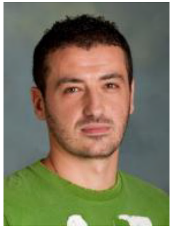
The Sine-Gordon and related models — Overview

Roland Bauerschmidt (Cambridge)

Dedicated to Pierluigi Falco (1977–2014)



Models and their connections

- Discrete Gaussian model
- Lattice Sine-Gordon model
- Lattice Coulomb gas
- Rotator model

} lattice

- Continuum Sine-Gordon model
- Continuum Coulomb gas
- Massive Thirring Model

} continuum

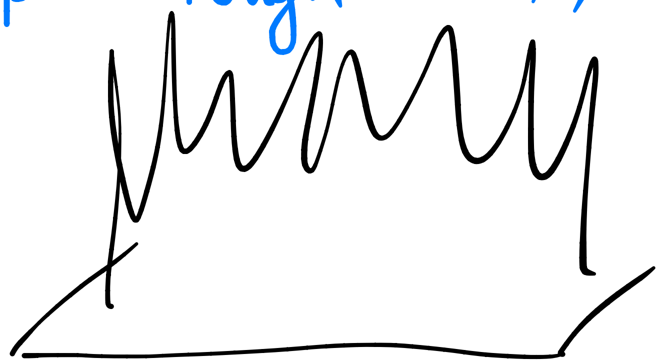
The Discrete Gaussian model ($\mathbb{Z}$-ferromagnet)

$\varphi : \Lambda \longrightarrow \mathbb{Z}$ where $\Lambda \subset \mathbb{Z}^d$

$$\langle F \rangle \propto \sum_{\varphi \in \beta^{-\frac{1}{2}} \mathbb{Z}^\Lambda} e^{-\frac{1}{2}(\varphi, -\Delta \varphi)} F(\varphi)$$

"GFF cond. to be integer-valued"

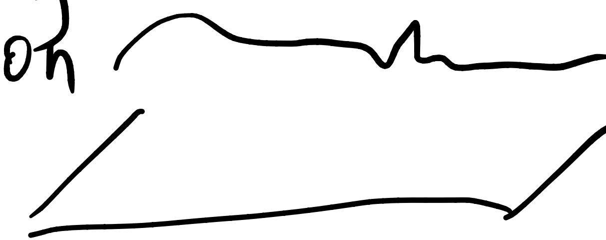
$\beta \gg 1$ rough ($\sim$GFF)



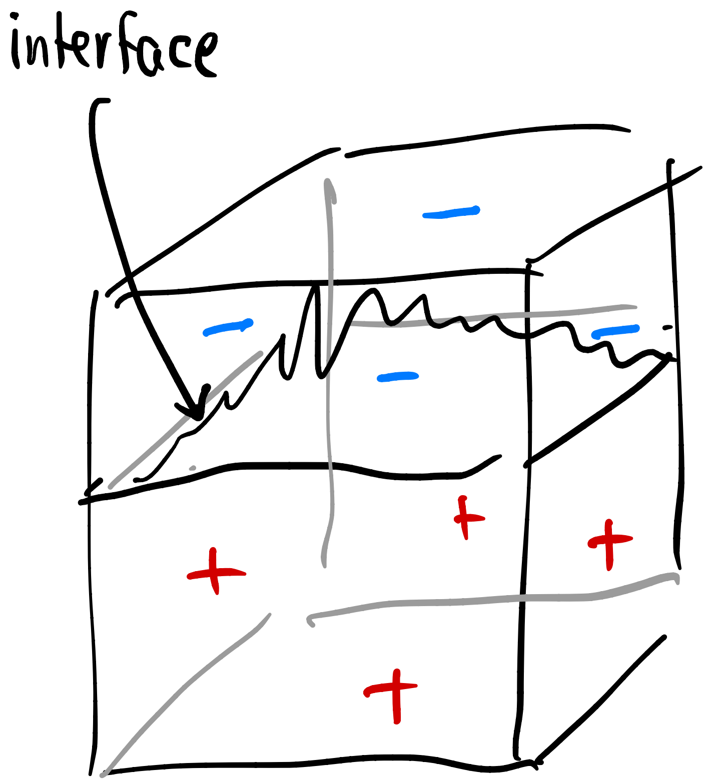
roughening transition

$d=2$

$\beta \ll 1$ flat



Discrete Gaussian as interface model



Ising model on 3D cube with Dobrushin boundary conditions (low temperature)

Solid on Solid model:

$$H(\varphi) = \sum_{x \sim y} |\varphi_x - \varphi_y|$$

Discrete Gaussian model:

$$H(\varphi) = \sum_{x \sim y} |\varphi_x - \varphi_y|^2$$

The lattice Sine-Gordon model

$$\sum_{\varphi\in\beta^{-1/2}\mathbb{Z}} \rightsquigarrow \int_{\mathbb{R}} d\varphi\, e^{+z\cos(\sqrt{\beta}\varphi)}$$

$\mathbb{Z}$ symmetry

$$\langle F\rangle \propto \int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2}(\varphi,-\Delta\varphi)-\sum_{x\in\Lambda} z\cos(\sqrt{\beta}\varphi_x)} F(\varphi)\, d\varphi$$

Expect (and often know) exactly the same behaviour as in DG.

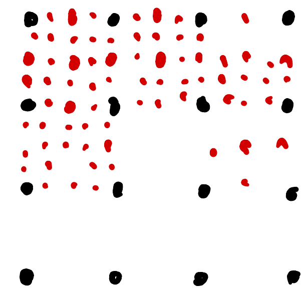
Effectively obtained from DG model after simple RG step.

The continuum Sine-Gordon model

$\Lambda \subset \mathbb{Z}^2 \rightsquigarrow \Lambda_\varepsilon \subset \mathbb{R}^2$

$\sum_{x\in\Lambda} \rightsquigarrow \varepsilon^2 \sum_{x\in\Lambda_\varepsilon}$

$\Delta\varphi_x \rightsquigarrow \Delta^\varepsilon \varphi_x = \varepsilon^{-2} \sum_{y\sim x} (\varphi_y - \varphi_x)$



formal mass term $m^2 \geq 0$

~ Wick renorm.

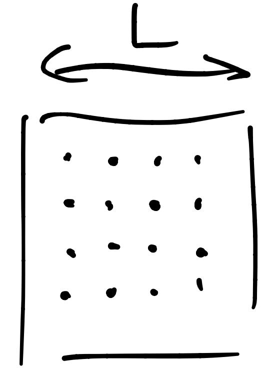
$$\langle F \rangle \propto \int_{\mathbb{R}^{\Lambda_\varepsilon}} \exp\Big(-\varepsilon^2 \sum_{x\in\Lambda_\varepsilon} \big(\varphi_x (-\Delta^\varepsilon \varphi)_x + z\,\varepsilon^{-\frac{\beta}{4\pi}} \cos(\sqrt{\beta}\,\varphi_x) + m^2 \varphi_x^2\big)\Big) F(\varphi)\, d\varphi$$

Under suitable conditions the limit $\varepsilon \to 0$ exists and is <u>non-Gaussian</u>.

~ For order 1 test functions looks like DG model on order 1 lattice.

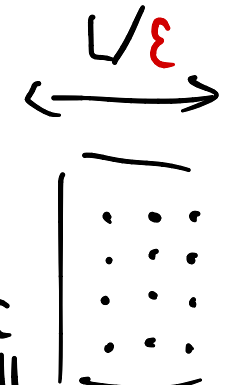
Difference between UV & IR problems

Lattice Sine-Gordon:



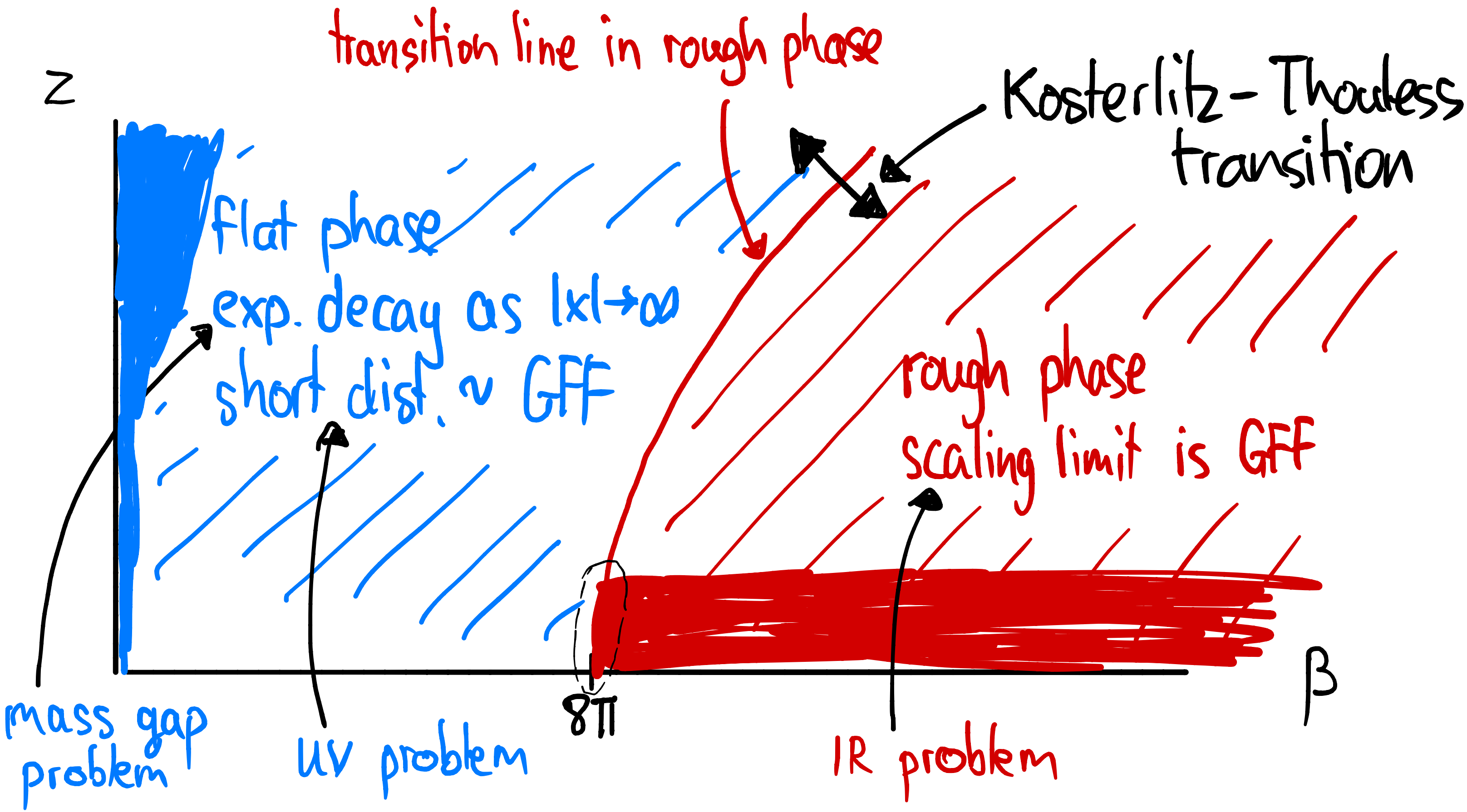
$$H(\varphi) = \sum_{x\in\Lambda_L} \left(\varphi_x(-\Delta\varphi)_x + z\cos(\sqrt{\beta}\,\varphi_x) + m^2\varphi_x^2 \right)$$

Continuum Sine-Gordon rescaled to unit lattice:



$$H(\varphi) = \sum_{x\in\Lambda_{L\varepsilon}} \left(\varphi_x(-\Delta\varphi)_x + \underbrace{\varepsilon^{2-\frac{\beta}{4\pi}} z}_{\text{microscopic coupling constant small}} \cos(\sqrt{\beta}\,\varphi_x) + \underbrace{\varepsilon^2 m^2}_{\text{microscopic mass small}} \varphi_x^2 \right)$$

Phase diagram (lattice Sine-Gordon model)

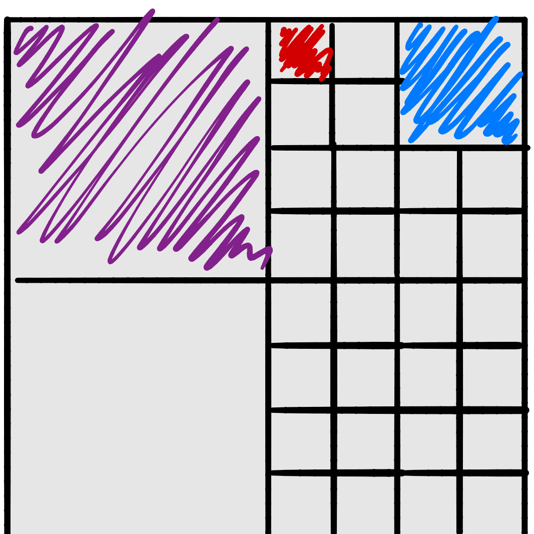


Scaling picture

smooth on scale 1

$$(-\Delta)^{-1}(x,y) \approx \sum_j \underbrace{\frac{\log L}{2\pi}\, u(L^{-j}x, L^{-j}y)}_{C_j(x,y)} \quad \text{where } u(0,0) \sim 1$$

covariance of GFF

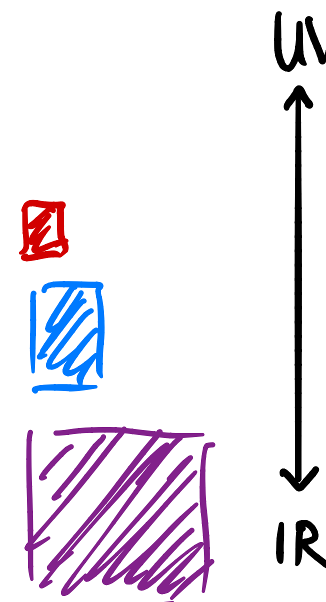


scale -1: ...

scale 0: measure on

scale 1: measure on

scale 2: measure on



Scaling picture

$$\mathbb{E}_{C_j} \sum_{x \in B_j^{\gamma}} \cos(\sqrt{\beta}(\varphi + \zeta_x)) = \underbrace{L^{2j} e^{-\frac{1}{2} C_j(0,0)}}_{L^{(2-\frac{\beta}{4\pi})j}} \cos(\sqrt{\beta}\varphi)$$

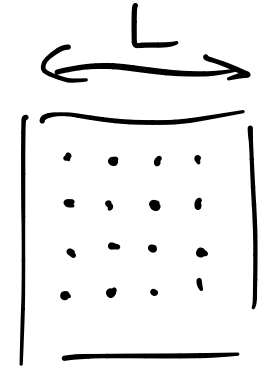
$\cos(\sqrt{\beta}\varphi)$ is $\begin{cases} \text{expanding} & \text{if } \beta < 8\pi \leftarrow \text{UV asympt. freedom} \\ \text{marginal} & \text{if } \beta = 8\pi \\ \text{contracting} & \text{if } \beta > 8\pi \leftarrow \text{IR asympt. freedom} \end{cases}$

relative to GFF

Second-order computation: KT flow equations

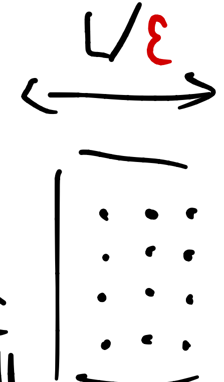
Difference between UV & IR problems

Lattice Sine-Gordon:



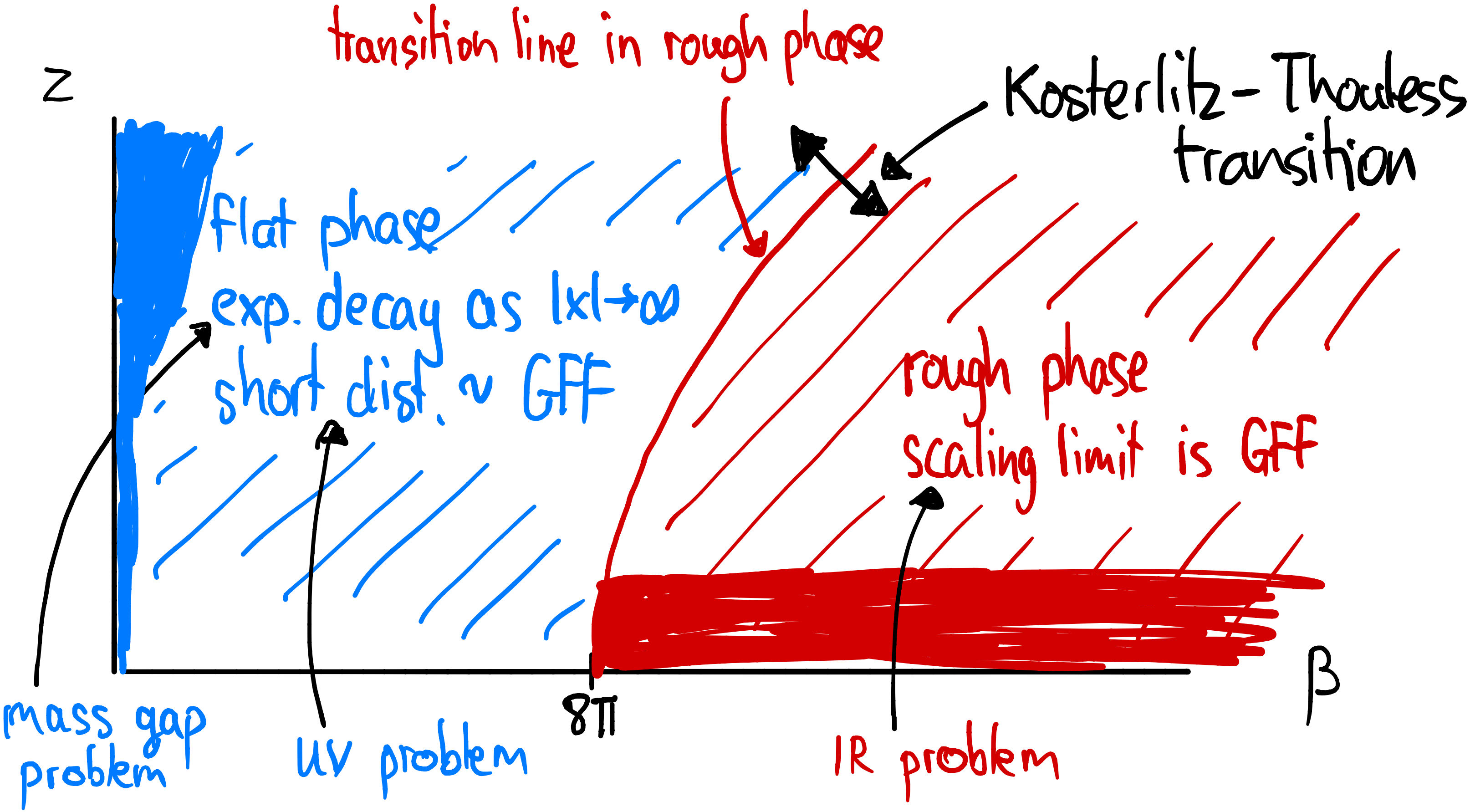
$$H(\varphi) = \sum_{x\in\Lambda_L} \left(\varphi_x(-\Delta\varphi)_x + z\cos(\sqrt{\beta}\,\varphi_x) + m^2\varphi_x^2 \right)$$

Continuum Sine-Gordon rescaled to unit lattice:



$$H(\varphi) = \sum_{x\in\Lambda_{L\varepsilon}} \left(\varphi_x(-\Delta\varphi)_x + \underbrace{\varepsilon^{2-\frac{\beta}{4\pi}} z}_{\text{microscopic coupling constant small}} \cos(\sqrt{\beta}\,\varphi_x) + \underbrace{\varepsilon^2 m^2}_{\text{microscopic mass small}} \varphi_x^2 \right)$$

Phase diagram (lattice Sine-Gordon model)



What is known: Discrete Gaussian model – flat phase

- $\beta \ll 1$: Flat interface (Peierls argument)

  large deviations (Bricmont-El-Mellouki-Fröhlich,
  and maximum Lubetzky-Martinelli-Sly)

- $d=3$: exponential decay for $\underline{\text{all } \beta}$ (Göpfert-Mack)

  $\beta \gg 1$: iterated Mayer expansion for Coulomb gas

  intermediate β: correlation inequality

What is known: Discrete Gaussian / Lattice Sine-Gordon models

- $d=2$, $\beta \gg 1$: log-fluctuations in $d=2$ (Fröhlich-Spencer) via Coulomb gas
  also for range of related models
- $d=2$, $\beta \gg 1$: scaling limit on torus is GFF (Dimock-Hurd)
  (not written like this, but should be implied)
- $d=2$, $\underbrace{\beta=\beta_c,\ z \text{ small}}$: $\langle e^{i\sqrt{\beta}\sigma(\varphi_x-\varphi_y)}\rangle \sim |x|^{c_1(\beta,\sigma)}(\log|x|)^{c_2(\beta,\sigma)}$ (Falco)

8π segment of crit. curve

What is not known (also applies to continuum limit)

- For $\beta < \beta_c$: correlations decay exponentially.
- As $\beta \uparrow \beta_c$: rate of exp. decay satisfies

$$m \sim \exp\left(-c/\sqrt{\beta_c - \beta}\right)$$ ← non-perturbative!

  Infinite order phase transition.

- Critical DG model.
- Scaling limit of DG model with boundary.
- ...

Coulomb gas

now β is inverse temp.
not temp.!

Charges $\xi_i = (x_i, \sigma_i) \in \Lambda \times \{\pm 1\}$

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{\substack{\xi_1, \dots, \xi_N \\ \text{neutral}}} e^{-\frac{\beta}{2}(q, (-\Delta)^{-1} q)}$$

where $q = \sum_{i=1}^{N} \sigma_i \delta_{x_i}$

$$\sum_{i=1}^{N} C_{x_i x_i} + \sum_{i \neq j} \sigma_i \sigma_j C_{x_i x_j}$$

grand canonical partition function of lattice Coulomb gas

self-energy renormalizes z (activity)

Coulomb gas ⟷ Sine-Gordon model

$$Z = \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{\substack{\xi_1,\dots,\xi_N \\ \text{neutral}}} \underbrace{e^{-\frac{\beta}{2}(q,(-\Delta)^{-1}q)}}_{\mathbb{E}^{GFF}\left(e^{i\sqrt{\beta}(\varphi,q)}\right)}$$

Fourier duality

$$= \mathbb{E}^{GFF}\left(\exp\left(\sum_{x\in\Lambda} z\cos(\sqrt{\beta}\varphi_x)\right)\right)$$

→ $e^{i\sqrt{\beta}\sigma\varphi_x}$ represents a charge σ at x

(fractional charges: $\sigma \notin \mathbb{Z}$)

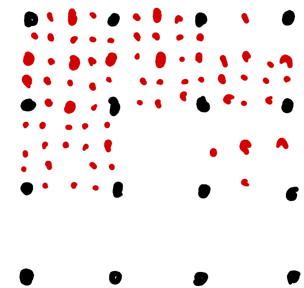
Variants of this duality are useful both ways.

The continuum Sine-Gordon model (SG)

$\Lambda \subset \mathbb{Z}^2 \rightsquigarrow \Lambda_\varepsilon \subset \mathbb{R}^2$

$\sum_{x\in\Lambda} \rightsquigarrow \varepsilon^2 \sum_{x\in\Lambda_\varepsilon}$

$\Delta\varphi_x \rightsquigarrow \Delta^\varepsilon \varphi_x = \varepsilon^{-2}\sum_{y\sim x}(\varphi_y-\varphi_x)$



formal mass term $m^2 \geq 0$

~ Wick renorm.

$$\langle F\rangle \propto \int_{\mathbb{R}^{\Lambda_\varepsilon}} \exp\Big(-\varepsilon^2\sum_{x\in\Lambda_\varepsilon}\Big(\varphi_x(-\Delta^\varepsilon\varphi)_x + z\,\varepsilon^{-\frac{\beta}{4\pi}}\cos(\sqrt{\beta}\,\varphi_x)+m^2\varphi_x^2\Big)\Big)\,F(\varphi)\,d\varphi$$

Under suitable conditions the limit $\varepsilon\to 0$ exists and is <u>non-Gaussian</u>.

Literature on continuum Sine-Gordon model

- Fröhlich-Seiler 1976
- Benfatto-Gallavotti-Nicolo 1982
- Brydges-Kennedy 1986
- Nicolo-Renn-Steinmann 1986
- Dimock-Hurd 1993

renormalised expansions

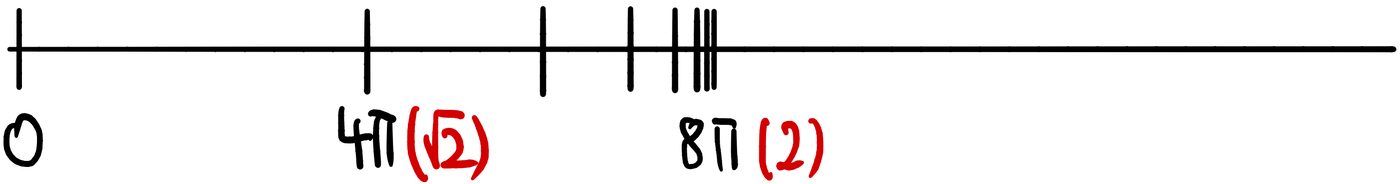
all for $m^2>0$ or finite volume

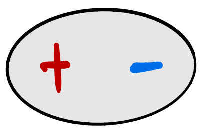
also applies to IR problem (harder)

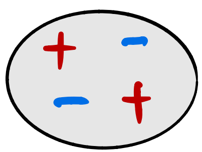
- Park / Fröhlich-Park: correlation inequalities

$\Rightarrow$ massless inf. vol. limit exists (in some sense) and satisfies OS axioms, but no information

Regions of collapse in the continuum SG model



$\beta > 4\pi$: 

$\beta > 6\pi$: 

$\vdots$

create divergences in partition function but not in the measure.

Are these thresholds physical?

The Massive Thirring Model (MTM)

Complex two-component fermionic field $\psi, \bar{\psi}$. Action:

$$S = \int \Big(\underbrace{\bar{\psi}(i\gamma^\mu \partial_\mu \psi) - m\bar{\psi}\psi}_{\text{free Dirac fermions}} - \frac{g}{2}(\underbrace{\bar{\psi}\gamma^\mu\psi}_{j^\mu})(\bar{\psi}\gamma_\mu\psi) \Big)\, dx$$

"Determinantal correlation functions"

where

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Massive Thirring Model

Massless Thirring model integrable in different ways.

Benfatto-Falco-Mastropietro: construction in infinite vol. (massive TM)

- $G(x,y) \sim |x-y|^{-1-\eta_g}$ as $|x-y| \to 0$
- $G(x,y) \lesssim |x-y|^{-1-\eta_g} \exp(-c_g \sqrt{|x-y|})$.

I expect that exponential is the truth (without $\sqrt{\cdot}$).

Coleman correspondence: cont. SG $\longleftrightarrow$ MTM

(Bosonisation)

$$z = g, \quad \frac{\beta}{4\pi} = \frac{1}{1 - g/\pi}$$

$$\cos(\sqrt{\beta}\varphi) \longleftrightarrow \bar{\psi}\psi, \quad \partial^\mu \varphi \longleftrightarrow \bar{\psi}\gamma^\mu\psi, \quad \dots$$

in the sense that all correlation functions are equal!

- Coleman: physics
- Fröhlich–Seiler: $m^2 > 0$, $\beta < 4\pi$
- Dimock: $m^2 > 0$, $\beta = 4\pi$ (free fermion point)
- Benfatto–Falco–Mastropietro: $m^2 > 0$, $\beta < 16/3\,\pi$
- massless case open even at $\beta = 4\pi$!

Problems: continuum Sine-Gordon model

- Coleman correspondence for $m=0$
  - seems open even at $\beta=4\pi$ where MTM is free
- Exp. decay for formally massless Sine-Gordon model
- Explicit formulas for mass and one point function

Recent developments

Junnila-Saksman-Webb: $\beta = 2\pi$, finite domain

$\Rightarrow$ $\cos(\sqrt{\beta}\varphi)$ has distribution related to critical XOR Ising model with scaled magnetic field.

Lacoin-Rhodes-Vargas: existence of 1D boundary version for all coupling parameters in finite volume.

Dynamics

- Wellposedness of SPDE for cont. Sine-Gordon model:

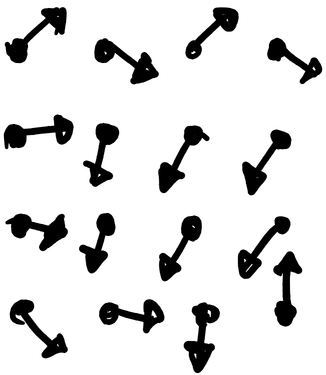
  $\beta < 16\pi/3$: Hairer-Shen

  $\beta < 8\pi$: Chandra-Hairer-Shen

- Log-Sobolev inequality for massive infinite volume continuum Sine-Gordon model $\beta < 6\pi$ (B.-Bodineau)

- Spectral gap for hierarchical Discrete Gaussian model and lattice Sine-Gordon model $\beta > 8\pi$ (B.-Bodineau)

The rotator model

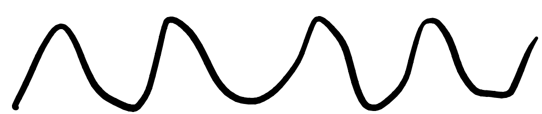


$\Lambda \subset \mathbb{Z}^d$

$\sigma : \Lambda \to S^1$

$$\langle F \rangle \propto \int_{[-\pi,\pi]^\Lambda} e^{\beta \sum_{xy} \underbrace{\cos(\theta_x - \theta_y)}_{\sigma_x \cdot \sigma_y}} F(\theta) \prod_x d\theta_x$$

Villain model: $e^{\beta \cos(\theta_x - \theta_y)} \rightsquigarrow \sum_{n \in \mathbb{Z}} e^{-\frac{\beta}{2}(\theta_x - \theta_y + n)^2}$

— both look like 

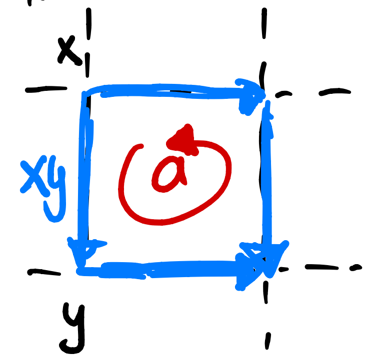
The rotator model

$$\langle F\rangle \propto \int_{[-\pi,\pi]^{V}} \sum_{n\in\mathbb{Z}^E} e^{-\frac{\beta}{2}\sum_{xy}(\underbrace{\theta_x-\theta_y}_{(\nabla\theta)_{xy}}+n_{xy})^2} F(\theta)\prod_x d\theta_x$$

$$\underbrace{\qquad\qquad}$$

$$= \sum_{\substack{q\in\mathbb{Z}^F\\ \nabla q=0}} \sum_{\substack{n\in\mathbb{Z}^E\\ \nabla n=q}}$$

$\nabla n = \sum_{e\in\partial a} n_e$



$$= \sum_{\substack{q\in\mathbb{Z}^F\\ \nabla q=0}} \sum_{\phi\in\mathbb{Z}^V}$$

$n = \nabla\phi + n_q$ where $\nabla n_q = q$ is some soln.

charge configurations on the faces

The rotator model

$$Z = \int_{\mathbb{R}^\Lambda} \sum_{\substack{q \in \mathbb{Z}^F \\ \nabla q = 0}} \underbrace{e^{-\frac{\beta}{2}(\nabla\varphi + n_q,\, \nabla\varphi + n_q)}}_{e^{-\frac{\beta}{2}(\nabla\varphi,\nabla\varphi)}\; e^{-\frac{\beta}{2}(q,-\Delta^{-1}q)}} \prod_{x\in\Lambda} d\varphi_x$$

spin waves

lattice Coulomb gas on faces

Sine-Gordon represenatation

The rotator model: open problems

- $\langle \sigma_x \cdot \sigma_y \rangle \sim C|x|^{-\frac{1}{8}} (\log|x|)^{-\frac{1}{8}}$

  when $\beta = \beta_c$

- low temperature phase asymptotics
- dynamics

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lattice

- Continuum Sine-Gordon model
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- Massive Thirring Model

continuum