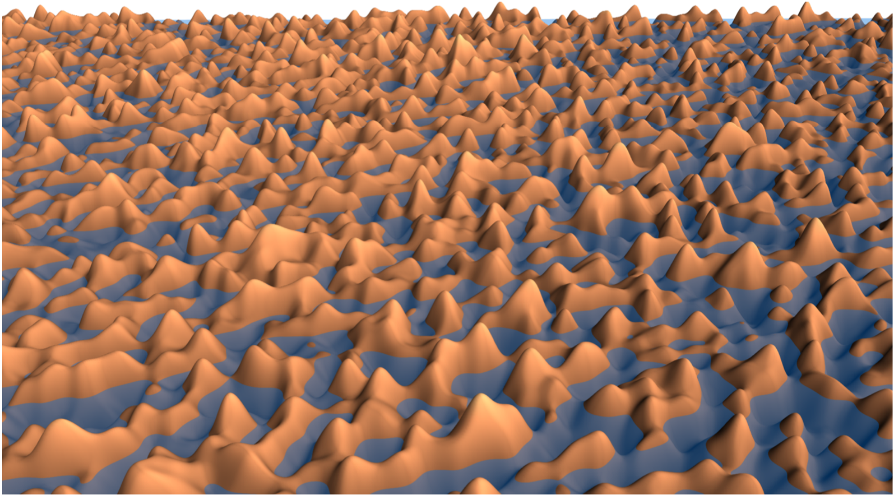
Percolation and Gaussian fields

V. Beffara and D. Gayet — Université Grenoble Alpes

Porquerolles, 17–21 June 2019



Plan for the mini-course

- Introduction etc (Monday)
- RSW for well decorrelated fields (Tuesday)
- Sharp thresholds and critical points (from Thursday)

Spherical harmonics / Laplacian eigenfunctions

Random eigenfunction of the Laplacian on the sphere

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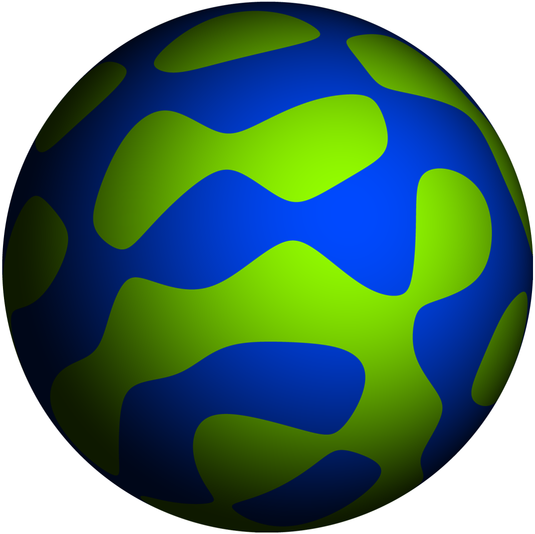

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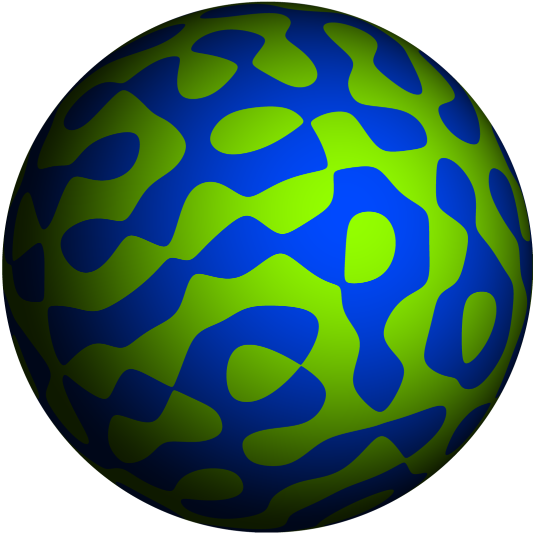

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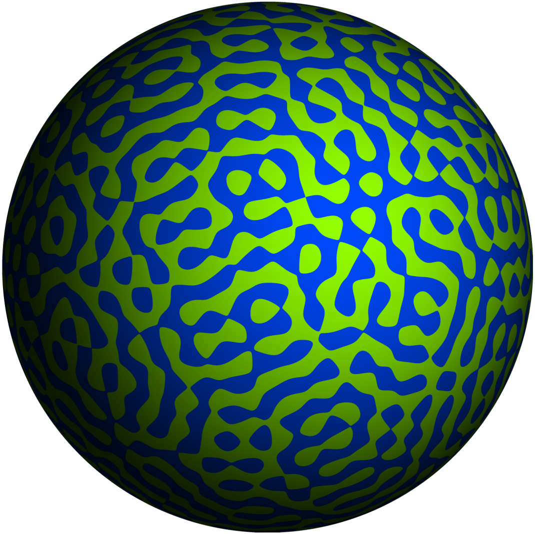

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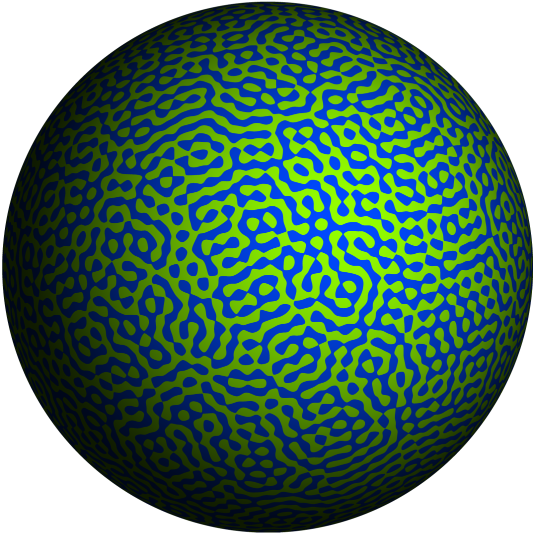

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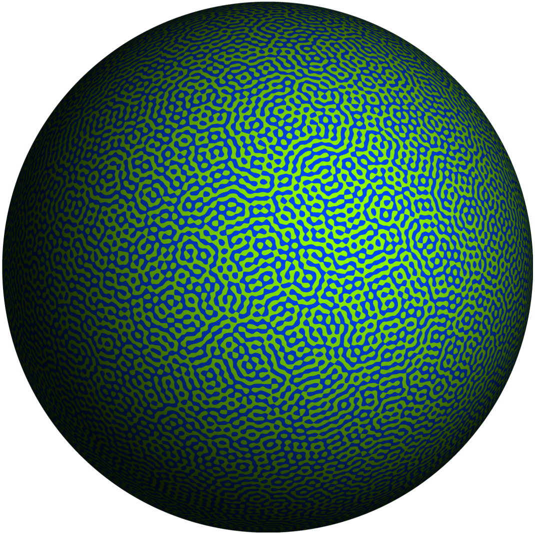

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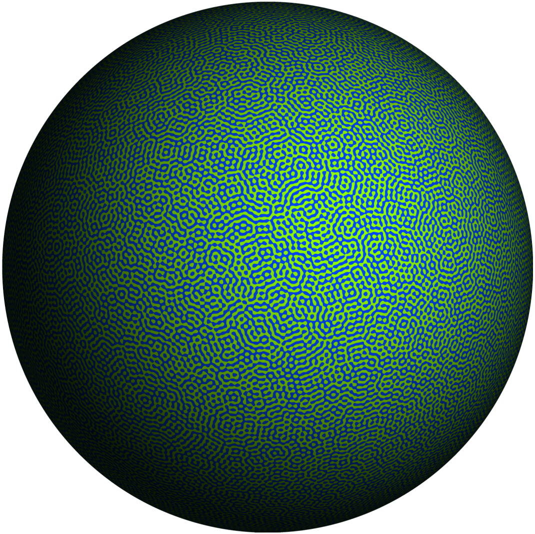

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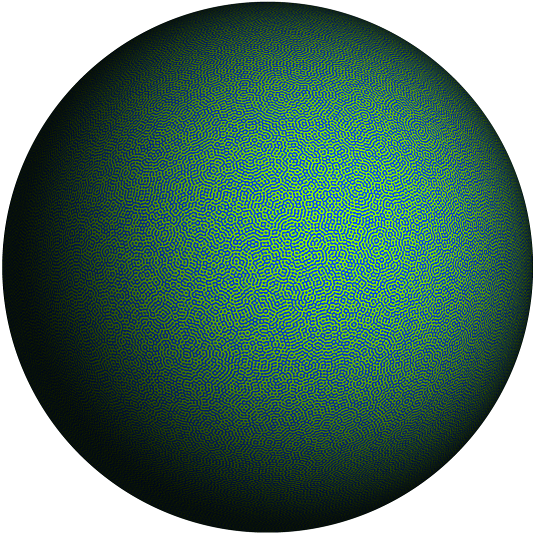


Plane waves on $\mathbb{R}^2$

Consider solutions of the equation

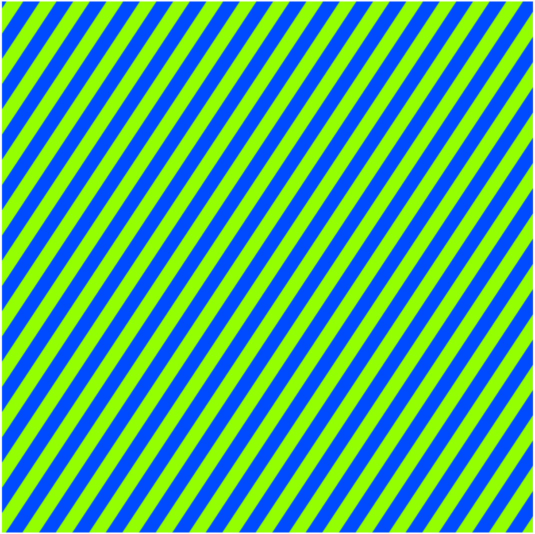
$$\Delta f + \lambda f = 0$$

on the plane. Particular solutions are given by

$$f_{\alpha,\beta}(x, y) = \cos(\alpha x + \beta y + \varphi)$$

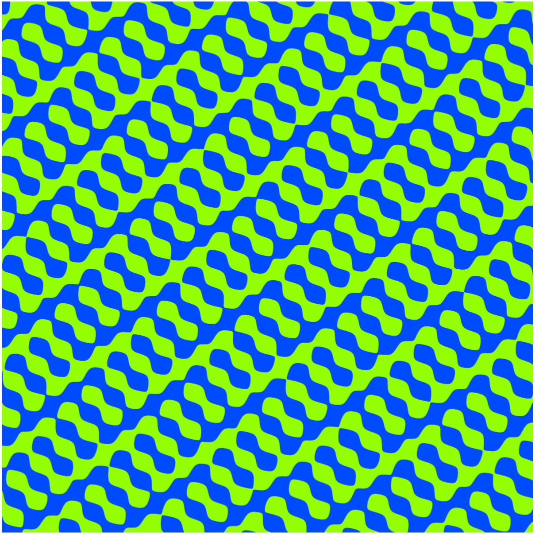
with $\alpha^2 + \beta^2 = \lambda$. By linearity, one can consider linear combinations of the $f_{\alpha,\beta}$.

Plane waves : one component

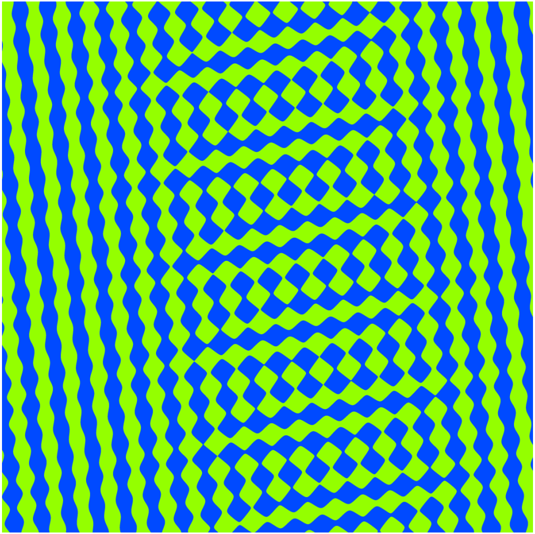


Plane waves : two components

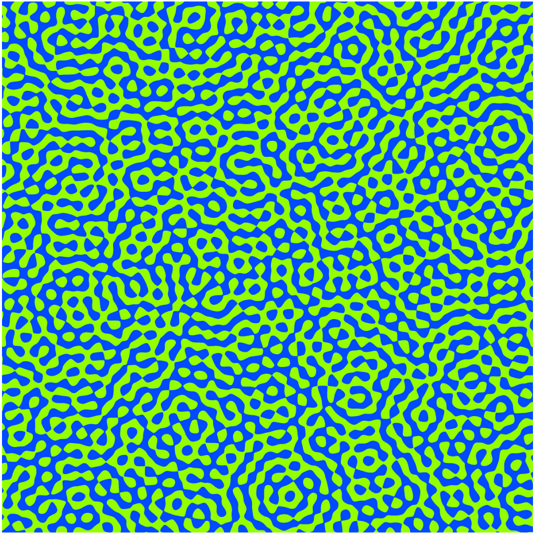
Plane waves : three components



Plane waves : four components



Infinitely many components / local limit on the sphere



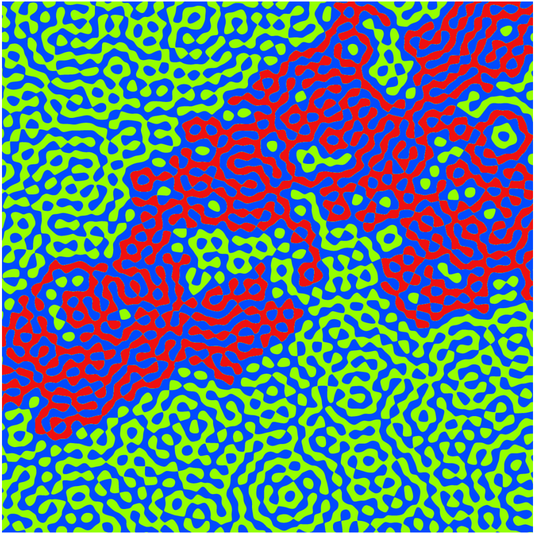
The limit as a Gaussian field

The local limit of random eigenfunctions of Δ as $\lambda \to \infty$ is given by a Gaussian field ϕ of covariance

$$Cov[\phi(x), \phi(y)] = J_0(\|y - x\|)$$

The covariance oscillates, and decays as $1/\sqrt{\|y - x\|}$.

One large connected component



Random polynomials / Kostlan ensemble

Random polynomial

Define a random homogeneous polynomial on $\mathbb{R}^3$ by

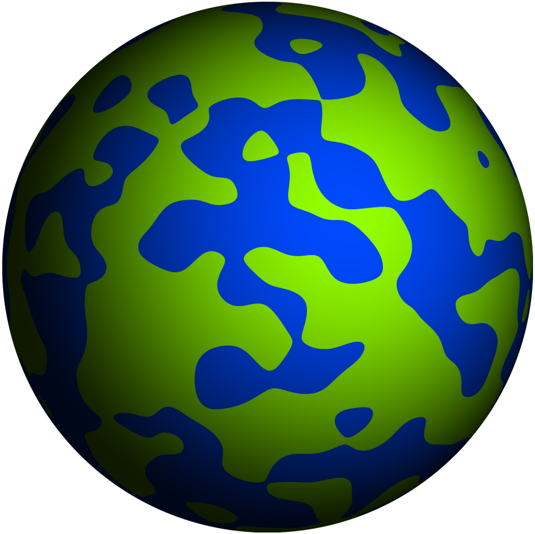
$$P_d(X) = \sum_{|I|=d} a_I \sqrt{\frac{(d+2)!}{I!}} X^I$$

where the a_I are i.i.d. Gaussians.

Restrict it to the unit sphere.

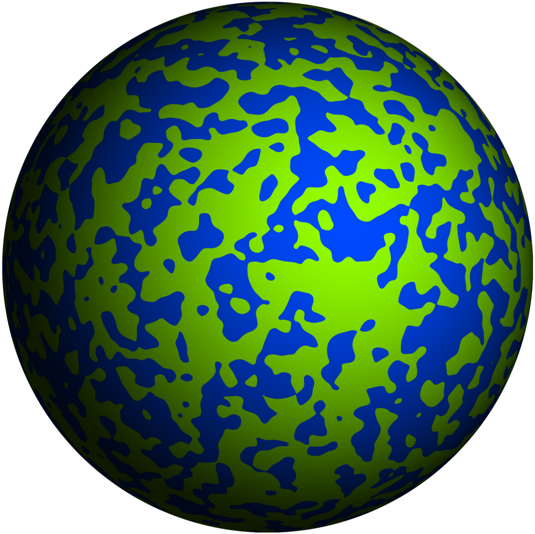
Restriction to the sphere (d=30)

Restriction to the sphere (d=100)

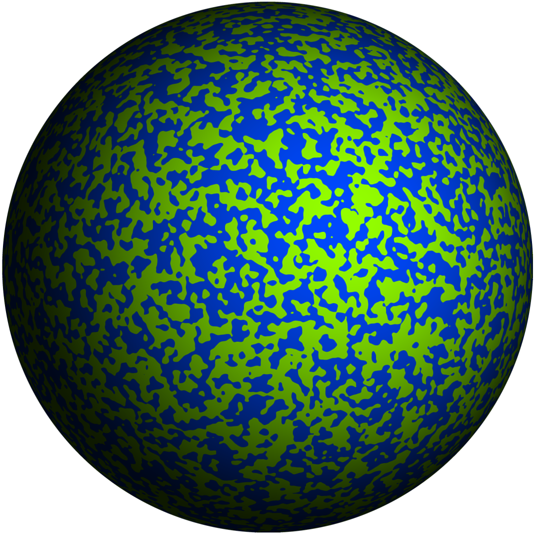


Restriction to the sphere (d=200)

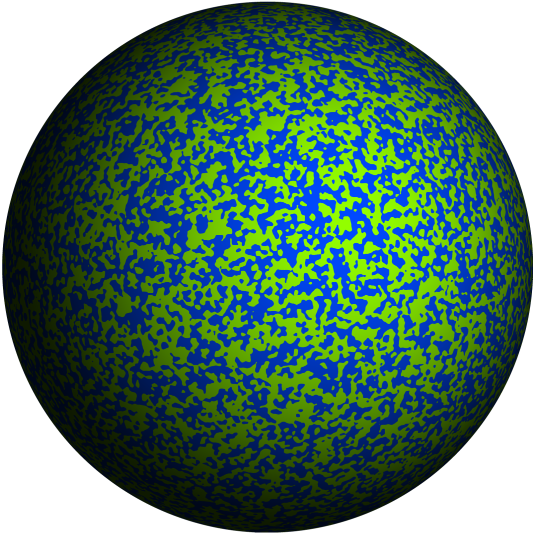
Restriction to the sphere (d=1000)



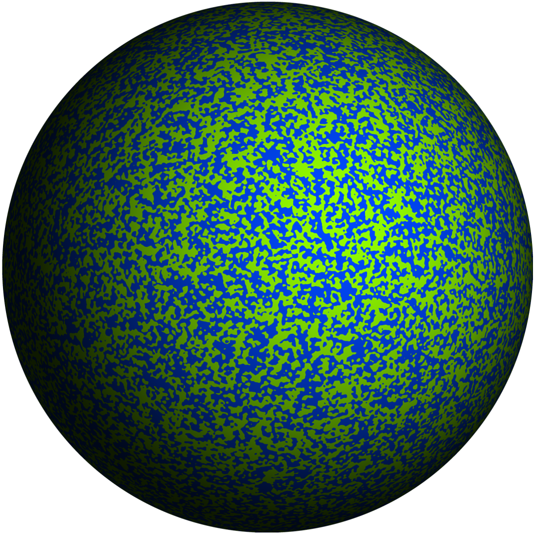
Restriction to the sphere (d=5000)



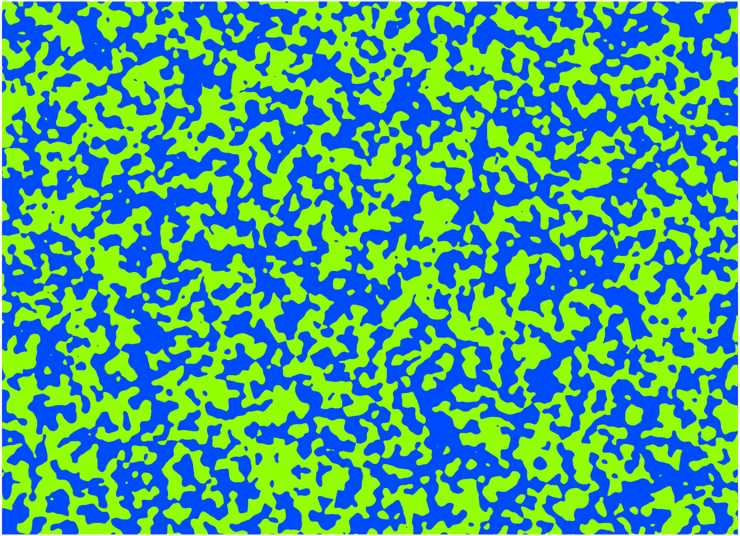
Restriction to the sphere (d=10000)



Restriction to the sphere (d=20000)



Local limit as $d \to \infty$



The limit as a Gaussian field

$$Q_d(x,y) = \sum_{i+j\leqslant d} a_{ij}\sqrt{\frac{(d+2)!}{i!j!(d-i-j)!}}x^i y^j$$

Rescale by a factor $\sqrt{d}$:

$$Q_d(x/\sqrt{d}, y/\sqrt{d}) \simeq \sum_{i+j\leqslant d} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

In the limit $d \to \infty$:

$$\psi(x,y) = \sum_{i,j\geqslant 0} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

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In the limit $d \to \infty$:

$$\psi(x,y) = e^{-(x^2+y^2)/2}\sum_{i,j\geqslant 0} \frac{a_{ij}}{\sqrt{i!j!}}x^i y^j$$

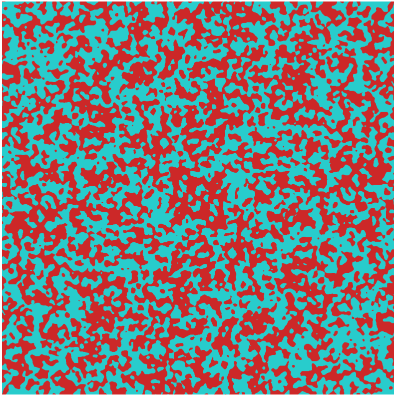
The limit as a Gaussian field

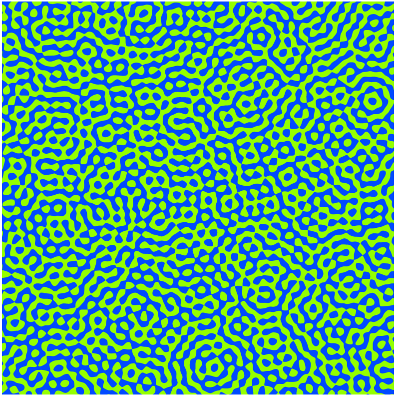
The limit is a stationary centered Gaussian field ψ on $\mathbb{R}^2$, with covariance given by

$$Cov[\psi(x), \psi(y)] = \exp(-\|y - x\|^2/2).$$

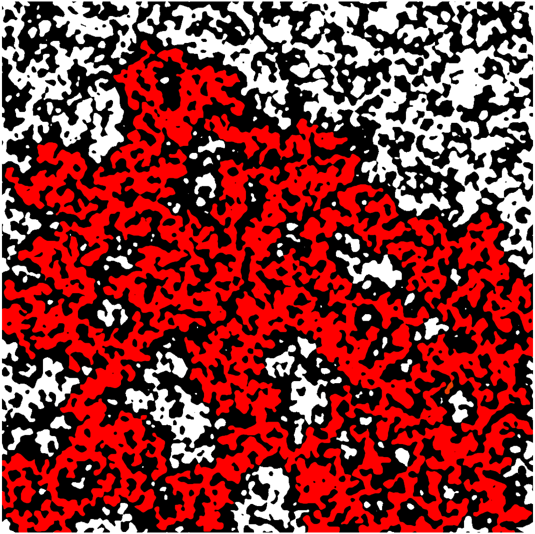
In particular, the covariance is positive and decays very fast.

Comparison between the two models

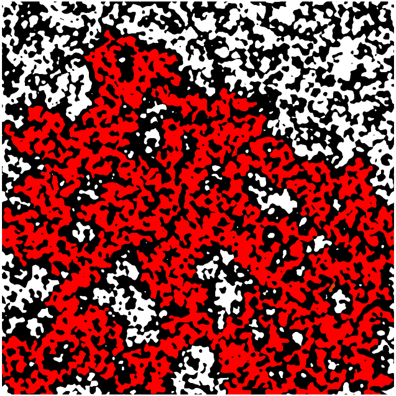


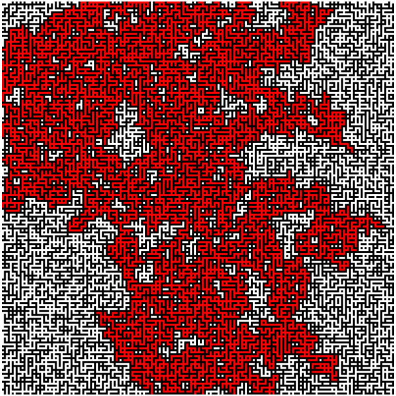


A large connected component in ψ



The same, and a critical percolation cluster





Percolation

Percolation : classical results

- Kesten (1980) : $p_c = 1/2$
- For $p < p_c$, sub-critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \exp(-\lambda_p \|x\|)$
  - Largest cluster in Λ_n has diameter $\approx \log n$
- For $p > p_c$, super-critical regime :
  - There exists a.s. a unique infinite cluster
  - $P[0 \longleftrightarrow x, |C(x)| < \infty] \approx \exp(-\lambda_p \|x\|)$
  - Largest *finite* cluster in Λ_n has diameter $\approx \log n$
- At $p = p_c$, critical regime :
  - All clusters are a.s. finite
  - $P[0 \longleftrightarrow x] \approx \|x\|^{-5/24}$
  - Largest cluster in Λ_n has diameter $\approx n$

Russo-Seymour-Welsh

Russo-Seymour-Welsh for critical percolation

Theorem (RSW)

For every $\lambda > 0$ there exists $c \in (0,1)$ such that for all n large enough,

$$c \leqslant P_{p_c}[LR(\lambda n, n)] \leqslant 1 - c.$$

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$$c \leqslant P_{p_c}[LR(\lambda n, n)] \leqslant 1 - c.$$

The case $\lambda = 1$ is easy by duality; it is enough to know how the estimate for one value of $\lambda > 1$ and then to glue the pieces.

Russo-Seymour-Welsh : proof ($\lambda = 3/2$)

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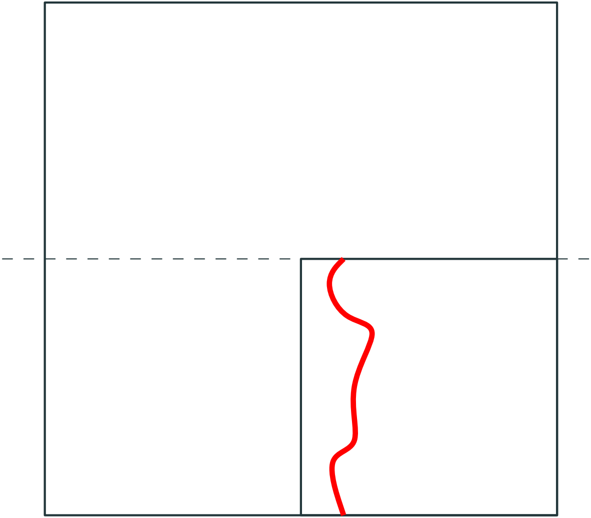

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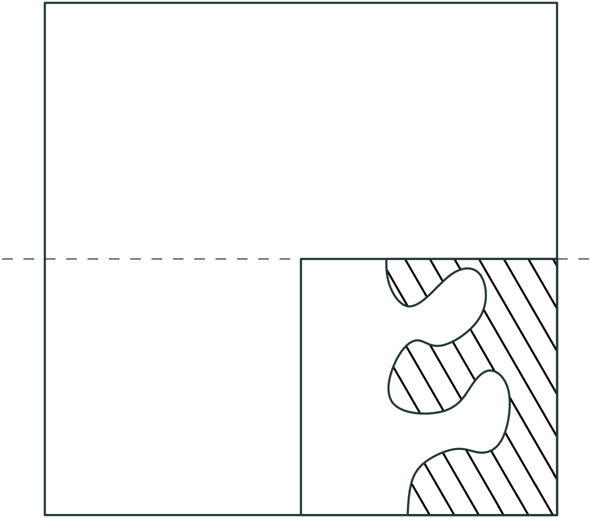

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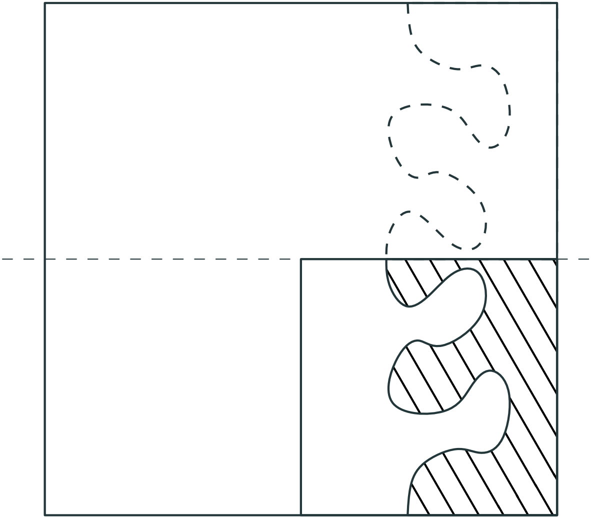

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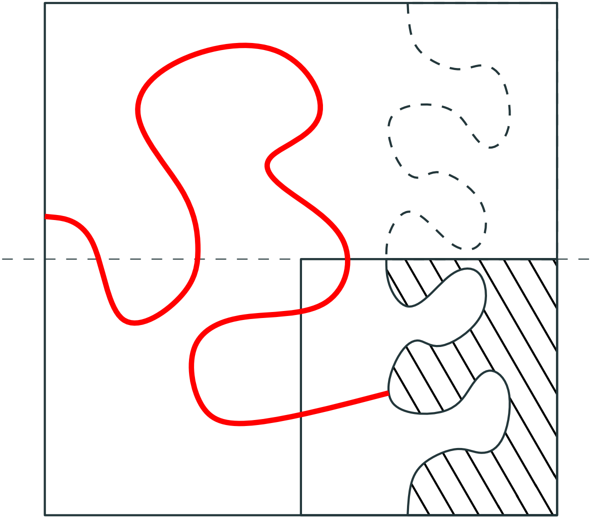

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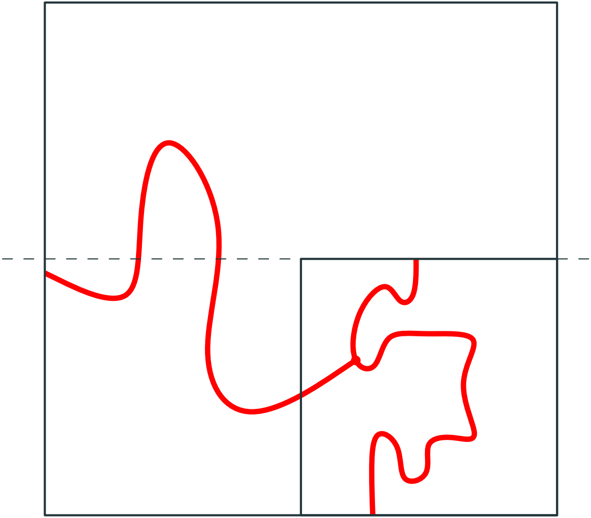

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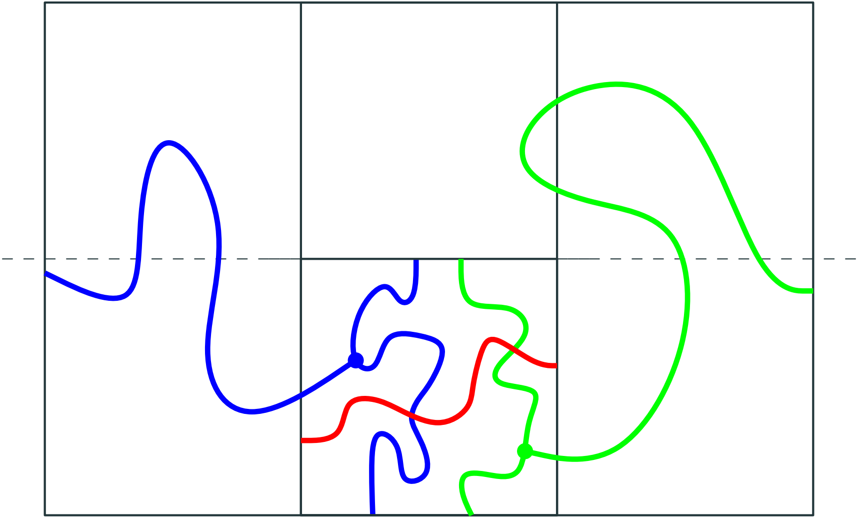

Russo-Seymour-Welsh for the field ψ

Main tools used were decorrelation and the FKG inequality.

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A few consequences:

- The set $\{z : \psi(z) > 0\}$ has no unbounded component
- Neither do $\{z : \psi(z) < 0\}$ and $\{z : \psi(z) = 0\}$
- The universal critical exponents are the same as for percolation
- $\psi = 0$ is the critical level [Rivera-Vanneuville]

A few words about the proof

The main obstacle is the analyticity of the field ψ, which goes against independence of its behavior in distant regions.

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To go around it, we discretize the field on the vertices of a triangular lattice with a small mesh δ, and look only at its sign on it, to get a dependent, discrete percolation model. The choice of δ is crucial:

- If δ is too large, the discretization does not catch all the topology;

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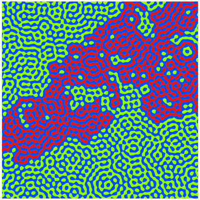
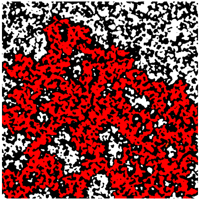
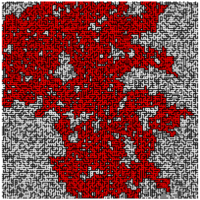
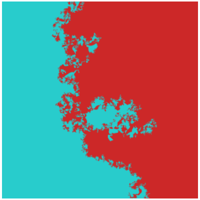
- If δ is too small, we lose in the decorrelation.

The case of the Laplacian eigenfunctions is bad on all respects: too slow decorrelation, no FKG inequality.

The Bogomolny-Schmidt conjecture

Conjecture

The nodal lines of ϕ (and ψ) converge, in the scaling limit, to the same conformally invariant object as interfaces of critical percolation; in particular, asymptotic crossing probabilities are given by Cardy's formula.

The proof of the RSW theorem for Bargman-Fock (BG improved by Belyaev-Muirhead and Rivera-Vanneuville)

Definitions and setup

- $\forall x_1, \cdots x_N \in^2$, any linear combination of the $(f(x_i))_{i=1,\cdots,N}$ is a Gaussian variable.
- Will always assume that f is centered and of variance 1.
- Characterized by $e(x,y) := E[f(x)f(y)] = k(\|x-y\|)$ with k symmetric and $k(0) = 1$.
- Big assumption: Almost surely, f is C^2. This is true if e is C^3.
- $Z_f = \{(x,y) : f(x,y) = 0\}$, $D_f = \{(x,y) : f(x,y) \geq 0\}$

ψ is the Bargman-Fock field with covariance $e^{-\|x-y\|^2/2}$.

Russo-Seymour-Welsh for the field ψ

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Natural idea: common features with Bernoulli percolation

- ☐ Symmetries
- ☐ Uniform crossing of squares
- ☐ (Asymptotic) independence
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Tassion's RSW theorem (2016)

If $f : \mathbb{R}^2 \to \{\pm 1\}$ satisfies these conditions, then it satisfies RSW.

Checking the assumptions for Bargmann-Fock

- ☒ Symmetries ok
- ☒ Uniform crossing of squares by duality
- ☐ (Asymptotic) independence almost?
- ☒ Positive correlation of positive crossings (FKG) Pitt

Discretization scheme

Theorem (Kac-Rice formula)

Let f be a Gaussian field on an interval $I \subset \mathbb{R}$, such that almost surely, f is C^1 and that for any $x \neq y \in I$, cov$(f(x), f(y))$ is definite. Then $E[N_I(N_I - 1)]$ is equal to

$$\int_{I^2} E\left[|f'(x)||f'(y)| \,\middle|\, f(x) = f(y) = 0\right] \phi_{(f(x),f(y))}(0,0)dxdy$$

where $\phi_X(u)$ is the Gaussian density of $X \in \mathbb{R}^2$ at $u \in \mathbb{R}^2$.

Corollary

If f is C^2 and $k'(0) \neq 0$, then

$$\mathbb{E}(N_I(N_I - 1)) \leq O(|I|^3).$$

Main step: discretization of the model

Discretize the sign of ψ on a Union Jack triangulation $\delta\mathcal{T}$ with mesh $\delta > 0$ (to be fixed later). If the field is smooth and if δ is small, we catch all the topology of ψ on the discretization:

Theorem (BG 2016)

There exists $C > 0$ such that for any $n > 1$, letting $\delta_n = n^{-3}$,

$$P\left[\forall R \subset B_n,\ f \text{ crosses } R \text{ iff } f_{\delta_n} \text{ crosses } R\right] \geq 1 - \frac{C}{n}.$$

Topological fact: Since $\mathcal{T}$ is a triangulation, it is enough to prove that $\{f = 0\}$ cuts all edges at most once.

The Kac-Rice first-moment formula

Theorem

$$\mathbb{E}[N_I] = \int_I \mathbb{E}\big(|f'(x)| \mid f(x) = 0\big)\phi_{f(x)}(0)dx.$$

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$$\mathbb{E}[N_I] = \int_I \mathbb{E}\big(|f'(x)| \mid f(x) = 0\big)\phi_{f(x)}(0)dx.$$

Proof:

- If f vanishes transversally on I,

$$N_I = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_I |f'(x)| \mathbf{1}_{|f| \leq \epsilon} dx,$$

- and this implies that

$$\mathbb{E}N_I = \int_I \mathbb{E}\Big(|f'(x)| \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathbf{1}_{|f| \leq \epsilon}\Big) dx.$$

Proof of the discretization theorem

By the topological fact, is enough to prove that with high probability,

$$\forall e \in \frac{1}{n^3}\mathcal{E} \cap B_n,\ N_e \leq 1.$$

By the Markov inequality and Kac-Rice,

$$\mathbb{P}\big[N_e > 1\big] = \mathbb{P}\big[N_e(N_e - 1) \geq 1\big] \leq C|e|^3.$$

Hence,

$$\begin{aligned}\mathbb{P}\Big[\forall e \in \frac{1}{n^3}\mathcal{E} \cap B_n,\ N_e \leq 1\Big] &\geq 1 - \#\{e \in \frac{1}{n^3}\mathcal{E} \cap B_n\}(C|e|^3) \\ &\geq 1 - Cn^2 n^6 \frac{1}{n^9} \to 1.\end{aligned}$$

Proof of the Corollary

By Kac-Rice, $E[N(N-1)]$ is equal to

$$\int_{I^2} \mathbb{E}\big[|f'(x)||f'(y)| \bigm| f(x)=f(y)=0\big]\phi_{(f(x),f(y))}(0,0)dxdy.$$

When $|I| \to 0$,

- $\int_{I^2} dxdy \sim |I|^2$;
- $f(x)=f(y)$ implies $|f'(x)||f'(y)| \leq |I|^2$;
- $\phi_{(f(x),f(y))}(0,0) \sim |I|^{-1}$ since $(f(x),f(y))$ degenerates.

This gives the $|I|^3$.

Back to the proof

Tassion's condition: dependence$\big(A(n, 2n), A(3n, n \log n)\big) \to_{n\to\infty} 0$.

If we discretize at mesh $(n \log n)^{-3}$ to apply the discretization scheme, then we get of the order of $n^8 \log^4 n$ points in the approximation. The covariance kernel across the annulus $A(2n, 3n)$ is tiny, but we need a quantitative bound.

A decorrelation inequality

Theorem

Let X and Y be two Gaussian vectors in $\mathbb{R}^{m+n}$, of covariances

$$\Sigma_X = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix} \quad \text{and} \quad \Sigma_Y = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where $\Sigma_1 \in M_m(\mathbb{R})$ and $\Sigma_2 \in M_n(\mathbb{R})$ have all diagonal entries equal to 1. Denote by μ_X (resp. μ_Y) the law of the signs of the coordinates of X (resp. Y), and by η the largest absolute value of the entries of Σ_{12}. Then,

$$d_{TV}(\mu_X, \mu_Y) \leqslant C(m+n)^{8/5}\eta^{1/5}.$$

Another decorrelation inequality

Theorem

Let $X = (x_i)$ be a centered Gaussian vector in $\mathbb{R}^n$ with covariance matrix $A = (a_{ij})_{1 \le i,j \le n}$ satisfying $\forall 1 \le i \le n$, $a_{ii} = 1$, and let $\delta \in (0, 1/n)$. Then, the shifted truncation

$$B = (b_{ij}) \quad \textit{where} \quad b_{ij} := a_{ij} 1_{|a_{ij}| > \delta} + (n\delta)^{3/5} 1_{i=j}$$

is a positive matrix, and there exists a coupling of X with another centered Gaussian vector $Y = (y_i)$ with covariance matrix B such that

$$P\left[\forall 1 \le i \le n, \quad x_i y_i > 0\right] \geqslant 1 - 3n^{6/5}\delta^{1/5}.$$

Corollary: coupling with a finitely correlated field.

A sharper inequality

Theorem (Piterbarg 1982)

Let $f : \mathcal{V} \to \mathbb{R}$ be a centered centered symmetric Gaussian over a finite set. Then, there exists $C > 0$, such that for any R, S two disjoint open sets in $\mathcal{V}$,

$$\text{dependence}(R, S) := \max_{\substack{A \text{ in } R \\ B \text{ in } S}} |\mathbb{P}(A \text{ and } B) - \mathbb{P}(A)\mathbb{P}(B)|$$

$$\leq$$

$$C|R \cup S|^2 \max_{\substack{x \in R \\ y \in S}} \frac{|e(x, y)|}{\sqrt{1 - e(x, y)^2}}.$$

The Plackett-Piterbarg method (Biometrika 1954)

Let

$$U := \big(f(x)\big)_{x \in R} \qquad V := \big(f(y)\big)_{y \in S}$$
$$X_1 := (U, V) \qquad X_0 := (U, V)_{ind}$$

where

- X_0 and X_1 are independent, and
- for X_0, V is an independent copy of f.

We want a bound for

$$\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B] = \mathbb{E}_{X_1}(\mathbf{1}_{A \cap B}) - \mathbb{E}_{X_0}(\mathbf{1}_{A \cap B}).$$

The Plackett-Piterbarg method

Interpolate $X_t := \sqrt{t}X_1 + \sqrt{1-t}X_0$. Then X_t has covariance

$$\Sigma_t = \begin{pmatrix} (U,U) & t\ \mathrm{cov}(U,V) \\ t\ \mathrm{cov}(U,V)^T & \mathrm{cov}(V,V) \end{pmatrix}$$

with

$$\mathrm{cov}\ (U,V) = (e(x,y))_{x\in R, y\in S}.$$

Then we can rewrite

$$\begin{aligned} \mathbb{E}_{X_1}(\mathbf{1}_{A\cap B}) - \mathbb{E}_{X_0}(\mathbf{1}_{A\cap B}) &= \int_0^1 \frac{d}{dt}\mathbb{E}_{X_t}(\mathbf{1}_{A\cap B})dt \\ &= \int_0^1 dt \int_{(u,v)\in A\times B} \frac{d\phi_{X_t}}{dt}(u,v)d(u,v) \\ &= \sum_{i\le j}\int_0^1 dt \int_{A\times B} \frac{d\sigma_{t,ij}}{dt}\frac{\partial \phi_{X_t}}{\partial \sigma_{t,ij}} d(u,v) \end{aligned}$$

The Plackett-Piterbarg method

$$\mathbb{E}_{X_1}(\mathbf{1}_{A\cap B}) - \mathbb{E}_{X_0}(\mathbf{1}_{A\cap B}) = \sum_{i\leq j} \int_0^1 dt \int_{A\times B} \frac{d\sigma_{t,ij}}{dt} \frac{\partial \phi_{X_t}}{\partial \sigma_{t,ij}} d(u,v)$$

with $\frac{d\sigma_{t,ij}}{dt} = e(x,y)$ if $i = x \in R$ and $j = y \in S$ and 0 otherwise.

Lemma (A Gaussian equality)

$$\forall i \neq j,\ \frac{\partial \phi_X}{\partial \sigma_{ij}} = \frac{\partial^2 \phi_X}{\partial u_i \partial u_j}.$$

Proof: Use $\phi_X(u) = \int_{\xi\in\mathbb{R}^N} e^{i\langle u,\xi\rangle} e^{-\frac{1}{2}\langle \Sigma\xi,\xi\rangle} \frac{d\xi}{\sqrt{2\pi}^N}$.

Then

$$\mathbb{P}[A\cap B] - \mathbb{P}[A]\mathbb{P}[B] = \sum_{\substack{x\in R\\ y\in S}} e(x,y) \int_0^1 dt \int_{A\times B} \frac{\partial^2 \phi_{X_t}}{\partial u_x \partial v_y} d(u,v).$$

The Plackett-Piterbarg method

$$\mathbb{P}[A \cap B] - \mathbb{P}[A]\mathbb{P}[B] = \sum_{\substack{x \in R \\ y \in S}} e(x,y) \int_0^1 dt \int_{A \times B} \frac{\partial^2 \phi_{X_t}}{\partial u_x \partial v_y} d(u,v).$$

Recall that A depends only on the signs of $f(x) = u_x$, and B on the signs of $f(y) = v_y$. Integrating par parts gives the bound

$$(\#R)(\#S) \max_{\substack{x \in R \\ y \in S}} \frac{|e(x,y)|}{\sqrt{1 - e(x,y)^2}}.$$

Quasi-independence

Let $\mathcal{D}_p = \{(x, y) : \psi(x, y) \geq -p\}$, and fix two families $(\mathcal{E}_i)_{i \leq k}$ and $(\mathcal{E}'_i)_{i \leq k'}$ of rectangles or annuli.

Theorem (Rivera-Vanneuville)

Uniformly for A (resp. B) defined in terms of crossings of the $\mathcal{E}_i$ (resp. $\mathcal{E}'_i$) by $\mathcal{D}_p$, $|P[A \cap B] - P[A]P[B]|$ is bounded above by

$$\frac{C(p)\eta}{\sqrt{1-\eta^2}}\Big(k + \sum |E_i| + \sum |\partial E_i|\Big)\Big(k' + \sum |E'_i| + \sum |\partial E'_i|\Big)$$

where $\eta := \sup\{e(x, y) : x \in \bigcup \mathcal{E}_i, y \in \bigcup \mathcal{E}'_i\}$.

Consequence: this is enough to obtain RSW estimates for a positively correlated field with covariance smaller than d^{-4}.

Quasi-independence: sketch of proof

- Prove a general Quasi-independence of “threshold events” for Gaussian vectors
- Discretize the events (but the key point is that the bounds will be independent of the discretization)
- Control the probability that a point is pivotal is small enough (this involves a percolation type argument and the Kac-Rice formula)

Quasi-independence for Gaussian vectors

Let $\mathrm{Piv}_i(U) = \{x \in \mathbb{R}^n : \exists y, y' \in \mathbb{R} : x^{i \leftarrow y} \in U, x^{i \leftarrow y'} \in U^c\}$.

Theorem

Let X and Y be two Gaussian vectors in $\mathbb{R}^{k+k'}$, of covariances $\Sigma_X = \begin{bmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_2 \end{bmatrix}$ *and* $\Sigma_Y = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$*, where Σ_1 and Σ_2 have all diagonal entries equal to 1. Let $q \in \mathbb{R}^{k+k'}$ and U and V be in the σ-field of the $\{x_i \geq q_i\}$ for $i \leq k$ (resp. $i > k$). Then* $|P[X \in U \cap V] - P[Y \in U \cap V]| \leq$

$$\sum_{i \leq k, j > k} \frac{\Sigma_{ij} e^{-(q_i^2 + q_j^2)/2}}{2\pi\sqrt{1 - \Sigma_{ij}}} \times$$

$$\int_0^1 P[X_t \in Piv_i(U) \cap Piv_j(V) | X_t(i) = q_i, X_t(j) = q_j].$$

Quasi-independence for the discretized field

Discretize at scale $\delta > 0$:

Theorem (Rivera-Vanneuville)

Uniformly for A^δ (resp. B^δ) defined in terms of crossings of the $\mathcal{E}_i$ (resp. $\mathcal{E}'_i$) by $\mathcal{D}_p^\delta$, $|P[A^\delta \cap B^\delta] - P[A^\delta]P[B^\delta]|$ is bounded above by

$$\frac{C(p)\eta}{\sqrt{1-\eta^2}}\left(k + \sum |E_i| + \sum |\partial E_i|\right)\left(k' + \sum |E'_i| + \sum |\partial E'_i|\right)$$

where $\eta := \sup\{e(x,y) : x \in \bigcup \mathcal{E}_i, y \in \bigcup \mathcal{E}'_i\}$.

Main idea of the proof: A vertex is pivotal with small probability; conditionally on the value of the field there, small means ε^2.

The critical threshold for Bargmann-Fock percolation (following Rivera-Vanneuville)

The setup and the statement

- Recall that ψ is the Bargmann-Fock Gaussian field in the plane, with covariance function $\exp(-\|x-y\|^2/2)$.
- We are interested in the level sets

$$\mathcal{D}_p = \{(x,y) : \psi(x,y) \geq -p\}.$$

- Easy to see that $\theta(p) := P_p(0 \longleftrightarrow \infty)$ is non-decreasing.

Theorem (Rivera-Vanneuville, 2019)

The critical level is equal to 0. More precisely,

- *If $p \leq 0$, then $\mathcal{D}_p$ a.s. has no unbounded component, while*
- *If $p \geq 0$, then $\mathcal{D}_p$ a.s. has a unique unbounded component.*

Moreover, exponential decay away from $p = 0$.

Warm-up: Bernoulli percolation

Theorem (Kesten)

For Bernoulli site percolation on $\mathcal{T}$, $p_c = 1/2$.

Sketch of the proof (classical style):

- At $p = 1/2$ we have the box-crossing property
- Whenever the BXP holds, get many pivotal points
- Sharp threshold for large boxes obtained by Russo's formula:

$$\partial_p P_p[A] = \sum P_p[\mathrm{Piv}_i(A)]$$

- Glue larger and larger rectangles to build an infinite cluster

Kahn-Kalai-Linial theorem

This is a more manageable tool to obtain a sharp threshold: rather than proving that each point is pivotal with large probability, show that the largest influence is small: for a product of Bernoulli variables,

$$\sum P_p[\mathrm{Piv}_i(A)] \geq c P_p[A] P_p[A^c] \, \log \frac{1}{\max P_p[\mathrm{Piv}_i(A)]}.$$

Easier to show that the probability that a vertex is pivotal is small.

Phase transition for Bargmann-Fock: overall strategy

- Discretize the model in the box $2R \times R$ at mesh $\delta_R > 0$, and show that $P_p^\delta[LR(2R, R)]$ is close to 1 when R is large. This turns out to work well if

$$\delta_R \geq (\log R)^{-1/2+\epsilon}.$$

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- Those are incompatible! Instead, sprinkling to obtain that $P[LR_p(2R,R)|LR^\delta_{p/2}(2R,R)] \simeq 1$. This works if

$$\delta_R \leq (\log R)^{-1/4-\epsilon}.$$

Sharp threshold for dependent Gaussian vectors

Define the geometric influence of a vector $v \in \mathbb{R}^n$ ona Borel set A under a measure μ as

$$I_{v,\mu}(A) := \liminf \frac{\mu(A + [-r, r]v) - \mu(A)}{2r}.$$

Theorem (Rivera-Vanneuville 2017)

For every increasing event $A \subset \mathbb{R}^n$,

$$\sum_{i=1}^{n} I_{i,\mu}(A) \geq c \left\| \sqrt{\Sigma} \right\|^{-1} \mu(A)\mu(A^c) \times \sqrt{\log_+ \frac{1}{\left\| \sqrt{\Sigma} \right\| \max I_{i,\mu}(A)}}.$$

Smaller δ makes $\left\| \sqrt{\Sigma} \right\|$ larger, hence lower bound on δ.

Bounding the terms in the KKL estimate

Upper bound on the influences

$$P\Big[\mathrm{Piv}_x(LR_p^{\epsilon}(2R,R))\;\Big|\;\psi(x)=-p\Big]\leq CR^{-\eta}$$

Upper bound on the operator norm

$$\left\|\sqrt{\Sigma}\right\|\leq C\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$$

Sprinkling to relate discrete and continuous

In the first step, we had to choose δ not too small, so the discrete and continuous crossing events are too independent.

Theorem (Rivera-Vanneuville 2017)

For small enough δ, and for every $R > 1$,

$$P\left[LR^{\delta}_{p/2}(2R, R) \ \setminus \ LR_p(2R, R)\right] \leq CR^2\delta^{-2}\exp(-c\delta^{-4}).$$

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This is based on the following estimate. If $e = (x,y)$ is an edge of length δ, define

$$\text{Fold}(e) = \{\psi(x) > -p/2, \psi(y) > -p/2, \inf_e \psi < -p\}.$$

Lemma

$$P[Fold(e)] \leq C\exp(-c\delta^{-4}).$$

A few open problems

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- Bogomolny-Schmidt conjecture: do we have convergence to SLE in the scaling limit for Bargmann-Fock?
- How much can one weaken the tail decay condition? For slow enough decay, can one obtain another scaling limit?
- Is it possible to handle negatively correlated fields?

A few open problems: 3d

- Dynamical version: exceptional times and so on

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- Dynamical version: exceptional times and so on
- Is it the case that $h_c > 0$?

That’s all Folks!