On the $c \leq 1$ analytic continuation of Liouville path integral

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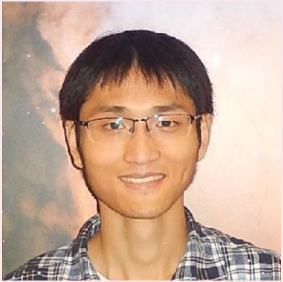
Outline

- (Space-like) Liouville path integral and its bootstrap solution
- The analytic continuation of bootstrap solution. Crossing simmetric correlation functions at $c \leq 1$.
- Timelike (or imaginary) Liouville path integral: the Lefshetz-Pham thimble theory framework
- Insights from a two complex variable toy model



Xiangyu Cao

Romain Usciati

Liouville quantum field theory (on the sphere)

$$\left\langle \prod_i e^{\alpha_i \varphi(x_i)} \right\rangle = \int_{\{\varphi(x)\}\in\mathbb{R}} \mathcal{D}\varphi \ e^{-\mathcal{S}(\varphi)}$$

$$\mathcal{S} = \int_{\hat{\mathbb{C}}} \mathrm{d}x \left(\frac{1}{16\pi} (\nabla\varphi)^2 + \mu \ e^{b\varphi} + \varphi \left(Q \ \delta(x-\infty) - \sum_i \alpha_i \delta(x - x_i) \right) \right)$$

$$Q = b + b^{-1}$$

Interested in the behavior of this integral by varying the parameters b and $\{\alpha_i\} \in \mathbb{C}$

The space-like sector: $b \in \mathbb{R} \quad (c \geq 25)$

$$\int_{-\infty}^{\infty} d\phi_0 \int_{\{\phi(x)\}\in\mathbb{R}} \mathcal{D}\phi \; e^{-\mathcal{S}(\phi_0+\phi)} \propto \Gamma\left(-\frac{n}{b}\right) \left\langle \mathcal{Z}^{\frac{n}{b}} e^{-Q\phi(\infty)} \prod_i e^{\alpha_i \, \phi(x_i)} \right\rangle_{\mathrm{GFF}_\phi}$$

$$\mathcal{Z} = \int_{\hat{\mathbb{C}}} \mathrm{d}x \; e^{b\phi(x)}$$

Dorn-Otto ('94), David, Kuppianen, Rhodes, Vargas ('14)

"Good" parameter Region (Seiberg Bounds)

$$\mathrm{Re}\,(\alpha_i) < \frac{Q}{2}, \quad \frac{n}{b} = \frac{Q - \sum_i \alpha_i}{b} < 0$$

Poles for $n/b \in \mathbb{N}$. Residues: Coulomb Gas integrals

Bootstrap solutions: Three-point function

Dorn,Otto ('94), Zamolodchikov,Zamolodchikov ('94)

$$\left\langle \prod_{i=1}^{3} e^{\alpha_i \varphi} \right\rangle = C^{\mathrm{DOZZ}} = \text{Product of superfactorials}$$

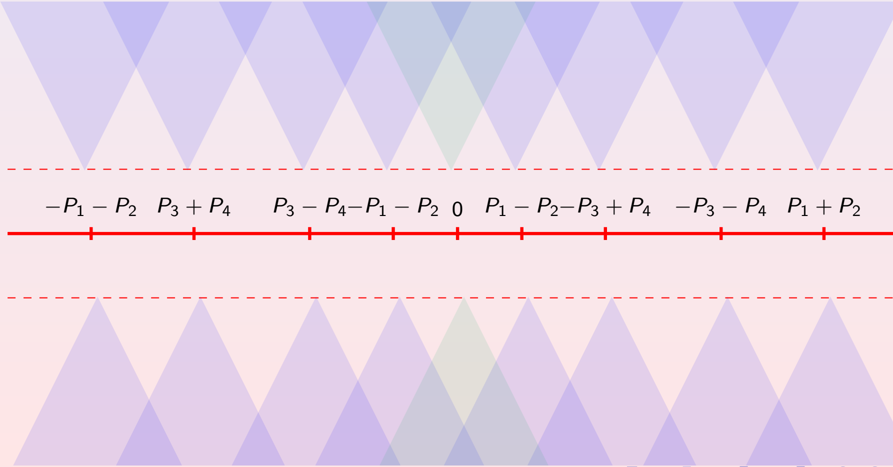
$$\Upsilon_b(x+b) = b^{1-2bx} \frac{\Gamma(x)}{\Gamma(1-x)} \Upsilon_b(x+b)$$

$$\lim_{b \to i\beta} C^{\mathrm{DOZZ}} \quad \text{does not exist for } \beta^2 \in \mathbb{R}/\mathbb{Q}$$

Bootstrap solutions: Four-point function

Zamolodchikov,Zamolodchikov ('94)

$$\left\langle \prod_{i=1}^{4} e^{\alpha_i \, \varphi(x_i)} \right\rangle = \int_{\alpha=\frac{Q}{2}+iP} dP \; C^{\mathrm{DOZZ}} C^{\mathrm{DOZZ}} \; |\mathcal{F}|^2$$



A straightforward application of

- DKRV formula

$$\left\langle P_q = \left(\frac{e^{b\phi}}{\sum_x e^{b\phi}} \right)^q \right\rangle_{\mathrm{GFF}_\phi} \leftrightarrow \left\langle \prod e^{\alpha\varphi} \right\rangle$$

- and the α continuation of bootstrap solution

predicts different regimes for P_q, when $qb <, =, > Q/2$, consistent with analogous results (in REM models for instance) by pure statistical approaches.

Cao,Le-Doussal,Rosso, R.S. ('16)-('18)

$$\alpha \to i\alpha(\alpha \in \mathbb{R}),\ b \to i\beta\ (c \le 1)$$

$$C^{c\le 1} = \text{product of superfactorials}$$

Schomerus ('04), Kostov,Petkova ('05),Zamolodchikov ('05)

$$C^{c\le 1} \text{ plays crucial role in } CLE_{\kappa(c)} \text{ models}$$

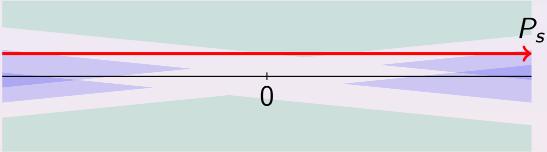
Delfino, Viti('11), Picco, R.S ('13','21), Estienne,Ykhlef, Saleur,Jacobsen ('15), Ang, Sun ('21)

$$\frac{C^{\mathrm{DOZZ}}}{C^{c\le 1}} = \text{elliptic functions. Periods: } (b, b^{-1})$$

$$C^{c\le 1} = \left\langle \prod_{i=1}^{3} e^{i\alpha_i\phi} \right\rangle ??$$

$$\alpha \to i\alpha(\alpha \in \mathbb{R}),\ b \to i\beta\ (c \leq 1)$$

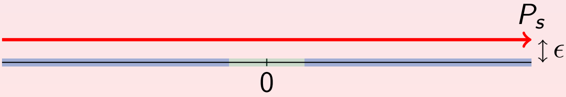
Poles collapse on a the real P axis



New bootstrap solution:

Ribault,R.S. ('15), Gavrilenko,R.S.('18)

$$\int_{\alpha=\frac{Q}{2}+iP} dP\ C^{c\leq 1} C^{c\leq 1}\ |\mathcal{F}|^2\ \left(= \left\langle \prod_{i=1}^{4} e^{i\alpha_i\ \varphi(x_i)} \right\rangle ?? \right)$$



Other crossing symmetric correlation functions at $c \leq 1$

Coulomb Gas (CG) integrals:

$$S^{CG} = \int_{\hat{\mathcal{C}}} d\ x\ \ (\nabla\phi)^2 + \mu_+\ e^{i\beta\phi} + \mu_-\ e^{-i\ \beta^{-1}\phi} + i(\beta - \beta^{-1})\phi\delta(x - \infty)$$

$$\left\langle \prod_i^4 e^{i\alpha_i\phi} \right\rangle_{\mathrm{CG}} = \frac{\delta_{\sum_i \alpha_i + m\beta - n\beta^{-1}, Q}}{\Gamma(1+n)\Gamma(1+m)} \left\langle \prod_i^4 e^{i\alpha_i\phi} \mathcal{Z}_+^n \mathcal{Z}_-^m \right\rangle_{\mathrm{GFF}_\phi}, \ \ n, m \in \mathbb{N}$$

$$\sum_{p=0}^{m} \sum_{q=0}^{n} C^{c\leq 1}(\alpha_1, \alpha_2, \alpha_{p,q}) C^{c\leq 1}(\alpha_{p,q}, \alpha_3, \alpha_4) |\mathcal{F}|^2$$

where $\alpha_{p,q} = \alpha_1 + \alpha_2 + p\beta - q/\beta \quad (p, q) = [0, m] \times [0, n]$.

Other crossing symmetric correlation functions at $c \leq 1$

BPZ functions:

$$\alpha_1 = \frac{Q}{2} + \frac{m}{2}\beta - \frac{n}{2}\beta, \quad n, m \in \mathbb{N}$$

$$\sum_{p=0}^{m} \sum_{q=0}^{n} C^{c\leq 1}(\alpha_1, \alpha_2, \alpha_{p,q}) C^{c\leq 1}(\alpha_{p,q}, \alpha_3, \alpha_4) |\mathcal{F}|^2$$

where $\alpha_{p,q} = \alpha_1 + \alpha_2 + p\beta - q/\beta \quad (p, q) = [0, m] \times [0, n]$.

Completely consistent theory (for general c)

Generalized Minimal Model $\in$ CG $\cap$ BPZ

RS solution

O(n)/Potts solutions

Picco, Ribault, R.S ('16),('18), Ribault,Rongvoram, Yfei, Jacobsen, Saleur (18'-22)

Questions:

- Can we define a consistent analitic continuation to $b \to i\beta$ of the Liouville Path integral

- In that case, does it correspond to known bootstrap solutions?

Lefschetz-Pham approach to analitic continuation to path integrals

Witten ('10)

$$I_{\mathcal{T}} = \int_{\mathcal{T}} \mathcal{D}\,\varphi\; e^{-S(\varphi)}$$

- Consider $\varphi \in \mathbb{C}$
- Set of critical points $\{\varphi_j^c\}$: $\left.\frac{\delta S}{\delta \varphi}\right|_{\varphi=\varphi_j^c} = 0$
- Lefschetz Thimbles: $\mathcal{T}_{\text{critical point}}$

Generalization of the stationary phases contours in 1-complex variable integrals:

$\text{Im}(S)$ on $\mathcal{T}_{\text{critical point}}$ is constant

Lefschetz-Pham approach to analitic continuation to path integrals

Witten ('10)

$$I_{\mathcal{T}} = \int_{\mathcal{T}} \mathcal{D}\,\varphi\; e^{-S(\varphi)}$$

$$I_{\mathcal{T}} = \sum_{\text{critical point}} m_{\text{critical point}}(\text{parameters})\; I_{\mathcal{T}_{\text{critical point}}}$$

$$m_{\text{critical point}} \in \mathbb{Z}$$

Stokes Phenomena:

Under variation of parameters, abrupt exchange of the multiplicities $m_{\text{critical point}}$.

1-dimensional complex variable: the theory of the Γ function

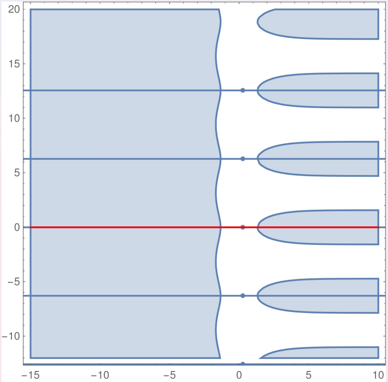
$$I_\gamma = \int_\gamma d\phi_0 \; e^{\,-n\;\phi_0 - \mu\;\mathcal{Z}\;e^{b\phi_0}} = b^{-1}\,(\mu\mathcal{Z})^{\frac{n}{b}}\;\int_\gamma d\phi_0\; e^{\,-\frac{n}{b}\;\phi_0 -\;e^{\phi_0}}$$

One important relevant parameter: n/b

Critical points: $\quad \phi_0^{(c)} = \log\left(-\frac{n}{b}\right) + 2\pi i\,\mathbb{Z}$

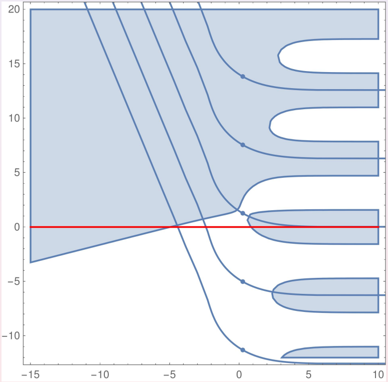
Stationary phase path $\mathcal{T}_m$: $\quad \frac{d\phi_0}{dt} = \overline{\frac{d\;S}{d\phi_0}}, \quad \phi_0(0) = \phi_0^{(c,m)}$

$$\frac{n}{b} = \left|\frac{n}{b}\right| e^{i(\theta-\pi)}, \quad \theta = 0$$



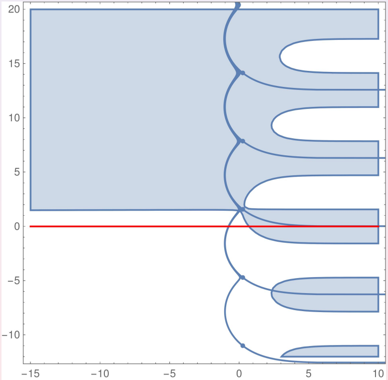
$$\mathbb{R} = \mathcal{T}_0, \; I_{\mathbb{R}} = I_{\mathcal{T}_0} = \Gamma\left(-\frac{n}{b}\right)$$

$$\frac{n}{b} = \left|\frac{n}{b}\right| e^{i(\theta-\pi)}, \quad \theta = \frac{2}{5}\pi$$



$$\mathbb{R} \neq \mathbb{T}_0, \; I_{\mathbb{R}} = I_{\mathcal{T}_0} = \Gamma\left(-\frac{n}{b}\right)$$

$$\frac{n}{b} = \left|\frac{n}{b}\right| e^{i(\theta-\pi)}, \quad \theta = \frac{1}{2}\pi - \epsilon$$



$$\textcolor{red}{I_{\mathbb{R}}} = I_{\mathbb{T}_0} = \Gamma\left(-\frac{n}{b}\right)$$

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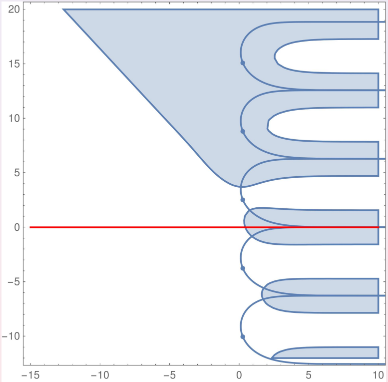
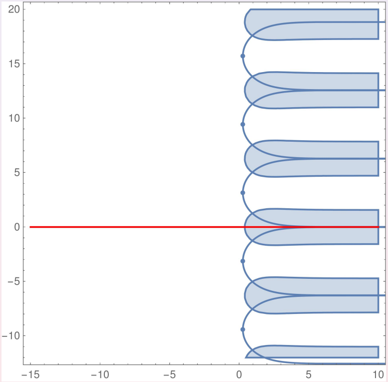


$$I_{\mathbb{R}} = \sum_{m=0}^{\infty} I_{\mathcal{T}_m} = \Gamma\left(-\frac{n}{b}\right)$$

$$\frac{n}{b} = \left|\frac{n}{b}\right| e^{i(\theta-\pi)}, \quad \theta = \pi$$



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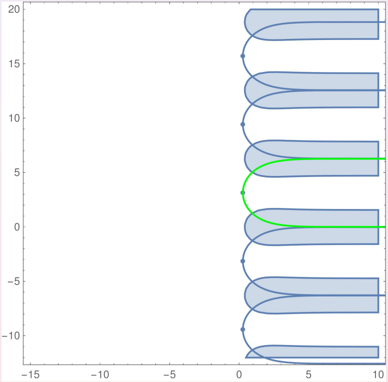
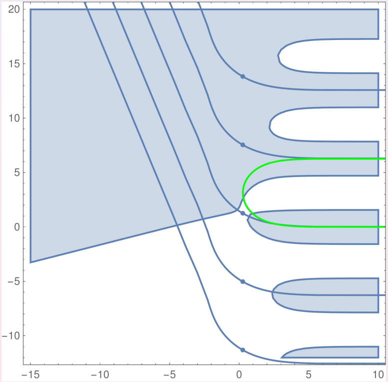


$$I = \frac{2i\pi e^{i\pi\frac{n}{b}}}{\Gamma\left(1+\frac{n}{b}\right)} = I_{\mathcal{T}_0}$$

$$\frac{n}{b} = \left|\frac{n}{b}\right| e^{i(\theta-\pi)}, \quad \theta = \pi$$



$$I = \frac{2i\pi e^{i\pi \frac{n}{b}}}{\Gamma\left(1+\frac{n}{b}\right)} = I_{\mathcal{T}_0} - I_{\mathcal{T}_1}$$

$$I_{\mathbb{R}} = \Gamma(-\frac{n}{b}) = \begin{cases} I_{\mathcal{T}_0} & 0 \leq \theta < \frac{\pi}{2} \\ \sum_{m>0} I_{\mathcal{T}_m} = (1 - e^{2i\pi\frac{n}{b}})^{-1} \; I_{\mathcal{T}_0} & \frac{\pi}{2} < \theta \leq \pi \end{cases}$$

$$I = \frac{2i\pi e^{i\pi\frac{n}{b}}}{\Gamma\left(1 + \frac{n}{b}\right)} = \begin{cases} I_{\mathcal{T}_0} & \frac{\pi}{2} < \theta \leq \pi \\ I_{\mathcal{T}_0} - I_{\mathcal{T}_1} = (1 - e^{2i\pi\frac{n}{b}}) \; I_{\mathcal{T}_0} & 0 \leq \theta < \frac{\pi}{2} \end{cases}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad \left(\frac{C^{\mathrm{DOZZ}}}{C^{c\leq 1}} = \text{elliptic functions..} \right)$$

A naif trial on imaginary Liouville...

$$b \to i\beta, \quad \alpha_i \to i\alpha_i, \quad \frac{n}{b} = \frac{(\beta - \beta^{-1} - \sum_i \alpha_i)}{\beta}$$

$$\left\langle \prod_i e^{i\alpha_i \varphi} \right\rangle \propto \Gamma\left(-\frac{n}{b}\right) \left\langle \left(\int d\, x\, e^{i\beta\phi} \right)^{\frac{n}{b}} e^{-Q\phi(\infty)} \prod_i e^{i\alpha_i\, \phi} \right\rangle_{\mathrm{GFF}_\phi}$$

$$\left\langle \prod_i e^{i\alpha_i \varphi} \right\rangle \propto \frac{1}{\Gamma\left(1 + \frac{n}{b}\right)} \left\langle \left(\int d\, x\, e^{i\beta\phi} \right)^{\frac{n}{b}} e^{-Q\phi(\infty)} \prod_i e^{i\alpha_i\, \phi} \right\rangle_{\mathrm{GFF}_\phi}$$

Note: random variables oscillating around the branch point!

Semi-classical analysis of Liouville field theory

Harlow, Malts, Witten ('11)

$$\lim_{b\to 0}\left\langle \prod_i e^{\alpha_i\varphi(x_i)}\right\rangle \sim \sum_{\{\varphi^{(c)}\}} m_{c.p.}\, e^{-S(\varphi^{(c)})}\,(1+O(b))$$

A_m : Uni-valued solution of the Liouville equation $+\ \dfrac{2\pi i}{b}\, m,\ \ m\in\mathbb{Z}$

These are not enough to explain the behabior of bootstrap solutions. One needs to include multi-valued complex solutions

Semi-classical analysis of Liouville field theory

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A_m : Uni-valued solution of the Liouville equation $+\dfrac{2\pi i}{b}\, m,\ m\in\mathbb{Z}$

$$b\in\mathbb{R}$$

Region I:

$$\alpha_i<\frac{Q}{2},\qquad \frac{n}{b}=\frac{Q-\sum_i\alpha_i}{b}<0$$

Principal thimble $\mathcal{T}_{A_0}$, corresponding to $\varphi\in\mathbb{R}$

Semi-classical analysis of Liouville field theory

Harlow, Malts, Witten ('11)

$$\lim_{b\to 0}\left\langle \prod_i e^{\alpha_i \varphi(x_i)} \right\rangle \sim \sum_{\{\varphi^{(c)}\}} m_{c.p.} e^{-S(\varphi^{(c)})} (1 + O(b))$$

$$A_m : \text{Uni-valued solution of the Liouville equation} + \frac{2\pi i}{b} m, \ m \in \mathbb{Z}$$

$$b \in \mathbb{R}$$

Region II:

$$\alpha_i < \frac{Q}{2}, \quad \frac{n}{b} = \frac{Q - \sum_i \alpha_i}{b} > 0$$

$$\sum_{m \geq 0} \mathcal{T}_{A_m}$$

Semi-classical analysis of Liouville field theory

Harlow, Malts, Witten ('11)

$$\lim_{b\to 0}\left\langle \prod_i e^{\alpha_i\varphi(x_i)}\right\rangle \sim \sum_{\{\varphi^{(c)}\}} m_{c.p.}e^{-S(\varphi^{(c)})}\left(1+O(b)\right)$$

A_m : Uni-valued solution of the Liouville equation $+\,\dfrac{2\pi i}{b}\,m,\;\; m\in\mathbb{Z}$

$$\beta\in\mathbb{R},\alpha_i\to i\alpha_i$$

Region II:

$$\alpha_i<\frac{\beta-\beta^{-1}}{2},\qquad \frac{n}{b}=\frac{\beta-\beta{-}1-\sum_i\alpha_i}{\beta}>0$$

Principal thimble $\mathcal{T}_{A_0}$

Semi-classical analysis of Liouville field theory

Harlow, Malts, Witten ('11)

$$\lim_{b\to 0}\left\langle \prod_i e^{\alpha_i \varphi(x_i)} \right\rangle \sim \sum_{\{\varphi^{(c)}\}} m_{c.p.} e^{-S(\varphi^{(c)})} (1 + O(b))$$

$$A_m : \text{Uni-valued solution of the Liouville equation} + \frac{2\pi i}{b} m,\ m \in \mathbb{Z}$$

$$\beta \in \mathbb{R}, \alpha_i \to i\alpha_i$$

Region I:

$$\alpha_i < \frac{\beta - \beta^{-1}}{2}, \quad \frac{n}{b} = \frac{\beta - \beta{-1} - \sum_i \alpha_i}{\beta} < 0$$

Two thimbles $\mathcal{T}_{A_0}$ and $\mathcal{T}_{A_1}$

Two-complex variables model

Cao, R.S.,Usciati ('22)

$$S(\phi_1,\phi_2) = -\frac{1}{4}(\phi_1-\phi_2)^2 + \alpha_1\,\phi_1 + \alpha_2\,\phi_2 - e^{-\beta\phi_1} - e^{-\beta\phi_2}$$

$$\alpha_1 = \alpha_2 = n,\ \phi_0 = \phi_1+\phi_2, \quad \phi_d = \phi_1-\phi_2,$$

$$S(\phi_0,\phi_d) = -\frac{1}{4}\phi_d^2 + n\,\phi_0 - 2\cosh\left(\beta\frac{\phi_d}{2}\right)e^{-\beta\frac{\phi_0}{2}}$$

Two parameters: n and β

Coulomb gas integration manifolds

Spacelike sector: $\beta = ib, \quad n/\beta > 0$

$$\Sigma_1 = i\mathbb{R} \times (\mathbb{R} + 2\pi i|\beta|^{-1})$$

$$I_{\Sigma_1} = \frac{2}{\beta}\Gamma\left(2\frac{n}{\beta}\right) e^{2i\pi\frac{n}{\beta}} \int_{-\infty}^{\infty} d\,(\mathrm{Im}\ \phi_d)\left(2\lambda\ \cos\left(\frac{\beta\ \mathrm{Im}\ \phi_d}{2}\right)\right)^{-2\frac{n}{\beta}} e^{-\frac{(\mathrm{Im}\ \phi_d)^2}{4}}$$

Coulomb gas integration manifolds

Timelike sector: $\beta, \quad n/\beta < 0$

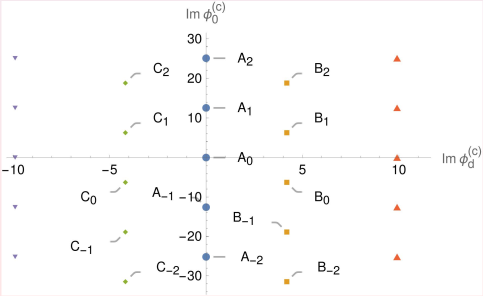
$$\Sigma_2 = \bigcup_{\phi_d \in i\mathbb{R}} \mathcal{T}_0^{(\phi_d)}$$

$$I_{\Sigma_2} = \frac{4\pi}{\beta} \frac{1}{\Gamma\left(1 - 2\frac{n}{\beta}\right)} \int_{-\infty}^{\infty} d\,(\mathrm{Im}\;\phi_d) \left(2\;\cos\left(\frac{\beta\;\mathrm{Im}\;\phi_d}{2}\right)\right)^{-2\frac{n}{\beta}} e^{-\frac{(\mathrm{Im}\;\phi_d)^2}{4}}$$

Critical points:

$$\phi_0^{(c)} = -\frac{2}{\beta}\log\left(\frac{n}{\beta}\right) + \frac{2}{\beta}\log\left(\cosh\left(\beta\frac{\phi_d^{(c)}}{2}\right)\right) + \frac{4i\pi}{\beta}\left(m+\frac{1}{2}\right),$$

$$\phi_d^{(c)} = 2n\tanh\left(\beta\frac{\phi_d^{(c)}}{2}\right), \qquad m\in\mathbb{Z}$$



The method:

$$I_{\mathcal{T}}(\lambda) = \int_{\mathcal{T}} dz \wedge d\omega \; e^{-\;S(z,\omega)}$$

for any $(\xi_1, \xi_2) \in \mathcal{D}, (z(\xi_1, \xi_2), \omega(\xi_1, \xi_2)) \in \mathcal{T}$

$$I_{\mathcal{T}} = \int_{\mathcal{D}} d\xi_1 d\xi_2 \; e^{-\;S(z(\xi_1,\xi_2),\omega(\xi_1,\xi_2))} \,\text{det}\,[\text{Jacobian}]$$

$$S[z,\omega] = S[z_c,\omega_c] + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots$$

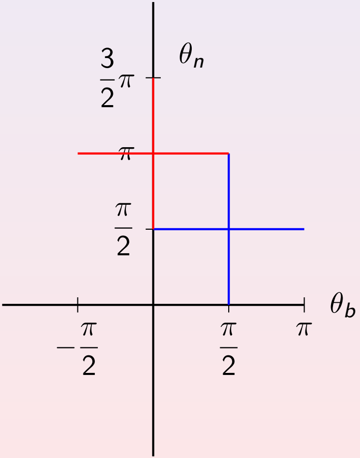
Close to the critical point:

$$\mathcal{T}_{(z_c,\omega_c)} : \mathrm{Im}(x_i) = 0$$

$$\frac{dx_i}{dt} = \overline{\frac{\partial S(x_1,x_2)}{\partial x_i}}, \quad x_1(0) = \epsilon\cos(\theta), x_2(0) = \epsilon\sin(\theta)$$

$$\frac{d}{dt}\,\frac{dx_i}{d\theta} = \frac{1}{\lambda_i}\overline{H_{ij}\,x_j}, \quad \frac{dx_1}{d\theta}(0) = -\epsilon\sin(\theta), \frac{dx_2}{d\theta}(0) = \epsilon\cos(\theta)$$

$$n = |n| e^{i\theta_n}, \beta = |\beta| e^{i\theta_b}$$

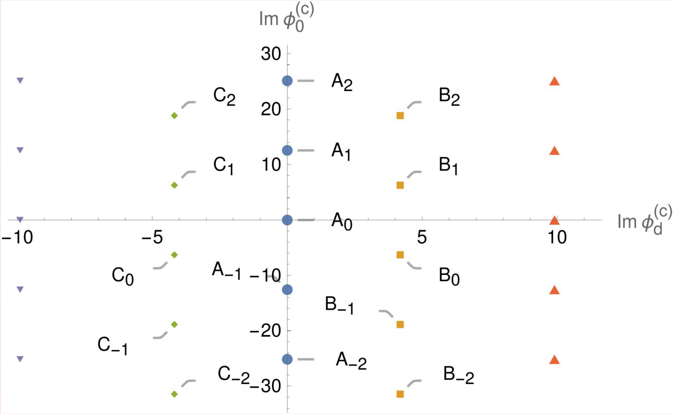


$$I_{\Sigma_2} = I_{\mathcal{T}_{A_0}} + I_{\mathcal{T}_{C_0}} + I_{\mathcal{T}_{B_1}} + \ldots$$

$$I_{\Sigma_1} = I_{\mathcal{T}_{A_0}}$$

$$n/\beta < 0, \quad \beta \in \mathbb{R}$$

$$I_{\Sigma_2} = \frac{4\pi}{\beta} \frac{1}{\Gamma\left(1 - 2\frac{n}{\beta}\right)} \int_{-\infty - i\epsilon}^{\infty - i\epsilon} d\left(\mathrm{Im}\,\phi_d\right) \left(2\, \cos\left(\frac{\beta\, \mathrm{Im}\,\phi_d}{2}\right)\right)^{-2\frac{n}{\beta}} e^{-\frac{(\mathrm{Im}\,\phi_d)^2}{4}}$$



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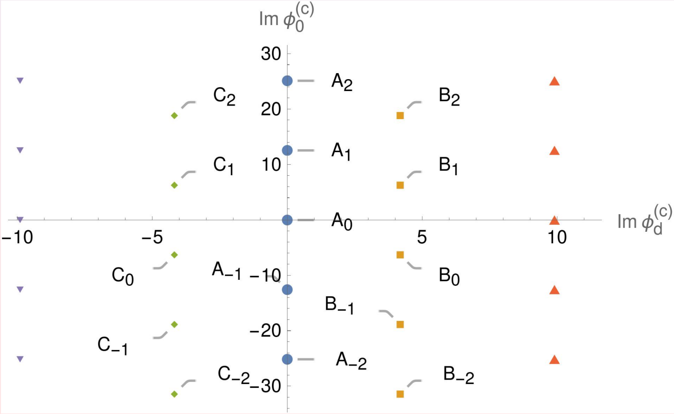

Conclusions

- Very complicated pattern of Stokes phenomena behind the continuation to imaginary Liouville.
- Probably the analytical continuation from the Liouville path integral gets contribution from a number of infinite thimbles. The phase ambiguity can be understood in terms of different thimble decompositions.
- Bootstrap solutions can be accounted for different choice of thimble decomposition. We would like to check if the RS solution is the one-thimble one.