sigma(\underline{a}, \underline{\lambda}) \simeq V_\Sigma(\underline{a}', \underline{\lambda}'), \quad w/$$

$$\underline{a}' = (a_1 - a_m, a_{m+1}) \quad \&$$
$$\underline{\lambda}' = (\lambda_1 - \lambda_m, \lambda_0).$$

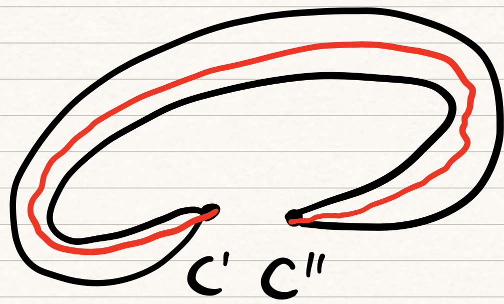
• (Factorisation)

Can have singular points (not marked). Namely nodes:



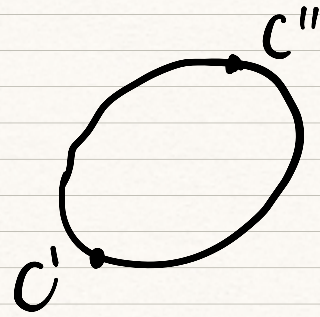
Desingularisation yields :

$\Sigma \rightsquigarrow$



$\simeq$

$\simeq$



Σ'

Then :

$$V_{\Sigma}(\underline{a}, \underline{\lambda}) \simeq \bigoplus_{\mu \in \Lambda} V_{\Sigma'}(\underline{a}', \underline{\lambda}'(\mu))$$

w/ $\underline{a}' = \underline{a} \cup \{c', c''\}$,

$\underline{\lambda}'(\mu) = \underline{\lambda} \cup \{\mu, \mu^{\vee}\}$.

• (Normalisation / nontriviality)

$$V_{\mathbb{CP}^1}(\emptyset, \emptyset) = \mathbb{C}.$$

This can be achieved by the following construction (in WZNW)!

Let $\mathfrak{g}$ be a simple Lie algebra, fin. dim. over $\mathbb{C}$.

E.g. | Take $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{End}(\mathbb{C}^2) \;\middle|\; a+d=0 \right\}.$$

Here $\mathfrak{g} = \mathbb{C}\cdot\{E, H, F\}$, w/

$$E = \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & \\ 1 & 0 \end{pmatrix}.$$

Lie bracket $\mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}$ = commutator.

Choose an ad_g-invariant 'scalar product'

$$g \otimes g \to \mathbb{C}, \quad (X, Y) \mapsto (X|Y).$$

<u>E.g.</u> | $(X|Y) := \mathrm{Tr}(XY)$ above.

We now need the <u>affine</u> Lie algebra associated to $(g, (\cdot|\cdot))$.

First let: $[X \otimes z^m, Y \otimes z^n] = [X, Y] \otimes z^{m+n}$

$$L_g = g((z)) := g \otimes \mathbb{C}((z))$$

$$= \left\{ \sum_{i \geq m} X_i \otimes z^i \;\middle|\; m \in \mathbb{Z}, X_i \in g \right\}.$$

(Think: (formal) Laurent expansions of maps $S^1 \to g$).

$\Rightarrow$ Loop algebra of g ($\leftrightarrow$ Witt).

Then we use a central extension:

$$0 \to \mathbb{C}k \to \hat{g} \to \mathcal{L}g \to 0,$$ w.l Lie bracket

$$\begin{cases} [X_m, Y_n] = [X, Y]_{m+n} \\ \qquad\qquad + m\delta_{m+n,0}(X|Y)k; \\ [\mathcal{L}g, k] = (0), \end{cases}$$

where $X_m := X \otimes z^m$ for $X \in g$, $m \in \mathbb{Z}$.

$\Rightarrow$ Affine Lie algebra of $(g, (\cdot|\cdot))$

$(\leftrightarrow$ Virasoro$)$.

E.g. | $\mathrm{Tr}(EF) = 1$, so

$$[E_m, F_n] = H_{m+n} + m\,\delta_{m+n,0}\,K.$$

Main point | $V_\Sigma(\underline{a}, \underline{\lambda})$ comes from a tensor product

$$\hat{V}_{\underline{\lambda}} = \bigotimes_{i=1}^{m} \hat{L}_{\lambda_i, \kappa}, \text{ of } \hat{\mathfrak{g}}\text{-modules.}$$

To construct the modules...

Use a 'triangular' decomposition

$$\mathfrak{g} = \underbrace{\mathfrak{r}}_{\text{Cartan}} \oplus \underbrace{\mathfrak{n}^+ \oplus \mathfrak{n}^-}_{\text{nilpotent}}.$$

Use root systems.

E.g. $\mathfrak{r} = \mathbb{C}\cdot H$, $\mathfrak{n}^+ = \mathbb{C}\cdot E$, $\mathfrak{n}^- = \mathbb{C}\cdot F$.

Choose then $\kappa \in \mathbb{C}$ and $\lambda \in \mathfrak{t}^\vee$ (a 'level' and a 'weight'): there are integrable highest-weight $\hat{\mathfrak{g}}$-modules

$\hat{L} = \hat{L}_{\lambda,\kappa}$, such that:

- $\dim(\hat{L}) = +\infty$;
- K acts as $\kappa \cdot \mathrm{Id}_{\hat{L}}$; $\quad \mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$
- $\hat{L}$ contains the simple $\mathfrak{g}$-module L_λ of h.w. $\lambda \in \mathfrak{t}^\vee$.

[• $\mathcal{L}\mathfrak{g}$ acts locally nilpotent.]

Rk: For this we need $\kappa \in \mathbb{Z}_{\geq 0}$ and $\lambda \in P_\kappa^+ \subseteq \mathfrak{t}^\vee$: dominant integral of level k.

(I.e. P^+_k plays the role of the set of labels).

<u>E.g.</u> $P^+_k = \{0, \dots, k\} \subseteq \mathbb{C}$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathbb{R}$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \mathfrak{t}^\vee$

for $\mathfrak{sl}_2$. Then L_λ is the spin-$\lambda/2$ representation of $\mathfrak{g}$ (and of $G = SL_2(\mathbb{C})$):

$\dim L_\lambda = \lambda + 1$, and L_λ is generated by a (h.w.) vector $|\lambda\rangle \in L_\lambda$ <u>s.t.</u>:

$H|\lambda\rangle = \lambda|\lambda\rangle$, $E|\lambda\rangle = 0$.

$\ulcorner$ Thus $L_\lambda = \bigoplus_{i=0}^{n} \mathbb{C} \cdot F^i |\lambda\rangle$. $\lrcorner$

Now $\hat{V}_{\underline{\lambda},\kappa} = \bigotimes_i \hat{L}_{\lambda_i,\kappa}$ is defined.

But indep. of $(\Sigma, \underline{a})$...

Idea: introduce symmetries! Consider

$$\mathfrak{g}_\Sigma(\underline{a}) := \{ f : \Sigma \to \mathfrak{g} \mid$$

$$\mid f \text{ meromorphic w/ poles @ } \underline{a} \subseteq \Sigma \}.$$

It's a Lie algebra:

$$[f, f'](x) = [f(x), f'(x)] \in \mathfrak{g},$$

$$\text{for } x \in \Sigma.$$

We have Laurent expansions:

$$\iota_{a_i} : \mathfrak{g}_\Sigma(\underline{a}) \to \mathfrak{g}((z_i)), \quad \text{w/}$$

$$z_i(a_i) = 0, \quad i = 1 \text{ — } m.$$

(But there is a <u>coordinate-free</u> construction.)

Ah: this acts thus on $\hat{V}_{\underline{\lambda}}$ via:

$$\iota(f).(v_1 \otimes \text{—} \otimes v_m) := \sum_{i=1}^{m} v_1 \otimes \text{—} \otimes v_{i-1} \otimes \iota_{a_i}(f) v_i \otimes v_{i+1} \otimes \text{—} \otimes v_m, \text{ for}$$

$$f \in \mathfrak{g}_\Sigma(\underline{a}), \quad v_i \in \hat{L}_{\lambda_i, \kappa}.$$

Lemma: The map

$$\tau : \mathfrak{g}_\Sigma(\underline{a}) \to \mathrm{End}(\hat{V}_{\underline{\lambda}})$$ is a

Lie-algebra morphism.

I.e.

$$\tau([f, f']) = [\tau(f), \tau(f')].$$

Main definition:

$$V^\dagger_\Sigma(\underline{a}, \underline{\lambda}) := \hat{V}_{\underline{\lambda}} / \mathfrak{g}_\Sigma(\underline{a}) \cdot \hat{V}_{\underline{\lambda}} \;;$$

$$V_\Sigma(\underline{a}, \underline{\lambda}) := \mathrm{Hom}_{\mathfrak{g}_\Sigma(\underline{a})}(\hat{V}_{\underline{\lambda}}, \mathbb{C})$$

$$= \{ \psi : \hat{V}_{\underline{\lambda}} \to \mathbb{C} \mid \psi(\tau(f).v) = 0 \text{ for } f \in \mathfrak{g}_\Sigma(\underline{a}),\ v \in \hat{V}_{\underline{\lambda}} \}.$$

//

$\boxed{CB}$

What about the bundles? Let's see for $g=0$. Cplx. structure on $\Sigma = \mathbb{CP}^1$ is fixed, can deform marked points:

$$\mathbb{B} = C_m(\Sigma) := \Sigma^m \setminus \underbrace{\Delta}_{\text{diagonals}} .$$

Consider

$$E := \hat{V}_{\underline{\lambda}} \times \mathbb{B} \to \mathbb{B} \quad (\text{trivial}).$$

We said the meromorphic fields should yield (local) independence on $(\Sigma, \underline{a})$: here it's just $L_{-1} \in \mathrm{Witt}$.

Fact 1 (Segal–Sugawara). The sum

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{j \in \mathbb{Z}} \sum_{X \in B} :X_j X_{n-j}:$$

yields a well-defined element of $\mathrm{End}(\hat{L}_{\lambda,k})$ for all $k \neq -h^\vee$, $\lambda \in \mathfrak{t}^\vee$.

Here $B \subseteq \mathfrak{g}$ is a $(\cdot|\cdot)$-ONB, and the 'normal' order puts

$\mathfrak{g}[[z]] \subseteq \mathcal{L}\mathfrak{g}$ to the right.

For the standard global coord. $(t_1 - t_m)$ on $C_m(\mathbb{C}) \subseteq \mathbb{B}$ we then define the covariant derivative

$$\hat{\nabla}_{\partial t_i} \Psi := \left(\partial_{t_i} + L_{-1}^{(i)}\right)\Psi$$

action on

$$\hat{L}_{\lambda_i, n} \hookrightarrow \hat{V}_{\underline{\lambda}},$$

where $\psi : C_m(\mathbb{C}) \to \hat{V}_{\underline{\lambda}}$.

Thm 1 (Cf. Kac's book)

This is compatible w/ the $g_\Sigma(\underline{a})$ - action:

$$\hat{\nabla}_{\partial_{t_i}}(\tau(f).\psi) = \tau(\partial_{t_i}(f)).\psi$$

$$+ \tau(f).\hat{\nabla}_{\partial_{t_i}}\psi, \quad \text{for}$$

$f \in g_\Sigma(\underline{a})$ (sheaf) and $i = 1 \_ m$.

$\Longrightarrow$ There is an induced connection on $V_\lambda \longrightarrow \mathbb{B}$: KZ. (!)

More explicit? | Makes sense to consider restrictions:

$$\begin{array}{c} \varphi : \bigotimes_{i=1}^{m} L_{\lambda_i} \longrightarrow \mathbb{C} . \\ \rotatebox{90}{$\subseteq$} \\ \hat{V}_{\underline{\lambda}} \end{array}$$

One can express KZ for such functionals! Result:

$$\partial_{t_i} \varphi = \frac{1}{k - h^\vee} \left(\sum_{j \neq i} \frac{\Omega^{(ij)}}{t_i - t_j} \right) \varphi ,$$

where

$$\Omega = \sum_{X \in B} X \otimes X \in \mathfrak{g} \otimes \mathfrak{g} \quad \text{(think:}$$

truncation of L_0).

E.g. $\Omega = \frac{H \otimes H}{2} + E \otimes F + F \otimes E$

for $\mathfrak{g} = \mathfrak{sl}_2$.

(Monodromy of KZ $\Rightarrow$ Kohno-Drinfel'd thm. ('87/'89))

THANK YOU FOR YOUR ATTENTION!

♡

References

Kohno: CFT & topology.

Felder-R. : Singular modules for affine Lie algebras [...]