Liouville CFT
from probabilistic construction to bootstrap solution

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Based on joint works with G. Baverez, C. Guillarmou, A. Kupiainen, R. Rhodes

Context

Statistical physics model in 2D at criticality

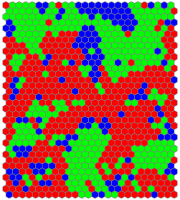
Conformal Field Theories

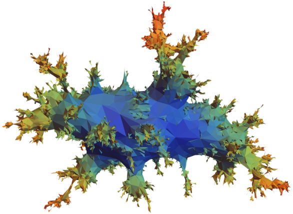
Belavin-Polyakov-Zamolodchikov (1984): Conformal Bootstrap

Gawedski, Kontsevich, Segal, Borcherds-Frenkel,...

Context

Discrete statistical physics models $$F \mapsto \sum_{\sigma \text{ config}} F(\sigma) e^{-H_\beta(\sigma)}$$



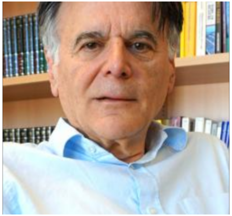


Scaling limit $$F \mapsto \int F(\Phi) e^{-S(\Phi)} D\Phi =: \langle F \rangle$$

Field $x \in \Sigma \mapsto V_\alpha(\Phi, x) := V_\alpha(x)$ indexed by α

Correlation functions:

$$\langle V_{\alpha_1}(x_1) \dots V_{\alpha_m}(x_m) \rangle$$



" *The manuscript that follows was written fifteen years ago...I just wanted to justify my proposed definition by checking that all know examples of CFTs did fit the definition. This task held me up....*"

Graeme Segal, *The definition of Conformal Field Theory* (2004).

Our work: Segal's axioms for Liouville CFT/Conformal Bootstrap

Segal's axioms

Correspondence

- Circles:

$$\text{Disjoint unions of parametrized circles } \sqcup_{i=1}^{b} \mathcal{C}_i$$

$$\downarrow$$

$$L^2(M^b, \mu_0^{\otimes^b}),\ \mathcal{H} = L^2(M, \mu_0) \text{ Hilbert space } (\emptyset \to \mathbb{C})$$

- Surfaces:

Riemann surface (Σ, g) with parametrized geodesic boundary $\partial\Sigma = \sqcup_{i=1}^{b} \mathcal{C}_i$ and marked points $(\mathbf{x}, \boldsymbol{\alpha})$

$$\downarrow$$

$$\text{Amplitude } \mathcal{A}_{\Sigma, g, \mathbf{x}, \boldsymbol{\alpha}}(\varphi_1, \ldots, \varphi_b) \in L^2(M^b, \mu_0^{\otimes^b}).$$

Geometric rules

- **diffeomorphism invariance** :

$$\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha}} = \mathcal{A}_{\psi(\Sigma),\psi_* g,\psi(\mathbf{x}),\boldsymbol{\alpha}}$$

- **Weyl covariance** :

$$\mathcal{A}_{\Sigma,e^{\omega}g,\mathbf{x},\boldsymbol{\alpha}} = e^{\frac{c_L}{96\pi}\int_{\Sigma}(|d\omega|_g^2+2K_g\omega)\mathrm{d}v_g} \Big(\prod_{i=1}^{m} e^{-\Delta_{\alpha_i}\omega(x_i)}\Big)\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha}}$$

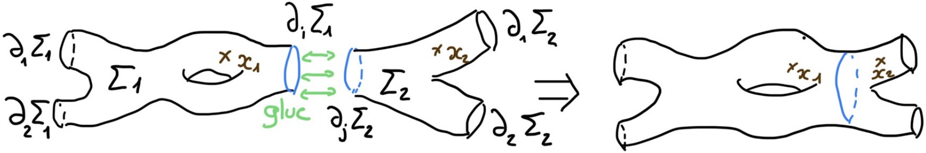
where

- $\mathbf{c}_L$ is the central charge
- $\Delta_{\alpha} \in \mathbb{C}$ conformal weight of V_{α}

Gluing rule

- Gluing rule: gluing (Σ_1, g_1) with (Σ_2, g_2) along $\partial_i\Sigma_1 \sim \partial_j\Sigma_2$ yields

$$\boxed{\mathcal{A}_{\Sigma_1\sharp\Sigma_2,g,\mathbf{x},\boldsymbol{\alpha}}(\varphi_1,\varphi_2,\varphi_1',\varphi_2') = \int_M \mathcal{A}_{\Sigma_1,g_1,\mathbf{x}_1,\boldsymbol{\alpha}_1}(\varphi_1,\varphi_2,\varphi)\mathcal{A}_{\Sigma_2,g_2,\mathbf{x}_2,\boldsymbol{\alpha}_2}(\varphi_1',\varphi_2',\varphi)d\mu_0(\varphi)}$$



Physics heuristics

Hilbert space:

$$\mathcal{H} = L^2(H^{-\epsilon}(\mathbb{S}^1), \mu_0)$$

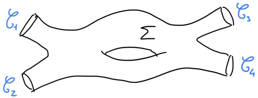
with

$$H^{-\epsilon}(\mathbb{S}^1) := \{\varphi = \sum_n a_n e^{in\theta} \mid \sum_n (1 + |n|^2)^{-\epsilon} |a_n|^2 < \infty\}$$

Amplitudes:

When $\partial\Sigma \neq \emptyset$

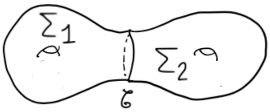
$$\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha}}(\varphi) := \int_{\substack{\{\Phi:\Sigma\to\mathbb{R},\\ \Phi|_{\mathcal{C}_i}=\varphi_i\}}} V_{\alpha_1}(x_1)\dots V_{\alpha_m}(x_m) e^{-S_\Sigma(\Phi,g)} D\Phi$$



When $\partial\Sigma = \emptyset$

$$\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha}} = \langle V_{\alpha_1}(x_1)\dots V_{\alpha_m}(x_m)\rangle_{\Sigma,g}$$

Physics heuristics



Assume the action is local

$$S_{\Sigma}(\Phi, g) = S_{\Sigma_1}(\Phi|_{\Sigma_1}, g) + S_{\Sigma_2}(\Phi|_{\Sigma_2}, g).$$

Conditioning on $\mathcal{C}$

$$\int_{\{\Phi:\Sigma\to\mathbb{R}\}} V_{\alpha_1}(x_1)\dots V_{\alpha_m}(x_m)e^{-S_{\Sigma}(\Phi,g)}D\Phi = \int_{\{\varphi:\mathcal{C}\to\mathbb{R}\}} \mathcal{A}_{\Sigma_1,g_1,\mathbf{x}_1,\boldsymbol{\alpha}_1}(\varphi)\mathcal{A}_{\Sigma_2,g_2,\mathbf{x}_2,\boldsymbol{\alpha}_2}(\varphi)D\varphi$$

Path integral for Liouville CFT

Riemann surface Σ, metric g

$$F \mapsto \int F(\Phi) e^{-S_\Sigma(\Phi,g)} D\Phi$$

Liouville action

$$S_\Sigma(\Phi,g) = \frac{1}{4\pi} \int_\Sigma (|d\Phi|_g^2 + QK_g\Phi + \mu e^{\gamma\Phi}) \mathrm{dv}_g$$

Parameters

$$\underline{\gamma \in (0,2)}, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2} \quad , \mu > 0$$

Hilbert space

Trigonometric series

$$\varphi(\theta) = c + \sum_{n\geq 1} \varphi_n e^{in\theta} + \bar{\varphi}_n e^{-in\theta}, \quad \varphi_n = \frac{1}{2}\frac{x_n + iy_n}{\sqrt{n}}, \; n > 0$$

equipped with

$$\mathbb{P} = \prod_{n\geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2+y_n^2)} dx_n dy_n.$$

Set $\mu_0 = dc \otimes \mathbb{P}$ then

$$\boxed{\mathcal{H} := L^2(H^{-\varepsilon}(\mathbb{S}^1), \mu_0)}$$

Hilbert space: the Hamiltonian

For $F \in \mathcal{H}$

$$\mathbf{H}F(\varphi) = -\frac{1}{2}\partial_c^2 F + \frac{Q^2}{2}F + 2\sum_{n\geq 1}(A_n^* A_n + \tilde{A}_n^* \tilde{A}_n)F + \mu \int_0^{2\pi} e^{\gamma\varphi(\theta)} d\theta F$$

where

$$A_n = \frac{i}{2}\partial_{\varphi_n}, \qquad \tilde{A}_n = \frac{i}{2}\partial_{\bar{\varphi}_n}$$

and $e^{\gamma\varphi(\theta)} d\theta$ is Gaussian Multiplicative chaos (Kahane '85):

$$e^{\gamma\varphi(\theta)} d\theta := \lim_{\epsilon\to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma\varphi_\epsilon(\theta)} d\theta$$

Gaussian Free Field on (Σ, g):

$$X_g(x) = \sqrt{2\pi} \sum_{n \geq 0} \frac{\alpha_n}{\sqrt{\lambda_n}} e_n(x)$$

with

- $(\alpha_n)_n$ iid standard Gaussians
- $(e_n)_n$ o.n.b. of eigenfunctions of Laplacian Δ_g with Dirichlet b.c.

$$\Delta_g e_n = \lambda_n e_n$$

- Covariance $\mathbb{E}[X_g(x)X_g(x')] = G_g(x, x')$ Green function of the Laplacian.

Gaussian integral:

$$\boxed{\int_{\{\Phi|_{\partial_i \Sigma} = \varphi_i\}} F(\Phi) e^{-\frac{1}{4\pi}\int_\Sigma |d\Phi|_g^2 \mathrm{dv}_g} D\Phi = (\det{}'(\Delta_g))^{-1/2} \mathbb{E}_{\varphi}\Big[F(X_g + P\varphi)\Big]}$$

Liouville amplitudes for $\partial\Sigma \neq \emptyset$:
Fix $x_1, \dots, x_n \in \Sigma$ and real weights $\alpha_1, \dots, \alpha_n < Q$

$$\mathcal{A}_{\Sigma,g,\mathbf{x},\boldsymbol{\alpha}}(\varphi) := \mathcal{A}^0_{\Sigma,g}(\varphi)\mathbb{E}_\varphi\Big[\prod_{j=1}^m e^{\alpha_j \Phi(x_j)} e^{-\frac{1}{4\pi}\int_\Sigma (QK_g\Phi + \mu e^{\gamma\Phi})\mathrm{dv}_g}\Big]$$

where $\Phi = X_g + P\varphi$.

- $\mu > 0$, $\gamma \in (0, 2)$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$

- Gaussian Multiplicative chaos:

$$e^{\gamma X_g(x)}\mathrm{dv}_g(x) := \lim_{\epsilon \to 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\epsilon}(x)}\mathrm{dv}_g(x)$$

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where $\Phi = X_g + P\varphi$.

Theorem (GKRV '21): If

$$\sum_{j=1}^{m} \alpha_j > \chi(\Sigma) Q$$

then the above amplitudes satisfy Segal's axioms.

Riemann sphere and structure constants

3 point correlation function

$$\langle V_{\alpha_1}(x_1) V_{\alpha_2}(x_2) V_{\alpha_3}(x_3) \rangle_{\hat{\mathbb{C}}, g_0}$$

Structure constant

$$C(\alpha_1, \alpha_2, \alpha_3) := \langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(\infty) \rangle_{\hat{\mathbb{C}}, g_0}$$

Theorem (KRV '17): DOZZ formula

$$C(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\,\mu\,\ell(\frac{\gamma^2}{4})\,(\frac{\gamma}{2})^{2-\gamma^2/2}\right)^{\frac{2Q-\bar\alpha}{\gamma}} \times \frac{\Upsilon'_{\frac{\gamma}{2}}(0)\Upsilon_{\frac{\gamma}{2}}(\alpha_1)\Upsilon_{\frac{\gamma}{2}}(\alpha_2)\Upsilon_{\frac{\gamma}{2}}(\alpha_3)}{\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2Q}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_1}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_2}{2})\Upsilon_{\frac{\gamma}{2}}(\frac{\bar\alpha-2\alpha_3}{2})}$$

with

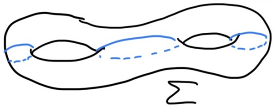
$$\bar\alpha = \alpha_1+\alpha_2+\alpha_3, \quad \ell(x) = \Gamma(x)/\Gamma(1-x)$$

and the $\Upsilon_{\frac{\gamma}{2}}$ function defined as analytic continuation of the following integral defined for $0 < \Re(z) < Q$

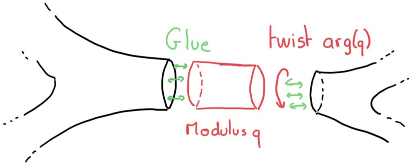
$$\ln \Upsilon_{\frac{\gamma}{2}}(z) = \int_0^\infty \left(\left(\frac{Q}{2} - z\right)^2 e^{-t} - \frac{(\sinh((\frac{Q}{2}-z)\frac{t}{2}))^2}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})} \right) \frac{dt}{t}$$

Moduli space and plumbing coordinates

- Pants

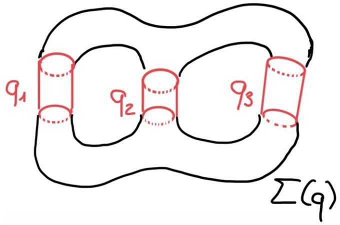


- Gluing annuli



- Local coordinates on $\mathcal{M}_{h,m}$

$$q \in \mathbb{D}^{3h-3+m} \mapsto \Sigma(q)$$



Conformal Bootstrap

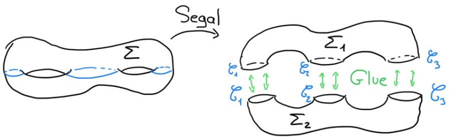
Theorem (GKRV '21):

$$\boxed{\langle V_{\alpha_1}(x_1)\dots V_{\alpha_m}(x_m)\rangle_{\Sigma(q),g} = \int_{\mathbb{R}_+^{3\mathbf{h}-3+m}} \rho(p,\alpha)|\mathcal{F}_{c_L,p,\Delta_\alpha}(q)|^2 \mathrm{d}p}$$

- $\rho(p,\alpha)$ product of structure constants
- $\mathcal{F}_{c_L,p,\Delta_\alpha}(q)$ holomorphic series: conformal blocks

Idea of the proof

- $\langle 1 \rangle_{\Sigma,g} = \langle A_{\Sigma_1,g}, A_{\Sigma_2,g} \rangle_{\mathcal{H}^{\otimes 3}}$



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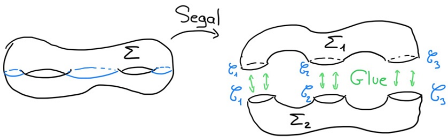

- Orthogonal basis $\Psi_{Q+ip} := A_{\mathbb{D},g,0,Q+ip}$, for $p \in \mathbb{R}_+$ (disk amplitudes)

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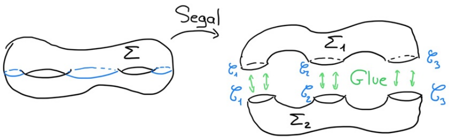

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- Plancherel

$$\langle 1 \rangle_{\Sigma,g} = \int_{(\mathbb{R}^+)^3} \langle A_{\Sigma_1,g}, \otimes_{j=1}^3 \Psi_{Q+ip_j} \rangle_{\mathcal{H}^{\otimes 3}} \langle \otimes_{j=1}^3 \Psi_{Q+ip_j}, A_{\Sigma_2,g} \rangle_{\mathcal{H}^{\otimes 3}} \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3$$

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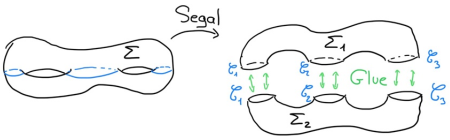

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- Segal

$$\langle A_{\Sigma_1,g}, \otimes_{j=1}^3 \Psi_{Q+ip_j} \rangle_{\mathcal{H}^{\otimes 3}} = C(Q+ip_1, Q+ip_2, Q+ip_3)$$

Full spectrum of Liouville CFT

In fact, one must take into account extra terms in $\mathcal{H}$:

Theorem (GKRV '20):

- Complete family of eigenstates $\Psi_{Q+ip,\nu,\tilde{\nu}}$, $p \in \mathbb{R}_+$ and Young diagrams $\nu, \tilde{\nu}$:

$$\mathbf{H}\Psi_{Q+ip,\nu,\tilde{\nu}} = \Big(\frac{Q^2}{2} + \frac{p^2}{2} + |\nu| + |\tilde{\nu}|\Big)\Psi_{Q+ip,\nu,\tilde{\nu}}.$$

- Plancherel formula:

$$\langle u_1, u_2\rangle = \sum_{|\nu'|=|\nu|} \sum_{|\tilde{\nu}'|=|\tilde{\nu}|} \int_0^\infty \langle u_1, \Psi_{Q+ip,\nu,\tilde{\nu}}\rangle \langle \Psi_{Q+ip,\nu',\tilde{\nu}'}, u_2\rangle \mathcal{Q}^{-1}_{Q+ip}(\nu,\nu')\mathcal{Q}^{-1}_{Q+ip}(\tilde{\nu},\tilde{\nu}')\, dp$$

  with $\mathcal{Q}_{Q+ip}(\nu,\tilde{\nu})$ Schapovalov form.

References

Structure constants

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Spectrum of Liouville CFT

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Segal's axioms and bootstrap

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Flow of deformations

- G. Baverez, C. Guillarmou, A. Kupiainen, R. Rhodes, V. Vargas: *The Virasoro structure and the scattering matrix for Liouville CFT*, arXiv:2204.02745